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Analysis of Algorithms: An Example of a Sort Algorithm

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Summary. We analyse three algorithms: exponentiation by squaring, calculation of maximum, and sorting by exchanging in terms of program algebra over an algebra.

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The notation and terminology used in this paper have been introduced in the following articles: [37], [1], [2], [17], [3], [4], [13], [18], [34], [23], [29], [19], [20], [15], [5], [33], [6], [27], [38], [28], [30], [14], [7], [8], [31], [16], [24], [26], [35], [9], [21], [32], [39], [36], [10], [11], [25], [12], and [22].

1. Exponentiation by Squaring Revisited

Now we state the propositions:

(1) (i) $1 \bmod 2 = 1$, and

(ii) $2 \bmod 2 = 0$.

(2) Let us consider a non empty non void many sorted signature Σ, an algebra $\mathfrak{A}$ over Σ, a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, a sort symbol s of Σ, and a set a. Suppose $a \in$ (the sorts of $\mathfrak{B}$)(s). Then $a \in$ (the sorts of $\mathfrak{A}$)(s).

(3) Let us consider a non empty set I, sets a, b, c, and an element i of I. Then $c \in (i\text{-singleton}\, a)(b)$ if and only if $b = i$ and $c = a$.

(4) Let us consider a non empty set I, sets a, b, c, d, and elements i, j of I. Then $c \in (i\text{-singleton}\, a \cup j\text{-singleton}\, d)(b)$ if and only if $b = i$ and $c = a$ or $b = j$ and $c = d$. The theorem is a consequence of (3).

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Let Σ be a boolean correct non empty non void boolean signature with integers with connectives from 4 and the sort at 1 and $\mathfrak{A}$ be a non-empty algebra over Σ. We say that $\mathfrak{A}$ is integer if and only if

(Def. 1) There exists an image $\mathfrak{C}$ of $\mathfrak{A}$ such that $\mathfrak{C}$ is a boolean correct algebra over Σ with integers with connectives from 4 and the sort at 1.

Now we state the propositions:

(5) Let us consider a non empty non void many sorted signature Σ and a non-empty algebra $\mathfrak{A}$ over Σ. Then $\operatorname{Im} \mathrm{id}_{\alpha} =$ the algebra of $\mathfrak{A}$, where α is the sorts of $\mathfrak{A}$.

(6) Let us consider a non empty non void many sorted signature Σ. Then every non-empty algebra over Σ is an image of $\mathfrak{A}$. The theorem is a consequence of (5). PROOF: $\mathfrak{A}$ is $\mathfrak{A}$-image. $\square$

Let Σ be a boolean correct non empty non void boolean signature with integers with connectives from 4 and the sort at 1. One can verify that there exists a non-empty algebra over Σ which is integer.

Let $\mathfrak{A}$ be an integer non-empty algebra over Σ. Note that there exists an image of $\mathfrak{A}$ which is boolean correct.

Let us note that there exists a boolean correct image of $\mathfrak{A}$ which has integers with connectives from 4 and the sort at 1.

Now we state the proposition:

(7) Let us consider a non empty non void many sorted signature Σ, a non-empty algebra $\mathfrak{A}$ over Σ, an operation symbol o of Σ, a set a, and a sort symbol r of Σ. Suppose o is of type $a \to r$. Then
   (i) $\mathrm{Den}(o, \mathfrak{A})$ is a function from (the sorts of $\mathfrak{A})^{\#}(a)$ into (the sorts of $\mathfrak{A})(r)$, and
   (ii) $\mathrm{Args}(o, \mathfrak{A}) =$ (the sorts of $\mathfrak{A})^{\#}(a)$, and
   (iii) $\mathrm{Result}(o, \mathfrak{A}) =$ (the sorts of $\mathfrak{A})(r)$.

Let Σ be a boolean correct non empty non void boolean signature and $\mathfrak{A}$ be a boolean correct non-empty algebra over Σ. Observe that every non-empty subalgebra of $\mathfrak{A}$ is boolean correct.

Let Σ be a boolean correct non empty non void boolean signature with integers with connectives from 4 and the sort at 1 and $\mathfrak{A}$ be a boolean correct non-empty algebra over Σ with integers with connectives from 4 and the sort at 1. Note that every non-empty subalgebra of $\mathfrak{A}$ has integers with connectives from 4 and the sort at 1.

Let X be a non-empty many sorted set indexed by the carrier of Σ. Let us observe that $\mathfrak{F}_{\Sigma}(X)$ is integer as a non-empty algebra over Σ.

Now we state the proposition:

(8) Let us consider a non empty non void many sorted signature Σ, algebras $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{B}_1$ over Σ, and a non-empty algebra $\mathfrak{B}_2$ over Σ. Suppose

(i) the algebra of $\mathfrak{A}_1$ = the algebra of $\mathfrak{A}_2$, and

(ii) the algebra of $\mathfrak{B}_1$ = the algebra of $\mathfrak{B}_2$.

Let us consider a many sorted function h_1 from $\mathfrak{A}_1$ into $\mathfrak{B}_1$ and a many sorted function h_2 from $\mathfrak{A}_2$ into $\mathfrak{B}_2$. Suppose

(iii) $h_1 = h_2$, and

(iv) h_1 is an epimorphism of $\mathfrak{A}_1$ onto $\mathfrak{B}_1$.

Then h_2 is an epimorphism of $\mathfrak{A}_2$ onto $\mathfrak{B}_2$.

Let Σ be a boolean correct non empty non void boolean signature with integers with connectives from 4 and the sort at 1 and X be a non-empty many sorted set indexed by the carrier of Σ. Let us note that there exists an including Σ-terms over X non-empty free variable algebra over Σ which is vf-free and integer.

Let Σ be a non empty non void many sorted signature. Let $\mathfrak{T}$ be an including Σ-terms over X non-empty algebra over Σ. The functor FreeGenerator($\mathfrak{T}$) yielding a non-empty generator set of $\mathfrak{T}$ is defined by the term

(Def. 2) FreeGenerator(X).

Let X_0 be a countable non-empty many sorted set indexed by the carrier of Σ and $\mathfrak{T}$ be an including Σ-terms over X_0 non-empty algebra over Σ. Let us observe that FreeGenerator($\mathfrak{T}$) is Equations($\Sigma, \mathfrak{T}$)-free and non-empty.

Let X be a non-empty many sorted set indexed by the carrier of Σ, $\mathfrak{T}$ be an including Σ-terms over X algebra over Σ, and G be a generator set of $\mathfrak{T}$. We say that G is basic if and only if

(Def. 3) FreeGenerator($\mathfrak{T}$) $\subseteq G$.

Let s be a sort symbol of Σ and x be an element of $G(s)$. We say that x is pure if and only if

(Def. 4) $x \in$ (FreeGenerator($\mathfrak{T}$))(s).

Observe that FreeGenerator($\mathfrak{T}$) is basic.

Note that there exists a non-empty generator set of $\mathfrak{T}$ which is basic.

Let G be a basic generator set of $\mathfrak{T}$ and s be a sort symbol of Σ. One can check that there exists an element of $G(s)$ which is pure.

Now we state the proposition:

(9) Let us consider a non empty non void many sorted signature Σ, a non-empty many sorted set X indexed by the carrier of Σ, an including Σ-terms over X algebra $\mathfrak{T}$ over Σ, a basic generator set G of $\mathfrak{T}$, a sort symbol s of Σ, and a set a. Then a is a pure element of $G(s)$ if and only if $a \in$ (FreeGenerator($\mathfrak{T}$))(s).

Let Σ be a non empty non void many sorted signature, X be a non-empty many sorted set indexed by the carrier of Σ, $\mathfrak{T}$ be an including Σ-terms over X algebra over Σ, and G be a generator system over Σ, X, and $\mathfrak{T}$. We say that G is basic if and only if

(Def. 5) The generators of G are basic.

Observe that there exists a generator system over Σ, X, and $\mathfrak{T}$ which is basic.

Let G be a basic generator system over Σ, X, and $\mathfrak{T}$. Note that the generators of G are basic.

In this paper Σ denotes a boolean correct non empty non void boolean signature with integers with connectives from 4 and the sort at 1, X denotes a non-empty many sorted set indexed by the carrier of Σ, $\mathfrak{T}$ denotes a vf-free including Σ-terms over X integer non-empty free variable algebra over Σ, $\mathfrak{C}$ denotes a boolean correct non-empty image of $\mathfrak{T}$ with integers with connectives from 4 and the sort at 1, G denotes a basic generator system over Σ, X, and $\mathfrak{T}$, $\mathfrak{A}$ denotes a if-while algebra over the generators of G, I denotes an integer sort symbol of Σ, x, y, z, m denote pure elements of (the generators of G)(I), b denotes a pure element of (the generators of G)((the boolean sort of Σ)), τ, τ_1, τ_2 denote elements of $\mathfrak{T}$ from I, P denotes an algorithm of $\mathfrak{A}$, and s, s_1, s_2 denote elements of $\mathfrak{C}$-States(the generators of G).

Let Σ be a boolean correct non empty non void boolean signature and $\mathfrak{A}$ be a non-empty algebra over Σ. The functor $\mathrm{false}_{\mathfrak{A}}$ yielding an element of $\mathfrak{A}$ from the boolean sort of Σ is defined by the term

(Def. 6) $\neg\,\mathrm{true}_{\mathfrak{A}}$.

In this paper f denotes an execution function of $\mathfrak{A}$ over $\mathfrak{C}$-States(the generators of G) and $\mathrm{States}_{b \not\to \mathrm{false}_{\mathfrak{C}}}$(the generators of G).

Now we state the proposition:

(10) $\mathrm{false}_{\mathfrak{C}} = \mathit{false}$.

