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Coproducts in Categories without Uniqueness of cod and dom

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Summary. The paper introduces coproducts in categories without uniqueness of cod and dom . It is proven that set-theoretical disjoint union is the coproduct in the category Ens [9].

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The notation and terminology used in this paper have been introduced in the following articles: [10], [7], [6], [1], [11], [2], [3], [8], [4], [12], [14], [13], and [5].

From now on I denotes a set and E denotes a non empty set.

Let I be a non empty set, A be a many sorted set indexed by I , and i be an element of I . Let us observe that $\text{coprod}(i, A)$ is relation-like and function-like.

Let C be a non empty category structure, o be an object of C , I be a set, and f be an objects family of I and C . A morphisms family of f and o is a many sorted set indexed by I and is defined by

(Def. 1) Let us consider an element i . Suppose $i \in I$. Then there exists an object o_1 of C such that

- (i) $o_1 = f(i)$, and
- (ii) $it(i)$ is a morphism from o_1 to o .

Let I be a non empty set. Let us note that a morphisms family of f and o can equivalently be formulated as follows:

(Def. 2) Let us consider an element i of I . Then $it(i)$ is a morphism from $f(i)$ to o .

Let M be a morphisms family of f and o and i be an element of I . Note that the functor $M(i)$ yields a morphism from $f(i)$ to o . Let C be a functional non empty category structure. Let I be a set. Let us note that every morphisms family of f and o is function yielding.

Now we state the proposition:

- (1) Let us consider a non empty category structure C , an object o of C , and an objects family f of \emptyset and C . Then \emptyset is a morphisms family of f and o .

Let C be a non empty category structure, I be a set, A be an objects family of I and C , B be an object of C , and P be a morphisms family of A and B . We say that P is feasible if and only if

- (Def. 3) Let us consider a set i . Suppose $i \in I$. Then there exists an object o of C such that

- (i) $o = A(i)$, and
- (ii) $P(i) \in \langle o, B \rangle$.

Let I be a non empty set. Let us observe that P is feasible if and only if the condition (Def. 4) is satisfied.

- (Def. 4) Let us consider an element i of I . Then $P(i) \in \langle A(i), B \rangle$.

Let C be a category and I be a set. We say that P is coprojection morphisms if and only if

- (Def. 5) Let us consider an object X of C and a morphisms family F of A and X . Suppose F is feasible. Then there exists a morphism f from B to X such that

- (i) $f \in \langle B, X \rangle$, and
- (ii) for every set i such that $i \in I$ there exists an object s_i of C and there exists a morphism P_i from s_i to B such that $s_i = A(i)$ and $P_i = P(i)$ and $F(i) = f \cdot P_i$, and
- (iii) for every morphism f_1 from B to X such that for every set i such that $i \in I$ there exists an object s_i of C and there exists a morphism P_i from s_i to B such that $s_i = A(i)$ and $P_i = P(i)$ and $F(i) = f_1 \cdot P_i$ holds $f = f_1$.

Let I be a non empty set. Let us note that P is coprojection morphisms if and only if the condition (Def. 6) is satisfied.

- (Def. 6) Let us consider an object X of C and a morphisms family F of A and X . Suppose F is feasible. Then there exists a morphism f from B to X such that

- (i) $f \in \langle B, X \rangle$, and
- (ii) for every element i of I , $F(i) = f \cdot P(i)$, and
- (iii) for every morphism f_1 from B to X such that for every element i of I , $F(i) = f_1 \cdot P(i)$ holds $f = f_1$.

Let A be an objects family of \emptyset and C . Note that every morphisms family of A and B is feasible.

Now we state the propositions:

- (2) Let us consider a category C , an objects family A of \emptyset and C , and an object B of C . Suppose B is initial. Then there exists a morphisms family P of A and B such that P is empty and coprojection morphisms. The theorem is a consequence of (1).
- (3) Let us consider an objects family A of I and $\text{Ens}_{\{\emptyset\}}$ and an object o of $\text{Ens}_{\{\emptyset\}}$. Then $I \mapsto \emptyset$ is a morphisms family of A and o .
- (4) Let us consider an objects family A of I and $\text{Ens}_{\{\emptyset\}}$, an object o of $\text{Ens}_{\{\emptyset\}}$, and a morphisms family P of A and o . If $P = I \mapsto \emptyset$, then P is feasible and coprojection morphisms. PROOF: P is feasible by [11, (7)]. Reconsider $f = \emptyset$ as a morphism from o to Y . For every set i such that $i \in I$ there exists an object s_i of C and there exists a morphism P_i from s_i to o such that $s_i = A(i)$ and $P_i = P(i)$ and $F(i) = f \cdot P_i$ by [11, (7)]. \square

Let C be a category. We say that C has coproducts if and only if

- (Def. 7) Let us consider a set I and an objects family A of I and C . Then there exists an object B of C and there exists a morphisms family P of A and B such that P is feasible and coprojection morphisms.

Note that $\text{Ens}_{\{\emptyset\}}$ has coproducts and there exists a category which is strict and has products and coproducts.

Let C be a category, I be a set, A be an objects family of I and C , and B be an object of C . We say that B is A -category coproduct-like if and only if

- (Def. 8) There exists a morphisms family P of A and B such that P is feasible and coprojection morphisms.

Let C be a category with coproducts. Let us observe that there exists an object of C which is A -category coproduct-like.

Let C be a category and A be an objects family of \emptyset and C . Note that every object of C which is A -category coproduct-like is also initial.

Now we state the propositions:

- (5) Let us consider a category C , an objects family A of \emptyset and C , and an object B of C . If B is initial, then B is A -category coproduct-like. The theorem is a consequence of (2).
- (6) Let us consider a category C , an objects family A of I and C , and objects C_1, C_2 of C . Suppose
 - (i) C_1 is A -category coproduct-like, and
 - (ii) C_2 is A -category coproduct-like.

Then C_1, C_2 are iso.

From now on A denotes an objects family of I and Ens_E .

Let us consider I , E , and A . Assume $\bigcup \text{coprod}(A) \in E$. The functor $\coprod A$ yielding an object of Ens_E is defined by the term

(Def. 9) $\bigcup \text{coprod}(A)$.

The functor $\text{Coproduct}(A)$ yielding a many sorted set indexed by I is defined by

(Def. 10) Let us consider an element i . Suppose $i \in I$. Then there exists a function F from $A(i)$ into $\bigcup \text{coprod}(A)$ such that

(i) $it(i) = F$, and

(ii) for every element x such that $x \in A(i)$ holds $F(x) = \langle x, i \rangle$.

Observe that $\text{Coproduct}(A)$ is function yielding.

Assume $\bigcup \text{coprod}(A) \in E$. The functor $\coprod_{i \in I} A(i)$ yielding a morphisms family

of A and $\coprod A$ is defined by the term

(Def. 11) $\text{Coproduct}(A)$.

Now we state the propositions:

(7) If $\bigcup \text{coprod}(A) = \emptyset$, then $\text{Coproduct}(A)$ is empty yielding.

(8) If $\bigcup \text{coprod}(A) = \emptyset$, then A is empty yielding.

(9) If $\bigcup \text{coprod}(A) \in E$ and $\bigcup \text{coprod}(A) = \emptyset$, then $\coprod_{i \in I} A(i) = I \mapsto \emptyset$. The

theorem is a consequence of (7).

(10) If $\bigcup \text{coprod}(A) \in E$, then $\coprod_{i \in I} A(i)$ is feasible and coprojection morphisms.

The theorem is a consequence of (7) and (8).

(11) If $\bigcup \text{coprod}(A) \in E$, then $\coprod A$ is A -category coproduct-like. The theorem is a consequence of (10).

(12) If for every I and A , $\bigcup \text{coprod}(A) \in E$, then Ens_E has coproducts. The theorem is a consequence of (10).

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Formulation of Cell Petri Nets

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Summary. Based on the Petri net definitions and theorems already formalized in the Mizar article [13], in this article we were able to formalize the definition of cell Petri nets. It is based on [12]. Colored Petri net has already been defined in [11]. In addition, the conditions of the firing rule and the colored set to this definition, that defines the cell Petri nets are further extended to CPNT.i further. The synthesis of two Petri nets was introduced in [11] and in this work the definition is extended to produce the synthesis of a family of colored Petri nets. Specifically, the extension to a CPNT family is performed by specifying how to link the outbound transitions of each colored Petri net to the place elements of other nets to form a neighborhood relationship. Finally, the activation of colored Petri nets was formalized.

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The notation and terminology used in this paper have been introduced in the following articles: [1], [15], [10], [5], [6], [7], [17], [2], [3], [4], [8], [16], [13], [11], [19], [14], [18], and [9].

1. PRELIMINARIES

Let I be a non empty set and C_1 be a many sorted set indexed by I . We say that C_1 is colored Petri net family-like if and only if

(Def. 1) Let us consider an element i of I . Then $C_1(i)$ is a colored place/transition net.

Note that there exists a many sorted set indexed by I which is colored Petri net family-like.

A colored Petri net family of I is a colored Petri net family-like many sorted set indexed by I . Let C_1 be a colored Petri net family of I and i be an element

of I . One can check that the functor $C_1(i)$ yields a colored place/transition net. Let C_2 be a colored Petri net family of I . We say that C_2 is disjoint valued if and only if

- (Def. 2) Let us consider elements i, j of I . Suppose $i \neq j$. Then
- (i) the carrier of $C_2(i)$ misses the carrier of $C_2(j)$, and
 - (ii) the carrier' of $C_2(i)$ misses the carrier' of $C_2(j)$.

Now we state the propositions:

- (1) Let us consider a set I and many sorted sets F, D, R indexed by I . Suppose

- (i) for every element i such that $i \in I$ there exists a function f such that $f = F(i)$ and $\text{dom } f = D(i)$ and $\text{rng } f = R(i)$, and
- (ii) for every elements i, j and for every functions f, g such that $i, j \in I$ and $i \neq j$ and $f = F(i)$ and $g = F(j)$ holds $\text{dom } f$ misses $\text{dom } g$.

Then there exists a function G such that

- (iii) $G = \bigcup \text{rng } F$, and
- (iv) $\text{dom } G = \bigcup \text{rng } D$, and
- (v) $\text{rng } G = \bigcup \text{rng } R$, and
- (vi) for every elements i, x and for every function f such that $i \in I$ and $f = F(i)$ and $x \in \text{dom } f$ holds $G(x) = f(x)$.

PROOF: For every element z such that $z \in \bigcup \text{rng } F$ there exist elements x, y, i such that $z = \langle x, y \rangle$ and $z \in F(i)$ and $i \in I$. For every element z such that $z \in \bigcup \text{rng } F$ there exist elements x, y such that $z = \langle x, y \rangle$. Reconsider $G = \bigcup \text{rng } F$ as a binary relation. G is a function. For every element x , $x \in \text{dom } G$ iff $x \in \bigcup \text{rng } D$ by [5, (3)]. For every element x , $x \in \text{rng } G$ iff $x \in \bigcup \text{rng } R$ by [5, (3)]. For every elements i, x and for every function f such that $i \in I$ and $f = F(i)$ and $x \in \text{dom } f$ holds $G(x) = f(x)$ by [5, (1), (3)]. \square

- (2) Let us consider a set I and many sorted sets Y, Z indexed by I . Suppose elements i, j . If $i, j \in I$ and $i \neq j$, then $Y(i) \cap Z(j) = \emptyset$. Then $\bigcup(Y \setminus Z) = \bigcup Y \setminus \bigcup Z$. PROOF: Set $X = Y \setminus Z$. For every element x , $x \in \bigcup \text{rng } X$ iff $x \in \bigcup \text{rng } Y \setminus \bigcup \text{rng } Z$ by [5, (3)]. \square

- (3) Let us consider a set I and many sorted sets X, Y, Z indexed by I . Suppose

- (i) $X \subseteq Y \setminus Z$, and
- (ii) for every elements i, j such that $i, j \in I$ and $i \neq j$ holds $Y(i) \cap Z(j) = \emptyset$.

Then $\bigcup X \subseteq \bigcup Y \setminus \bigcup Z$. The theorem is a consequence of (2).

2. SYNTHESIS OF CPNT AND I

Let I be a non trivial set. The functor $\text{XorDelta } I$ yielding a non empty set is defined by the term

(Def. 3) $\{\langle i, j \rangle\}$, where i, j are elements of $I : i \neq j$.

Now we state the proposition:

(4) Let us consider a non trivial finite set I and a colored Petri net family C_2 of I . Then $\bigcup\{(\text{the carrier of } C_2(j))^{\text{Outbds}(C_2(i))}\}$, where i, j are elements of $I : i \neq j$ is not empty.

