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Differentiable Functions on Normed Linear Spaces[1]

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Summary. In this article, we formalize differentiability of functions on normed linear spaces. Partial derivative, mean value theorem for vector-valued functions, continuous differentiability, etc. are formalized. As it is well known, there is no exact analog of the mean value theorem for vector-valued functions. However a certain type of generalization of the mean value theorem for vector-valued functions is obtained as follows: If $||f'(x+t\cdot h)||$ is bounded for t between 0 and 1 by some constant M, then $||f(x+t\cdot h)-f(x)|| \leq M\cdot||h||$. This theorem is called the mean value theorem for vector-valued functions. By this theorem, the relation between the (total) derivative and the partial derivatives of a function is derived [23].

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The notation and terminology used here have been introduced in the following papers: [28], [29], [9], [4], [30], [12], [10], [25], [11], [1], [2], [26], [7], [3], [5], [8], [17], [22], [20], [27], [21], [31], [14], [24], [18], [16], [15], [19], [13], and [6].

1. Preliminaries

In this paper r is a real number and S, T are non trivial real normed spaces.

Next we state several propositions:

(1) Let R be a function from $\mathbb{R}$ into S. Then R is rest-like if and only if for every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every real number z such that $z \neq 0$ and $|z| < d$ holds $|z|^{-1} \cdot \|R_z\| < r$.

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(2) Let R be a rest of S. Suppose $R_0 = 0_S$. Let e be a real number. Suppose $e > 0$. Then there exists a real number d such that $d > 0$ and for every real number h such that $|h| < d$ holds $\|R_h\| \leq e \cdot |h|$.

(3) For every rest R of S and for every bounded linear operator L from S into T holds $L \cdot R$ is a rest of T.

(4) Let R_1 be a rest of S. Suppose $(R_1)_0 = 0_S$. Let R_2 be a rest of S, T. If $(R_2)_{0_S} = 0_T$, then for every linear L of S holds $R_2 \cdot (L + R_1)$ is a rest of T.

(5) Let R_1 be a rest of S. Suppose $(R_1)_0 = 0_S$. Let R_2 be a rest of S, T. Suppose $(R_2)_{0_S} = 0_T$. Let L_1 be a linear of S and L_2 be a bounded linear operator from S into T. Then $L_2 \cdot R_1 + R_2 \cdot (L_1 + R_1)$ is a rest of T.

(6) Let x_0 be an element of $\mathbb{R}$ and g be a partial function from $\mathbb{R}$ to the carrier of S. Suppose g is differentiable in x_0. Let f be a partial function from the carrier of S to the carrier of T. Suppose f is differentiable in g_{x_0}. Then $f \cdot g$ is differentiable in x_0 and $(f \cdot g)'(x_0) = f'(g_{x_0})(g'(x_0))$.

(7) Let S be a real normed space, x_1 be a finite sequence of elements of S, and y_1 be a finite sequence of elements of $\mathbb{R}$. Suppose $\operatorname{len} x_1 = \operatorname{len} y_1$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} x_1$ holds $y_1(i) = \|(x_1)_i\|$. Then $\|\sum x_1\| \leq \sum y_1$.

(8) Let S be a real normed space, x be a point of S, and N_1, N_2 be neighbourhoods of x. Then $N_1 \cap N_2$ is a neighbourhood of x.

(9) For every non-empty finite sequence X and for every set x such that $x \in \prod X$ holds x is a finite sequence.

Let G be a real norm space sequence. One can verify that $\prod G$ is constituted finite sequences.

Let G be a real linear space sequence, let z be an element of $\prod \overline{G}$, and let j be an element of $\operatorname{dom} G$. Then $z(j)$ is an element of $G(j)$.

One can prove the following propositions:

(10) The carrier of $\prod G = \prod \overline{G}$.

(11) Let i be an element of $\operatorname{dom} G$, r be a set, and x be a function. If $r \in$ the carrier of $G(i)$ and $x \in \prod \overline{G}$, then $x +\cdot (i, r) \in$ the carrier of $\prod G$.

