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On L^1 Space Formed by Complex-Valued Partial Functions

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Summary. In this article, we formalized L^1 space formed by complex-valued partial functions [11], [15]. The real-valued case was formalized in [22] and this article is its generalization.

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The notation and terminology used here have been introduced in the following papers: [4], [10], [5], [19], [17], [6], [7], [1], [22], [3], [18], [13], [16], [8], [14], [23], [24], [12], [20], [21], [2], and [9].

1. Preliminaries of Complex Linear Space

Let D be a non empty set and let E be a complex-membered set. One can verify that every element of $D \dot{\rightarrow} E$ is complex-valued.

Let D be a non empty set, let E be a complex-membered set, and let F_1, F_2 be elements of $D \dot{\rightarrow} E$. Then $F_1 + F_2$ is an element of $D \dot{\rightarrow} \mathbb{C}$. Then $F_1 - F_2$ is an element of $D \dot{\rightarrow} \mathbb{C}$. Then $F_1 \cdot F_2$ is an element of $D \dot{\rightarrow} \mathbb{C}$. Then F_1 / F_2 is an element of $D \dot{\rightarrow} \mathbb{C}$.

Let D be a non empty set, let E be a complex-membered set, let F be an element of $D \dot{\rightarrow} E$, and let a be a complex number. Then $a \cdot F$ is an element of $D \dot{\rightarrow} \mathbb{C}$.

Let V be a non empty CLS structure and let V_1 be a subset of V. We say that V_1 is multiplicatively closed if and only if:

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(Def. 1) For every complex number a and for every vector v of V such that $v \in V_1$ holds $a \cdot v \in V_1$.

Next we state the proposition

(1) Let V be a complex linear space and V_1 be a subset of V. Then V_1 is linearly closed if and only if V_1 is add closed and multiplicatively closed.

Let V be a non empty CLS structure. One can verify that there exists a non empty subset of V which is add closed and multiplicatively closed.

Let X be a non empty CLS structure and let X_1 be a multiplicatively closed non empty subset of X. The functor $\cdot_{(X_1)}$ yields a function from $\mathbb{C} \times X_1$ into X_1 and is defined by:

(Def. 2) $\cdot_{(X_1)} =$ (the external multiplication of X)$\restriction(\mathbb{C} \times X_1)$.

In the sequel a, b, r denote complex numbers and V denotes a complex linear space.

We now state two propositions:

(2) Let V be an Abelian add-associative right zeroed vector distributive scalar distributive scalar associative scalar unital non empty CLS structure, V_1 be a non empty subset of V, d_1 be an element of V_1, A be a binary operation on V_1, and M be a function from $\mathbb{C} \times V_1$ into V_1. Suppose $d_1 = 0_V$ and $A =$ (the addition of V) $\restriction$ (V_1) and $M =$ (the external multiplication of V)$\restriction(\mathbb{C} \times V_1)$. Then $\langle V_1, d_1, A, M\rangle$ is Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.

(3) Let V be an Abelian add-associative right zeroed vector distributive scalar distributive scalar associative scalar unital non empty CLS structure and V_1 be an add closed multiplicatively closed non empty subset of V. Suppose $0_V \in V_1$. Then $\langle V_1, 0_V(\in V_1), \mathrm{add}\,|(V_1, V), \cdot_{(V_1)}\rangle$ is Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.

2. Quasi-Complex Linear Space of Partial Functions

We follow the rules: A, B are non empty sets and f, g, h are elements of $A \dot{\rightarrow} \mathbb{C}$.

Let us consider A. The functor multcpfunc A yielding a binary operation on $A \dot{\rightarrow} \mathbb{C}$ is defined as follows:

(Def. 3) For all elements f, g of $A \dot{\rightarrow} \mathbb{C}$ holds $(\mathrm{multcpfunc}\,A)(f, g) = f \cdot g$.

Let us consider A. The functor multcomplexcpfunc A yielding a function from $\mathbb{C} \times (A \dot{\rightarrow} \mathbb{C})$ into $A \dot{\rightarrow} \mathbb{C}$ is defined by:

(Def. 4) For every complex number a and for every element f of $A \dot{\rightarrow} \mathbb{C}$ holds $(\mathrm{multcomplexcpfunc}\,A)(a, f) = a \cdot f$.