Let I be a non trivial finite set and C_2 be a colored Petri net family of I . A connecting mapping of C_2 is a many sorted set indexed by $\text{XorDelta } I$ and is defined by

(Def. 4) (i) $\text{rng } it \subseteq \bigcup\{(\text{the carrier of } C_2(j))^{\text{Outbds}(C_2(i))}\}$,
 where i, j are elements of $I : i \neq j$, and
 (ii) for every elements i, j of I such that $i \neq j$ holds $it(\langle i, j \rangle)$ is a function from $\text{Outbds}(C_2(i))$ into the carrier of $C_2(j)$.

Now we state the proposition:

(5) Let us consider colored place/transition nets C_4, C_5 , a function O_1 from $\text{Outbds } C_4$ into the carrier of C_5 , and a function q_1 . Suppose

(i) $\text{dom } q_1 = \text{Outbds } C_4$, and
 (ii) for every transition t_1 of C_4 such that t_1 is outbound holds $q_1(t_1)$ is a function from the thin cylinders of the colored set of C_4 and $^*\{t_1\}$ into the thin cylinders of the colored set of C_4 and $O_1 \circ t_1$.

Then $q_1 \in (\bigcup\{(\text{the thin cylinders of the colored set of } C_4 \text{ and } O_1 \circ t_1)^\alpha\})^{\text{Outbds } C_4}$, where α is the thin cylinders of the colored set of C_4 and $^*\{t_1\}$.

Let I be a non trivial finite set, C_2 be a colored Petri net family of I , and O be a connecting mapping of C_2 . A connecting firing rule of O is a many sorted set indexed by $\text{XorDelta } I$ and is defined by

(Def. 5) Let us consider elements i, j of I . Suppose $i \neq j$. Then there exists a function O_2 from $\text{Outbds}(C_2(i))$ into the carrier of $C_2(j)$ and there exists a function q_2 such that $q_2 = it(\langle i, j \rangle)$ and $O_2 = O(\langle i, j \rangle)$ and $\text{dom } q_2 = \text{Outbds}(C_2(i))$ and for every transition t_1 of $C_2(i)$ such that t_1 is outbound holds $q_2(t_1)$ is a function from the thin cylinders of the colored set of $C_2(i)$ and $^*\{t_1\}$ into the thin cylinders of the colored set of $C_2(i)$ and $O_2 \circ t_1$.

3. EXTENSION TO A FAMILY OF COLORED PETRI NETS

Let I be a non trivial finite set, C_2 be a colored Petri net family of I , O be a connecting mapping of C_2 , and q be a connecting firing rule of O . Assume C_2 is disjoint valued and for every elements i, j_1, j_2 of I such that $i \neq j_1$ and $i \neq j_2$ and there exist elements x, y_1, y_2 such that $\langle x, y_1 \rangle \in q(\langle i, j_1 \rangle)$ and $\langle x, y_2 \rangle \in q(\langle i, j_2 \rangle)$ holds $j_1 = j_2$. The functor synthesis q yielding a strict colored place/transition net is defined by

- (Def. 6) There exist many sorted sets P, T, S_1, T_1, C_3, F indexed by I and there exist functions U, U_1 such that for every element i of I , $P(i)$ = the carrier of $C_2(i)$ and $T(i)$ = the carrier' of $C_2(i)$ and $S_1(i)$ = the S-T arcs of $C_2(i)$ and $T_1(i)$ = the T-S arcs of $C_2(i)$ and $C_3(i)$ = the colored set of $C_2(i)$ and $F(i)$ = the firing rule of $C_2(i)$ and $U = \bigcup \text{rng } F$ and $U_1 = \bigcup \text{rng } q$ and the carrier of $it = \bigcup \text{rng } P$ and the carrier' of $it = \bigcup \text{rng } T$ and the S-T arcs of $it = \bigcup \text{rng } S_1$ and the T-S arcs of $it = \bigcup \text{rng } T_1 \cup \bigcup \text{rng } O$ and the colored set of $it = \bigcup \text{rng } C_3$ and the firing rule of $it = U + \cdot U_1$.

4. DEFINITION OF CELL PETRI NETS

Let I be a non empty finite set and C_2 be a colored Petri net family of I . We say that C_2 is cell Petri nets if and only if

- (Def. 7) There exists a function N from I into $2^{\text{rng } C_2}$ such that for every element i of I , $N(i) = \{C_2(j), \text{ where } j \text{ is an element of } I : j \neq i\}$.

Let N be a function from I into $2^{\text{rng } C_2}$ and O be a connecting mapping of C_2 . We say that (N, O) is cell Petri nets if and only if

- (Def. 8) Let us consider an element i of I . Then $N(i) = \{C_2(j), \text{ where } j \text{ is an element of } I : j \neq i \text{ and there exists a transition } t \text{ of } C_2(i) \text{ and there exists an element } s \text{ such that } \langle t, s \rangle \in O(\langle i, j \rangle)\}$.

Now we state the proposition:

- (6) Let us consider a non trivial finite set I , a colored Petri net family C_2 of I , a function N from I into $2^{\text{rng } C_2}$, and a connecting mapping O of C_2 . Suppose

- (i) C_2 is one-to-one, and
- (ii) (N, O) is cell Petri nets.

Let us consider an element i of I . Then $C_2(i) \notin N(i)$.

5. ACTIVATION OF PETRI NETS

Let C_6 be a colored place/transition net structure. We say that C_6 has nontrivial colored set if and only if

(Def. 9) The colored set of C_6 is not trivial.

One can verify that there exists a strict colored-PT-net-like colored Petri net which has nontrivial colored set.

Let C_2 be a colored place/transition net with nontrivial colored set. One can verify that the colored set of C_2 is non trivial.

Let C_6 be a colored place/transition net with nontrivial colored set, S be a subset of the carrier of C_6 , and D be a thin cylinder of the colored set of C_6 and S . A color threshold of D is a function from $\text{loc } D$ into the colored set of C_6 . Let C_6 be a colored place/transition net. A color count of C_6 is a function from the colored set of C_6 into \mathbb{N} . The colored states of C_6 yielding a non empty set is defined by the term

(Def. 10) the set of all e where e is a color count of C_6 .

A colored state of C_6 is a function from C_6 into the colored states of C_6 . From now on C_6 denotes a colored place/transition net with nontrivial colored set, m denotes a colored state of C_6 , and t denotes an element of the carrier' of C_6 .

Let C_6 be a colored place/transition net with nontrivial colored set, m be a colored state of C_6 , and p be a place of C_6 . Observe that the functor $m(p)$ yields a color count of C_6 . Let m_1 be a color count of C_6 and x be an element. Let us observe that the functor $m_1(x)$ yields an element of \mathbb{N} . Let us consider C_6 , m , and t . Let D be a thin cylinder of the colored set of C_6 and $\ast\{t\}$ and C_7 be a color threshold of D . We say that t is firable on m and C_7 if and only if

(Def. 11) (i) (the firing rule of C_6)($\langle t, D \rangle$) $\neq \emptyset$, and

(ii) for every place p of C_6 such that $p \in \text{loc } D$ holds $1 \leq m(p)(C_7(p))$.

The firable set on m and t yielding a set is defined by the term

(Def. 12) $\{D, \text{ where } D \text{ is a thin cylinder of the colored set of } C_6 \text{ and } \ast\{t\} : \text{ there exists a color threshold } C_7 \text{ of } D \text{ such that } t \text{ is firable on } m \text{ and } C_7\}$.

Now we state the proposition:

(7) Let us consider a thin cylinder D of the colored set of C_6 and $\ast\{t\}$. Then there exists a color threshold C_7 of D such that t is firable on m and C_7 if and only if $D \in$ the firable set on m and t .

Let us consider C_6 , m , and t . Let D be a thin cylinder of the colored set of C_6 and $\ast\{t\}$, C_7 be a color threshold of D , and p be an element of C_6 . Assume t is firable on m and C_7 . The Petri subtraction(C_7, m, p) yielding a function from the colored set of C_6 into \mathbb{N} is defined by

(Def. 13) Let us consider an element x of the colored set of C_6 . Then

- (i) if $p \in \text{loc } D$ and $x = C_7(p)$, then $it(x) = m(p)(x) - 1$, and
- (ii) if it is not true that $p \in \text{loc } D$ and $x = C_7(p)$, then $it(x) = m(p)(x)$.

Let D be a thin cylinder of the colored set of C_6 and $\overline{\{t\}}$. The Petri addition (C_7, m, p) yielding a function from the colored set of C_6 into \mathbb{N} is defined by

(Def. 14) Let us consider an element x of the colored set of C_6 . Then

- (i) if $p \in \text{loc } D$ and $x = C_7(p)$, then $it(x) = m(p)(x) + 1$, and
- (ii) if it is not true that $p \in \text{loc } D$ and $x = C_7(p)$, then $it(x) = m(p)(x)$.

Let D be a thin cylinder of the colored set of C_6 and $^*\{t\}$ and E be a thin cylinder of the colored set of C_6 and $\overline{\{t\}}$. Let C_{10} be a color threshold of E . The firing result (C_7, C_{10}, m, p) yielding a function from the colored set of C_6 into \mathbb{N} is defined by the term

$$(\text{Def. 15}) \quad \left\{ \begin{array}{ll} \text{the Petri subtraction}(C_7, m, p), & \text{if } t \text{ is firable on } m \text{ and } C_7, \text{ and } p \in \text{loc } D \setminus \text{loc } E, \\ \text{the Petri addition}(C_{10}, m, p), & \text{if } t \text{ is firable on } m \text{ and } C_7, \text{ and } p \in \text{loc } E \setminus \text{loc } D, \\ m(p), & \text{otherwise.} \end{array} \right.$$

Let us consider a thin cylinder D_1 of the colored set of C_6 and $^*\{t\}$, a thin cylinder D_2 of the colored set of C_6 and $\overline{\{t\}}$, a color threshold C_8 of D_1 , a color threshold C_9 of D_2 , an element x of the colored set of C_6 , and an element p of C_6 . Now we state the propositions:

- (8) $m(p)(x) - 1 \leq (\text{the firing result}(C_8, C_9, m, p))(x) \leq m(p)(x) + 1$.
- (9) If t is outbound, then $m(p)(x) - 1 \leq (\text{the firing result}(C_8, C_9, m, p))(x) \leq m(p)(x)$.

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Isometric Differentiable Functions on Real Normed Space¹

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Summary. In this article, we formalize isometric differentiable functions on real normed space [17], and their properties.

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The notation and terminology used in this paper have been introduced in the following articles: [3], [2], [8], [4], [5], [18], [10], [11], [19], [14], [16], [1], [6], [9], [15], [23], [24], [21], [22], [13], [25], and [7].

1. PRELIMINARIES

From now on S, T, W, Y denote real normed spaces, f, f_1, f_2 denote partial functions from S to T , Z denotes a subset of S , and i, n denote natural numbers.

Now we state the propositions:

- (1) Let us consider a set X and functions I, f . Then $(f \upharpoonright X) \cdot I = (f \cdot I) \upharpoonright I^{-1}(X)$.
- (2) Let us consider real normed spaces S, T , a linear operator L from S into T , and points x, y of S . Then $L(x) - L(y) = L(x - y)$.

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(3) Let us consider real normed spaces X, Y, W , a function I from X into Y , and partial functions f_1, f_2 from Y to W . Then

(i) $(f_1 + f_2) \cdot I = f_1 \cdot I + f_2 \cdot I$, and

(ii) $(f_1 - f_2) \cdot I = f_1 \cdot I - f_2 \cdot I$.

PROOF: Set $D_1 =$ the carrier of X . For every element s of D_1 , $s \in \text{dom}((f_1 + f_2) \cdot I)$ iff $s \in \text{dom}(f_1 \cdot I + f_2 \cdot I)$ by [4, (11)]. For every element z of D_1 such that $z \in \text{dom}((f_1 + f_2) \cdot I)$ holds $((f_1 + f_2) \cdot I)(z) = (f_1 \cdot I + f_2 \cdot I)(z)$ by [4, (11), (12)]. For every element s of D_1 , $s \in \text{dom}((f_1 - f_2) \cdot I)$ iff $s \in \text{dom}(f_1 \cdot I - f_2 \cdot I)$ by [4, (11)]. For every element z of D_1 such that $z \in \text{dom}((f_1 - f_2) \cdot I)$ holds $((f_1 - f_2) \cdot I)(z) = (f_1 \cdot I - f_2 \cdot I)(z)$ by [4, (11), (12)]. \square

(4) Let us consider real normed spaces X, Y, W , a function I from X into Y , a partial function f from Y to W , and a real number r . Then $r \cdot (f \cdot I) = (r \cdot f) \cdot I$. PROOF: Set $D_1 =$ the carrier of X . For every element s of D_1 , $s \in \text{dom}((r \cdot f) \cdot I)$ iff $s \in \text{dom}(f \cdot I)$ by [4, (11)]. For every element s of D_1 , $s \in \text{dom}(r \cdot (f \cdot I))$ iff $I(s) \in \text{dom}(r \cdot f)$ by [4, (11)]. For every element z of D_1 such that $z \in \text{dom}(r \cdot (f \cdot I))$ holds $(r \cdot (f \cdot I))(z) = ((r \cdot f) \cdot I)(z)$ by [4, (12)]. \square

(5) Let us consider a partial function f from T to W , a function g from S into T , and a point x of S . Suppose

(i) $x \in \text{dom } g$, and

(ii) $g_x \in \text{dom } f$, and

(iii) g is continuous in x , and

(iv) f is continuous in g_x .