Let G be a real norm space sequence. We say that G is nontrivial if and only if:

(Def. 1) For every element j of $\operatorname{dom} G$ holds $G(j)$ is non trivial.

Let us mention that there exists a real norm space sequence which is nontrivial.

Let G be a nontrivial real norm space sequence and let i be an element of $\operatorname{dom} G$. Note that $G(i)$ is non trivial.

Let G be a nontrivial real norm space sequence. Note that $\prod G$ is non trivial.

The following propositions are true:

(12) Let G be a real norm space sequence, p, q be points of $\prod G$, and r_0, p_0, q_0 be elements of $\prod \overline{G}$. Suppose $p = p_0$ and $q = q_0$. Then $p+q = r_0$ if and only if for every element i of $\operatorname{dom} G$ holds $r_0(i) = p_0(i) + q_0(i)$.

(13) Let G be a real norm space sequence, p be a point of $\prod G$, r be a real number, and r_0, p_0 be elements of $\prod \overline{G}$. Suppose $p = p_0$. Then $r \cdot p = r_0$ if and only if for every element i of $\operatorname{dom} G$ holds $r_0(i) = r \cdot p_0(i)$.

(14) Let G be a real norm space sequence and p_0 be an element of $\prod \overline{G}$. Then $0_{\prod G} = p_0$ if and only if for every element i of $\operatorname{dom} G$ holds $p_0(i) = 0_{G(i)}$.

(15) Let G be a real norm space sequence, p, q be points of $\prod G$, and r_0, p_0, q_0 be elements of $\prod \overline{G}$. Suppose $p = p_0$ and $q = q_0$. Then $p-q = r_0$ if and only if for every element i of $\operatorname{dom} G$ holds $r_0(i) = p_0(i) - q_0(i)$.

2. Mean Value Theorem for Vector-Valued Functions

Let S be a real linear space and let p, q be points of S. The functor $]p,q[$ yielding a subset of S is defined as follows:

(Def. 2) $]p,q[= \{p + t\cdot(q-p); t$ ranges over real numbers: $0 < t \ \wedge\ t < 1\}$.

Let S be a real linear space and let p, q be points of S. We introduce $[p,q]$ as a synonym of $\mathcal{L}(p,q)$.

Next we state several propositions:

(16) For every real linear space S and for all points p, q of S holds $]p,q[\subseteq [p,q]$.

(17) Let T be a non trivial real normed space and R be a partial function from $\mathbb{R}$ to T. Suppose R is total. Then R is rest-like if and only if for every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every real number z such that $z \neq 0$ and $|z| < d$ holds $\frac{\|R_z\|}{|z|} < r$.

(18) Let R be a function from $\mathbb{R}$ into $\mathbb{R}$. Then R is rest-like if and only if for every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every real number z such that $z \neq 0$ and $|z| < d$ holds $\frac{|R(z)|}{|z|} < r$.

(19) Let S, T be non trivial real normed spaces, f be a partial function from S to T, p, q be points of S, and M be a real number. Suppose that

(i) $[p,q] \subseteq \operatorname{dom} f$,

(ii) for every point x of S such that $x \in [p,q]$ holds f is continuous in x,

(iii) for every point x of S such that $x \in\ ]p,q[$ holds f is differentiable in x, and

(iv) for every point x of S such that $x \in\ ]p,q[$ holds $\|f'(x)\| \leq M$.

Then $\|f_q - f_p\| \leq M \cdot \|q-p\|$.

(20) Let S, T be non trivial real normed spaces, f be a partial function from S to T, p, q be points of S, M be a real number, and L be a point of the real norm space of bounded linear operators from S into T. Suppose that

(i) $[p,q] \subseteq \operatorname{dom} f$,

(ii) for every point x of S such that $x \in [p,q]$ holds f is continuous in x,

(iii) for every point x of S such that $x \in]p,q[$ holds f is differentiable in x, and

(iv) for every point x of S such that $x \in]p,q[$ holds $\|f'(x) - L\| \leq M$.

Then $\|f_q - f_p - L(q-p)\| \leq M \cdot \|q-p\|$.