Let Σ be a boolean correct non empty non void boolean signature, X be a non-empty many sorted set indexed by the carrier of Σ, $\mathfrak{T}$ be an including Σ-terms over X algebra over Σ, G be a generator system over Σ, X, and $\mathfrak{T}$, b be an element of (the generators of G)((the boolean sort of Σ)), $\mathfrak{C}$ be an image of $\mathfrak{T}$, $\mathfrak{A}$ be a pre-if-while algebra, f be an execution function of $\mathfrak{A}$ over $\mathfrak{C}$-States(the generators of G) and $\mathrm{States}_{b \not\to \mathrm{false}_{\mathfrak{C}}}$(the generators of G), s be an element of $\mathfrak{C}$-States(the generators of G), and P be an algorithm of $\mathfrak{A}$. Note that the functor $f(s, P)$ yields an element of $\mathfrak{C}$-States(the generators of G). Let Σ be a non empty non void many sorted signature, $\mathfrak{T}$ be a non-empty algebra over Σ, G be a non-empty generator set of $\mathfrak{T}$, s be a sort symbol of Σ, and x be an element of $G(s)$. The functor ${}^{@}x$ yielding an element of $\mathfrak{T}$ from s is defined by the term

(Def. 7) x.

Let us consider Σ, X, $\mathfrak{T}$, G, $\mathfrak{A}$, b, I, τ_1, and τ_2. The functors $b\,\mathrm{leq}(\tau_1, \tau_2, \mathfrak{A})$ and $b\,\mathrm{gt}(\tau_1, \tau_2, \mathfrak{A})$ yielding algorithms of $\mathfrak{A}$ are defined by the terms, respectively.

(Def. 8) $b{:=}_{\mathfrak{A}}(\mathrm{leq}(\tau_1, \tau_2))$.

(Def. 9) $b :=_{\mathfrak{A}} (\neg \mathrm{leq}(\tau_1, \tau_2))$.

The functor $2^I_{\mathfrak{T}}$ yielding an element of $\mathfrak{T}$ from I is defined by the term

(Def. 10) $1^I_{\mathfrak{T}} + 1^I_{\mathfrak{T}}$.

Let us consider G, $\mathfrak{A}$, and b. Let us consider τ. The functors τ is odd$(b, \mathfrak{A})$ and τ is even$(b, \mathfrak{A})$ yielding algorithms of $\mathfrak{A}$ are defined by the terms, respectively.

(Def. 11) $b\,\mathrm{gt}(\tau \bmod 2^I_{\mathfrak{T}}, 0^I_{\mathfrak{T}}, \mathfrak{A})$.

(Def. 12) $b\,\mathrm{leq}(\tau \bmod 2^I_{\mathfrak{T}}, 0^I_{\mathfrak{T}}, \mathfrak{A})$.

Let us consider $\mathfrak{C}$. Let us consider s. Let x be an element of (the generators of G)(I). Let us note that $s(I)(x)$ is integer.

Let us consider τ. Let us note that τ value at$(\mathfrak{C}, s)$ is integer.

In the sequel u denotes a many sorted function from FreeGenerator$(\mathfrak{T})$ into the sorts of $\mathfrak{C}$.

Let us consider Σ, X, $\mathfrak{T}$, $\mathfrak{C}$, I, u, and τ. One can verify that τ value at$(\mathfrak{C}, u)$ is integer.

Let us consider G. Let us consider s. Let τ be an element of $\mathfrak{T}$ from the boolean sort of Σ. One can verify that τ value at$(\mathfrak{C}, s)$ is boolean.

Let us consider u. One can check that τ value at$(\mathfrak{C}, u)$ is boolean.

Let us consider an operation symbol o of Σ. Now we state the propositions:

(11) Suppose $o =$ (the connectives of Σ)$(1)(\in$ (the carrier' of Σ)). Then
- (i) $o =$ (the connectives of Σ)(1), and
- (ii) Arity$(o) = \emptyset$, and
- (iii) the result sort of $o =$ the boolean sort of Σ.

(12) Suppose $o =$ (the connectives of Σ)$(2)(\in$ (the carrier' of Σ)). Then
- (i) $o =$ (the connectives of Σ)(2), and
- (ii) Arity$(o) = \langle$the boolean sort of $\Sigma\rangle$, and
- (iii) the result sort of $o =$ the boolean sort of Σ.

(13) Suppose $o =$ (the connectives of Σ)$(3)(\in$ (the carrier' of Σ)). Then
- (i) $o =$ (the connectives of Σ)(3), and
- (ii) Arity$(o) = \langle$the boolean sort of Σ, the boolean sort of $\Sigma\rangle$, and
- (iii) the result sort of $o =$ the boolean sort of Σ.

(14) Suppose $o =$ (the connectives of Σ)$(4)(\in$ (the carrier' of Σ)). Then
- (i) Arity$(o) = \emptyset$, and
- (ii) the result sort of $o = I$.

(15) Suppose $o =$ (the connectives of Σ)$(5)(\in$ (the carrier' of Σ)). Then
- (i) Arity$(o) = \emptyset$, and
- (ii) the result sort of $o = I$.

(16) Suppose $o =$ (the connectives of Σ)(6)($\in$ (the carrier' of Σ)). Then

(i) Arity(o) $= \langle I \rangle$, and

(ii) the result sort of $o = I$.

(17) Suppose $o =$ (the connectives of Σ)(7)($\in$ (the carrier' of Σ)). Then

(i) Arity(o) $= \langle I, I \rangle$, and

(ii) the result sort of $o = I$.

(18) Suppose $o =$ (the connectives of Σ)(8)($\in$ (the carrier' of Σ)). Then

(i) Arity(o) $= \langle I, I \rangle$, and

(ii) the result sort of $o = I$.

(19) Suppose $o =$ (the connectives of Σ)(9)($\in$ (the carrier' of Σ)). Then

(i) Arity(o) $= \langle I, I \rangle$, and

(ii) the result sort of $o = I$.

(20) Suppose $o =$ (the connectives of Σ)(10)($\in$ (the carrier' of Σ)). Then

(i) Arity(o) $= \langle I, I \rangle$, and

(ii) the result sort of $o =$ the boolean sort of Σ.

(21) Let us consider a non empty non void many sorted signature Σ and an operation symbol o of Σ. Suppose Arity(o) $= \emptyset$. Let us consider an algebra $\mathfrak{A}$ over Σ. Then Args($o, \mathfrak{A}$) $= \{\emptyset\}$.

(22) Let us consider a non empty non void many sorted signature Σ, a sort symbol a of Σ, and an operation symbol o of Σ. Suppose Arity(o) $= \langle a \rangle$. Let us consider an algebra $\mathfrak{A}$ over Σ. Then Args($o, \mathfrak{A}$) $= \prod\langle$(the sorts of $\mathfrak{A}$)(a)$\rangle$.

(23) Let us consider a non empty non void many sorted signature Σ, sort symbols a, b of Σ, and an operation symbol o of Σ. Suppose Arity(o) $= \langle a, b \rangle$. Let us consider an algebra $\mathfrak{A}$ over Σ. Then Args($o, \mathfrak{A}$) $= \prod\langle$(the sorts of $\mathfrak{A}$)(a), (the sorts of $\mathfrak{A}$)(b)$\rangle$.

(24) Let us consider a non empty non void many sorted signature Σ, sort symbols a, b, c of Σ, and an operation symbol o of Σ. Suppose Arity(o) $= \langle a, b, c \rangle$. Let us consider an algebra $\mathfrak{A}$ over Σ. Then Args($o, \mathfrak{A}$) $= \prod\langle$(the sorts of $\mathfrak{A}$)(a), (the sorts of $\mathfrak{A}$)(b), (the sorts of $\mathfrak{A}$)(c)$\rangle$.

(25) Let us consider a non empty non void many sorted signature Σ, non-empty algebras $\mathfrak{A}$, $\mathfrak{B}$ over Σ, a sort symbol s of Σ, an element a of $\mathfrak{A}$ from s, a many sorted function h from $\mathfrak{A}$ into $\mathfrak{B}$, and an operation symbol o of Σ. Suppose Arity(o) $= \langle s \rangle$. Let us consider an element p of Args($o, \mathfrak{A}$). If $p = \langle a \rangle$, then $h\#p = \langle h(s)(a) \rangle$.

(26) Let us consider a non empty non void many sorted signature Σ, non-empty algebras $\mathfrak{A}$, $\mathfrak{B}$ over Σ, sort symbols s_1, s_2 of Σ, an element a of $\mathfrak{A}$ from s_1, an element b of $\mathfrak{A}$ from s_2, a many sorted function h from $\mathfrak{A}$ into $\mathfrak{B}$, and an operation symbol o of Σ. Suppose $\mathrm{Arity}(o) = \langle s_1, s_2\rangle$. Let us consider an element p of $\mathrm{Args}(o, \mathfrak{A})$. Suppose $p = \langle a, b\rangle$. Then $h\#p = \langle h(s_1)(a), h(s_2)(b)\rangle$.

(27) Let us consider a non empty non void many sorted signature Σ, non-empty algebras $\mathfrak{A}$, $\mathfrak{B}$ over Σ, sort symbols s_1, s_2, s_3 of Σ, an element a of $\mathfrak{A}$ from s_1, an element b of $\mathfrak{A}$ from s_2, an element c of $\mathfrak{A}$ from s_3, a many sorted function h from $\mathfrak{A}$ into $\mathfrak{B}$, and an operation symbol o of Σ. Suppose $\mathrm{Arity}(o) = \langle s_1, s_2, s_3\rangle$. Let us consider an element p of $\mathrm{Args}(o, \mathfrak{A})$. Suppose $p = \langle a, b, c\rangle$. Then $h\#p = \langle h(s_1)(a), h(s_2)(b), h(s_3)(c)\rangle$.