Let D be a non empty set. The functor addcpfunc D yields a binary operation on $D \dot{\to} \mathbb{C}$ and is defined as follows:

(Def. 5) For all elements F_1, F_2 of $D \dot{\to} \mathbb{C}$ holds $(\text{addcpfunc}\, D)(F_1, F_2) = F_1 + F_2$.

Let A be a set. The functor CPFuncZero A yields an element of $A \dot{\to} \mathbb{C}$ and is defined by:

(Def. 6) $\text{CPFuncZero}\, A = A \longmapsto 0_{\mathbb{C}}$.

Let A be a set. The functor CPFuncUnit A yielding an element of $A \dot{\to} \mathbb{C}$ is defined as follows:

(Def. 7) $\text{CPFuncUnit}\, A = A \longmapsto 1_{\mathbb{C}}$.

The following propositions are true:

(4) $h = (\text{addcpfunc}\, A)(f, g)$ iff $\text{dom}\, h = \text{dom}\, f \cap \text{dom}\, g$ and for every element x of A such that $x \in \text{dom}\, h$ holds $h(x) = f(x) + g(x)$.

(5) $h = (\text{multcpfunc}\, A)(f, g)$ iff $\text{dom}\, h = \text{dom}\, f \cap \text{dom}\, g$ and for every element x of A such that $x \in \text{dom}\, h$ holds $h(x) = f(x) \cdot g(x)$.

(6) $\text{CPFuncZero}\, A \neq \text{CPFuncUnit}\, A$.

(7) $h = (\text{multcomplexcpfunc}\, A)(a, f)$ iff $\text{dom}\, h = \text{dom}\, f$ and for every element x of A such that $x \in \text{dom}\, f$ holds $h(x) = a \cdot f(x)$.

Let us consider A. Note that addcpfunc A is commutative and associative.

Observe that multcpfunc A is commutative and associative.

One can prove the following propositions:

(8) CPFuncUnit A is a unity w.r.t. multcpfunc A.

(9) CPFuncZero A is a unity w.r.t. addcpfunc A.

(10) $(\text{addcpfunc}\, A)(f, (\text{multcomplexcpfunc}\, A)(-1_{\mathbb{C}}, f)) = \text{CPFuncZero}\, A \restriction \text{dom}\, f$.

(11) $(\text{multcomplexcpfunc}\, A)(1_{\mathbb{C}}, f) = f$.

(12) $(\text{multcomplexcpfunc}\, A)(a, (\text{multcomplexcpfunc}\, A)(b, f)) = (\text{multcomplexcpfunc}\, A)(a \cdot b, f)$.

(13) $(\text{addcpfunc}\, A)((\text{multcomplexcpfunc}\, A)(a, f), (\text{multcomplexcpfunc}\, A)(b, f)) = (\text{multcomplexcpfunc}\, A)(a + b, f)$.

(14) $(\text{multcpfunc}\, A)(f, (\text{addcpfunc}\, A)(g, h)) = (\text{addcpfunc}\, A)((\text{multcpfunc}\, A)(f, g), (\text{multcpfunc}\, A)(f, h))$.

(15) $(\text{multcpfunc}\, A)((\text{multcomplexcpfunc}\, A)(a, f), g) = (\text{multcomplexcpfunc}\, A)(a, (\text{multcpfunc}\, A)(f, g))$.

Let us consider A. The functor CLSp PFunct A yields a non empty CLS structure and is defined as follows:

(Def. 8) $\text{CLSp PFunct}\, A = \langle A \dot{\to} \mathbb{C}, \text{CPFuncZero}\, A, \text{addcpfunc}\, A, \text{multcomplexcpfunc}\, A \rangle$.

In the sequel u, v, w are vectors of CLSp PFunct A.

Note that CLSp PFunct A is strict, Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.

3. Quasi-Complex Linear Space of Integrable Functions

For simplicity, we use the following convention: X is a non empty set, x is an element of X, S is a σ-field of subsets of X, M is a σ-measure on S, E, A are elements of S, and f, g, h, f_1, g_1 are partial functions from X to $\mathbb{C}$.

Let us consider X and let f be a partial function from X to $\mathbb{C}$. Note that $|f|$ is non-negative.

Next we state the proposition

(16) Let f be a partial function from X to $\mathbb{C}$. Suppose $\operatorname{dom} f \in S$ and for every x such that $x \in \operatorname{dom} f$ holds $0 = f(x)$. Then f is integrable on M and $\int f \,\mathrm{d}M = 0$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor L_1CFunctions M yielding a non empty subset of CLSp PFunct X is defined by the condition (Def. 9).

