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Semantics of MML Query - Ordering

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Summary. Semantics of order directives of MML Query is presented. The formalization is done according to [1].

MML identifier: `MMLQUER2`, version: `8.1.01 5.7.1169`

The notation and terminology used in this paper have been introduced in the following articles: [2], [7], [13], [9], [10], [8], [3], [4], [5], [11], [17], [19], [18], [6], [15], [16], [14], and [12].

1. Preliminaries

In this paper X denotes a set, R, R_1, R_2 denote binary relations, x, y, z denote sets, and n, m, k denote natural numbers.

Let us consider a binary relation R on X. Now we state the propositions:

(1) $\operatorname{field} R \subseteq X$.

(2) If $x, y \in R$, then x, $y \in X$.

Now we state the propositions:

(3) Let us consider sets X, Y. Then $(\mathrm{id}_X)^\circ Y = X \cap Y$.

(4) $\langle x, y \rangle \in R\,|^2\, X$ if and only if x, $y \in X$ and $\langle x, y \rangle \in R$.

(5) $\operatorname{dom}(X \upharpoonright R) \subseteq \operatorname{dom} R$.

(6) Let us consider a total reflexive binary relation R on X and a subset S of X. Then $R\,|^2\, S$ is a total reflexive binary relation on S. The theorem is a consequence of (4). Proof: Set $Q = R\,|^2\, S$. $\operatorname{dom} Q = S$. $\square$

(7) Let us consider transfinite sequences f, g. Then $\operatorname{rng}(f ^\frown g) = \operatorname{rng} f \cup \operatorname{rng} g$.

Let us consider R. Let us note that R is transitive if and only if the condition (Def. 1) is satisfied.

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(Def. 1) If $x, y \in R$ and $y, z \in R$, then $x, z \in R$.

One can verify that R is antisymmetric if and only if the condition (Def. 2) is satisfied.

(Def. 2) If $x, y \in R$ and $y, x \in R$, then $x = y$.

Now we state the proposition:

(8) Let us consider a non empty set X, a total connected binary relation R on X, and elements x, y of X. If $x \neq y$, then $x, y \in R$ or $y, x \in R$.

2. Composition of Orders

Let R_1, R_2 be binary relations. The functor R_1, R_2 yielding a binary relation is defined by the term

(Def. 3) $R_1 \cup (R_2 \setminus R_1{}^{\smile})$.

Now we state the propositions:

(9) $x, y \in R_1$, R_2 if and only if $x, y \in R_1$ or $y, x \notin R_1$ and $x, y \in R_2$.

(10) $\mathrm{field}(R_1, R_2) = \mathrm{field}\, R_1 \cup \mathrm{field}\, R_2$. The theorem is a consequence of (9).

(11) R_1, $R_2 \subseteq R_1 \cup R_2$. The theorem is a consequence of (9).

Let X be a set and R_1, R_2 be binary relations on X. Note that the functor R_1, R_2 yields a binary relation on X. Let R_1, R_2 be reflexive binary relations. One can verify that R_1, R_2 is reflexive.

Let R_1, R_2 be antisymmetric binary relations. Note that R_1, R_2 is antisymmetric.

Let X be a set and R be a binary relation on X. We say that R is β-transitive if and only if

(Def. 4) Let us consider elements x, y of X. If $x, y \notin R$, then for every element z of X such that $x, z \in R$ holds $y, z \in R$.

Observe that every binary relation on X which is connected total and transitive is also β-transitive.

Let us observe that there exists an order in X which is connected.

Let R_1 be a β-transitive transitive binary relation on X and R_2 be a transitive binary relation on X. Observe that R_1, R_2 is transitive.

Let R_1 be a binary relation on X and R_2 be a total reflexive binary relation on X. Let us note that R_1, R_2 is total and reflexive as a binary relation on X.

Let R_2 be a total connected reflexive binary relation on X. One can verify that R_1, R_2 is connected.

Now we state the propositions:

(12) $(R, R_1), R_2 = R, (R_1, R_2)$. The theorem is a consequence of (9).