Let us consider a many sorted function h from $\mathfrak{T}$ into $\mathfrak{C}$, a sort symbol a of Σ, and an element τ of $\mathfrak{T}$ from a. Now we state the propositions:

(28) If h is a homomorphism of $\mathfrak{T}$ into $\mathfrak{C}$, then $\tau\,\mathrm{value\ at}(\mathfrak{C}, h \restriction \mathrm{FreeGenerator}(\mathfrak{T})) = h(a)(\tau)$.

(29) Suppose h is a homomorphism of $\mathfrak{T}$ into $\mathfrak{C}$ and $s = h \restriction$ the generators of G. Then $\tau\,\mathrm{value\ at}(\mathfrak{C}, s) = h(a)(\tau)$.

(30) $\mathrm{true}_{\mathfrak{T}}\,\mathrm{value\ at}(\mathfrak{C}, s) = \mathit{true}$. The theorem is a consequence of (11) and (21).

(31) Let us consider an element τ of $\mathfrak{T}$ from the boolean sort of Σ. Then $\neg\tau\,\mathrm{value\ at}(\mathfrak{C}, s) = \neg(\tau\,\mathrm{value\ at}(\mathfrak{C}, s))$. The theorem is a consequence of (29), (12), (22), and (25).

(32) Let us consider a boolean set a and an element τ of $\mathfrak{T}$ from the boolean sort of Σ. Then $\neg\tau\,\mathrm{value\ at}(\mathfrak{C}, s) = \neg a$ if and only if $\tau\,\mathrm{value\ at}(\mathfrak{C}, s) = a$. The theorem is a consequence of (31).

(33) Let us consider an element a of $\mathfrak{C}$ from the boolean sort of Σ and a boolean set x. Then $\neg a = \neg x$ if and only if $a = x$.

(34) $\mathrm{false}_{\mathfrak{T}}\,\mathrm{value\ at}(\mathfrak{C}, s) = \mathit{false}$. The theorem is a consequence of (31) and (30).

(35) Let us consider elements τ_1, τ_2 of $\mathfrak{T}$ from the boolean sort of Σ. Then $(\tau_1 \wedge \tau_2)\,\mathrm{value\ at}(\mathfrak{C}, s) = (\tau_1\,\mathrm{value\ at}(\mathfrak{C}, s)) \wedge (\tau_2\,\mathrm{value\ at}(\mathfrak{C}, s))$. The theorem is a consequence of (29), (13), (23), and (26).

(36) $0^I_{\mathfrak{T}}\,\mathrm{value\ at}(\mathfrak{C}, s) = 0$. The theorem is a consequence of (14) and (21).

(37) $1^I_{\mathfrak{T}}\,\mathrm{value\ at}(\mathfrak{C}, s) = 1$. The theorem is a consequence of (15) and (21).

(38) $(-\tau)\,\mathrm{value\ at}(\mathfrak{C}, s) = -\tau\,\mathrm{value\ at}(\mathfrak{C}, s)$. The theorem is a consequence of (16), (22), and (25).

(39) $(\tau_1 + \tau_2)\,\mathrm{value\ at}(\mathfrak{C}, s) = \tau_1\,\mathrm{value\ at}(\mathfrak{C}, s) + \tau_2\,\mathrm{value\ at}(\mathfrak{C}, s)$. The theorem is a consequence of (17), (23), and (26).

(40) $2^I_{\mathfrak{T}}\,\mathrm{value\ at}(\mathfrak{C}, s) = 2$. The theorem is a consequence of (37) and (39).

(41) $(\tau_1 - \tau_2)$ value at$(\mathfrak{C}, s) = \tau_1$ value at$(\mathfrak{C}, s) - \tau_2$ value at$(\mathfrak{C}, s)$. The theorem is a consequence of (39) and (38).

(42) $(\tau_1 \cdot \tau_2)$ value at$(\mathfrak{C}, s) = (\tau_1$ value at$(\mathfrak{C}, s)) \cdot (\tau_2$ value at$(\mathfrak{C}, s))$. The theorem is a consequence of (29), (18), (23), and (26).

(43) $(\tau_1 \operatorname{div} \tau_2)$ value at$(\mathfrak{C}, s) = \tau_1$ value at$(\mathfrak{C}, s) \operatorname{div} \tau_2$ value at$(\mathfrak{C}, s)$. The theorem is a consequence of (19), (23), and (26).

(44) $(\tau_1 \bmod \tau_2)$ value at$(\mathfrak{C}, s) = \tau_1$ value at$(\mathfrak{C}, s) \bmod \tau_2$ value at$(\mathfrak{C}, s)$. The theorem is a consequence of (41), (42), and (43).

(45) leq(τ_1, τ_2) value at$(\mathfrak{C}, s) =$ leq$(\tau_1$ value at$(\mathfrak{C}, s), \tau_2$ value at$(\mathfrak{C}, s))$. The theorem is a consequence of (20), (23), and (26).

(46) $\mathrm{true}_{\mathfrak{T}}$ value at$(\mathfrak{C}, u) = $ *true*. The theorem is a consequence of (11) and (21).

(47) Let us consider an element τ of $\mathfrak{T}$ from the boolean sort of Σ. Then $\neg\tau$ value at$(\mathfrak{C}, u) = \neg(\tau$ value at$(\mathfrak{C}, u))$. The theorem is a consequence of (28), (12), (22), and (25).

(48) Let us consider a boolean set a and an element τ of $\mathfrak{T}$ from the boolean sort of Σ. Then $\neg\tau$ value at$(\mathfrak{C}, u) = \neg a$ if and only if τ value at$(\mathfrak{C}, u) = a$. The theorem is a consequence of (47).

(49) $\mathrm{false}_{\mathfrak{T}}$ value at$(\mathfrak{C}, u) = $ *false*. The theorem is a consequence of (47) and (46).

(50) Let us consider elements τ_1, τ_2 of $\mathfrak{T}$ from the boolean sort of Σ. Then $(\tau_1 \wedge \tau_2)$ value at$(\mathfrak{C}, u) = (\tau_1$ value at$(\mathfrak{C}, u)) \wedge (\tau_2$ value at$(\mathfrak{C}, u))$. The theorem is a consequence of (28), (13), (23), and (26).

(51) $0^I_{\mathfrak{T}}$ value at$(\mathfrak{C}, u) = 0$. The theorem is a consequence of (14) and (21).

(52) $1^I_{\mathfrak{T}}$ value at$(\mathfrak{C}, u) = 1$. The theorem is a consequence of (15) and (21).

(53) $(-\tau)$ value at$(\mathfrak{C}, u) = -\tau$ value at$(\mathfrak{C}, u)$. The theorem is a consequence of (16), (22), and (25).

(54) $(\tau_1 + \tau_2)$ value at$(\mathfrak{C}, u) = \tau_1$ value at$(\mathfrak{C}, u) + \tau_2$ value at$(\mathfrak{C}, u)$. The theorem is a consequence of (17), (23), and (26).

(55) $2^I_{\mathfrak{T}}$ value at$(\mathfrak{C}, u) = 2$. The theorem is a consequence of (52) and (54).

(56) $(\tau_1 - \tau_2)$ value at$(\mathfrak{C}, u) = \tau_1$ value at$(\mathfrak{C}, u) - \tau_2$ value at$(\mathfrak{C}, u)$. The theorem is a consequence of (54) and (53).

(57) $(\tau_1 \cdot \tau_2)$ value at$(\mathfrak{C}, u) = (\tau_1$ value at$(\mathfrak{C}, u)) \cdot (\tau_2$ value at$(\mathfrak{C}, u))$. The theorem is a consequence of (28), (18), (23), and (26).

(58) $(\tau_1 \operatorname{div} \tau_2)$ value at$(\mathfrak{C}, u) = \tau_1$ value at$(\mathfrak{C}, u) \operatorname{div} \tau_2$ value at$(\mathfrak{C}, u)$. The theorem is a consequence of (19), (23), and (26).

(59) $(\tau_1 \bmod \tau_2)$ value at$(\mathfrak{C}, u) = \tau_1$ value at$(\mathfrak{C}, u) \bmod \tau_2$ value at$(\mathfrak{C}, u)$. The theorem is a consequence of (56), (57), and (58).

(60) $\mathrm{leq}(\tau_1, \tau_2)$ value $\mathrm{at}(\mathfrak{C}, u) = \mathrm{leq}(\tau_1$ value $\mathrm{at}(\mathfrak{C}, u), \tau_2$ value $\mathrm{at}(\mathfrak{C}, u))$.
The theorem is a consequence of (20), (23), and (26).

(61) Let us consider a sort symbol a of Σ and an element x of (the generators of G)(a). Then ${}^{@}x$ value $\mathrm{at}(\mathfrak{C}, s) = s(a)(x)$. The theorem is a consequence of (29).

(62) Let us consider a sort symbol a of Σ, a pure element x of (the generators of G)(a), and a many sorted function u from FreeGenerator($\mathfrak{T}$) into the sorts of $\mathfrak{C}$. Then ${}^{@}x$ value $\mathrm{at}(\mathfrak{C}, u) = u(a)(x)$.

Let us consider integers i, j and elements a, b of $\mathfrak{C}$ from I. Now we state the propositions:

(63) If $a = i$ and $b = j$, then $a - b = i - j$.

(64) If $a = i$ and $b = j$ and $j \neq 0$, then $a \bmod b = i \bmod j$.

(65) Suppose G is $\mathfrak{C}$-supported and $f \in \mathfrak{C}\text{-Execution}_{b \not\to \mathrm{false}_{\mathfrak{C}}}(\mathfrak{A})$. Then let us consider a sort symbol a of Σ, a pure element x of (the generators of G)(a), and an element τ of $\mathfrak{T}$ from a. Then

(i) $f(s, x{:=}_{\mathfrak{A}}\tau)(a)(x) = \tau$ value $\mathrm{at}(\mathfrak{C}, s)$, and

(ii) for every pure element z of (the generators of G)(a) such that $z \neq x$ holds $f(s, x{:=}_{\mathfrak{A}}\tau)(a)(z) = s(a)(z)$, and

(iii) for every sort symbol b of Σ such that $a \neq b$ for every pure element z of (the generators of G)(b), $f(s, x{:=}_{\mathfrak{A}}\tau)(b)(z) = s(b)(z)$.