Then $f \cdot g$ is continuous in x . PROOF: Set $h = f \cdot g$. For every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of S such that $x_1 \in \text{dom } h$ and $\|x_1 - x\| < s$ holds $\|h_{x_1} - h_x\| < r$ by [14, (7)], [12, (3), (4)]. \square

Let X, Y be real normed spaces and x be an element of $X \times Y$. The functor $\text{reproj1}(x)$ yielding a function from X into $X \times Y$ is defined by

(Def. 1) Let us consider an element r of X . Then $it(r) = \langle r, x_2 \rangle$.

The functor $\text{reproj2}(x)$ yielding a function from Y into $X \times Y$ is defined by

(Def. 2) Let us consider an element r of Y . Then $it(r) = \langle x_1, r \rangle$.

2. ISOMETRIES

Now we state the propositions:

(6) Let us consider a linear operator I from S into T and a point x of S . If I is isometric, then I is continuous in x .

- (7) Let us consider real normed spaces S , T and a linear operator f from S into T . Then f is isometric if and only if for every element x of S , $\|f(x)\| = \|x\|$. The theorem is a consequence of (2).
- (8) Let us consider a linear operator I from S into T and a subset Z of S . If I is isometric, then I is continuous on Z . The theorem is a consequence of (6).
- (9) Let us consider a linear operator I from S into T . Suppose I is one-to-one, onto, and isometric. Then there exists a linear operator J from T into S such that
- (i) $J = I^{-1}$, and
 - (ii) J is one-to-one, onto, and isometric.

The theorem is a consequence of (7). PROOF: Reconsider $J = I^{-1}$ as a function from T into S . For every points v , w of T , $J(v+w) = J(v) + J(w)$ by [5, (113)], [4, (34)]. For every point v of T and for every real number r , $J(r \cdot v) = r \cdot J(v)$ by [5, (113)], [4, (34)]. For every point v of T , $\|J(v)\| = \|v\|$ by [5, (113)], [4, (34)]. \square

Let us consider a linear operator I from S into T and a sequence s_1 of S . Now we state the propositions:

- (10) If I is isometric and s_1 is convergent, then $I \cdot s_1$ is convergent and $\lim(I \cdot s_1) = I(\lim s_1)$.
- (11) If I is one-to-one, onto, and isometric, then s_1 is convergent iff $I \cdot s_1$ is convergent.

Let us consider a linear operator I from S into T and a subset Z of S . Now we state the propositions:

- (12) If I is one-to-one, onto, and isometric, then Z is closed iff $I^\circ Z$ is closed.
- (13) If I is one-to-one, onto, and isometric, then Z is open iff $I^\circ Z$ is open.
- (14) If I is one-to-one, onto, and isometric, then Z is compact iff $I^\circ Z$ is compact.

Now we state the propositions:

- (15) Let us consider a partial function f from T to W and a linear operator I from S into T . Suppose I is one-to-one, onto, and isometric. Let us consider a point x of S . Suppose $I(x) \in \text{dom } f$. Then $f \cdot I$ is continuous in x if and only if f is continuous in $I(x)$. The theorem is a consequence of (9), (6), and (5).
- (16) Let us consider a partial function f from T to W , a linear operator I from S into T , and a set X . Suppose
- (i) $X \subseteq \text{the carrier of } T$, and
 - (ii) I is one-to-one, onto, and isometric.

Then f is continuous on X if and only if $f \cdot I$ is continuous on $I^{-1}(X)$. The theorem is a consequence of (15) and (1). PROOF: For every point y of T such that $y \in X$ holds $f|_X$ is continuous in y by [5, (113)], [23, (57)]. \square

Let X, Y be real normed spaces. The functor $\text{IsoCPNrSP}(X, Y)$ yielding a linear operator from $X \times Y$ into $\prod\langle X, Y \rangle$ is defined by

(Def. 3) Let us consider a point x of X and a point y of Y . Then $it(x, y) = \langle x, y \rangle$.

Now we state the proposition:

(17) Let us consider real normed spaces X, Y . Then $0_{\prod\langle X, Y \rangle} = (\text{IsoCPNrSP}(X, Y))(0_{X \times Y})$.

Let X, Y be real normed spaces. Observe that $\text{IsoCPNrSP}(X, Y)$ is one-to-one onto and isometric.

Let us note that there exists a linear operator from $X \times Y$ into $\prod\langle X, Y \rangle$ which is one-to-one, onto, and isometric.

Let f be a one-to-one onto isometric linear operator from $X \times Y$ into $\prod\langle X, Y \rangle$. Let us note that the functor f^{-1} yields a linear operator from $\prod\langle X, Y \rangle$ into $X \times Y$. One can verify that f^{-1} is one-to-one onto and isometric as a linear operator from $\prod\langle X, Y \rangle$ into $X \times Y$.

Observe that there exists a linear operator from $\prod\langle X, Y \rangle$ into $X \times Y$ which is one-to-one, onto, and isometric.

Now we state the propositions:

(18) Let us consider real normed spaces X, Y , a point x of X , and a point y of Y . Then $(\text{IsoCPNrSP}(X, Y))^{-1}(\langle x, y \rangle) = \langle x, y \rangle$. PROOF: Set $I = \text{IsoCPNrSP}(X, Y)$. Set $J = I^{-1}$. For every point x of X and for every point y of Y , $J(\langle x, y \rangle) = \langle x, y \rangle$ by [4, (34)]. \square

(19) Let us consider real normed spaces X, Y . Then $(\text{IsoCPNrSP}(X, Y))^{-1}(0_{\prod\langle X, Y \rangle}) = 0_{X \times Y}$. The theorem is a consequence of (17).

(20) Let us consider real normed spaces X, Y and a subset Z of $X \times Y$. Then $\text{IsoCPNrSP}(X, Y)$ is continuous on Z .

(21) Let us consider real normed spaces X, Y and a subset Z of $\prod\langle X, Y \rangle$. Then $(\text{IsoCPNrSP}(X, Y))^{-1}$ is continuous on Z .

(22) Let us consider real normed spaces S, T, W , a point f of the real norm space of bounded linear operators from S into W , a point g of the real norm space of bounded linear operators from T into W , and a linear operator I from S into T . Suppose

- (i) I is one-to-one, onto, and isometric, and
- (ii) $f = g \cdot I$.

Then $\|f\| = \|g\|$. The theorem is a consequence of (9) and (7). PROOF: Consider J being a linear operator from T into S such that $J = I^{-1}$ and

J is one-to-one, onto, and isometric. Reconsider $g_0 = g$ as a Lipschitzian linear operator from T into W . Reconsider $g_3 = g \cdot I$ as a Lipschitzian linear operator from S into W . For every element x , $x \in \{\|g_0(t)\|\}$, where t is a vector of $T : \|t\| \leq 1\}$ iff $x \in \{\|g_3(w)\|\}$, where w is a vector of $S : \|w\| \leq 1\}$ by [4, (13), (35)]. \square

Let us consider S and T . One can verify that every linear operator from S into T which is isometric is also Lipschitzian.

3. ISOMETRIC DIFFERENTIABLE FUNCTIONS ON REAL NORMED SPACE

Let us consider a real norm space sequence G , a real normed space F , a set i , partial functions f, g from $\prod G$ to F , and a subset X of $\prod G$. Now we state the propositions:

- (23) Suppose X is open and $i \in \text{dom } G$ and f is partially differentiable on X w.r.t. i and g is partially differentiable on X w.r.t. i . Then
- (i) $f + g$ is partially differentiable on X w.r.t. i , and
 - (ii) $(f + g)|^i X = (f|^i X) + (g|^i X)$.
- (24) Suppose X is open and $i \in \text{dom } G$ and f is partially differentiable on X w.r.t. i and g is partially differentiable on X w.r.t. i . Then
- (i) $f - g$ is partially differentiable on X w.r.t. i , and
 - (ii) $(f - g)|^i X = (f|^i X) - (g|^i X)$.

Now we state the propositions:

- (25) Let us consider a real norm space sequence G , a real normed space F , a set i , a partial function f from $\prod G$ to F , a real number r , and a subset X of $\prod G$. Suppose
- (i) X is open, and
 - (ii) $i \in \text{dom } G$, and
 - (iii) f is partially differentiable on X w.r.t. i .

Then

- (iv) $r \cdot f$ is partially differentiable on X w.r.t. i , and
- (v) $r \cdot f|^i X = r \cdot (f|^i X)$.

PROOF: Set $h = r \cdot f$. For every point x of $\prod G$ such that $x \in X$ holds h is partially differentiable in x w.r.t. i and $\text{partdiff}(h, x, i) = r \cdot \text{partdiff}(f, x, i)$ by [19, (24), (30)]. Set $f_3 = f|^i X$. For every point x of $\prod G$ such that $x \in X$ holds $(r \cdot f_3)_x = \text{partdiff}(h, x, i)$. \square

- (26) Let us consider real normed spaces S, T , a Lipschitzian linear operator L from S into T , and a point x_0 of S . Then

- (i) L is differentiable in x_0 , and
- (ii) $L'(x_0) = L$.

The theorem is a consequence of (2). PROOF: Reconsider $L_0 = L$ as a point of the real norm space of bounded linear operators from S into T . Reconsider $R = (\text{the carrier of } S) \mapsto 0_T$ as a partial function from S to T . Set $N =$ the neighbourhood of x_0 . For every point x of S such that $x \in N$ holds $L_{0x} - L_{0x_0} = L(x - x_0) + R_{x-x_0}$ by [20, (7)], [21, (4)]. \square

- (27) Let us consider a partial function f from T to W , a Lipschitzian linear operator I from S into T , and a point I_0 of the real norm space of bounded linear operators from S into T . Suppose $I_0 = I$. Let us consider a point x of S . Suppose f is differentiable in $I(x)$. Then

- (i) $f \cdot I$ is differentiable in x , and
- (ii) $(f \cdot I)'(x) = f'(I(x)) \cdot I_0$.

The theorem is a consequence of (26).

- (28) Let us consider a partial function f from T to W and a linear operator I from S into T . Suppose

- (i) I is one-to-one and onto, and
- (ii) I is isometric.

Let us consider a point x of S . Then $f \cdot I$ is differentiable in x if and only if f is differentiable in $I(x)$. The theorem is a consequence of (9), (26), and (27).

- (29) Let us consider a partial function f from T to W , a linear operator I from S into T , and a set X . Suppose

- (i) $X \subseteq$ the carrier of T , and
- (ii) I is one-to-one and onto, and
- (iii) I is isometric.

Then f is differentiable on X if and only if $f \cdot I$ is differentiable on $I^{-1}(X)$. The theorem is a consequence of (28) and (1). PROOF: For every point y of T such that $y \in X$ holds $f \upharpoonright X$ is differentiable in y by [5, (113)]. \square

- (30) Let us consider real normed spaces X, Y , a partial function f from $\coprod\langle X, Y \rangle$ to W , and a subset D of $\coprod\langle X, Y \rangle$. Suppose f is differentiable on D . Let us consider a point z of $\coprod\langle X, Y \rangle$. Suppose $z \in \text{dom } f'_{\upharpoonright D}$. Then $f'_{\upharpoonright D}(z) = ((f \cdot \text{IsoCPNrSP}(X, Y))'_{\upharpoonright (\text{IsoCPNrSP}(X, Y)^{-1}(D))} (\text{IsoCPNrSP}(X, Y)^{-1}(z))) (\text{IsoCPNrSP}(X, Y)^{-1})$. The theorem is a consequence of (17), (29), and (27). PROOF: Set $I = \text{IsoCPNrSP}(X, Y)$. Set $J = (\text{IsoCPNrSP}(X, Y))^{-1}$. Set $g = f \cdot I$. Set $E = I^{-1}(D)$. For every point z of $\coprod\langle X, Y \rangle$ such that $z \in \text{dom } f'_{\upharpoonright D}$ holds $f'_{\upharpoonright D}(z) = (g'_{\upharpoonright E})_{J(z)} \cdot I^{-1}$ by [10, (31)], [5, (113)], [23, (36)]. \square

- (31) Let us consider real normed spaces X, Y , a partial function f from $X \times Y$ to W , and a subset D of $X \times Y$. Suppose f is differentiable on D . Let us consider a point z of $X \times Y$. Suppose $z \in \text{dom } f'_{|D}$. Then $f'_{|D}(z) = ((f \cdot (\text{IsoCPNrSP}(X, Y))^{-1})'_{|((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}(D)})_{(\text{IsoCPNrSP}(X, Y))(z)} \cdot ((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}$. The theorem is a consequence of (18), (19), (17), (29), and (27). PROOF: Set $I = (\text{IsoCPNrSP}(X, Y))^{-1}$. Set $J = \text{IsoCPNrSP}(X, Y)$. Set $g = f \cdot I$. Set $E = I^{-1}(D)$. For every point z of $X \times Y$ such that $z \in \text{dom } f'_{|D}$ holds $f'_{|D}(z) = (g'_{|E})_{J(z)} \cdot I^{-1}$ by [10, (31)], [5, (113)], [23, (36)]. \square
- (32) Let us consider real normed spaces X, Y and a point z of $X \times Y$. Then
- (i) $\text{reproj1}(z) = (\text{IsoCPNrSP}(X, Y))^{-1} \cdot \text{reproj}(1(\in \text{dom}\langle X, Y \rangle), (\text{IsoCPNrSP}(X, Y))(z))$, and
 - (ii) $\text{reproj2}(z) = (\text{IsoCPNrSP}(X, Y))^{-1} \cdot \text{reproj}(2(\in \text{dom}\langle X, Y \rangle), (\text{IsoCPNrSP}(X, Y))(z))$.