3. Partial Derivative of a Function of Several Variables

Let G be a real norm space sequence and let i be an element of $\operatorname{dom} G$. The projection onto i yielding a function from $\prod G$ into $G(i)$ is defined by:

(Def. 3) For every element x of $\prod \overline{G}$ holds (the projection onto i)$(x) = x(i)$.

Let G be a real norm space sequence, let i be an element of $\operatorname{dom} G$, and let x be an element of $\prod G$. The functor $\operatorname{reproj}(i,x)$ yielding a function from $G(i)$ into $\prod G$ is defined by:

(Def. 4) For every element r of $G(i)$ holds $(\operatorname{reproj}(i,x))(r) = x +\cdot (i,r)$.

Let G be a nontrivial real norm space sequence and let j be a set. Let us assume that $j \in \operatorname{dom} G$. The functor $\operatorname{modetrans}(G,j)$ yields an element of $\operatorname{dom} G$ and is defined by:

(Def. 5) $\operatorname{modetrans}(G,j) = j$.

Let G be a nontrivial real norm space sequence, let F be a non trivial real normed space, let i be a set, let f be a partial function from $\prod G$ to F, and let x be an element of $\prod G$. We say that f is partially differentiable in x w.r.t. i if and only if:

(Def. 6) $f \cdot \operatorname{reproj}(\operatorname{modetrans}(G,i),x)$ is differentiable in (the projection onto $\operatorname{modetrans}(G,i))(x)$.

Let G be a nontrivial real norm space sequence, let F be a non trivial real normed space, let i be a set, let f be a partial function from $\prod G$ to F, and let x be a point of $\prod G$. The functor $\operatorname{partdiff}(f,x,i)$ yielding a point of the real norm space of bounded linear operators from $G(\operatorname{modetrans}(G,i))$ into F is defined as follows:

(Def. 7) $\operatorname{partdiff}(f,x,i) = (f \cdot \operatorname{reproj}(\operatorname{modetrans}(G,i),x))'$((the projection onto $\operatorname{modetrans}(G,i))(x))$.

4. Linearity of Partial Differential Operator

For simplicity, we adopt the following rules: G denotes a nontrivial real norm space sequence, F denotes a non trivial real normed space, i denotes an element of $\operatorname{dom} G$, f, f_1, f_2 denote partial functions from $\prod G$ to F, x denotes a point of $\prod G$, and X denotes a set.

Let G be a nontrivial real norm space sequence, let F be a non trivial real normed space, let i be a set, let f be a partial function from $\prod G$ to F, and let X be a set. We say that f is partially differentiable on X w.r.t. i if and only if:

(Def. 8) $X \subseteq \operatorname{dom} f$ and for every point x of $\prod G$ such that $x \in X$ holds $f{\restriction}X$ is partially differentiable in x w.r.t. i.

Next we state several propositions:

(21) For every element x_2 of $G(i)$ holds $\|(\mathrm{reproj}(i, 0_{\prod G}))(x_2)\| = \|x_2\|$.

(22) Let G be a nontrivial real norm space sequence, i be an element of $\operatorname{dom} G$, x be a point of $\prod G$, and r be a point of $G(i)$. Then $(\mathrm{reproj}(i, x))(r) - x = (\mathrm{reproj}(i, 0_{\prod G}))(r - (\text{the projection onto } i)(x))$ and $x - (\mathrm{reproj}(i, x))(r) = (\mathrm{reproj}(i, 0_{\prod G}))((\text{the projection onto } i)(x) - r)$.

(23) Let G be a nontrivial real norm space sequence, i be an element of $\operatorname{dom} G$, x be a point of $\prod G$, and Z be a subset of $\prod G$. Suppose Z is open and $x \in Z$. Then there exists a neighbourhood N of (the projection onto i)(x) such that for every point z of $G(i)$ if $z \in N$, then $(\mathrm{reproj}(i, x))(z) \in Z$.