(66) Suppose G is $\mathfrak{C}$-supported and $f \in \mathfrak{C}\text{-Execution}_{b \not\to \mathrm{false}_{\mathfrak{C}}}(\mathfrak{A})$. Then

(i) τ_1 value $\mathrm{at}(\mathfrak{C}, s) < \tau_2$ value $\mathrm{at}(\mathfrak{C}, s)$ iff $f(s, b\,\mathrm{gt}(\tau_2, \tau_1, \mathfrak{A})) \in \mathrm{States}_{b \not\to \mathrm{false}_{\mathfrak{C}}}$(the generators of G), and

(ii) τ_1 value $\mathrm{at}(\mathfrak{C}, s) \leqslant \tau_2$ value $\mathrm{at}(\mathfrak{C}, s)$ iff $f(s, b\,\mathrm{leq}(\tau_1, \tau_2, \mathfrak{A})) \in \mathrm{States}_{b \not\to \mathrm{false}_{\mathfrak{C}}}$(the generators of G), and

(iii) for every x, $f(s, b\,\mathrm{gt}(\tau_1, \tau_2, \mathfrak{A}))(I)(x) = s(I)(x)$ and $f(s, b\,\mathrm{leq}(\tau_1, \tau_2, \mathfrak{A}))(I)(x) = s(I)(x)$, and

(iv) for every pure element c of (the generators of G)((the boolean sort of Σ)) such that $c \neq b$ holds $f(s, b\,\mathrm{gt}(\tau_1, \tau_2, \mathfrak{A}))$((the boolean sort of Σ))(c) $= s$((the boolean sort of Σ))(c) and $f(s, b\,\mathrm{leq}(\tau_1, \tau_2, \mathfrak{A}))$((the boolean sort of Σ))(c) $= s$((the boolean sort of Σ))(c).

The theorem is a consequence of (31), (45), and (33).

Let i, j be real numbers and a, b be boolean sets. One can verify that $(i > j \to a, b)$ is boolean.

Now we state the proposition:

(67) Suppose G is $\mathfrak{C}$-supported and $f \in \mathfrak{C}\text{-Execution}_{b \not\to \mathrm{false}_{\mathfrak{C}}}(\mathfrak{A})$. Then

(i) $f(s, \tau \text{ is odd}(b, \mathfrak{A}))$((the boolean sort of Σ))(b) $= \tau$ value $\mathrm{at}(\mathfrak{C}, s) \bmod 2$, and

(ii) $f(s, \tau \text{ is even}(b, \mathfrak{A}))((\text{the boolean sort of } \Sigma))(b) = (\tau \text{ value at}(\mathfrak{C}, s) + 1) \bmod 2$, and

(iii) for every z, $f(s, \tau \text{ is odd}(b, \mathfrak{A}))(I)(z) = s(I)(z)$ and $f(s, \tau \text{ is even}(b, \mathfrak{A}))(I)(z) = s(I)(z)$.

The theorem is a consequence of (36), (40), (64), (31), (45), (44), and (1).

Let us consider Σ, X, $\mathfrak{T}$, G, and $\mathfrak{A}$. We say that $\mathfrak{A}$ is elementary if and only if

(Def. 13) rng the assignments of $\mathfrak{A} \subseteq \text{ElementaryInstructions}_{\mathfrak{A}}$.

Now we state the proposition:

(68) Suppose $\mathfrak{A}$ is elementary. Then let us consider a sort symbol a of Σ, an element x of (the generators of G)(a), and an element τ of $\mathfrak{T}$ from a. Then $x{:=}_{\mathfrak{A}}\tau \in \text{ElementaryInstructions}_{\mathfrak{A}}$.

Let us consider Σ, X, $\mathfrak{T}$, and G. One can verify that there exists a strict if-while algebra over the generators of G which is elementary.

Let $\mathfrak{A}$ be an elementary if-while algebra over the generators of G, a be a sort symbol of Σ, x be an element of (the generators of G)(a), and τ be an element of $\mathfrak{T}$ from a. Let us observe that $x{:=}_{\mathfrak{A}}\tau$ is absolutely-terminating.

Now let Γ denotes the program

```
y:=𝔄 1^I_𝔗;
while b gt(@m, 0^I_𝔗, 𝔄) do
  if @m is odd(b, 𝔄) then
    y:=𝔄 @y · @x
  fi;
  m:=𝔄 @m div 2^I_𝔗;
  x:=𝔄 @x · @x
done
```

Then we state the propositions:

(69) Let us consider an elementary if-while algebra $\mathfrak{A}$ over the generators of G and an execution function f of $\mathfrak{A}$ over $\mathfrak{C}$-States(the generators of G) and $\text{States}_{b \not\to \text{false}_{\mathfrak{C}}}$(the generators of G). Suppose

(i) G is $\mathfrak{C}$-supported, and

(ii) $f \in \mathfrak{C}\text{-Execution}_{b \not\to \text{false}_{\mathfrak{C}}}(\mathfrak{A})$, and

(iii) there exists a function d such that $d(x) = 1$ and $d(y) = 2$ and $d(m) = 3$.

Then Γ is terminating w.r.t. f and $\{s : s(I)(m) \geqslant 0\}$. The theorem is a consequence of (66), (36), (61), (65), (40), and (43). Proof: Set $ST = \mathfrak{C}$-States(the generators of G). Set $TV = \text{States}_{b \not\to \text{false}_{\mathfrak{C}}}$(the generators of G). Set $P = \{s : s(I)(m) \geqslant 0\}$. Set $W = b\,\text{gt}({}^{@}m, 0^{I}_{\mathfrak{T}}, \mathfrak{A})$. Define $\mathcal{F}(\text{element of } ST) = \$_1(I)(m) (\in \mathbb{N})$. Define $\mathcal{R}[\text{element of } ST] \equiv \$_1(I)(m) >$

0. Set $K = \text{if } {}^{@}m \text{ is odd}(b, \mathfrak{A}) \text{ then}(y{:=}_{\mathfrak{A}}({}^{@}y \cdot {}^{@}x))$.
Set $J = (K; m{:=}_{\mathfrak{A}}({}^{@}m \operatorname{div} 2^{I}_{\mathfrak{T}})); x{:=}_{\mathfrak{A}}({}^{@}x \cdot {}^{@}x)$. P is invariant w.r.t. W and f. For every element s of ST such that $s \in P$ and $f(f(s, J), W) \in TV$ holds $f(s, J) \in P$. P is invariant w.r.t. $y{:=}_{\mathfrak{A}}(1^{I}_{\mathfrak{T}})$ and f. For every s such that $f(s, W) \in P$ holds iteration of f started in $J; W$ terminates w.r.t. $f(s, W)$. □

(70) Suppose G is $\mathfrak{C}$-supported and there exists a function d such that $d(b) = 0$ and $d(x) = 1$ and $d(y) = 2$ and $d(m) = 3$. Then let us consider an element s of $\mathfrak{C}$-States(the generators of G) and a natural number n. Suppose $n = s(I)(m)$. If $f \in \mathfrak{C}\text{-Execution}_{b \not\to \text{false}_{\mathfrak{C}}}(\mathfrak{A})$, then $f(s, \Gamma)(I)(y) = s(I)(x)^n$. The theorem is a consequence of (65), (66), (36), (61), (37), (40), (43), (67), (10), and (42). Proof: Set $\Sigma = \mathfrak{C}$-States(the generators of G). Set $W = \mathfrak{T}$. Set $g = f$. Set $\mathfrak{T} = \text{States}_{b \not\to \text{false}_{\mathfrak{C}}}$(the generators of G). Set $s0 = f(s, y{:=}_{\mathfrak{A}}(1^{I}_{W}))$. Define $\mathcal{R}[\text{element of } \Sigma] \equiv \$_1(I)(m) > 0$. Set $\mathfrak{C} = b \operatorname{gt}({}^{@}m, 0^{I}_{W}, \mathfrak{A})$. Define $\mathcal{P}[\text{element of } \Sigma] \equiv s(I)(x)^n = \$_1(I)(y) \cdot \$_1(I)(x)^{\$_1(I)(m)}$ and $\$_1(I)(m) \geqslant 0$. Define $\mathcal{F}(\text{element of } \Sigma) = \$_1(I)(m) (\in \mathbb{N})$. Set $I = \text{if } {}^{@}m \text{ is odd}(b, \mathfrak{A}) \text{ then}(y{:=}_{\mathfrak{A}}({}^{@}y \cdot {}^{@}x))$.
Set $J = (I; m{:=}_{\mathfrak{A}}({}^{@}m \operatorname{div} 2^{Y}_{W})); x{:=}_{\mathfrak{A}}({}^{@}x \cdot {}^{@}x)$. For every element s of Σ such that $\mathcal{P}[s]$ holds $\mathcal{P}[(g(s, \mathfrak{C})$ **qua** element of $\Sigma)]$ and $g(s, \mathfrak{C}) \in \mathfrak{T}$ iff $\mathcal{R}[(g(s, \mathfrak{C})$ **qua** element of $\Sigma)]$. Set $s_1 = g(s0, \mathfrak{C})$. For every element s of Σ such that $\mathcal{R}[s]$ holds $\mathcal{R}[(g(s, J; \mathfrak{C})$ **qua** element of $\Sigma)]$ iff $g(s, J; \mathfrak{C}) \in \mathfrak{T}$ and $\mathcal{F}((g(s, J; \mathfrak{C})$ **qua** element of $\Sigma)) < \mathcal{F}(s)$. Set $q = s$. For every element s of Σ such that $\mathcal{P}[s]$ and $s \in \mathfrak{T}$ and $\mathcal{R}[s]$ holds $\mathcal{P}[(g(s, J)$ **qua** element of $\Sigma)]$. □

2. Calculation of Maximum

Let X be a non empty set, f be a finite sequence of elements of X^ω, and x be a natural number. Let us observe that $f(x)$ is transfinite sequence-like finite function-like and relation-like.