The theorem is a consequence of (18).

Let X, Y be real normed spaces and z be a point of $X \times Y$. Let us note that the functor z_1 yields a point of X . One can verify that the functor z_2 yields a point of Y . Let X, Y, W be real normed spaces. Let f be a partial function from $X \times Y$ to W . We say that f is partially differentiable in z w.r.t. 1 if and only if

(Def. 4) $f \cdot \text{reproj1}(z)$ is differentiable in z_1 .

We say that f is partially differentiable in z w.r.t. 2 if and only if

(Def. 5) $f \cdot \text{reproj2}(z)$ is differentiable in z_2 .

Now we state the propositions:

- (33) Let us consider real normed spaces X, Y and a point z of $X \times Y$. Then
- (i) $z_1 = \text{the projection onto } 1(\in \text{dom}\langle X, Y \rangle)((\text{IsoCPNrSP}(X, Y))(z))$, and
 - (ii) $z_2 = \text{the projection onto } 2(\in \text{dom}\langle X, Y \rangle)((\text{IsoCPNrSP}(X, Y))(z))$.
- (34) Let us consider real normed spaces X, Y, W , a point z of $X \times Y$, and a partial function f from $X \times Y$ to W . Then
- (i) f is partially differentiable in z w.r.t. 1 iff $f \cdot (\text{IsoCPNrSP}(X, Y))^{-1}$ is partially differentiable in $(\text{IsoCPNrSP}(X, Y))(z)$ w.r.t. 1, and
 - (ii) f is partially differentiable in z w.r.t. 2 iff $f \cdot (\text{IsoCPNrSP}(X, Y))^{-1}$ is partially differentiable in $(\text{IsoCPNrSP}(X, Y))(z)$ w.r.t. 2.

The theorem is a consequence of (32) and (33).

Let X, Y, W be real normed spaces, z be a point of $X \times Y$, and f be a partial function from $X \times Y$ to W . The functor $\text{partdiff}(f, z)$ w.r.t. 1 yielding a point of the real norm space of bounded linear operators from X into W is defined by the term

(Def. 6) $(f \cdot \text{reproj1}(z))'(z_1)$.

The functor $\text{partdiff}(f, z)$ w.r.t. 2 yielding a point of the real norm space of bounded linear operators from Y into W is defined by the term

(Def. 7) $(f \cdot \text{reproj2}(z))'(z_2)$.

Now we state the proposition:

(35) Let us consider real normed spaces X, Y, W , a point z of $X \times Y$, and a partial function f from $X \times Y$ to W . Then

- (i) $\text{partdiff}(f, z)$ w.r.t. 1 = $\text{partdiff}(f \cdot (\text{IsoCPNrSP}(X, Y))^{-1}, (\text{IsoCPNrSP}(X, Y))(z), 1)$, and
- (ii) $\text{partdiff}(f, z)$ w.r.t. 2 = $\text{partdiff}(f \cdot (\text{IsoCPNrSP}(X, Y))^{-1}, (\text{IsoCPNrSP}(X, Y))(z), 2)$.

The theorem is a consequence of (32) and (33).

Let us consider real normed spaces X, Y, W , a point z of $X \times Y$, and partial functions f_1, f_2 from $X \times Y$ to W . Now we state the propositions:

(36) Suppose f_1 is partially differentiable in z w.r.t. 1 and f_2 is partially differentiable in z w.r.t. 1. Then

- (i) $f_1 + f_2$ is partially differentiable in z w.r.t. 1, and
- (ii) $\text{partdiff}(f_1 + f_2, z)$ w.r.t. 1 = $\text{partdiff}(f_1, z)$ w.r.t. 1 + $\text{partdiff}(f_2, z)$ w.r.t. 1, and
- (iii) $f_1 - f_2$ is partially differentiable in z w.r.t. 1, and
- (iv) $\text{partdiff}(f_1 - f_2, z)$ w.r.t. 1 = $\text{partdiff}(f_1, z)$ w.r.t. 1 - $\text{partdiff}(f_2, z)$ w.r.t. 1.

(37) Suppose f_1 is partially differentiable in z w.r.t. 2 and f_2 is partially differentiable in z w.r.t. 2. Then

- (i) $f_1 + f_2$ is partially differentiable in z w.r.t. 2, and
- (ii) $\text{partdiff}(f_1 + f_2, z)$ w.r.t. 2 = $\text{partdiff}(f_1, z)$ w.r.t. 2 + $\text{partdiff}(f_2, z)$ w.r.t. 2, and
- (iii) $f_1 - f_2$ is partially differentiable in z w.r.t. 2, and
- (iv) $\text{partdiff}(f_1 - f_2, z)$ w.r.t. 2 = $\text{partdiff}(f_1, z)$ w.r.t. 2 - $\text{partdiff}(f_2, z)$ w.r.t. 2.

Let us consider real normed spaces X, Y, W , a point z of $X \times Y$, a real number r , and a partial function f from $X \times Y$ to W . Now we state the propositions:

(38) Suppose f is partially differentiable in z w.r.t. 1. Then

- (i) $r \cdot f$ is partially differentiable in z w.r.t. 1, and
- (ii) $\text{partdiff}(r \cdot f, z)$ w.r.t. 1 = $r \cdot \text{partdiff}(f, z)$ w.r.t. 1.

(39) Suppose f is partially differentiable in z w.r.t. 2. Then

- (i) $r \cdot f$ is partially differentiable in z w.r.t. 2, and
- (ii) $\text{partdiff}(r \cdot f, z)$ w.r.t. 2 = $r \cdot \text{partdiff}(f, z)$ w.r.t. 2.

Let X, Y, W be real normed spaces, Z be a set, and f be a partial function from $X \times Y$ to W . We say that f is partially differentiable on Z w.r.t. 1 if and only if

- (Def. 8) (i) $Z \subseteq \text{dom } f$, and
- (ii) for every point z of $X \times Y$ such that $z \in Z$ holds $f \upharpoonright Z$ is partially differentiable in z w.r.t. 1.

We say that f is partially differentiable on Z w.r.t. 2 if and only if

- (Def. 9) (i) $Z \subseteq \text{dom } f$, and
- (ii) for every point z of $X \times Y$ such that $z \in Z$ holds $f \upharpoonright Z$ is partially differentiable in z w.r.t. 2.

Now we state the proposition:

- (40) Let us consider real normed spaces X, Y, W , a subset Z of $X \times Y$, and a partial function f from $X \times Y$ to W . Then

- (i) f is partially differentiable on Z w.r.t. 1 iff $f \cdot (\text{IsoCPNrSP}(X, Y))^{-1}$ is partially differentiable on $((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)$ w.r.t. 1, and
- (ii) f is partially differentiable on Z w.r.t. 2 iff $f \cdot (\text{IsoCPNrSP}(X, Y))^{-1}$ is partially differentiable on $((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)$ w.r.t. 2.

The theorem is a consequence of (18), (19), (17), (34), and (1). PROOF: Set $I = (\text{IsoCPNrSP}(X, Y))^{-1}$. Set $g = f \cdot I$. Set $E = I^{-1}(Z)$. f is partially differentiable on Z w.r.t. 1 iff g is partially differentiable on E w.r.t. 1 by [5, (113)], [4, (34)], [5, (38)]. f is partially differentiable on Z w.r.t. 2 iff g is partially differentiable on E w.r.t. 2 by [5, (113)], [4, (34)], [5, (38)]. \square

Let X, Y, W be real normed spaces, Z be a set, and f be a partial function from $X \times Y$ to W . Assume f is partially differentiable on Z w.r.t. 1. The functor $f \upharpoonright^1 Z$ yielding a partial function from $X \times Y$ to the real norm space of bounded linear operators from X into W is defined by

- (Def. 10) (i) $\text{dom } it = Z$, and
- (ii) for every point z of $X \times Y$ such that $z \in Z$ holds $it_z = \text{partdiff}(f, z)$ w.r.t. 1.

Assume f is partially differentiable on Z w.r.t. 2. The functor $f \upharpoonright^2 Z$ yielding a partial function from $X \times Y$ to the real norm space of bounded linear operators from Y into W is defined by

- (Def. 11) (i) $\text{dom } it = Z$, and
- (ii) for every point z of $X \times Y$ such that $z \in Z$ holds $it_z = \text{partdiff}(f, z)$ w.r.t. 2.

Let us consider real normed spaces X, Y, W , a subset Z of $X \times Y$, and a partial function f from $X \times Y$ to W . Now we state the propositions:

- (41) Suppose f is partially differentiable on Z w.r.t. 1. Then $f \upharpoonright^1 Z = (f \cdot (\text{IsoCPNrSP}(X, Y))^{-1} \upharpoonright^1 ((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)) \cdot \text{IsoCPNrSP}(X, Y)$.
- (42) Suppose f is partially differentiable on Z w.r.t. 2. Then $f \upharpoonright^2 Z = (f \cdot (\text{IsoCPNrSP}(X, Y))^{-1} \upharpoonright^2 ((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)) \cdot \text{IsoCPNrSP}(X, Y)$.
- (43) Suppose Z is open. Then f is partially differentiable on Z w.r.t. 1 if and only if $Z \subseteq \text{dom } f$ and for every point x of $X \times Y$ such that $x \in Z$ holds f is partially differentiable in x w.r.t. 1.
- (44) Suppose Z is open. Then f is partially differentiable on Z w.r.t. 2 if and only if $Z \subseteq \text{dom } f$ and for every point x of $X \times Y$ such that $x \in Z$ holds f is partially differentiable in x w.r.t. 2.

Let us consider real normed spaces X, Y, W , a subset Z of $X \times Y$, and partial functions f, g from $X \times Y$ to W . Now we state the propositions:

- (45) Suppose Z is open and f is partially differentiable on Z w.r.t. 1 and g is partially differentiable on Z w.r.t. 1. Then
 - (i) $f + g$ is partially differentiable on Z w.r.t. 1, and
 - (ii) $(f + g) \upharpoonright^1 Z = (f \upharpoonright^1 Z) + (g \upharpoonright^1 Z)$.
- (46) Suppose Z is open and f is partially differentiable on Z w.r.t. 1 and g is partially differentiable on Z w.r.t. 1. Then
 - (i) $f - g$ is partially differentiable on Z w.r.t. 1, and
 - (ii) $(f - g) \upharpoonright^1 Z = (f \upharpoonright^1 Z) - (g \upharpoonright^1 Z)$.
- (47) Suppose Z is open and f is partially differentiable on Z w.r.t. 2 and g is partially differentiable on Z w.r.t. 2. Then
 - (i) $f + g$ is partially differentiable on Z w.r.t. 2, and
 - (ii) $(f + g) \upharpoonright^2 Z = (f \upharpoonright^2 Z) + (g \upharpoonright^2 Z)$.
- (48) Suppose Z is open and f is partially differentiable on Z w.r.t. 2 and g is partially differentiable on Z w.r.t. 2. Then
 - (i) $f - g$ is partially differentiable on Z w.r.t. 2, and
 - (ii) $(f - g) \upharpoonright^2 Z = (f \upharpoonright^2 Z) - (g \upharpoonright^2 Z)$.

Let us consider real normed spaces X, Y, W , a subset Z of $X \times Y$, a real number r , and a partial function f from $X \times Y$ to W . Now we state the propositions:

- (49) Suppose Z is open and f is partially differentiable on Z w.r.t. 1. Then
 - (i) $r \cdot f$ is partially differentiable on Z w.r.t. 1, and
 - (ii) $r \cdot f \upharpoonright^1 Z = r \cdot (f \upharpoonright^1 Z)$.
- (50) Suppose Z is open and f is partially differentiable on Z w.r.t. 2. Then

- (i) $r \cdot f$ is partially differentiable on Z w.r.t. 2, and
- (ii) $r \cdot f \upharpoonright^2 Z = r \cdot (f \upharpoonright^2 Z)$.