(24) Let G be a nontrivial real norm space sequence, T be a non trivial real normed space, i be a set, f be a partial function from $\prod G$ to T, and Z be a subset of $\prod G$. Suppose Z is open. Then f is partially differentiable on Z w.r.t. i if and only if $Z \subseteq \operatorname{dom} f$ and for every point x of $\prod G$ such that $x \in Z$ holds f is partially differentiable in x w.r.t. i.

(25) For every set i such that $i \in \operatorname{dom} G$ and f is partially differentiable on X w.r.t. i holds X is a subset of $\prod G$.

Let G be a nontrivial real norm space sequence, let S be a non trivial real normed space, and let i be a set. Let us assume that $i \in \operatorname{dom} G$. Let f be a partial function from $\prod G$ to S and let X be a set. Let us assume that f is partially differentiable on X w.r.t. i. The functor $f{\restriction}^{i}X$ yields a partial function from $\prod G$ to the real norm space of bounded linear operators from $G(\mathrm{modetrans}(G, i))$ into S and is defined by:

(Def. 9) $\operatorname{dom}(f{\restriction}^{i}X) = X$ and for every point x of $\prod G$ such that $x \in X$ holds $(f{\restriction}^{i}X)_x = \mathrm{partdiff}(f, x, i)$.

One can prove the following propositions:

(26) For every set i such that $i \in \operatorname{dom} G$ holds $(f_1 + f_2) \cdot \mathrm{reproj}(\mathrm{modetrans}(G, i), x) = f_1 \cdot \mathrm{reproj}(\mathrm{modetrans}(G, i), x) + f_2 \cdot$

reproj(modetrans$(G, i), x)$ and $(f_1 - f_2) \cdot$ reproj(modetrans$(G, i), x) =$ $f_1 \cdot$ reproj(modetrans$(G, i), x) - f_2 \cdot$ reproj(modetrans$(G, i), x)$.

(27) For every set i such that $i \in \operatorname{dom} G$ holds $r \cdot (f \cdot$ reproj(modetrans$(G, i), x)) =$ $(r \cdot f) \cdot$ reproj(modetrans$(G, i), x)$.

(28) Let i be a set. Suppose $i \in \operatorname{dom} G$ and f_1 is partially differentiable in x w.r.t. i and f_2 is partially differentiable in x w.r.t. i. Then f_1+f_2 is partially differentiable in x w.r.t. i and partdiff$(f_1 + f_2, x, i) =$ partdiff$(f_1, x, i) +$ partdiff(f_2, x, i).

(29) Let i be a set. Suppose $i \in \operatorname{dom} G$ and f_1 is partially differentiable in x w.r.t. i and f_2 is partially differentiable in x w.r.t. i. Then f_1-f_2 is partially differentiable in x w.r.t. i and partdiff$(f_1 - f_2, x, i) =$ partdiff$(f_1, x, i) -$ partdiff(f_2, x, i).

(30) Let i be a set. Suppose $i \in \operatorname{dom} G$ and f is partially differentiable in x w.r.t. i. Then $r \cdot f$ is partially differentiable in x w.r.t. i and partdiff$(r \cdot f, x, i) = r \cdot$ partdiff(f, x, i).

5. Continuous Differentiatibility of Partial Derivative

Next we state the proposition

(31) $\|$(the projection onto $i)(x)\| \leq \|x\|$.

Let G be a nontrivial real norm space sequence. One can verify that every point of $\prod G$ is len G-element.

We now state a number of propositions:

(32) Let G be a nontrivial real norm space sequence, T be a non trivial real normed space, i be a set, Z be a subset of $\prod G$, and f be a partial function from $\prod G$ to T. Suppose Z is open. Then f is partially differentiable on Z w.r.t. i if and only if $Z \subseteq \operatorname{dom} f$ and for every point x of $\prod G$ such that $x \in Z$ holds f is partially differentiable in x w.r.t. i.

(33) Let i, j be elements of dom G, x be a point of $G(i)$, and z be an element of $\prod \overline{G}$ such that $z = ($reproj$(i, 0_{\prod G}))(x)$. Then

(i) if $i = j$, then $z(j) = x$, and

(ii) if $i \neq j$, then $z(j) = 0_{G(j)}$.