Let us note that every finite sequence of elements of X^ω is function yielding.

Let i be a natural number, f be an i-based finite array, and a, x be sets. Note that $f +\cdot (a, x)$ is i-based finite and segmental.

Let X be a non empty set, f be an X-valued function, a be a set, and x be an element of X. Let us observe that $f +\cdot (a, x)$ is X-valued.

The scheme *Sch1* deals with a non empty set $\mathcal{X}$ and a natural number j and a set $\mathfrak{B}$ and a ternary functor $\mathcal{F}$ yielding a set and a unary functor $\mathfrak{A}$ yielding a set and states that

(Sch. 1) There exists a finite sequence f of elements of $\mathcal{X}^\omega$ such that $\operatorname{len} f = j$ and $f(1) = \mathfrak{B}$ or $j = 0$ and for every natural number i such that $1 \leqslant i < j$ holds $f(i+1) = \mathcal{F}(f(i), i, \mathfrak{A}(i))$

provided

- for every 0-based finite array a of $\mathcal{X}$ and for every natural number i such that $1 \leqslant i < j$ for every element x of $\mathcal{X}$, $\mathcal{F}(a, i, x)$ is a 0-based finite array of $\mathcal{X}$ and
- $\mathfrak{B}$ is a 0-based finite array of $\mathcal{X}$ and
- for every natural number i such that $i < j$ holds $\mathfrak{A}(i) \in \mathcal{X}$.

Now we state the propositions:

(71) Let us consider a non empty non void boolean signature Σ with arrays of type 1 with connectives from 11 and integers at 1, sets J, L, and a sort symbol K of Σ. Suppose (the connectives of Σ)(11) is of type $\langle J, L \rangle \to K$. Then

(i) $J =$ the array sort of Σ, and

(ii) for every integer sort symbol I of Σ, the array sort of $\Sigma \neq I$.

(72) Let us consider a 1-1-connectives 11-array correct boolean correct non empty non void boolean signature Σ with integers with connectives from 4 and the sort at 1 and arrays of type 1 with connectives from 11 and integers at 1, an integer sort symbol I of Σ, a boolean correct non-empty algebra $\mathfrak{A}$ over Σ with integers with connectives from 4 and the sort at 1 and arrays of type 1 with connectives from 11 and integers at 1, and elements a, b of $\mathfrak{A}$ from I. If $a = 0$, then init.array$(a, b) = \emptyset$.

(73) Let us consider an 11-array correct boolean correct non empty non void boolean signature Σ with arrays of type 1 with connectives from 11 and integers at 1 and an integer sort symbol I of Σ. Then

(i) the array sort of $\Sigma \neq I$, and

(ii) (the connectives of Σ)(11) is of type $\langle$the array sort of $\Sigma, I\rangle \to I$, and

(iii) (the connectives of Σ)$(11+1)$ is of type $\langle$the array sort of $\Sigma, I, I\rangle \to$ the array sort of Σ, and

(iv) (the connectives of Σ)$(11+2)$ is of type $\langle$the array sort of $\Sigma\rangle \to I$, and

(v) (the connectives of Σ)$(11+3)$ is of type $\langle I, I\rangle \to$ the array sort of Σ.

(74) Let us consider a 1-1-connectives 11-array correct boolean correct non empty non void boolean signature Σ with arrays of type 1 with connectives from 11 and integers at 1 and integers with connectives from 4 and the sort at 1, an integer sort symbol I of Σ, and a boolean correct non-empty algebra $\mathfrak{A}$ over Σ with arrays of type 1 with connectives from 11 and integers at 1 and integers with connectives from 4 and the sort at 1. Then

(i) (the sorts of $\mathfrak{A}$)(the array sort of Σ) $= \mathbb{Z}^{\omega}$, and

(ii) for every elements i, j of $\mathfrak{A}$ from I such that i is a non negative integer holds init.array$(i, j) = i \longmapsto j$, and

(iii) for every element a of (the sorts of $\mathfrak{A}$)(the array sort of Σ), $\mathrm{length}_I\, a = \overline{\overline{a}}$ and for every element i of $\mathfrak{A}$ from I and for every function f such that $f = a$ and $i \in \mathrm{dom}\, f$ holds $a(i) = f(i)$ and for every element x of $\mathfrak{A}$ from I, $a_{i \leftarrow x} = f +\cdot (i, x)$.

The theorem is a consequence of (71).

Let a be a 0-based finite array. Observe that length a is finite.

Let Σ be a 1-1-connectives 11-array correct boolean correct non empty non void boolean signature with integers with connectives from 4 and the sort at 1 and arrays of type 1 with connectives from 11 and integers at 1 and $\mathfrak{A}$ be a boolean correct non-empty algebra over Σ with arrays of type 1 with connectives from 11 and integers at 1 and integers with connectives from 4 and the sort at 1. Observe that every non-empty subalgebra of $\mathfrak{A}$ has arrays of type 1 with connectives from 11 and integers at 1.

Let $\mathfrak{A}$ be a non-empty algebra over Σ. We say that $\mathfrak{A}$ is integer array if and only if

(Def. 14) There exists an image $\mathfrak{C}$ of $\mathfrak{A}$ such that $\mathfrak{C}$ is a boolean correct algebra over Σ with integers with connectives from 4 and the sort at 1 and arrays of type 1 with connectives from 11 and integers at 1.

Let X be a non-empty many sorted set indexed by the carrier of Σ. One can verify that $\mathfrak{F}_\Sigma(X)$ is integer array as a non-empty algebra over Σ.

Note that every non-empty algebra over Σ which is integer array is also integer.

One can check that there exists an including Σ-terms over X non-empty strict free variable algebra over Σ which is vf-free and integer array.

One can check that there exists a non-empty algebra over Σ which is integer array.

Let $\mathfrak{A}$ be an integer array non-empty algebra over Σ. Observe that there exists a boolean correct image of $\mathfrak{A}$ which has integers with connectives from 4 and the sort at 1 and arrays of type 1 with connectives from 11 and integers at 1.

In this paper Σ denotes a 1-1-connectives 11-array correct boolean correct non empty non void boolean signature with integers with connectives from 4 and the sort at 1 and arrays of type 1 with connectives from 11 and integers at 1, X denotes a non-empty many sorted set indexed by the carrier of Σ, $\mathfrak{T}$ denotes a vf-free including Σ-terms over X integer array non-empty free variable algebra over Σ, $\mathfrak{C}$ denotes a boolean correct non-empty image of $\mathfrak{T}$ with arrays of type 1 with connectives from 11 and integers at 1 and integers with connectives from 4 and the sort at 1, G denotes a basic generator system over Σ, X, and $\mathfrak{T}$, $\mathfrak{A}$ denotes a if-while algebra over the generators of G, I denotes an integer sort symbol of Σ, x, y, m, i denote pure elements of (the generators of G)(I), M, N

denote pure elements of (the generators of G)(the array sort of Σ), b denotes a pure element of (the generators of G)((the boolean sort of Σ)), and s, s_1 denote elements of $\mathfrak{C}$-States(the generators of G).

Let us consider Σ. Let $\mathfrak{A}$ be a boolean correct non-empty algebra over Σ with arrays of type 1 with connectives from 11 and integers at 1. Observe that every element of (the sorts of $\mathfrak{A}$)(the array sort of Σ) is relation-like and function-like.

Note that every element of (the sorts of $\mathfrak{A}$)(the array sort of Σ) is finite and transfinite sequence-like.

Let us consider an operation symbol o of Σ. Now we state the propositions:

(75) Suppose $o =$ (the connectives of Σ)(11)($\in$ (the carrier' of Σ)). Then
   (i) Arity(o) = $\langle$the array sort of $\Sigma, I\rangle$, and
   (ii) the result sort of $o = I$.

(76) Suppose $o =$ (the connectives of Σ)(12)($\in$ (the carrier' of Σ)). Then
   (i) Arity(o) = $\langle$the array sort of $\Sigma, I, I\rangle$, and
   (ii) the result sort of $o =$ the array sort of Σ.

(77) Suppose $o =$ (the connectives of Σ)(13)($\in$ (the carrier' of Σ)). Then
   (i) Arity(o) = $\langle$the array sort of $\Sigma\rangle$, and
   (ii) the result sort of $o = I$.

(78) Suppose $o =$ (the connectives of Σ)(14)($\in$ (the carrier' of Σ)). Then
   (i) Arity(o) = $\langle I, I\rangle$, and
   (ii) the result sort of $o =$ the array sort of Σ.

(79) Let us consider an element τ of $\mathfrak{T}$ from the array sort of Σ and an element τ_1 of $\mathfrak{T}$ from I.
Then $\tau(\tau_1)$ value at($\mathfrak{C}, s$) = (τ value at($\mathfrak{C}, s$))(τ_1 value at($\mathfrak{C}, s$)). The theorem is a consequence of (29), (75), (23), and (26).

(80) Let us consider an element τ of $\mathfrak{T}$ from the array sort of Σ and elements τ_1, τ_2 of $\mathfrak{T}$ from I. Then $\tau_{\tau_1 \leftarrow \tau_2}$ value at($\mathfrak{C}, s$) = (τ value at($\mathfrak{C}, s$))$_{\tau_1 \text{ value at}(\mathfrak{C},s) \leftarrow \tau_2 \text{ value at}(\mathfrak{C},s)}$. The theorem is a consequence of (29), (76), (24), and (27).