Let us consider real normed spaces X, Y, W , a subset Z of $X \times Y$, and a partial function f from $X \times Y$ to W . Now we state the propositions:

- (51) Suppose f is differentiable on Z . Then $f'_{\upharpoonright Z}$ is continuous on Z if and only if $(f \cdot (\text{IsoCPNrSP}(X, Y))^{-1})'_{\upharpoonright ((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)}$ is continuous on $((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)$.
- (52) Suppose Z is open. Then f is partially differentiable on Z w.r.t. 1 and f is partially differentiable on Z w.r.t. 2 and $f \upharpoonright^1 Z$ is continuous on Z and $f \upharpoonright^2 Z$ is continuous on Z if and only if f is differentiable on Z and $f'_{\upharpoonright Z}$ is continuous on Z .

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Differential Equations on Functions from \mathbb{R} into Real Banach Space¹

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Summary. In this article, we describe the differential equations on functions from \mathbb{R} into real Banach space. The descriptions are based on the article [20]. As preliminary to the proof of these theorems, we proved some properties of differentiable functions on real normed space. For the proof we referred to descriptions and theorems in the article [21] and the article [32]. And applying the theorems of Riemann integral introduced in the article [22], we proved the ordinary differential equations on real Banach space. We referred to the methods of proof in [30].

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The notation and terminology used in this paper have been introduced in the following articles: [29], [5], [11], [3], [6], [7], [19], [13], [34], [31], [33], [1], [15], [25], [32], [18], [24], [23], [26], [27], [20], [2], [8], [14], [16], [28], [12], [37], [38], [9], [35], [36], [17], and [10].

1. SOME PROPERTIES OF DIFFERENTIABLE FUNCTIONS ON REAL NORMED SPACE

From now on Y denotes a real normed space.

Now we state the propositions:

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(1) Let us consider a real normed space Y , a function J from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into \mathbb{R} , a point x_0 of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, an element y_0 of \mathbb{R} , a partial function g from \mathbb{R} to Y , and a partial function f from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to Y . Suppose

- (i) $J = \text{proj}(1, 1)$, and
- (ii) $x_0 \in \text{dom } f$, and
- (iii) $y_0 \in \text{dom } g$, and
- (iv) $x_0 = \langle y_0 \rangle$, and
- (v) $f = g \cdot J$.

Then f is continuous in x_0 if and only if g is continuous in y_0 . PROOF: If f is continuous in x_0 , then g is continuous in y_0 by [14, (2)], [6, (39)], [37, (36)]. \square

(2) Let us consider a real normed space Y , a function I from \mathbb{R} into $\langle \mathcal{E}^1, \|\cdot\| \rangle$, a point x_0 of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, an element y_0 of \mathbb{R} , a partial function g from \mathbb{R} to Y , and a partial function f from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to Y . Suppose

- (i) $I = (\text{proj}(1, 1) \text{ qua function})^{-1}$, and
- (ii) $x_0 \in \text{dom } f$, and
- (iii) $y_0 \in \text{dom } g$, and
- (iv) $x_0 = \langle y_0 \rangle$, and
- (v) $f \cdot I = g$.

Then f is continuous in x_0 if and only if g is continuous in y_0 . PROOF: If f is continuous in x_0 , then g is continuous in y_0 by [14, (1)], [21, (33)], [26, (15)]. \square

(3) Let us consider a function I from \mathbb{R} into $\langle \mathcal{E}^1, \|\cdot\| \rangle$.

Suppose $I = (\text{proj}(1, 1) \text{ qua function})^{-1}$. Then

- (i) for every rest R of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, Y , $R \cdot I$ is a rest of Y , and
- (ii) for every linear operator L from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into Y , $L \cdot I$ is a linear of Y .

PROOF: For every rest R of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, Y , $R \cdot I$ is a rest of Y by [15, (23)], [5, (47)], [14, (3)]. Reconsider $L_0 = L$ as a function from \mathcal{R}^1 into Y . Reconsider $L_1 = L_0 \cdot I$ as a partial function from \mathbb{R} to Y . Reconsider $r = L_1(jj)$ as a point of Y . For every real number p , $L_{1p} = p \cdot r$ by [6, (13)], [14, (3)], [6, (12)]. \square

(4) Let us consider a function J from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into \mathbb{R} . Suppose $J = \text{proj}(1, 1)$. Then

- (i) for every rest R of Y , $R \cdot J$ is a rest of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, Y , and
- (ii) for every linear L of Y , $L \cdot J$ is a Lipschitzian linear operator from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into Y .

PROOF: For every rest R of Y , $R \cdot J$ is a rest of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, Y by [14, (4)], [15, (6)], [5, (47)]. Consider r being a point of Y such that for every real number p , $L_p = p \cdot r$. \square

- (5) Let us consider a function I from \mathbb{R} into $\langle \mathcal{E}^1, \|\cdot\| \rangle$, a point x_0 of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, an element y_0 of \mathbb{R} , a partial function g from \mathbb{R} to Y , and a partial function f from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to Y . Suppose
- (i) $I = (\text{proj}(1, 1) \text{ qua function})^{-1}$, and
 - (ii) $x_0 \in \text{dom } f$, and
 - (iii) $y_0 \in \text{dom } g$, and
 - (iv) $x_0 = \langle y_0 \rangle$, and
 - (v) $f \cdot I = g$, and
 - (vi) f is differentiable in x_0 .

Then

- (vii) g is differentiable in y_0 , and
- (viii) $g'(y_0) = f'(x_0)(\langle 1 \rangle)$, and
- (ix) for every element r of \mathbb{R} , $f'(x_0)(\langle r \rangle) = r \cdot g'(y_0)$.

The theorem is a consequence of (3). PROOF: Consider N_1 being a neighbourhood of x_0 such that $N_1 \subseteq \text{dom } f$ and there exists a point L of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into Y and there exists a rest R of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, Y such that for every point x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ such that $x \in N_1$ holds $f_x - f_{x_0} = L(x - x_0) + R_{x-x_0}$. Consider e being a real number such that $0 < e$ and $\{z, \text{ where } z \text{ is a point of } \langle \mathcal{E}^1, \|\cdot\| \rangle : \|z - x_0\| < e\} \subseteq N_1$. Consider L being a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into Y , R being a rest of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, Y such that for every point x_3 of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ such that $x_3 \in N_1$ holds $f_{x_3} - f_{x_0} = L(x_3 - x_0) + R_{x_3-x_0}$. Reconsider $R_0 = R \cdot I$ as a rest of Y . Reconsider $L_0 = L \cdot I$ as a linear of Y . Set $N = \{z, \text{ where } z \text{ is a point of } \langle \mathcal{E}^1, \|\cdot\| \rangle : \|z - x_0\| < e\}$. $N \subseteq \text{the carrier of } \langle \mathcal{E}^1, \|\cdot\| \rangle$. Set $N_0 = \{z, \text{ where } z \text{ is an element of } \mathbb{R} : |z - y_0| < e\}$. $]y_0 - e, y_0 + e[\subseteq N_0$ by [28, (1)]. $N_0 \subseteq]y_0 - e, y_0 + e[$ by [28, (1)]. For every real number y_1 such that $y_1 \in N_0$ holds $(f \cdot I)_{y_1} - (f \cdot I)_{y_0} = L_{0y_1-y_0} + R_{0y_1-y_0}$ by [6, (12)], [7, (35)], [14, (3)]. \square

- (6) Let us consider a function I from \mathbb{R} into $\langle \mathcal{E}^1, \|\cdot\| \rangle$, a point x_0 of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, a real number y_0 , a partial function g from \mathbb{R} to Y , and a partial function f from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to Y . Suppose
- (i) $I = (\text{proj}(1, 1) \text{ qua function})^{-1}$, and
 - (ii) $x_0 \in \text{dom } f$, and
 - (iii) $y_0 \in \text{dom } g$, and

- (iv) $x_0 = \langle y_0 \rangle$, and
- (v) $f \cdot I = g$.

Then f is differentiable in x_0 if and only if g is differentiable in y_0 . The theorem is a consequence of (5) and (4). **PROOF:** Reconsider $J = \text{proj}(1, 1)$ as a function from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into \mathbb{R} . Consider N_0 being a neighbourhood of y_0 such that $N_0 \subseteq \text{dom}(f \cdot I)$ and there exists a linear L of Y and there exists a rest R of Y such that for every real number y such that $y \in N_0$ holds $(f \cdot I)_y - (f \cdot I)_{y_0} = L_{y-y_0} + R_{y-y_0}$. Consider e_0 being a real number such that $0 < e_0$ and $N_0 =]y_0 - e_0, y_0 + e_0[$. Reconsider $e = e_0$ as an element of \mathbb{R} . Set $N = \{z, \text{ where } z \text{ is a point of } \langle \mathcal{E}^1, \|\cdot\| \rangle : \|z - x_0\| < e\}$. Consider L being a linear of Y , R being a rest of Y such that for every real number y_1 such that $y_1 \in N_0$ holds $(f \cdot I)_{y_1} - (f \cdot I)_{y_0} = L_{y_1-y_0} + R_{y_1-y_0}$. Reconsider $R_0 = R \cdot J$ as a rest of $\langle \mathcal{E}^1, \|\cdot\| \rangle, Y$. Reconsider $L_0 = L \cdot J$ as a Lipschitzian linear operator from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into Y . $N \subseteq$ the carrier of $\langle \mathcal{E}^1, \|\cdot\| \rangle$. For every point y of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ such that $y \in N$ holds $f_y - f_{x_0} = L_0(y - x_0) + R_{0y-x_0}$ by [6, (13)], [7, (35)], [14, (4)]. \square

- (7) Let us consider a function J from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into \mathbb{R} , a point x_0 of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, an element y_0 of \mathbb{R} , a partial function g from \mathbb{R} to Y , and a partial function f from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to Y . Suppose

- (i) $J = \text{proj}(1, 1)$, and
- (ii) $x_0 \in \text{dom } f$, and
- (iii) $y_0 \in \text{dom } g$, and
- (iv) $x_0 = \langle y_0 \rangle$, and
- (v) $f = g \cdot J$.

Then f is differentiable in x_0 if and only if g is differentiable in y_0 . The theorem is a consequence of (6).

- (8) Let us consider a function I from \mathbb{R} into $\langle \mathcal{E}^1, \|\cdot\| \rangle$, a point x_0 of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, an element y_0 of \mathbb{R} , a partial function g from \mathbb{R} to Y , and a partial function f from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to Y . Suppose

- (i) $I = (\text{proj}(1, 1) \text{ qua function})^{-1}$, and
- (ii) $x_0 \in \text{dom } f$, and
- (iii) $y_0 \in \text{dom } g$, and
- (iv) $x_0 = \langle y_0 \rangle$, and
- (v) $f \cdot I = g$, and
- (vi) f is differentiable in x_0 .

Then $\|g'(y_0)\| = \|f'(x_0)\|$. The theorem is a consequence of (5). **PROOF:** Reconsider $d_1 = f'(x_0)$ as a Lipschitzian linear operator from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into Y . Set $A = \text{PreNorms}(d_1)$. For every real number r such that $r \in A$ holds $r \leq \|g'(y_0)\|$ by [14, (1), (4)]. \square

Let us consider real numbers a, b, z and points p, q, x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$. Now we state the propositions:

- (9) Suppose $p = \langle a \rangle$ and $q = \langle b \rangle$ and $x = \langle z \rangle$. Then
- (i) if $z \in]a, b[$, then $x \in]p, q[$, and
 - (ii) if $x \in]p, q[$, then $a \neq b$ and if $a < b$, then $z \in]a, b[$ and if $a > b$, then $z \in]b, a[$.
- (10) Suppose $p = \langle a \rangle$ and $q = \langle b \rangle$ and $x = \langle z \rangle$. Then
- (i) if $z \in [a, b]$, then $x \in [p, q]$, and
 - (ii) if $x \in [p, q]$, then if $a \leq b$, then $z \in [a, b]$ and if $a \geq b$, then $z \in [b, a]$.

Now we state the propositions:

- (11) Let us consider real numbers a, b , points p, q of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, and a function I from \mathbb{R} into $\langle \mathcal{E}^1, \|\cdot\| \rangle$. Suppose
- (i) $p = \langle a \rangle$, and
 - (ii) $q = \langle b \rangle$, and
 - (iii) $I = (\text{proj}(1, 1) \text{ qua function})^{-1}$.

Then

- (iv) if $a \leq b$, then $I^\circ[a, b] = [p, q]$, and
- (v) if $a < b$, then $I^\circ]a, b[=]p, q[$.

The theorem is a consequence of (10) and (9).

- (12) Let us consider a real normed space Y , a partial function g from \mathbb{R} to the carrier of Y , and real numbers a, b, M . Suppose

- (i) $a \leq b$, and
- (ii) $[a, b] \subseteq \text{dom } g$, and
- (iii) for every real number x such that $x \in [a, b]$ holds g is continuous in x , and
- (iv) for every real number x such that $x \in]a, b[$ holds g is differentiable in x , and
- (v) for every real number x such that $x \in]a, b[$ holds $\|g'(x)\| \leq M$.