(34) For all points x, y of $G(i)$ holds $($reproj$(i, 0_{\prod G}))(x + y) =$ $($reproj$(i, 0_{\prod G}))(x) + ($reproj$(i, 0_{\prod G}))(y)$.

(35) Let x, y be points of $\prod G$. Then (the projection onto $i)(x + y) =$ (the projection onto $i)(x) +$ (the projection onto $i)(y)$.

(36) For all points x, y of $G(i)$ holds $($reproj$(i, 0_{\prod G}))(x - y) =$ $($reproj$(i, 0_{\prod G}))(x) - ($reproj$(i, 0_{\prod G}))(y)$.

(37) Let x, y be points of $\prod G$. Then (the projection onto i)$(x - y)$ = (the projection onto i)(x) − (the projection onto i)(y).

(38) For every point x of $G(i)$ such that $x \neq 0_{G(i)}$ holds $(\mathrm{reproj}(i, 0_{\prod G}))(x) \neq 0_{\prod G}$.

(39) For every point x of $G(i)$ and for every element a of $\mathbb{R}$ holds $(\mathrm{reproj}(i, 0_{\prod G}))(a \cdot x) = a \cdot (\mathrm{reproj}(i, 0_{\prod G}))(x)$.

(40) Let x be a point of $\prod G$ and a be an element of $\mathbb{R}$. Then (the projection onto i)$(a \cdot x) = a \cdot$ (the projection onto i)(x).

(41) Let G be a nontrivial real norm space sequence, S be a non trivial real normed space, f be a partial function from $\prod G$ to S, x be a point of $\prod G$, and i be a set. Suppose f is differentiable in x. Then f is partially differentiable in x w.r.t. i and $\mathrm{partdiff}(f, x, i) = f'(x) \cdot \mathrm{reproj}(\mathrm{modetrans}(G, i), 0_{\prod G})$.

(42) Let S be a real normed space and h, g be finite sequences of elements of S. Suppose $\mathrm{len}\, h = \mathrm{len}\, g + 1$ and for every natural number i such that $i \in \mathrm{dom}\, g$ holds $g_i = h_i - h_{i+1}$. Then $h_1 - h_{\mathrm{len}\, h} = \sum g$.

(43) Let G be a nontrivial real norm space sequence, x, y be elements of $\prod \overline{G}$, and Z be a set. Then $x {+\cdot} y {\restriction} Z$ is an element of $\prod \overline{G}$.

(44) Let G be a nontrivial real norm space sequence, x, y be points of $\prod G$, Z, x_0 be elements of $\prod \overline{G}$, and X be a set. If $Z = 0_{\prod G}$ and $x_0 = x$ and $y = Z {+\cdot} x_0 {\restriction} X$, then $\|y\| \leq \|x\|$.

(45) Let G be a nontrivial real norm space sequence, S be a non trivial real normed space, f be a partial function from $\prod G$ to S, and x, y be points of $\prod G$. Then there exists a finite sequence h of elements of $\prod G$ and there exists a finite sequence g of elements of S and there exist elements Z, y_0 of $\prod \overline{G}$ such that
$y_0 = y$ and $Z = 0_{\prod G}$ and $\mathrm{len}\, h = \mathrm{len}\, G + 1$ and $\mathrm{len}\, g = \mathrm{len}\, G$ and for every natural number i such that $i \in \mathrm{dom}\, h$ holds $h_i = Z {+\cdot} y_0 {\restriction} \mathrm{Seg}((\mathrm{len}\, G + 1) -' i)$ and for every natural number i such that $i \in \mathrm{dom}\, g$ holds $g_i = f_{x+h_i} - f_{x+h_{i+1}}$ and for every natural number i and for every point h_1 of $\prod G$ such that $i \in \mathrm{dom}\, h$ and $h_i = h_1$ holds $\|h_1\| \leq \|y\|$ and $f_{x+y} - f_x = \sum g$.