(81) Let us consider an element τ of $\mathfrak{T}$ from the array sort of Σ. Then length$_I$ τ value at($\mathfrak{C}, s$) = length$_I$(τ value at($\mathfrak{C}, s$)). The theorem is a consequence of (29), (77), (22), and (25).

(82) Let us consider elements τ_1, τ_2 of $\mathfrak{T}$ from I. Then init.array(τ_1, τ_2) value at($\mathfrak{C}, s$) = init.array(τ_1 value at($\mathfrak{C}, s$), τ_2 value at($\mathfrak{C}, s$)). The theorem is a consequence of (29), (78), (23), and (26).

In the sequel u denotes a many sorted function from FreeGenerator($\mathfrak{T}$) into the sorts of $\mathfrak{C}$.

Now we state the propositions:

(83) Let us consider an element τ of $\mathfrak{T}$ from the array sort of Σ and an element τ_1 of $\mathfrak{T}$ from I.
Then $\tau(\tau_1)$ value at$(\mathfrak{C}, u) = (\tau$ value at$(\mathfrak{C}, u))(\tau_1$ value at$(\mathfrak{C}, u))$. The theorem is a consequence of (28), (75), (23), and (26).

(84) Let us consider an element τ of $\mathfrak{T}$ from the array sort of Σ and elements τ_1, τ_2 of $\mathfrak{T}$ from I.
Then $\tau_{\tau_1 \leftarrow \tau_2}$ value at$(\mathfrak{C}, u) = (\tau \text{ value at}(\mathfrak{C}, u))_{\tau_1 \text{ value at}(\mathfrak{C},u) \leftarrow \tau_2 \text{ value at}(\mathfrak{C},u)}$.
The theorem is a consequence of (28), (76), (24), and (27).

(85) Let us consider an element τ of $\mathfrak{T}$ from the array sort of Σ. Then $\text{length}_I\, \tau$ value at$(\mathfrak{C}, u) = \text{length}_I(\tau$ value at$(\mathfrak{C}, u))$. The theorem is a consequence of (28), (77), (22), and (25).

(86) Let us consider elements τ_1, τ_2 of $\mathfrak{T}$ from I. Then init.array(τ_1, τ_2) value at$(\mathfrak{C}, u) =$ init.array$(\tau_1$ value at$(\mathfrak{C}, u), \tau_2$ value at$(\mathfrak{C}, u))$. The theorem is a consequence of (28), (78), (23), and (26).

Let us consider Σ, X, $\mathfrak{T}$, and I. Let i be an integer. The functor $i^I_{\mathfrak{T}}$ yielding an element of $\mathfrak{T}$ from I is defined by

(Def. 15) There exists a function f from $\mathbb{Z}$ into (the sorts of $\mathfrak{T})(I)$ such that

(i) $it = f(i)$, and

(ii) $f(0) = 0^I_{\mathfrak{T}}$, and

(iii) for every natural number j and for every element τ of $\mathfrak{T}$ from I such that $f(j) = \tau$ holds $f(j+1) = \tau + 1^I_{\mathfrak{T}}$ and $f(-(j+1)) = -(\tau + 1^I_{\mathfrak{T}})$.

Now we state the propositions:

(87) $0^I_{\mathfrak{T}} = 0^I_{\mathfrak{T}}$.

(88) Let us consider a natural number n. Then

(i) $(n+1)^I_{\mathfrak{T}} = n^I_{\mathfrak{T}} + 1^I_{\mathfrak{T}}$, and

(ii) $-(n+1)^I_{\mathfrak{T}} = -(n+1)^I_{\mathfrak{T}}$.

(89) $1^I_{\mathfrak{T}} = 0^I_{\mathfrak{T}} + 1^I_{\mathfrak{T}}$. The theorem is a consequence of (88) and (87).

(90) Let us consider an integer i. Then $i^I_{\mathfrak{T}}$ value at$(\mathfrak{C}, s) = i$. The theorem is a consequence of (87), (36), (37), (88), (39), and (38).

Let us consider Σ, X, $\mathfrak{T}$, G, I, and M. Let i be an integer. The functor $M(i, I)$ yielding an element of $\mathfrak{T}$ from I is defined by the term

(Def. 16) $({}^{@}M)(i^I_{\mathfrak{T}})$.

Let us consider $\mathfrak{C}$ and s. Note that s(the array sort of $\Sigma)(M)$ is function-like and relation-like.

Note that s(the array sort of $\Sigma)(M)$ is finite transfinite sequence-like and $\mathbb{Z}$-valued.

Observe that rng$(s$(the array sort of $\Sigma)(M))$ is finite and integer-membered.

Let us consider an integer j. Now we state the propositions:

(91) Suppose $j \in \operatorname{dom}(s(\text{the array sort of } \Sigma)(M))$ and $M(j, I) \in (\text{the generators of } G)(I)$. Then $s(\text{the array sort of } \Sigma)(M)(j) = s(I)(M(j, I))$.

(92) Suppose $j \in \operatorname{dom}(s(\text{the array sort of } \Sigma)(M))$ and $({}^{@}M)({}^{@}i) \in (\text{the generators of } G)(I)$ and $j = {}^{@}i$ value at$(\mathfrak{C}, s)$. Then $(s(\text{the array sort of } \Sigma)(M))({}^{@}i \text{ value at}(\mathfrak{C}, s)) = s(I)((({}^{@}M)({}^{@}i)))$.

Let X be a non empty set. One can verify that X^{ω} is infinite.

Now we state the propositions:

(93) Now let Γ denotes the program

```
m:=𝔄 0^I_𝔗;
for i:=𝔄 1^I_𝔗 until b gt(length_I @M, @i, 𝔄) step i:=𝔄 @i + 1^I_𝔗
do
  if b gt((@M)(@i), (@M)(@m), 𝔄) then
    m:=𝔄 @i
  fi
done
```

Let us consider an execution function f of $\mathfrak{A}$ over $\mathfrak{C}$-States(the generators of G) and $\text{States}_{b \not\to \text{false}_{\mathfrak{C}}}$(the generators of G). Suppose

(i) $f \in \mathfrak{C}\text{-Execution}_{b \not\to \text{false}_{\mathfrak{C}}}(\mathfrak{A})$, and

(ii) G is $\mathfrak{C}$-supported, and

(iii) $i \neq m$, and

(iv) $s(\text{the array sort of } \Sigma)(M) \neq \emptyset$.

Let us consider a natural number n. Suppose $f(s, \Gamma)(I)(m) = n$. Let us consider a non empty finite integer-membered set X. Suppose $X = \operatorname{rng}(s(\text{the array sort of } \Sigma)(M))$. Then $M(n, I)$ value at$(\mathfrak{C}, s) = \max X$. The theorem is a consequence of (65), (36), (37), (74), (71), (66), (81), (61), (39), (79), and (90). Proof: Set $ST = \mathfrak{C}$-States(the generators of G). Define $\mathcal{R}[\text{element of } ST] \equiv s(\text{the array sort of } \Sigma)(M) = \$_1(\text{the array sort of } \Sigma)(M)$. Reconsider $sm = s$ as a many sorted function from the generators of G into the sorts of $\mathfrak{C}$. Reconsider $z = sm$(the array sort of $\Sigma)(M)$ as a 0-based finite array of $\mathbb{Z}$. Define $\mathcal{P}[\text{element of } ST] \equiv \mathcal{R}[\$_1]$ and $\$_1(I)(i), \$_1(I)(m) \in \mathbb{N}$ and $\$_1(I)(i) \leqslant \operatorname{len} z$ and $\$_1(I)(m) < \$_1(I)(i)$ and $\$_1(I)(m) < \operatorname{len} z$ and for every integer mx such that $mx = \$_1(I)(m)$ for every natural number j such that $j < \$_1(I)(i)$ holds $z(j) \leqslant z(mx)$. Define $\mathcal{Q}[\text{element of } ST] \equiv \mathcal{R}[\$_1]$ and $\$_1(I)(i) < \text{length}_I {}^{@}M$ value at$(\mathfrak{C}, s)$. Set $s0 = s$. Set $s_1 = f(s, m{:=}_{\mathfrak{A}}(0^I_{\mathfrak{T}}))$. Set $s_2 = f(s_1, i{:=}_{\mathfrak{A}}(1^I_{\mathfrak{T}}))$. Consider $J1$, $K1$, $L1$ being elements of Σ such that $L1 = 1$ and $K1 = 1$ and $J1 \neq L1$ and $J1 \neq K1$ and (the connectives of Σ)(11) is of type $\langle J1, K1\rangle \to L1$ and (the connectives of Σ)(11 + 1) is of type $\langle J1, K1, L1\rangle \to J1$ and (the connectives of Σ)(11 + 2) is of type $\langle J1\rangle \to K1$ and (the connectives of Σ)(11 + 3) is of type $\langle K1, L1\rangle \to J1$. $\mathcal{P}[s_2]$. Define $\mathcal{F}$(element of ST) = (len($s0$(the array sort of Σ)(M)) $-$ $\$_1(I)(i)$)($\in \mathbb{N}$). $f(s_2, W) \in TV$ iff $\mathcal{Q}[f(s_2, W)]$. Now let Γ denotes the program

```
J;
K;
W
```

For every element s of ST such that $\mathcal{Q}[s]$ holds $\mathcal{Q}[f(s,\Gamma)]$ iff $f(s,\Gamma) \in TV$ and $\mathcal{F}(f(s,\Gamma)) < \mathcal{F}(s)$. For every element s of ST such that $\mathcal{P}[s]$ and $s \in TV$ and $\mathcal{Q}[s]$ holds $\mathcal{P}[f(s, J; K)]$. For every element s of ST such that $\mathcal{P}[s]$ holds $\mathcal{P}[f(s, W)]$ and $f(s, W) \in TV$ iff $\mathcal{Q}[f(s, W)]$. $M(n, I)$ value at($\mathfrak{C}, s$) is a upper bound of X. For every upper bound x of X, $M(n, I)$ value at($\mathfrak{C}, s$) $\leqslant x$. $\square$

(94) Now let Γ denotes the program

```
J;
i :=𝔄 @i + 1^I_𝔗
```

Now let Δ denotes the program

```
for i :=𝔄 τ0 until b gt(τ1, @i, 𝔄) step i :=𝔄 @i + 1^I_𝔗 do
  J
done
```

Let us consider an elementary if-while algebra $\mathfrak{A}$ over the generators of G and an execution function f of $\mathfrak{A}$ over $\mathfrak{C}$-States(the generators of G) and $\text{States}_{b \not\to \text{false}_{\mathfrak{C}}}$(the generators of G). Suppose

(i) $f \in \mathfrak{C}\text{-Execution}_{b \not\to \text{false}_{\mathfrak{C}}}(\mathfrak{A})$, and

(ii) G is $\mathfrak{C}$-supported.