Then $\|g_b - g_a\| \leq M \cdot |b - a|$. The theorem is a consequence of (11), (10), (1), (9), (7), and (8).

2. DIFFERENTIAL EQUATIONS

In the sequel X, Y denote real Banach spaces, Z denotes an open subset of \mathbb{R} , a, b, c, d, e, r, x_0 denote real numbers, y_0 denotes a vector of X , and G denotes a function from X into X .

Now we state the propositions:

- (13) Let us consider a real Banach space X , a partial function F from \mathbb{R} to the carrier of X , and a continuous partial function f from \mathbb{R} to the carrier of X . Suppose

(i) $[a, b] \subseteq \text{dom } f$, and

(ii) $]a, b[\subseteq \text{dom } F$, and

(iii) for every real number x such that $x \in]a, b[$ holds $F_x = \int_a^x f(x)dx$,

and

(iv) $x_0 \in]a, b[$, and

(v) f is continuous in x_0 .

Then

(vi) F is differentiable in x_0 , and

(vii) $F'(x_0) = f_{x_0}$.

- (14) Let us consider a partial function F from \mathbb{R} to the carrier of X and a continuous partial function f from \mathbb{R} to the carrier of X . Suppose

(i) $\text{dom } f = [a, b]$, and

(ii) $\text{dom } F = [a, b]$, and

(iii) for every real number t such that $t \in [a, b]$ holds $F_t = \int_a^t f(x)dx$.

Let us consider a real number x . If $x \in [a, b]$, then F is continuous in x .

- (15) Let us consider a continuous partial function f from \mathbb{R} to the carrier of X . If $a \in \text{dom } f$, then $\int_a^a f(x)dx = 0_X$.

Let us consider a continuous partial function f from \mathbb{R} to the carrier of X and a partial function g from \mathbb{R} to the carrier of X . Now we state the propositions:

- (16) Suppose $a \leq b$ and $\text{dom } f = [a, b]$ and for every real number t such that

$t \in [a, b]$ holds $g_t = y_0 + \int_a^t f(x)dx$. Then $g_a = y_0$.

- (17) Suppose $\text{dom } f = [a, b]$ and $\text{dom } g = [a, b]$ and $Z =]a, b[$ and for every real number t such that $t \in [a, b]$ holds $g_t = y_0 + \int_a^t f(x)dx$. Then
- (i) g is continuous and differentiable on Z , and
 - (ii) for every real number t such that $t \in Z$ holds $g'(t) = f_t$.

Let us consider a partial function f from \mathbb{R} to the carrier of X . Now we state the propositions:

- (18) Suppose $a \leq b$ and $[a, b] \subseteq \text{dom } f$ and for every real number x such that $x \in [a, b]$ holds f is continuous in x and f is differentiable on $]a, b[$ and for every real number x such that $x \in]a, b[$ holds $f'(x) = 0_X$. Then $f_b = f_a$.
- (19) Suppose $[a, b] \subseteq \text{dom } f$ and for every real number x such that $x \in [a, b]$ holds f is continuous in x and f is differentiable on $]a, b[$ and for every real number x such that $x \in]a, b[$ holds $f'(x) = 0_X$. Then $f|]a, b[$ is constant.

Now we state the propositions:

- (20) Let us consider a continuous partial function f from \mathbb{R} to the carrier of X . Suppose
- (i) $[a, b] = \text{dom } f$, and
 - (ii) $f|]a, b[$ is constant.

Let us consider a real number x . If $x \in [a, b]$, then $f_x = f_a$.

- (21) Let us consider continuous partial functions y, G_1 from \mathbb{R} to the carrier of X and a partial function g from \mathbb{R} to the carrier of X . Suppose
- (i) $a \leq b$, and
 - (ii) $Z =]a, b[$, and
 - (iii) $\text{dom } y = [a, b]$, and
 - (iv) $\text{dom } g = [a, b]$, and
 - (v) $\text{dom } G_1 = [a, b]$, and
 - (vi) y is differentiable on Z , and
 - (vii) $y_a = y_0$, and
 - (viii) for every real number t such that $t \in Z$ holds $y'(t) = G_{1t}$, and
 - (ix) for every real number t such that $t \in [a, b]$ holds $g_t = y_0 + \int_a^t G_1(x)dx$.

Then $y = g$. The theorem is a consequence of (17), (16), (19), and (20).

PROOF: Reconsider $h = y - g$ as a continuous partial function from \mathbb{R} to the carrier of X . For every real number x such that $x \in \text{dom } h$ holds $h_x = 0_X$ by [35, (15)]. For every element x of \mathbb{R} such that $x \in \text{dom } y$ holds $y(x) = g(x)$ by [35, (21)]. \square

Let X be a real Banach space, y_0 be a vector of X , G be a function from X into X , and a, b be real numbers. Assume $a \leq b$ and G is continuous on $\text{dom } G$. The functor $\text{Fredholm}(G, a, b, y_0)$ yielding a function from the \mathbb{R} -norm space of continuous functions of $[a, b]$ and X into the \mathbb{R} -norm space of continuous functions of $[a, b]$ and X is defined by

(Def. 1) Let us consider a vector x of the \mathbb{R} -norm space of continuous functions of $[a, b]$ and X . Then there exist continuous partial functions f, g, G_1 from \mathbb{R} to the carrier of X such that

- (i) $x = f$, and
- (ii) $it(x) = g$, and
- (iii) $\text{dom } f = [a, b]$, and
- (iv) $\text{dom } g = [a, b]$, and
- (v) $G_1 = G \cdot f$, and

(vi) for every real number t such that $t \in [a, b]$ holds $g_t = y_0 + \int_a^t G_1(x)dx$.

Now we state the propositions:

(22) Suppose $a \leq b$ and $0 < r$ and for every vectors y_1, y_2 of X , $\|G_{y_1} - G_{y_2}\| \leq r \cdot \|y_1 - y_2\|$. Let us consider vectors u, v of the \mathbb{R} -norm space of continuous functions of $[a, b]$ and X and continuous partial functions g, h from \mathbb{R} to the carrier of X . Suppose

- (i) $g = (\text{Fredholm}(G, a, b, y_0))(u)$, and
- (ii) $h = (\text{Fredholm}(G, a, b, y_0))(v)$.

Let us consider a real number t . Suppose $t \in [a, b]$. Then $\|g_t - h_t\| \leq (r \cdot (t - a)) \cdot \|u - v\|$. PROOF: Set $F = \text{Fredholm}(G, a, b, y_0)$. Consider f_1, g_1, G_3 being continuous partial functions from \mathbb{R} to the carrier of X such that $u = f_1$ and $F(u) = g_1$ and $\text{dom } f_1 = [a, b]$ and $\text{dom } g_1 = [a, b]$ and $G_3 = G \cdot f_1$ and for every real number t such that $t \in [a, b]$

holds $g_{1t} = y_0 + \int_a^t G_3(x)dx$. Consider f_2, g_2, G_5 being continuous partial

functions from \mathbb{R} to the carrier of X such that $v = f_2$ and $F(v) = g_2$ and $\text{dom } f_2 = [a, b]$ and $\text{dom } g_2 = [a, b]$ and $G_5 = G \cdot f_2$ and for every real

number t such that $t \in [a, b]$ holds $g_{2t} = y_0 + \int_a^t G_5(x)dx$. Set $G_4 = G_3 - G_5$.

For every real number x such that $x \in [a, t]$ holds $\|G_{4x}\| \leq r \cdot \|u - v\|$ by [20, (26)], [6, (12)]. \square

(23) Suppose $a \leq b$ and $0 < r$ and for every vectors y_1, y_2 of X , $\|G_{y_1} - G_{y_2}\| \leq r \cdot \|y_1 - y_2\|$. Let us consider vectors u, v of the \mathbb{R} -norm space of

continuous functions of $[a, b]$ and X , an element m of \mathbb{N} , and continuous partial functions g, h from \mathbb{R} to the carrier of X . Suppose

- (i) $g = (\text{Fredholm}(G, a, b, y_0))^{m+1}(u)$, and
- (ii) $h = (\text{Fredholm}(G, a, b, y_0))^{m+1}(v)$.

Let us consider a real number t . Suppose $t \in [a, b]$. Then $\|g_t - h_t\| \leq \frac{(r \cdot (t-a))^{m+1}}{(m+1)!} \cdot \|u - v\|$. The theorem is a consequence of (22). PROOF: Set $F = \text{Fredholm}(G, a, b, y_0)$. Define \mathcal{P} [natural number] \equiv for every continuous partial functions g, h from \mathbb{R} to the carrier of X such that $g = F^{\mathfrak{s}_1+1}(u_1)$ and $h = F^{\mathfrak{s}_1+1}(v_1)$ for every real number t such that $t \in [a, b]$ holds $\|g_t - h_t\| \leq \frac{(r \cdot (t-a))^{\mathfrak{s}_1+1}}{(\mathfrak{s}_1+1)!} \cdot \|u_1 - v_1\|$. $\mathcal{P}[0]$ by [4, (70)], [18, (5), (13)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [4, (71)], [6, (13)], [37, (27)]. For every natural number k , $\mathcal{P}[k]$ from [1, Sch. 2]. \square

(24) Let us consider a natural number m . Suppose

- (i) $a \leq b$, and
- (ii) $0 < r$, and
- (iii) for every vectors y_1, y_2 of X , $\|G_{y_1} - G_{y_2}\| \leq r \cdot \|y_1 - y_2\|$.

Let us consider vectors u, v of the \mathbb{R} -norm space of continuous functions of $[a, b]$ and X .

Then $\|(\text{Fredholm}(G, a, b, y_0))^{m+1}(u) - (\text{Fredholm}(G, a, b, y_0))^{m+1}(v)\| \leq \frac{(r \cdot (b-a))^{m+1}}{(m+1)!} \cdot \|u - v\|$. The theorem is a consequence of (23).

(25) If $a < b$ and G is Lipschitzian on the carrier of X , then there exists a natural number m such that $(\text{Fredholm}(G, a, b, y_0))^{m+1}$ is contraction. The theorem is a consequence of (24).

(26) If $a < b$ and G is Lipschitzian on the carrier of X , then $\text{Fredholm}(G, a, b, y_0)$ has unique fixpoint. The theorem is a consequence of (25).

(27) Let us consider continuous partial functions f, g from \mathbb{R} to the carrier of X . Suppose

- (i) $\text{dom } f = [a, b]$, and
- (ii) $\text{dom } g = [a, b]$, and
- (iii) $Z =]a, b[$, and
- (iv) $a < b$, and
- (v) G is Lipschitzian on the carrier of X , and
- (vi) $g = (\text{Fredholm}(G, a, b, y_0))(f)$.

Then

- (vii) $g_a = y_0$, and
- (viii) g is differentiable on Z , and

(ix) for every real number t such that $t \in Z$ holds $g'(t) = (G \cdot f)_t$.

The theorem is a consequence of (17) and (16).

(28) Let us consider a continuous partial function y from \mathbb{R} to the carrier of X . Suppose

- (i) $a < b$, and
- (ii) $Z =]a, b[$, and
- (iii) G is Lipschitzian on the carrier of X , and
- (iv) $\text{dom } y = [a, b]$, and
- (v) y is differentiable on Z , and
- (vi) $y_a = y_0$, and
- (vii) for every real number t such that $t \in Z$ holds $y'(t) = G(y_t)$.

Then y is a fixpoint of $\text{Fredholm}(G, a, b, y_0)$. The theorem is a consequence of (21). PROOF: Consider f, g, G_1 being continuous partial functions from \mathbb{R} to the carrier of X such that $y = f$ and $(\text{Fredholm}(G, a, b, y_0))(y) = g$ and $\text{dom } f = [a, b]$ and $\text{dom } g = [a, b]$ and $G_1 = G \cdot f$ and for every real number t such that $t \in [a, b]$ holds $g_t = y_0 + \int_a^t G_1(x)dx$. For every real number t such that $t \in Z$ holds $y'(t) = G_{1t}$ by [6, (13)]. \square

(29) Let us consider continuous partial functions y_1, y_2 from \mathbb{R} to the carrier of X . Suppose

- (i) $a < b$, and
- (ii) $Z =]a, b[$, and
- (iii) G is Lipschitzian on the carrier of X , and
- (iv) $\text{dom } y_1 = [a, b]$, and
- (v) y_1 is differentiable on Z , and
- (vi) $y_{1a} = y_0$, and
- (vii) for every real number t such that $t \in Z$ holds $y_1'(t) = G(y_{1t})$, and
- (viii) $\text{dom } y_2 = [a, b]$, and
- (ix) y_2 is differentiable on Z , and
- (x) $y_{2a} = y_0$, and
- (xi) for every real number t such that $t \in Z$ holds $y_2'(t) = G(y_{2t})$.