(Def. 9) $\mathrm{L}_1\mathrm{CFunctions}\, M = \{f; f$ ranges over partial functions from X to $\mathbb{C}$: $\bigvee_{N_1 : \text{element of } S} (M(N_1) = 0 \ \wedge\ \operatorname{dom} f = {N_1}^{\mathrm{c}} \ \wedge\ f$ is integrable on $M)\}$.

The following propositions are true:

(17) If $f, g \in \mathrm{L}_1\mathrm{CFunctions}\, M$, then $f + g \in \mathrm{L}_1\mathrm{CFunctions}\, M$.

(18) If $f \in \mathrm{L}_1\mathrm{CFunctions}\, M$, then $a \cdot f \in \mathrm{L}_1\mathrm{CFunctions}\, M$.

Note that L_1CFunctions M is multiplicatively closed and add closed.

The functor CLSp L_1Funct M yielding a non empty CLS structure is defined by:

(Def. 10) $\mathrm{CLSp}\ \mathrm{L}_1\mathrm{Funct}\, M = \langle \mathrm{L}_1\mathrm{CFunctions}\, M, 0_{\mathrm{CLSp\ PFunct}\, X}(\in \mathrm{L}_1\mathrm{CFunctions}\, M),$ $\mathrm{add}\,|(\mathrm{L}_1\mathrm{CFunctions}\, M, \mathrm{CLSp\ PFunct}\, X), \cdot_{\mathrm{L}_1\mathrm{CFunctions}\, M}\rangle$.

One can verify that CLSp L_1Funct M is strict, Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.

4. Quotient Space of Quasi-Complex Linear Space of Integrable Functions

In the sequel v, u are vectors of CLSp L_1Funct M.

Next we state two propositions:

(19) If $f = v$ and $g = u$, then $f + g = v + u$.

(20) If $f = u$, then $a \cdot f = a \cdot u$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f, g be partial functions from X to $\mathbb{C}$. We say that f a.e.cpfunc $= g$ and M if and only if:

(Def. 11) There exists an element E of S such that $M(E) = 0$ and $f \restriction E^{\rm c} = g \restriction E^{\rm c}$.

We now state several propositions:

(21) Suppose $f = u$. Then

(i) $u + (-1_{\mathbb{C}}) \cdot u = (X \longmapsto 0_{\mathbb{C}}) \restriction \operatorname{dom} f$, and

(ii) there exist partial functions v, g from X to $\mathbb{C}$ such that $v, g \in$ $\mathrm{L_1CFunctions}\, M$ and $v = u + (-1_{\mathbb{C}}) \cdot u$ and $g = X \longmapsto 0_{\mathbb{C}}$ and v a.e.cpfunc $= g$ and M.

(22) f a.e.cpfunc $= f$ and M.

(23) If f a.e.cpfunc $= g$ and M, then g a.e.cpfunc $= f$ and M.

(24) If f a.e.cpfunc $= g$ and M and g a.e.cpfunc $= h$ and M, then f a.e.cpfunc $= h$ and M.

(25) If f a.e.cpfunc $= f_1$ and M and g a.e.cpfunc $= g_1$ and M, then $f + g$ a.e.cpfunc $= f_1 + g_1$ and M.

(26) If f a.e.cpfunc $= g$ and M, then $a \cdot f$ a.e.cpfunc $= a \cdot g$ and M.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The almost zero cfunctions of M yields a non empty subset of CLSp $\mathrm{L_1Funct}\, M$ and is defined by the condition (Def. 12).

(Def. 12) The almost zero cfunctions of $M = \{f; f$ ranges over partial functions from X to $\mathbb{C}$: $f \in \mathrm{L_1CFunctions}\, M \ \wedge\ f$ a.e.cpfunc $= X \longmapsto 0_{\mathbb{C}}$ and $M\}$.

One can prove the following proposition

(27) $(X \longmapsto 0_{\mathbb{C}}) + (X \longmapsto 0_{\mathbb{C}}) = X \longmapsto 0_{\mathbb{C}}$ and $a \cdot (X \longmapsto 0_{\mathbb{C}}) = X \longmapsto 0_{\mathbb{C}}$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. One can check that the almost zero cfunctions of M is add closed and multiplicatively closed.