Let us consider elements τ_0, τ_1 of $\mathfrak{T}$ from I, an algorithm J of $\mathfrak{A}$, and a set P. Suppose

(iii) P is invariant w.r.t. $i{:=}_{\mathfrak{A}}\tau_0$ and f, invariant w.r.t. $b\,\text{gt}(\tau_1, {}^{@}i, \mathfrak{A})$ and f, invariant w.r.t. $i{:=}_{\mathfrak{A}}({}^{@}i + 1^{I}_{\mathfrak{T}})$ and f, and invariant w.r.t. J and f, and

(iv) J is terminating w.r.t. f and P, and

(v) for every s, $f(s, J)(I)(i) = s(I)(i)$ and $f(s, b\,\text{gt}(\tau_1, {}^{@}i, \mathfrak{A}))(I)(i) = s(I)(i)$ and τ_1 value at($\mathfrak{C}, f(s, b\,\text{gt}(\tau_1, {}^{@}i, \mathfrak{A}))$) = τ_1 value at($\mathfrak{C}, s$) and τ_1 value at($\mathfrak{C}, f(s,\Gamma)$) = τ_1 value at($\mathfrak{C}, s$).

Then Δ is terminating w.r.t. f and P. The theorem is a consequence of (61), (66), (65), (39), and (37). PROOF: Set $W = b\,\text{gt}(\tau_1, {}^{@}i, \mathfrak{A})$. Set $L = i{:=}_{\mathfrak{A}}({}^{@}i + 1^{I}_{\mathfrak{T}})$. Set $K = i{:=}_{\mathfrak{A}}\tau_0$. Set $ST = \mathfrak{C}$-States(the generators of G). Set $TV = \text{States}_{b \not\to \text{false}_{\mathfrak{C}}}$(the generators of G). Now let Γ denotes the program

```
J;
L;
W
```

For every s such that $f(s, W) \in P$ holds iteration of f started in Γ terminates w.r.t. $f(s, W)$. $\square$

(95) Now let Γ denotes the program

```
m:=𝔄 0^I_𝔗;
for i:=𝔄 1^I_𝔗 until b gt(length_I @M, @i, 𝔄) step i:=𝔄 @i + 1^I_𝔗
do
   if b gt((@M)(@i), (@M)(@m), 𝔄) then
      m:=𝔄 @i
   fi
done
```

Let us consider an elementary if-while algebra $\mathfrak{A}$ over the generators of G and an execution function f of $\mathfrak{A}$ over $\mathfrak{C}$-States(the generators of G) and $\mathrm{States}_{b \not\to \mathrm{false}_{\mathfrak{C}}}$(the generators of G). Suppose

(i) $f \in \mathfrak{C}\text{-}\mathrm{Execution}_{b \not\to \mathrm{false}_{\mathfrak{C}}}(\mathfrak{A})$, and

(ii) G is $\mathfrak{C}$-supported, and

(iii) $i \neq m$.

Then Γ is terminating w.r.t. f and $\{s : s(\text{the array sort of } \Sigma)(M) \neq \emptyset\}$. The theorem is a consequence of (74), (73), (65), (61), (81), and (94). PROOF: Set $J = m{:=}_{\mathfrak{A}}(0^I_{\mathfrak{T}})$. Set $K = i{:=}_{\mathfrak{A}}(1^I_{\mathfrak{T}})$. Set $W = b\,\mathrm{gt}(\mathrm{length}_I\, {}^{@}M, {}^{@}i, \mathfrak{A})$. Set $L = i{:=}_{\mathfrak{A}}({}^{@}i + 1^I_{\mathfrak{T}})$. Set $N = b\,\mathrm{gt}(({}^{@}M)({}^{@}i), ({}^{@}M)({}^{@}m), \mathfrak{A})$. Set $O = m{:=}_{\mathfrak{A}}({}^{@}i)$. Set $a =$ the array sort of Σ. Set $P = \{s : s(a)(M) \neq \emptyset\}$. P is invariant w.r.t. J and f. P is invariant w.r.t. K and f. P is invariant w.r.t. W and f. P is invariant w.r.t. L and f. P is invariant w.r.t. N and f. P is invariant w.r.t. O and f. Set $ST = \mathfrak{C}$-States(the generators of G). Set $TV = \mathrm{States}_{b \not\to \mathrm{false}_{\mathfrak{C}}}$(the generators of G). P is invariant w.r.t. if N then O and f. Now let Γ denotes the program

```
if N then
   O
fi;
L
```

For every s, $f(s, \text{if } N \text{ then } O)(I)(i) = s(I)(i)$ and $f(s, W)(I)(i) = s(I)(i)$ and $\mathrm{length}_I\, {}^{@}M$ value at$(\mathfrak{C}, f(s, W)) = \mathrm{length}_I\, {}^{@}M$ value at$(\mathfrak{C}, s)$ and $\mathrm{length}_I\, {}^{@}M$ value at$(\mathfrak{C}, f(s, \Gamma)) = \mathrm{length}_I\, {}^{@}M$ value at$(\mathfrak{C}, s)$. $\square$

3. Sorting by Exchanging

In this paper i_1, i_2 denote pure elements of (the generators of G)(I).

Let us consider Σ, X, $\mathfrak{T}$, and G. We say that G is integer array if and only if

(Def. 17) (i) $\{(^{@}M)(\tau)$ where τ is an element of $\mathfrak{T}$ from I : not contradiction$\} \subseteq$ (the generators of G)(I), and

(ii) for every M and for every element τ of $\mathfrak{T}$ from I and for every element g of G from I such that $g = (^{@}M)(\tau)$ there exists x such that $x \notin$ (vf τ)(I) and supp-var $g = x$ and (supp-term g)(the array sort of Σ)(M) $= (^{@}M)_{\tau \leftarrow {}^{@}x}$ and for every sort symbol s of Σ and for every y such that $y \in$ (vf g)(s) and if $s =$ the array sort of Σ, then $y \neq M$ holds (supp-term g)(s)(y) $= y$.

Now we state the proposition:

(96) If G is integer array, then for every element τ of $\mathfrak{T}$ from I, $(^{@}M)(\tau) \in$ (the generators of G)(I).

The functor $\langle \mathbb{Z}, \leqslant \rangle$ yielding a strict real non empty poset is defined by the term

(Def. 18) RealPoset $\mathbb{Z}$.

Let us consider Σ, X, $\mathfrak{T}$, and G. Let $\mathfrak{A}$ be an elementary if-while algebra over the generators of G, a be a sort symbol of Σ, and τ_1, τ_2 be elements of $\mathfrak{T}$ from a. Assume $\tau_1 \in$ (the generators of G)(a). The functor $\tau_1 :=_{\mathfrak{A}} \tau_2$ yielding an absolutely-terminating algorithm of $\mathfrak{A}$ is defined by the term

(Def. 19) (The assignments of $\mathfrak{A}$)($\langle \tau_1, \tau_2 \rangle$).

Now we state the proposition:

(97) Let us consider a countable non-empty many sorted set X indexed by the carrier of Σ, a vf-free including Σ-terms over X integer array non-empty free variable algebra $\mathfrak{T}$ over Σ, a basic generator system G over Σ, X, and $\mathfrak{T}$, a pure element M of (the generators of G)(the array sort of Σ), and pure elements i, x of (the generators of G)(I). Then $(^{@}M)(^{@}i) \neq x$. The theorem is a consequence of (73), (79), (61), and (74).

Let Σ be a non empty non void many sorted signature and $\mathfrak{A}$ be a disjoint valued algebra over Σ. Note that the sorts of $\mathfrak{A}$ is disjoint valued.

Let us consider Σ and X. Let $\mathfrak{T}$ be an including Σ-terms over X algebra over Σ. We say that $\mathfrak{T}$ is array degenerated if and only if

(Def. 20) There exists I and there exists an element M of (FreeGenerator($\mathfrak{T}$))(the array sort of Σ) and there exists an element τ of $\mathfrak{T}$ from I such that $(^{@}M)(\tau) \neq$ Sym((the connectives of Σ)(11)($\in$ (the carrier' of Σ)), X)-tree($\langle M, \tau \rangle$).

Observe that $\mathfrak{F}_\Sigma(X)$ is non array degenerated.

Observe that there exists an including Σ-terms over X algebra over Σ which is non array degenerated.