Then $y_1 = y_2$. The theorem is a consequence of (26) and (28).

(30) Suppose $a < b$ and $Z =]a, b[$ and G is Lipschitzian on the carrier of X . Then there exists a continuous partial function y from \mathbb{R} to the carrier of X such that

- (i) $\text{dom } y = [a, b]$, and
- (ii) y is differentiable on Z , and
- (iii) $y_a = y_0$, and
- (iv) for every real number t such that $t \in Z$ holds $y'(t) = G(y_t)$.

The theorem is a consequence of (26) and (27).

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Submodule of free \mathbb{Z} -module¹

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Summary. In this article, we formalize a free \mathbb{Z} -module and its property. In particular, we formalize the vector space of rational field corresponding to a free \mathbb{Z} -module and prove formally that submodules of a free \mathbb{Z} -module are free. \mathbb{Z} -module is necessary for lattice problems - LLL (Lenstra, Lenstra and Lovász) base reduction algorithm and cryptographic systems with lattice [20]. Some theorems in this article are described by translating theorems in [11] into theorems of \mathbb{Z} -module, however their proofs are different.

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The notation and terminology used in this paper have been introduced in the following articles: [6], [1], [24], [22], [5], [12], [7], [8], [16], [25], [19], [23], [21], [3], [4], [9], [17], [30], [32], [31], [26], [29], [18], [27], [28], [33], [10], [13], [14], and [15].

1. VECTOR SPACE OF RATIONAL FIELD GENERATED BY A FREE \mathbb{Z} -MODULE

From now on V denotes a \mathbb{Z} -module and W, W_1, W_2 denote submodules of V .

Let us consider a \mathbb{Z} -module V , submodules W_1, W_2 of V , and submodules W_5, W_6 of $W_1 + W_2$. Now we state the propositions:

- (1) If $W_5 = W_1$ and $W_6 = W_2$, then $W_1 + W_2 = W_5 + W_6$.
- (2) If $W_5 = W_1$ and $W_6 = W_2$, then $W_1 \cap W_2 = W_5 \cap W_6$.

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Let V be a \mathbb{Z} -module. Note that (the carrier of V) $\times (\mathbb{Z} \setminus \{0\})$ is non empty.

Assume V is cancelable on multiplication. The functor $\text{EQRZM}(V)$ yielding an equivalence relation of (the carrier of V) $\times (\mathbb{Z} \setminus \{0\})$ is defined by

- (Def. 1) Let us consider elements S, T . Then $\langle S, T \rangle \in \text{it}$ if and only if $S, T \in$ (the carrier of V) $\times (\mathbb{Z} \setminus \{0\})$ and there exist elements z_1, z_2 of V and there exist integers i_1, i_2 such that $S = \langle z_1, i_1 \rangle$ and $T = \langle z_2, i_2 \rangle$ and $i_1 \neq 0$ and $i_2 \neq 0$ and $i_2 \cdot z_1 = i_1 \cdot z_2$.

Now we state the proposition:

- (3) Let us consider a \mathbb{Z} -module V , elements z_1, z_2 of V , and integers i_1, i_2 . Suppose V is cancelable on multiplication. Then $\langle \langle z_1, i_1 \rangle, \langle z_2, i_2 \rangle \rangle \in \text{EQRZM}(V)$ if and only if $i_1 \neq 0$ and $i_2 \neq 0$ and $i_2 \cdot z_1 = i_1 \cdot z_2$.

Let V be a \mathbb{Z} -module. Assume V is cancelable on multiplication. The functor $\text{addCoset } V$ yielding a binary operation on $\text{Classes EQRZM}(V)$ is defined by

- (Def. 2) Let us consider elements A, B . Suppose $A, B \in \text{Classes EQRZM}(V)$. Let us consider elements z_1, z_2 of V and integers i_1, i_2 . Suppose

- (i) $i_1 \neq 0$, and
- (ii) $i_2 \neq 0$, and
- (iii) $A = [\langle z_1, i_1 \rangle]_{\text{EQRZM}(V)}$, and
- (iv) $B = [\langle z_2, i_2 \rangle]_{\text{EQRZM}(V)}$.

Then $\text{it}(A, B) = [\langle i_2 \cdot z_1 + i_1 \cdot z_2, i_1 \cdot i_2 \rangle]_{\text{EQRZM}(V)}$.

Assume V is cancelable on multiplication. The functor $\text{zeroCoset } V$ yielding an element of $\text{Classes EQRZM}(V)$ is defined by

- (Def. 3) Let us consider an integer i . Suppose $i \neq 0$. Then $\text{it} = [\langle 0_V, i \rangle]_{\text{EQRZM}(V)}$.

Assume V is cancelable on multiplication. The functor $\text{lmultCoset } V$ yielding a function from (the carrier of $\mathbb{F}_{\mathbb{Q}}$) $\times \text{Classes EQRZM}(V)$ into $\text{Classes EQRZM}(V)$ is defined by

- (Def. 4) Let us consider an element q and an element A . Suppose

- (i) $q \in \mathbb{Q}$, and
- (ii) $A \in \text{Classes EQRZM}(V)$.

Let us consider integers m, n, i and an element z of V . Suppose

- (iii) $n \neq 0$, and
- (iv) $q = \frac{m}{n}$, and
- (v) $i \neq 0$, and
- (vi) $A = [\langle z, i \rangle]_{\text{EQRZM}(V)}$.

Then $\text{it}(q, A) = [\langle m \cdot z, n \cdot i \rangle]_{\text{EQRZM}(V)}$.

Now we state the propositions:

(4) Let us consider a \mathbb{Z} -module V , an element z of V , and integers i, n .
Suppose

- (i) $i \neq 0$, and
- (ii) $n \neq 0$, and
- (iii) V is cancelable on multiplication.

Then $[\langle z, i \rangle]_{\text{EQRZM}(V)} = [\langle n \cdot z, n \cdot i \rangle]_{\text{EQRZM}(V)}$. The theorem is a consequence of (3).

(5) Let us consider a \mathbb{Z} -module V and an element v of $\langle \text{Classes EQRZM}(V), \text{addCoset } V, \text{zeroCoset } V, \text{lmultCoset } V \rangle$. Suppose V is cancelable on multiplication. Then there exists an integer i and there exists an element z of V such that $i \neq 0$ and $v = [\langle z, i \rangle]_{\text{EQRZM}(V)}$.

Let V be a \mathbb{Z} -module. Assume V is cancelable on multiplication. The functor $\text{ZMQVectSp}(V)$ yielding a vector space over $\mathbb{F}_{\mathbb{Q}}$ is defined by the term

(Def. 5) $\langle \text{Classes EQRZM}(V), \text{addCoset } V, \text{zeroCoset } V, \text{lmultCoset } V \rangle$.

Assume V is cancelable on multiplication. The functor $\text{MorphsZQ}(V)$ yielding a function from V into $\text{ZMQVectSp}(V)$ is defined by

- (Def. 6)
- (i) it is one-to-one, and
 - (ii) for every element v of V , $it(v) = [\langle v, 1 \rangle]_{\text{EQRZM}(V)}$, and
 - (iii) for every elements v, w of V , $it(v + w) = it(v) + it(w)$, and
 - (iv) for every element v of V and for every integer i and for every element q of $\mathbb{F}_{\mathbb{Q}}$ such that $i = q$ holds $it(i \cdot v) = q \cdot it(v)$, and
 - (v) $it(0_V) = 0_{\text{ZMQVectSp}(V)}$.

Now we state the propositions:

(6) Let us consider a \mathbb{Z} -module V . Suppose V is cancelable on multiplication. Let us consider a finite sequence s of elements of V and a finite sequence t of elements of $\text{ZMQVectSp}(V)$. Suppose

- (i) $\text{len } s = \text{len } t$, and
- (ii) for every element i of \mathbb{N} such that $i \in \text{dom } s$ there exists a vector s_1 of V such that $s_1 = s(i)$ and $t(i) = (\text{MorphsZQ}(V))(s_1)$.

Then $\sum t = (\text{MorphsZQ}(V))(\sum s)$. PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence s of elements of V for every finite sequence t of elements of $\text{ZMQVectSp}(V)$ such that $\text{len } s = \mathbb{S}_1$ and $\text{len } s = \text{len } t$ and for every element i of \mathbb{N} such that $i \in \text{dom } s$ there exists a vector s_1 of V such that $s_1 = s(i)$ and $t(i) = (\text{MorphsZQ}(V))(s_1)$ holds $\sum t = (\text{MorphsZQ}(V))(\sum s)$. $\mathcal{P}[0]$ by [26, (43)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$ by [5, (59)], [3, (11)], [5, (4)]. For every natural number k , $\mathcal{P}[k]$ from [3, Sch. 2]. \square

- (7) Let us consider a \mathbb{Z} -module V , a subset I of V , a subset I_6 of $\text{ZMQVectSp}(V)$, a z linear combination l of I , and a linear combination l_5 of I_6 . Suppose

- (i) V is cancelable on multiplication, and
- (ii) $I_6 = (\text{MorphsZQ}(V))^\circ I$, and
- (iii) $l = l_5 \cdot \text{MorphsZQ}(V)$.

Then $\sum l_5 = (\text{MorphsZQ}(V))(\sum l)$. The theorem is a consequence of (6).

- (8) Let us consider a \mathbb{Z} -module V , a subset I_6 of $\text{ZMQVectSp}(V)$, and a linear combination l_5 of I_6 . Then there exists an integer m and there exists an element a of $\mathbb{F}_\mathbb{Q}$ such that $m \neq 0$ and $m = a$ and $\text{rng}(a \cdot l_5) \subseteq \mathbb{Z}$. PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every linear combination l_5 of I_6 such that $\overline{\text{the support of } l_5} = \$_1$ there exists an integer m and there exists an element a of $\mathbb{F}_\mathbb{Q}$ such that $m \neq 0$ and $m = a$ and $\text{rng}(a \cdot l_5) \subseteq \mathbb{Z}$. $\mathcal{P}[0]$ by [27, (28)], [8, (113)], [27, (3)]. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [2, (44)], [10, (31)], [2, (42)]. For every natural number n , $\mathcal{P}[n]$ from [3, Sch. 2]. \square

- (9) Let us consider a \mathbb{Z} -module V , a subset I of V , a subset I_6 of $\text{ZMQVectSp}(V)$, and a linear combination l_5 of I_6 . Suppose

- (i) V is cancelable on multiplication, and
- (ii) $I_6 = (\text{MorphsZQ}(V))^\circ I$.

Then there exists an integer m and there exists an element a of $\mathbb{F}_\mathbb{Q}$ and there exists a z linear combination l of I such that $m \neq 0$ and $m = a$ and $l = (a \cdot l_5) \cdot \text{MorphsZQ}(V)$ and $(\text{MorphsZQ}(V))^{-1}$ (the support of l_5) = the support of l . The theorem is a consequence of (8). PROOF: Consider m being an integer, a being an element of $\mathbb{F}_\mathbb{Q}$ such that $m \neq 0$ and $m = a$ and $\text{rng}(a \cdot l_5) \subseteq \mathbb{Z}$. Reconsider $l = (a \cdot l_5) \cdot \text{MorphsZQ}(V)$ as an element of $\mathbb{Z}^{\text{the carrier of } V}$. Set $T = \{v, \text{ where } v \text{ is an element of } V : l(v) \neq 0\}$. Set $F = \text{MorphsZQ}(V)$. $T \subseteq F^{-1}$ (the support of l_5) by [7, (13)], [8, (38)]. F^{-1} (the support of l_5) $\subseteq T$ by [8, (38)], [7, (13)]. \square

- (10) Let us consider a \mathbb{Z} -module V , a subset I of V , a subset I_6 of $\text{ZMQVectSp}(V)$, a linear combination l_5 of I_6 , an integer m , an element a of $\mathbb{F}_\mathbb{Q}$, and a z linear combination l of I . Suppose

- (i) V is cancelable on multiplication, and
- (ii) $I_6 = (\text{MorphsZQ}(V))^\circ I$, and
- (iii) $m \neq 0$, and
- (iv) $m = a$, and
- (v) $l = (a \cdot l_5) \cdot \text{MorphsZQ}(V)$.

Then $a \cdot \sum l_5 = (\text{MorphsZQ}(V))(\sum l)$. The theorem is a consequence of (7).

(11) Let us consider a \mathbb{Z} -module V , a subset I of V , and a subset I_6 of $\text{ZMQVectSp}(V)$. Suppose

- (i) V is cancelable on multiplication, and
- (ii) $I_6 = (\text{MorphaZQ}(V))^\circ I$, and
- (iii) I is linearly independent.

Then I_6 is linearly independent. The theorem is a consequence of (9) and (10).

(12) Let us consider a \mathbb{Z} -module V , a subset I of V , a \mathbb{Z} linear combination l of I , and a subset I_6 of $\text{ZMQVectSp}(V)$. Suppose

- (i) V is cancelable on multiplication, and
- (ii) $I_6 = (\text{MorphaZQ}(V))^\circ I$.