(13) Let us consider a connected reflexive total binary relation R on X and a binary relation R_2 on X. Then $R, R_2 = R$. The theorem is a consequence of (9) and (2).

3. `number of` Ordering

Let X be a set and f be a function from X into $\mathbb{N}$. The functor `number of` f yielding a binary relation on X is defined by

(Def. 5) $x, y \in it$ if and only if x, $y \in X$ and $f(x) < f(y)$.

Let us note that `number of` f is antisymmetric transitive and β-transitive.

Let X be a finite set and O be an operation of X. The functor `value of` O yielding a function from X into $\mathbb{N}$ is defined by

(Def. 6) Let us consider an element x of X. Then $it(x) = \overline{\overline{x(O)}}$.

Now we state the proposition:

(14) Let us consider a finite set X, an operation O of X, and elements x, y of X. Then $x, y \in$ `number of value of` O if and only if $\overline{\overline{x(O)}} < \overline{\overline{y(O)}}$.

Let us consider X. Let O be an operation of X. The functor `first` O yielding a binary relation on X is defined by

(Def. 7) Let us consider elements x, y of X. Then $x, y \in it$ if and only if $x(O) \neq \emptyset$ and $y(O) = \emptyset$.

Let us observe that `first` O is antisymmetric transitive and β-transitive.

4. Ordering by Resources

Let A be a finite sequence and x be an element. The functor $A \leftarrow x$ yielding a set is defined by the term

(Def. 8) $\bigcap(A^{-1}(\{x\}))$.

Let us consider x. Note that $A \leftarrow x$ is natural.

Let us consider a finite sequence A. Now we state the propositions:

(15) If $x \notin \operatorname{rng} A$, then $A \leftarrow x = 0$.

(16) If $x \in \operatorname{rng} A$, then $A \leftarrow x \in \operatorname{dom} A$ and $x = A(A \leftarrow x)$.

(17) If $A \leftarrow x = 0$, then $x \notin \operatorname{rng} A$.

Let us consider X. Let A be a finite sequence and f be a function. The functor `resource`(X, A, f) yielding a binary relation on X is defined by

(Def. 9) $x, y \in it$ if and only if x, $y \in X$ and $A \leftarrow (f(x)) \neq 0$ and $A \leftarrow (f(x)) < A \leftarrow (f(y))$ or $A \leftarrow (f(y)) = 0$.

Let us observe that `resource`(X, A, f) is antisymmetric transitive and β-transitive.

5. Ordering by Number of Iteration

Let us consider X. Let R be a binary relation on X and n be a natural number. One can check that the functor R^n yields a binary relation on X. Now we state the propositions:

(18) If $(R^n)^\circ X = \emptyset$ and $m \geqslant n$, then $(R^m)^\circ X = \emptyset$.

(19) If for every n, $(R^n)^\circ X \neq \emptyset$ and X is finite, then there exists x such that $x \in X$ and for every n, $(R^n)^\circ x \neq \emptyset$. The theorem is a consequence of (18). Proof: Define $\mathcal{P}[\text{element}, \text{element}] \equiv$ there exists n such that $\$_2 = n$ and $(R^n)^\circ \$_1 = \emptyset$. For every element x such that $x \in X$ there exists an element y such that $y \in \mathbb{N}$ and $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = X$ and $\operatorname{rng} f \subseteq \mathbb{N}$ and for every element x such that $x \in X$ holds $\mathcal{P}[x, f(x)]$. Consider n such that $\operatorname{rng} f \subseteq \mathbb{Z}_n$. $\{\{x\}$ where x is an element of $X : x \in X\} \subseteq 2^X$. Reconsider $Y = \{\{x\}$ where x is an element of $X : x \in X\}$ as a family of subsets of X. $X = \bigcup Y$. $\{(R^n)^\circ y$ where y is a subset of $X : y \in Y\} \subseteq \{\emptyset\}$. □