(46) Let G be a nontrivial real norm space sequence, i be an element of $\mathrm{dom}\, G$, x, y be points of $\prod G$, and x_2 be a point of $G(i)$. If $y = (\mathrm{reproj}(i, x))(x_2)$, then (the projection onto i)$(y) = x_2$.

(47) Let G be a nontrivial real norm space sequence, i be an element of $\mathrm{dom}\, G$, y be a point of $\prod G$, and q be a point of $G(i)$. If q = (the projection onto i)(y), then $y = (\mathrm{reproj}(i, y))(q)$.

(48) Let G be a nontrivial real norm space sequence, i be an element of $\mathrm{dom}\, G$, x, y be points of $\prod G$, and x_2 be a point of $G(i)$. If $y = (\mathrm{reproj}(i, x))(x_2)$, then $\mathrm{reproj}(i, x) = \mathrm{reproj}(i, y)$.

(49) Let G be a nontrivial real norm space sequence, i, j be elements of $\operatorname{dom} G$, x, y be points of $\prod G$, and x_2 be a point of $G(i)$. Suppose $y = (\mathrm{reproj}(i,x))(x_2)$ and $i \neq j$. Then (the projection onto $j)(x) =$ (the projection onto $j)(y)$.

(50) Let G be a nontrivial real norm space sequence, F be a non trivial real normed space, i be an element of $\operatorname{dom} G$, x be a point of $\prod G$, x_2 be a point of $G(i)$, f be a partial function from $\prod G$ to F, and g be a partial function from $G(i)$ to F. If (the projection onto $i)(x) = x_2$ and $g = f \cdot \mathrm{reproj}(i,x)$, then $g'(x_2) = \mathrm{partdiff}(f,x,i)$.

(51) Let G be a nontrivial real norm space sequence, F be a non trivial real normed space, f be a partial function from $\prod G$ to F, x be a point of $\prod G$, i be a set, M be a real number, L be a point of the real norm space of bounded linear operators from $G(\mathrm{modetrans}(G,i))$ into F, and p, q be points of $G(\mathrm{modetrans}(G,i))$. Suppose that

(i) $i \in \operatorname{dom} G$,

(ii) for every point h of $G(\mathrm{modetrans}(G,i))$ such that $h \in]p,q[$ holds $\|\mathrm{partdiff}(f,(\mathrm{reproj}(\mathrm{modetrans}(G,i),x))(h),i) - L\| \leq M$,

(iii) for every point h of $G(\mathrm{modetrans}(G,i))$ such that $h \in [p,q]$ holds $(\mathrm{reproj}(\mathrm{modetrans}(G,i),x))(h) \in \operatorname{dom} f$, and

(iv) for every point h of $G(\mathrm{modetrans}(G,i))$ such that $h \in [p,q]$ holds f is partially differentiable in $(\mathrm{reproj}(\mathrm{modetrans}(G,i),x))(h)$ w.r.t. i.

Then $\|f_{(\mathrm{reproj}(\mathrm{modetrans}(G,i),x))(q)} - f_{(\mathrm{reproj}(\mathrm{modetrans}(G,i),x))(p)} - L(q-p)\| \leq M \cdot \|q-p\|$.

(52) Let G be a nontrivial real norm space sequence, x, y, z, w be points of $\prod G$, i be an element of $\operatorname{dom} G$, d be a real number, and p, q, r be points of $G(i)$. Suppose $\|y-x\| < d$ and $\|z-x\| < d$ and $p =$ (the projection onto $i)(y)$ and $z = (\mathrm{reproj}(i,y))(q)$ and $r \in [p,q]$ and $w = (\mathrm{reproj}(i,y))(r)$. Then $\|w-x\| < d$.