One can prove the following proposition

(28) $0_{\mathrm{CLSp\ L_1Funct}\, M} = X \longmapsto 0_{\mathbb{C}}$ and $0_{\mathrm{CLSp\ L_1Funct}\, M} \in$ the almost zero cfunctions of M.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The clsp almost zero functions of M yields a non empty CLS structure and is defined by the condition (Def. 13).

(Def. 13) The clsp almost zero functions of $M = \langle$the almost zero cfunctions of M, $0_{\mathrm{CLSp\ L_1Funct}\, M} (\in$ the almost zero cfunctions of $M)$, add $\restriction$(the almost zero cfunctions of M, CLSp $\mathrm{L_1Funct}\, M$), $\cdot_{\text{the almost zero cfunctions of } M}\rangle$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. One can check that CLSp $\mathrm{L_1Funct}\, M$ is strict, Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.

In the sequel v, u are vectors of the clsp almost zero functions of M.

One can prove the following proposition

(29) If $f = v$ and $g = u$, then $f + g = v + u$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\mathbb{C}$. The functor a.e-Ceq-class(f, M) yields a subset of $\mathrm{L}_1\mathrm{CFunctions}\, M$ and is defined as follows:

(Def. 14) a.e-Ceq-class$(f, M) = \{g; g$ ranges over partial functions from X to $\mathbb{C}$: $g \in \mathrm{L}_1\mathrm{CFunctions}\, M \ \wedge\ f \in \mathrm{L}_1\mathrm{CFunctions}\, M \ \wedge\ f$ a.e.cpfunc $= g$ and $M\}$.

Next we state several propositions:

(30) If f, $g \in \mathrm{L}_1\mathrm{CFunctions}\, M$, then g a.e.cpfunc $= f$ and M iff $g \in$ a.e-Ceq-class(f, M).

(31) If $f \in \mathrm{L}_1\mathrm{CFunctions}\, M$, then $f \in$ a.e-Ceq-class(f, M).

(32) If $f, g \in \mathrm{L}_1\mathrm{CFunctions}\, M$, then a.e-Ceq-class$(f, M) =$ a.e-Ceq-class(g, M) iff f a.e.cpfunc $= g$ and M.

(33) If $f, g \in \mathrm{L}_1\mathrm{CFunctions}\, M$, then a.e-Ceq-class$(f, M) =$ a.e-Ceq-class(g, M) iff $g \in$ a.e-Ceq-class(f, M).

(34) If f, f_1, g, $g_1 \in \mathrm{L}_1\mathrm{CFunctions}\, M$ and a.e-Ceq-class$(f, M) =$ a.e-Ceq-class(f_1, M) and a.e-Ceq-class$(g, M) =$ a.e-Ceq-class(g_1, M), then a.e-Ceq-class$(f + g, M) =$ a.e-Ceq-class$(f_1 + g_1, M)$.

(35) If $f, g \in \mathrm{L}_1\mathrm{CFunctions}\, M$ and a.e-Ceq-class$(f, M) =$ a.e-Ceq-class(g, M), then a.e-Ceq-class$(a \cdot f, M) =$ a.e-Ceq-class$(a \cdot g, M)$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor CCosetSet M yields a non empty family of subsets of $\mathrm{L}_1\mathrm{CFunctions}\, M$ and is defined by:

(Def. 15) CCosetSet $M = \{$a.e-Ceq-class$(f, M); f$ ranges over partial functions from X to $\mathbb{C}$: $f \in \mathrm{L}_1\mathrm{CFunctions}\, M\}$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor addCCoset M yields a binary operation on CCosetSet M and is defined by the condition (Def. 16).

(Def. 16) Let A, B be elements of CCosetSet M and a, b be partial functions from X to $\mathbb{C}$. If $a \in A$ and $b \in B$, then $(\mathrm{addCCoset}\, M)(A, B) =$ a.e-Ceq-class$(a + b, M)$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor zeroCCoset M yielding an element of CCosetSet M is defined by:

(Def. 17) zeroCCoset $M =$ a.e-Ceq-class$(X \longmapsto 0_{\mathbb{C}}, M)$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor lmultCCoset M yields a function from $\mathbb{C} \times$ CCosetSet M into CCosetSet M and is defined by the condition (Def. 18).