(20) If R is reversely well founded and irreflexive and X is finite and R is finite, then there exists n such that $(R^n)^\circ X = \emptyset$. The theorem is a consequence of (19). Proof: Define $\mathcal{Q}[\text{element}] \equiv$ for every n, $(R^n)^\circ \$_1 \neq \emptyset$. Consider $x0$ being a set such that $x0 \in X$ and $\mathcal{Q}[x0]$. Define $\mathcal{P}[\text{element}, \text{element}, \text{element}] \equiv$ if $\mathcal{Q}[\$_2]$, then $\$_3 \in R^\circ \$_2$ and $\mathcal{Q}[\$_3]$. For every natural number n and for every set x, there exists a set y such that $\mathcal{P}[n, x, y]$. Consider f being a function such that $\operatorname{dom} f = \mathbb{N}$ and $f(0) = x0$ and for every natural number n, $\mathcal{P}[n, f(n), f(n+1)]$. Define $\mathcal{R}[\text{natural number}] \equiv \mathcal{Q}[f(\$_1)]$. $\operatorname{rng} f \subseteq \operatorname{field} R$. Consider z being an element such that $z \in \operatorname{rng} f$ and for every element x such that $x \in \operatorname{rng} f$ and $z \neq x$ holds $\langle z, x \rangle \notin R$. Consider y being an element such that $y \in \mathbb{N}$ and $z = f(y)$. □

Let us consider X. Let O be an operation of X. Assume O is reversely well founded, irreflexive, and finite. The functor `iteration of` O yielding a binary relation on X is defined by

(Def. 10) There exists a function f from X into $\mathbb{N}$ such that

(i) $it =$ `number of` f, and

(ii) for every element x of X such that $x \in X$ there exists n such that $f(x) = n$ and $x(O^n) \neq \emptyset$ or $n = 0$ and $x(O^n) = \emptyset$ and $x(O^{n+1}) = \emptyset$.

Let us note that every binary relation which is empty is also irreflexive and reversely well founded.

Let us consider X. Let us note that there exists an operation of X which is empty.

Let O be a reversely well founded irreflexive finite operation of X. One can check that `iteration of` O is antisymmetric transitive and β-transitive.

6. `value of` ORDERING

Let X be a finite set. Let us observe that every order in X is well founded. Note that every connected order in X is well-ordering.

Let us consider X. Let R be a connected order in X and S be a finite subset of X. The functor $\mathrm{order}(S, R)$ yielding a finite 0-sequence of X is defined by

(Def. 11) (i) $\mathrm{rng}\, it = S$, and

(ii) it is one-to-one, and

(iii) for every natural numbers i, j such that $i, j \in \mathrm{dom}\, it$ holds $i \leqslant j$ iff $\langle it(i), it(j) \rangle \in R$.

Now we state the proposition:

(21) Let us consider finite subsets S_1, S_2 of X and a connected order R in X. Then $\mathrm{order}(S_1 \cup S_2, R) = \mathrm{order}(S_1, R) \frown \mathrm{order}(S_2, R)$ if and only if for every x and y such that $x \in S_1$ and $y \in S_2$ holds $x \neq y$ and $\langle x, y \rangle \in R$. The theorem is a consequence of (7). PROOF: Set $o1 = \mathrm{order}(S_1, R)$. Set $o2 = \mathrm{order}(S_2, R)$. $\mathrm{order}(S_1, R) \frown \mathrm{order}(S_2, R)$ is one-to-one. $\square$

Let X be a finite set, O be an operation of X, and R be a connected order in X. The functor `value of`(O, R) yielding a binary relation on X is defined by

(Def. 12) Let us consider elements x, y of X. Then $\langle x, y \rangle \in it$ if and only if $x(O) \neq \emptyset$ and $y(O) = \emptyset$ or $y(O) \neq \emptyset$ and $\langle (\mathrm{order}(x(O), R))_0, (\mathrm{order}(y(O), R))_0 \rangle \in R$ and $(\mathrm{order}(x(O), R))_0 \neq (\mathrm{order}(y(O), R))_0$.

Let R_1 be a connected order in X. One can check that `value of`(O, R_1) is antisymmetric transitive and β-transitive.

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