Now we state the propositions:

(98) Suppose $\mathfrak{T}$ is non array degenerated. Then $\mathrm{vf}((^{@}M)(^{@}i)) = I$-singleton $i \cup$ (the array sort of Σ)-singleton M. The theorem is a consequence of (73). Proof: Set $\tau = (^{@}M)(^{@}i)$. Reconsider $N = M$ as an element of (FreeGenerator($\mathfrak{T}$))(the array sort of Σ). Consider m being a set such that $m \in X$(the array sort of Σ) and $M =$ the root tree of $\langle m$, the array sort of $\Sigma\rangle$. Consider j being a set such that $j \in X(I)$ and $i =$ the root tree of $\langle j, I\rangle$. $\{M\} = (\mathrm{vf}\,\tau)$(the array sort of Σ). $\{i\} = (\mathrm{vf}\,\tau)(I)$. For every sort symbol s of Σ such that $s \neq$ the array sort of Σ and $s \neq I$ holds $\emptyset = (\mathrm{vf}\,\tau)(s)$. $\square$

(99) Let us consider an elementary if-while algebra $\mathfrak{A}$ over the generators of G and an execution function f of $\mathfrak{A}$ over $\mathfrak{C}$-States(the generators of G) and $\mathrm{States}_{b \not\to \mathrm{false}_{\mathfrak{C}}}$(the generators of G). Suppose

(i) G is integer array and $\mathfrak{C}$-supported, and

(ii) $f \in \mathfrak{C}\text{-Execution}_{b \not\to \mathrm{false}_{\mathfrak{C}}}(\mathfrak{A})$, and

(iii) X is countable, and

(iv) $\mathfrak{T}$ is non array degenerated.

Let us consider an element τ of $\mathfrak{T}$ from I. Then $f(s, (^{@}M)(^{@}i) :=_{\mathfrak{A}} \tau) = f(s, M :=_{\mathfrak{A}} ((^{@}M)_{^{@}i \leftarrow \tau}))$. The theorem is a consequence of (96), (98), (97), (4), (3), (62), (73), (61), (84), (65), and (80). Proof: Reconsider $H =$ FreeGenerator($\mathfrak{T}$) as a many sorted subset of the generators of G. Set $v = \tau$ value at$(\mathfrak{C}, s)$. Reconsider $p = (^{@}M)(^{@}i)$ as an element of G from I. Reconsider $g = s$ as a many sorted function from the generators of G into the sorts of $\mathfrak{C}$. Reconsider $g1 = f(s, (^{@}M)(^{@}i) :=_{\mathfrak{A}} \tau)$, $g2 = f(s, M :=_{\mathfrak{A}} ((^{@}M)_{^{@}i \leftarrow \tau}))$ as a many sorted function from the generators of G into the sorts of $\mathfrak{C}$. Reconsider $Mi = (^{@}M)(^{@}i)$ as an element of (the generators of G)(I). Reconsider $m = M$ as an element of G from the array sort of Σ. Consider x such that $x \notin (\mathrm{vf}\,^{@}i)(I)$ and supp-var $p = x$ and (supp-term p)(the array sort of Σ)$(M) = (^{@}M)_{^{@}i \leftarrow ^{@}x}$ and for every sort symbol s of Σ and for every y such that $y \in (\mathrm{vf}\,p)(s)$ and if $s =$ the array sort of Σ, then $y \neq M$ holds (supp-term p)$(s)(y) = y$. $g1 = g2$. $\square$

Let us consider Σ, X, $\mathfrak{T}$, G, $\mathfrak{C}$, s, and b. Let us observe that s((the boolean sort of Σ))(b) is boolean.

Now we state the proposition:

(100) Now let Γ denotes the program

```
while J do
   y:=𝔄(@M)(@i1);
   (@M)(@i1):=𝔄(@M)(@i2);
   (@M)(@i2):=𝔄@y
done
```

Let us consider an elementary if-while algebra $\mathfrak{A}$ over the generators of G and an execution function f of $\mathfrak{A}$ over $\mathfrak{C}$-States(the generators of G) and $\text{States}_{b \not\to \text{false}_{\mathfrak{C}}}$(the generators of G). Suppose

(i) G is integer array and $\mathfrak{C}$-supported, and

(ii) $f \in \mathfrak{C}\text{-Execution}_{b \not\to \text{false}_{\mathfrak{C}}}(\mathfrak{A})$, and

(iii) $\mathfrak{T}$ is non array degenerated, and

(iv) X is countable.

Let us consider an algorithm J of $\mathfrak{A}$. Suppose

(v) $f(s, J)$(the array sort of Σ)$(M) = s$(the array sort of Σ)(M), and

(vi) for every array D of $\langle \mathbb{Z}, \leqslant \rangle$ such that $D = s$(the array sort of Σ)(M) holds if $D \neq \emptyset$, then $f(s, J)(I)(i_1)$, $f(s, J)(I)(i_2) \in \operatorname{dom} D$ and if inversions $D \neq \emptyset$, then $\langle f(s, J)(I)(i_1), f(s, J)(I)(i_2)\rangle \in$ inversions D and $f(s, J)$((the boolean sort of Σ))$(b) = \mathit{true}$ iff inversions $D \neq \emptyset$.

Let us consider a 0-based finite array D of $\langle \mathbb{Z}, \leqslant \rangle$. Suppose

(vii) $D = s$(the array sort of Σ)(M), and

(viii) $y \neq i_1$, and

(ix) $y \neq i_2$.

Then

(x) $f(s, \Gamma)$(the array sort of Σ)(M) is an ascending permutation of D, and

(xi) if J is absolutely-terminating, then Γ is terminating w.r.t. f and $\{s_1 : s_1$(the array sort of Σ)$(M) \neq \emptyset\}$.

The theorem is a consequence of (73), (10), (61), (65), (99), (80), (74), and (79). PROOF: Define $\mathcal{F}$(natural number, element of $\mathfrak{C}$-States(the generators of G)) $= f(\$_2, ((J; y{:=}_{\mathfrak{A}}((^{@}M)(^{@}i_1))); (^{@}M)(^{@}i_1){:=}_{\mathfrak{A}}((^{@}M)(^{@}i_2)));$ $(^{@}M)(^{@}i_2){:=}_{\mathfrak{A}}(^{@}y))$. Set $ST = \mathfrak{C}$-States(the generators of G). Consider g being a function from $\mathbb{N}$ into ST such that $g(0) = s$ and for every natural number i, $g(i+1) = \mathcal{F}(i, (g(i)$ **qua** element of $ST))$. Define $\mathcal{G}$(element) $= g(\$_1(\in \mathbb{N}))$(the array sort of Σ)(M). Consider h being a function from $\mathbb{N}$ into $\mathbb{Z}^{\omega}$ such that for every element i such that $i \in \mathbb{N}$ holds $h(i) = \mathcal{G}(i)$. For every ordinal number a such that $a \in \operatorname{dom} g$ holds $h(a)$ is an array of $\langle \mathbb{Z}, \leqslant \rangle$. Set $TV = \text{States}_{b \not\to \text{false}_{\mathfrak{C}}}$(the generators of G). Consider s_1 such that $s = s_1$ and s_1(the array sort of Σ)$(M) \neq \emptyset$. Reconsider

$D = s$(the array sort of Σ)(M) as a 0-based finite non empty array of $\langle \mathbb{Z}, \leqslant \rangle$. Consider g being a function from $\mathbb{N}$ into ST such that $g(0) = s$ and for every natural number i, $g(i+1) = \mathcal{F}(i, (g(i)$ **qua** element of $ST))$. Define $\mathcal{G}$(element) $= g(\$_1(\in \mathbb{N}))$(the array sort of Σ)(M). Consider h being a function from $\mathbb{N}$ into $\mathbb{Z}^\omega$ such that for every element i such that $i \in \mathbb{N}$ holds $h(i) = \mathcal{G}(i)$. For every ordinal number a such that $a \in \operatorname{dom} g$ holds $h(a)$ is an array of $\langle \mathbb{Z}, \leqslant \rangle$. Define $\mathfrak{T}$[natural number] $\equiv h(\$_1) \neq \emptyset$. For every natural number i such that $\mathfrak{T}[i]$ holds $\mathfrak{T}[i+1]$. For every natural number a and for every array R of $\langle \mathbb{Z}, \leqslant \rangle$ such that $R = h(a)$ for every s such that $g(a) = s$ there exist sets x, y such that $x = f(s, J)(I)(i_1)$ and $y = f(s, J)(I)(i_2)$ and x, $y \in \operatorname{dom} R$ and $h(a+1) = \operatorname{Swap}(R, x, y)$. Define $\mathcal{Q}$[natural number] $\equiv h(\$_1)$ is a permutation of D. Define $\mathcal{P}$[natural number] $\equiv g(\$_1)$(the array sort of Σ)(M) is an ascending permutation of D. There exists a natural number i such that $\mathcal{P}[i]$. Consider $\mathfrak{B}$ being a natural number such that $\mathcal{P}[\mathfrak{B}]$ and for every natural number i such that $\mathcal{P}[i]$ holds $\mathfrak{B} \leqslant i$. Reconsider $c = h \upharpoonright \operatorname{succ} \mathfrak{B}$ as an array of $\mathbb{Z}^\omega$. Set $TV = \text{States}_{b \not\to \text{false}_{\mathfrak{C}}}$(the generators of G). Define $\mathcal{H}$(natural number) $= f(g(\$_1 - 1), J)$. Consider r being a finite sequence such that $\operatorname{len} r = \mathfrak{B} + 1$ and for every natural number i such that $i \in \operatorname{dom} r$ holds $r(i) = \mathcal{H}(i)$. $\operatorname{rng} r \subseteq ST$. Reconsider $R = g(\mathfrak{B})$(the array sort of Σ)(M) as an ascending permutation of D. Now let Γ denotes the program

$y :=_{\mathfrak{A}} ({}^{@}M)({}^{@}i_1);$
$({}^{@}M)({}^{@}i_1) :=_{\mathfrak{A}} ({}^{@}M)({}^{@}i_2);$
$({}^{@}M)({}^{@}i_2) :=_{\mathfrak{A}} {}^{@}y;$
J

For every natural number i such that $1 \leqslant i < \operatorname{len} r$ holds $r(i) \in TV$ and $r(i+1) = f(r(i), \Gamma)$. □

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