(Def. 18) Let z be a complex number, A be an element of CCosetSet M, and f be a partial function from X to $\mathbb{C}$. If $f \in A$, then (lmultCCoset M)$(z, A) =$ a.e-Ceq-class$(z \cdot f, M)$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor Pre-L-CSpace M yields a strict Abelian add-associative right zeroed right complementable vector distributive scalar distributive scalar associative scalar unital non empty CLS structure and is defined by the conditions (Def. 19).

(Def. 19)(i) The carrier of Pre-L-CSpace M = CCosetSet M,

(ii) the addition of Pre-L-CSpace M = addCCoset M,

(iii) $0_{\text{Pre-L-CSpace}\,M}$ = zeroCCoset M, and

(iv) the external multiplication of Pre-L-CSpace M = lmultCCoset M.

5. Complex Normed Space of Integrable Functions

Next we state several propositions:

(36) If f, $g \in$ $\mathrm{L_1CFunctions}\,M$ and f a.e.cpfunc $= g$ and M, then $\int f \, \mathrm{d}M = \int g \, \mathrm{d}M$.

(37) If f is integrable on M, then $\int f \, \mathrm{d}M \in \mathbb{C}$ and $\int |f| \, \mathrm{d}M \in \mathbb{R}$ and $|f|$ is integrable on M.

(38) If f, $g \in$ $\mathrm{L_1CFunctions}\,M$ and f a.e.cpfunc $= g$ and M, then $|f| =^{M}_{\text{a.e.}} |g|$ and $\int |f| \, \mathrm{d}M = \int |g| \, \mathrm{d}M$.

(39) If there exists a vector x of Pre-L-CSpace M such that f, $g \in x$, then f a.e.cpfunc $= g$ and M and f, $g \in$ $\mathrm{L_1CFunctions}\,M$.

(40) There exists a function N_2 from the carrier of Pre-L-CSpace M into $\mathbb{R}$ such that for every point x of Pre-L-CSpace M holds there exists a partial function f from X to $\mathbb{C}$ such that $f \in x$ and $N_2(x) = \int |f| \, \mathrm{d}M$.

In the sequel x is a point of Pre-L-CSpace M.

The following two propositions are true:

(41) If $f \in x$, then f is integrable on M and $f \in$ $\mathrm{L_1CFunctions}\,M$ and $|f|$ is integrable on M.

(42) If f, $g \in x$, then f a.e.cpfunc $= g$ and M and $\int f \, \mathrm{d}M = \int g \, \mathrm{d}M$ and $\int |f| \, \mathrm{d}M = \int |g| \, \mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor L-1-CNorm M yields a function from the carrier of Pre-L-CSpace M into $\mathbb{R}$ and is defined by:

(Def. 20) For every point x of Pre-L-CSpace M there exists a partial function f from X to $\mathbb{C}$ such that $f \in x$ and (L-1-CNorm M)$(x) = \int |f| \, \mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor L-1-CSpace M yields a non empty complex normed space structure and is defined as follows:

(Def. 21) L-1-CSpace M = $\langle$the carrier of Pre-L-CSpace M, the zero of Pre-L-CSpace M, the addition of Pre-L-CSpace M, the external multiplication of Pre-L-CSpace M, L-1-CNorm $M\rangle$.

In the sequel x denotes a point of L-1-CSpace M.

Next we state several propositions:

(43)(i) There exists a partial function f from X to $\mathbb{C}$ such that $f \in$ L_1CFunctions M and $x =$ a.e-Ceq-class(f, M) and $\|x\| = \int |f| \, \mathrm{d}M$, and

(ii) for every partial function f from X to $\mathbb{C}$ such that $f \in x$ holds $\int |f| \, \mathrm{d}M = \|x\|$.

(44) If $f \in x$, then $x =$ a.e-Ceq-class(f, M) and $\|x\| = \int |f| \, \mathrm{d}M$.

(45) If $f \in x$ and $g \in y$, then $f + g \in x + y$ and if $f \in x$, then $a \cdot f \in a \cdot x$.

(46) If $f \in \mathrm{L}_1$CFunctions M and $\int |f| \, \mathrm{d}M = 0$, then f a.e.cpfunc $= X \longmapsto 0_{\mathbb{C}}$ and M.

(47) If f, $g \in \mathrm{L}_1$CFunctions M, then $\int |f + g| \, \mathrm{d}M \leq \int |f| \, \mathrm{d}M + \int |g| \, \mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. One can check that L-1-CSpace M is complex normed space-like, vector distributive, scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, and right complementable.

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