Then there exists a linear combination l_5 of I_6 such that

- (iii) $l = l_5 \cdot \text{MorphaZQ}(V)$, and
- (iv) the support of $l_5 = (\text{MorphaZQ}(V))^\circ$ the support of l .

PROOF: Reconsider $I_0 =$ the support of l as a finite subset of V . Reconsider $I_7 = (\text{MorphaZQ}(V))^\circ I_0$ as a finite subset of $\text{ZMQVectSp}(V)$. Define $\mathcal{P}[\text{element}, \text{element}] \equiv \mathcal{S}_1 \in I_7$ and there exists an element v of V such that $v \in I_0$ and $\mathcal{S}_1 = (\text{MorphaZQ}(V))(v)$ and $\mathcal{S}_2 = l(v)$ or $\mathcal{S}_1 \notin I_7$ and $\mathcal{S}_2 = 0_{\mathbb{F}_Q}$. For every element x such that $x \in$ the carrier of $\text{ZMQVectSp}(V)$ there exists an element y such that $y \in \mathbb{Q}$ and $\mathcal{P}[x, y]$ by [8, (64)]. Consider l_5 being a function from the carrier of $\text{ZMQVectSp}(V)$ into \mathbb{Q} such that for every element x such that $x \in$ the carrier of $\text{ZMQVectSp}(V)$ holds $\mathcal{P}[x, l_5(x)]$ from [8, Sch. 1]. The support of $l_5 \subseteq I_7$. For every element x such that $x \in \text{dom } l$ holds $l(x) = (l_5 \cdot \text{MorphaZQ}(V))(x)$ by [8, (35), (19)], [7, (12)]. $I_7 \subseteq$ the support of l_5 by [8, (64)], [7, (12)], [14, (8)]. \square

(13) Let us consider a free \mathbb{Z} -module V , a subset I of V , a subset I_6 of $\text{ZMQVectSp}(V)$, a \mathbb{Z} linear combination l of I , and an integer i . Suppose

- (i) $i \neq 0$, and
- (ii) $I_6 = (\text{MorphaZQ}(V))^\circ I$.

Then $[\{\sum l, i\}]_{\text{EQRZM}(V)} \in \text{Lin}(I_6)$. The theorem is a consequence of (12) and (7).

Let us consider a free \mathbb{Z} -module V , a subset I of V , and a subset I_6 of $\text{ZMQVectSp}(V)$. Now we state the propositions:

- (14) If $I_6 = (\text{MorphaZQ}(V))^\circ I$, then $\overline{I} = \overline{I_6}$.
- (15) If $I_6 = (\text{MorphaZQ}(V))^\circ I$ and I is a basis of V , then I_6 is a basis of $\text{ZMQVectSp}(V)$.

Let V be a finite-rank free \mathbb{Z} -module. Note that $\text{ZMQVectSp}(V)$ is finite dimensional.

Now we state the propositions:

- (16) Let us consider a finite-rank free \mathbb{Z} -module V . Then $\text{rank } V = \dim(\text{ZMQVectSp}(V))$. The theorem is a consequence of (15) and (14).
- (17) Let us consider a free \mathbb{Z} -module V and finite subsets I, A of V . Suppose
- (i) I is a basis of V , and
 - (ii) $\overline{I} + 1 = \overline{A}$.

Then A is linearly dependent. The theorem is a consequence of (15), (11), and (14).

- (18) Let us consider a free \mathbb{Z} -module V and subsets A, B of V . If A is linearly dependent and $A \subseteq B$, then B is linearly dependent.
- (19) Let us consider a free \mathbb{Z} -module V and subsets D, A of V . Suppose
- (i) D is basis of V and finite, and
 - (ii) $\overline{D} \subset \overline{A}$.

Then A is linearly dependent. The theorem is a consequence of (17) and (18).

- (20) Let us consider a free \mathbb{Z} -module V and subsets I, A of V . Suppose
- (i) I is basis of V and finite, and
 - (ii) A is linearly independent.

Then $\overline{A} \subseteq \overline{I}$.

2. SUBMODULE OF FREE \mathbb{Z} -MODULE

Now we state the proposition:

- (21) Let us consider a \mathbb{Z} -module V . If Ω_V is free, then V is free.

Let us consider a \mathbb{Z} -module V , submodules W_1, W_2 of V , and strict submodules W_3, W_4 of V . Now we state the propositions:

- (22) If $W_3 = \Omega_{W_1}$ and $W_4 = \Omega_{W_2}$, then $W_3 + W_4 = W_1 + W_2$.
- (23) If $W_3 = \Omega_{W_1}$ and $W_4 = \Omega_{W_2}$, then $W_3 \cap W_4 = W_1 \cap W_2$.

Now we state the propositions:

- (24) Let us consider a \mathbb{Z} -module V and a strict submodule W of V . Suppose $W \neq \mathbf{0}_V$. Then there exists a vector v of V such that
- (i) $v \in W$, and
 - (ii) $v \neq 0_V$.

- (25) Let us consider a subset A of V and z linear combinations l_1, l_2 of A . Suppose $(\text{the support of } l_1) \cap (\text{the support of } l_2) = \emptyset$. Then the support of $l_1 + l_2 = (\text{the support of } l_1) \cup (\text{the support of } l_2)$. PROOF: $(\text{The support of } l_1) \cup (\text{the support of } l_2) \subseteq \text{the support of } l_1 + l_2$ by [14, (8)]. \square
- (26) Let us consider subsets A_1, A_2 of V and a z linear combination l of $A_1 \cup A_2$. Suppose $A_1 \cap A_2 = \emptyset$. Then there exists a z linear combination l_1 of A_1 and there exists a z linear combination l_2 of A_2 such that $l = l_1 + l_2$. PROOF: Define $\mathcal{P}[\text{element, element}] \equiv$ if $\$1$ is a vector of V , then $\$1 \in A_1$ and $\$2 = l(\$1)$ or $\$1 \notin A_1$ and $\$2 = 0$. For every element x such that $x \in$ the carrier of V there exists an element y such that $y \in \mathbb{Z}$ and $\mathcal{P}[x, y]$. There exists a function l_1 from the carrier of V into \mathbb{Z} such that for every element x such that $x \in$ the carrier of V holds $\mathcal{P}[x, l_1(x)]$ from [8, Sch. 1]. Consider l_1 being a function from the carrier of V into \mathbb{Z} such that for every element x such that $x \in$ the carrier of V holds $\mathcal{P}[x, l_1(x)]$. For every element x such that $x \in$ the support of l_1 holds $x \in A_1$ by [14, (8)]. Define $\mathcal{Q}[\text{element, element}] \equiv$ if $\$1$ is a vector of V , then $\$1 \in A_2$ and $\$2 = l(\$1)$ or $\$1 \notin A_2$ and $\$2 = 0$. For every element x such that $x \in$ the carrier of V there exists an element y such that $y \in \mathbb{Z}$ and $\mathcal{Q}[x, y]$. There exists a function l_2 from the carrier of V into \mathbb{Z} such that for every element x such that $x \in$ the carrier of V holds $\mathcal{Q}[x, l_2(x)]$ from [8, Sch. 1]. Consider l_2 being a function from the carrier of V into \mathbb{Z} such that for every element x such that $x \in$ the carrier of V holds $\mathcal{Q}[x, l_2(x)]$. For every element x such that $x \in$ the support of l_2 holds $x \in A_2$ by [14, (8)]. For every vector v of V , $l(v) = (l_1 + l_2)(v)$. \square
- (27) Let us consider a \mathbb{Z} -module V , free submodules W_1, W_2 of V , a basis I_1 of W_1 , and a basis I_2 of W_2 . If V is the direct sum of W_1 and W_2 , then $I_1 \cap I_2 = \emptyset$.

Let us consider a \mathbb{Z} -module V , free submodules W_1, W_2 of V , a basis I_1 of W_1 , a basis I_2 of W_2 , and a subset I of V . Now we state the propositions:

- (28) If V is the direct sum of W_1 and W_2 and $I = I_1 \cup I_2$, then $\text{Lin}(I) =$ the \mathbb{Z} -module structure of V .
- (29) If V is the direct sum of W_1 and W_2 and $I = I_1 \cup I_2$, then I is linearly independent.

Let us consider a \mathbb{Z} -module V and free submodules W_1, W_2 of V . Now we state the propositions:

- (30) If V is the direct sum of W_1 and W_2 , then V is free.
- (31) If $W_1 \cap W_2 = \mathbf{0}_V$, then $W_1 + W_2$ is free.

Let us consider a free \mathbb{Z} -module V , a basis I of V , and a vector v of V . Now we state the propositions:

- (32) If $v \in I$, then $\text{Lin}(I \setminus \{v\})$ is free and $\text{Lin}(\{v\})$ is free.

(33) If $v \in I$, then V is the direct sum of $\text{Lin}(I \setminus \{v\})$ and $\text{Lin}(\{v\})$.

Let V be a finite-rank free \mathbb{Z} -module. One can verify that every submodule of V is free.

Now we state the propositions:

(34) Let us consider a \mathbb{Z} -module V , a submodule W of V , and free submodules W_1, W_2 of V . Suppose

(i) $W_1 \cap W_2 = \mathbf{0}_V$, and

(ii) the \mathbb{Z} -module structure of $W = W_1 + W_2$.

Then W is free. The theorem is a consequence of (31).

(35) Let us consider a prime number p and a free \mathbb{Z} -module V .

If $\text{Z}_M\text{Q}_{\text{vectSp}}(V, p)$ is finite dimensional, then V is finite-rank.

(36) Let us consider a prime number p , a \mathbb{Z} -module V , an element s of V , an integer a , and an element b of $\text{GF}(p)$. Suppose $b = a \pmod{p}$. Then $b \cdot \text{ZMtoMQV}(V, p, s) = \text{ZMtoMQV}(V, p, a \cdot s)$.

(37) Let us consider a prime number p , a free \mathbb{Z} -module V , a subset I of V , a subset I_6 of $\text{Z}_M\text{Q}_{\text{vectSp}}(V, p)$, and a z linear combination l of I . Suppose $I_6 = \{\text{ZMtoMQV}(V, p, u), \text{ where } u \text{ is a vector of } V : u \in I\}$. Then $\text{ZMtoMQV}(V, p, \sum l) \in \text{Lin}(I_6)$.

(38) Let us consider a prime number p , a free \mathbb{Z} -module V , a subset I of V , and a subset I_6 of $\text{Z}_M\text{Q}_{\text{vectSp}}(V, p)$. Suppose

(i) $\text{Lin}(I) =$ the \mathbb{Z} -module structure of V , and

(ii) $I_6 = \{\text{ZMtoMQV}(V, p, u), \text{ where } u \text{ is a vector of } V : u \in I\}$.

Then $\text{Lin}(I_6) =$ the vector space structure of $\text{Z}_M\text{Q}_{\text{vectSp}}(V, p)$. The theorem is a consequence of (37). PROOF: For every element v_3 of $\text{Z}_M\text{Q}_{\text{vectSp}}(V, p)$, $v_3 \in \text{Lin}(I_6)$ by [15, (22)], [14, (64)]. \square

(39) Let us consider a finitely-generated free \mathbb{Z} -module V . Then there exists a finite subset A of V such that A is a basis of V . The theorem is a consequence of (38). PROOF: Set $p =$ the prime number. Consider B being a finite subset of V such that $\text{Lin}(B) =$ the \mathbb{Z} -module structure of V . Set $B_1 = \{\text{ZMtoMQV}(V, p, u), \text{ where } u \text{ is a vector of } V : u \in B\}$. Define $\mathcal{F}(\text{element of } V) = \text{ZMtoMQV}(V, p, \$_1)$. Consider f being a function from the carrier of V into $\text{Z}_M\text{Q}_{\text{vectSp}}(V, p)$ such that for every element x of V , $f(x) = \mathcal{F}(x)$ from [8, Sch. 4]. For every element y such that $y \in B_1$ there exists an element x such that $x \in \text{dom}(f \upharpoonright B)$ and $y = (f \upharpoonright B)(x)$ by [30, (62)], [7, (47)]. Consider I_6 being a basis of $\text{Z}_M\text{Q}_{\text{vectSp}}(V, p)$ such that $I_6 \subseteq B_1$. \square

One can verify that every finitely-generated free \mathbb{Z} -module is finite-rank and every finite-rank free \mathbb{Z} -module is finitely-generated.

Now we state the proposition:

- (40) Let us consider a finite-rank free \mathbb{Z} -module V and a subset A of V . If A is linearly independent, then A is finite. The theorem is a consequence of (19).

Let V be a \mathbb{Z} -module and W_1, W_2 be finite-rank free submodules of V . One can check that $W_1 \cap W_2$ is free.

Note that $W_1 \cap W_2$ is finite-rank.

Let V be a finite-rank free \mathbb{Z} -module. Note that every submodule of V is finite-rank.

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