(53) Let G be a nontrivial real norm space sequence, S be a non trivial real normed space, f be a partial function from $\prod G$ to S, X be a subset of $\prod G$, x, y, z be points of $\prod G$, i be a set, p, q be points of $G(\mathrm{modetrans}(G,i))$, and d, r be real numbers. Suppose that $i \in \operatorname{dom} G$ and X is open and $x \in X$ and $\|y-x\| < d$ and $\|z-x\| < d$ and $X \subseteq \operatorname{dom} f$ and for every point x of $\prod G$ such that $x \in X$ holds f is partially differentiable in x w.r.t. i and for every point z of $\prod G$ such that $\|z-x\| < d$ holds $z \in X$ and for every point z of $\prod G$ such that $\|z-x\| < d$ holds $\|\mathrm{partdiff}(f,z,i) - \mathrm{partdiff}(f,x,i)\| \leq r$ and $z = (\mathrm{reproj}(\mathrm{modetrans}(G,i),y))(p)$ and $q =$ (the projection onto $\mathrm{modetrans}(G,i))(y)$. Then $\|f_z - f_y - (\mathrm{partdiff}(f,x,i))(p-q)\| \leq \|p-q\| \cdot r$.

(54) Let G be a nontrivial real norm space sequence, h be a finite sequence of elements of $\prod G$, y, x be points of $\prod G$, y_0, Z be elements of $\prod \overline{G}$, and j be an element of $\mathbb{N}$. Suppose $y = y_0$ and $Z = 0_{\prod G}$ and

$\mathrm{len}\, h = \mathrm{len}\, G + 1$ and $1 \leq j \leq \mathrm{len}\, G$ and for every natural number i such that $i \in \mathrm{dom}\, h$ holds $h_i = Z{+\cdot}y_0 \restriction \mathrm{Seg}((\mathrm{len}\, G + 1) -' i)$. Then $x + h_j = (\mathrm{reproj}(\mathrm{modetrans}(G, (\mathrm{len}\, G+1) -' j), x + h_{j+1}))($(the projection onto $\mathrm{modetrans}(G, (\mathrm{len}\, G + 1) -' j))(x + y))$.

(55) Let G be a nontrivial real norm space sequence, h be a finite sequence of elements of $\prod G$, y, x be points of $\prod G$, y_0, Z be elements of $\prod \overline{G}$, and j be an element of $\mathbb{N}$. Suppose $y = y_0$ and $Z = 0_{\prod G}$ and $\mathrm{len}\, h = \mathrm{len}\, G + 1$ and $1 \leq j \leq \mathrm{len}\, G$ and for every natural number i such that $i \in \mathrm{dom}\, h$ holds $h_i = Z{+\cdot}y_0 \restriction \mathrm{Seg}((\mathrm{len}\, G + 1) -' i)$. Then (the projection onto $\mathrm{modetrans}(G, (\mathrm{len}\, G + 1) -' j))(x + y) -$ (the projection onto $\mathrm{modetrans}(G, (\mathrm{len}\, G + 1) -' j))(x + h_{j+1}) =$ (the projection onto $\mathrm{modetrans}(G, (\mathrm{len}\, G + 1) -' j))(y)$.

(56) Let G be a nontrivial real norm space sequence, S be a non trivial real normed space, f be a partial function from $\prod G$ to S, X be a subset of $\prod G$, and x be a point of $\prod G$. Suppose that

(i) X is open,

(ii) $x \in X$, and

(iii) for every set i such that $i \in \mathrm{dom}\, G$ holds f is partially differentiable on X w.r.t. i and $f{\restriction^i}X$ is continuous on X.

Then

(iv) f is differentiable in x, and

(v) for every point h of $\prod G$ there exists a finite sequence w of elements of S such that $\mathrm{dom}\, w = \mathrm{dom}\, G$ and for every set i such that $i \in \mathrm{dom}\, G$ holds $w(i) = (\mathrm{partdiff}(f, x, i))($(the projection onto $\mathrm{modetrans}(G, i))(h))$ and $f'(x)(h) = \sum w$.

(57) Let G be a nontrivial real norm space sequence, F be a non trivial real normed space, f be a partial function from $\prod G$ to F, and X be a subset of $\prod G$. Suppose X is open. Then for every set i such that $i \in \mathrm{dom}\, G$ holds f is partially differentiable on X w.r.t. i and $f{\restriction^i}X$ is continuous on X if and only if f is differentiable on X and $f'_{\restriction X}$ is continuous on X.

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