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More on Divisibility Criteria for Selected Primes

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Summary. This paper is a continuation of [19], where the divisibility criteria for initial prime numbers based on their representation in the decimal system were formalized. In the current paper we consider all primes up to 101 to demonstrate the method presented in [7].

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The notation and terminology used in this paper have been introduced in the following articles: [21], [25], [18], [1], [14], [12], [8], [9], [23], [17], [22], [2], [16], [19], [3], [4], [5], [6], [10], [15], [13], [26], [27], [24], and [11].

1. Preliminaries on Finite Sequences

In this paper n, k, b denote natural numbers and i denotes an integer.

Let us consider a non empty finite 0-sequence f. Now we state the propositions:

(1) $f \restriction 1 = \langle f(0) \rangle$.

(2) $f = \langle f(0) \rangle \frown f_{\lfloor 1}$.

Now we state the proposition:

(3) Let us consider a finite 0-sequence f. Then $\mathrm{mid}(f, 2, \mathrm{len}\, f) = f_{\lfloor 1}$.

Let us consider finite natural-membered sets X, Y. Now we state the propositions:

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(4) If X misses Y, then $\mathrm{dom}(\mathrm{Sgm}_0\, X \frown \mathrm{Sgm}_0\, Y) = \mathrm{dom}\,\mathrm{Sgm}_0(X \cup Y)$.

(5) $\mathrm{rng}(\mathrm{Sgm}_0\, X \frown \mathrm{Sgm}_0\, Y) = \mathrm{rng}\,\mathrm{Sgm}_0(X \cup Y)$.

Now we state the proposition:

(6) Let us consider a finite 0-sequence F and a set X. Then dom the X-subsequence of $F = \mathrm{dom}\,\mathrm{Sgm}_0(X \cap \mathrm{dom}\, F)$.

One can check that the functor $\mathbb{N}_{\mathrm{even}}$ is defined by the term

(Def. 1) $\{n$, where n is a natural number : n is even$\}$.

Note that the functor $\mathbb{N}_{\mathrm{odd}}$ is defined by the term

(Def. 2) $\{n$, where n is a natural number : n is odd$\}$.

Now we state the propositions:

(7) $\mathbb{N}_{\mathrm{even}}$ misses $\mathbb{N}_{\mathrm{odd}}$. PROOF: $\mathbb{N}_{\mathrm{even}} \cap \mathbb{N}_{\mathrm{odd}} \subseteq \emptyset$. $\square$

(8) $\mathbb{N}_{\mathrm{even}} \cup \mathbb{N}_{\mathrm{odd}} = \mathbb{N}$.

Let F be a transfinite sequence and P be a permutation of dom F. One can verify that $F \cdot P$ is transfinite sequence-like.

Now we state the propositions:

(9) Let us consider a finite 0-sequence F and sets X, Y. Suppose X misses Y. Then there exists a permutation P of dom the $X \cup Y$-subsequence of F such that (the $X \cup Y$-subsequence of F) $\cdot P$ = (the X-subsequence of F) $\frown$ (the Y-subsequence of F). The theorem is a consequence of (5), (4), and (6).

(10) Let us consider a complex-valued finite 0-sequence $\mathcal{F}$ and sets B_1, B_2. Suppose B_1 misses B_2. Then $\sum$ the $B_1 \cup B_2$-subsequence of $\mathcal{F}$ = $\sum$ the B_1-subsequence of $\mathcal{F}$ + $\sum$ the B_2-subsequence of $\mathcal{F}$. The theorem is a consequence of (9).

(11) Let us consider a finite 0-sequence F. Then F = the $\mathbb{N}$-subsequence of F.

Let us consider natural numbers N, i. Now we state the propositions:

(12) If $i \in \mathrm{dom}\,\mathrm{Sgm}_0(N \cap \mathbb{N}_{\mathrm{even}})$, then $(\mathrm{Sgm}_0(N \cap \mathbb{N}_{\mathrm{even}}))(i) = 2 \cdot i$.

(13) If $i \in \mathrm{dom}\,\mathrm{Sgm}_0(N \cap \mathbb{N}_{\mathrm{odd}})$, then $(\mathrm{Sgm}_0(N \cap \mathbb{N}_{\mathrm{odd}}))(i) = 2 \cdot i + 1$.

2. Lemmas on Some Divisibility Properties

Now we state the propositions:

(14) Let us consider integers i, j. Then $(i \bmod j) \bmod j = i \bmod j$.

(15) Let us consider integers i, j, k, l. Suppose $i \bmod l = j \bmod l$. Then $(k + i) \bmod l = (k + j) \bmod l$.

(16) Let us consider a finite 0-sequence d of $\mathbb{Z}$ and an integer n. Suppose a natural number i. If $i \in \mathrm{dom}\, d$, then $n \mid d(i)$. Then $n \mid \sum d$.

(17) Let us consider finite 0-sequences d, e of $\mathbb{Z}$ and an integer n. Suppose

(i) $\operatorname{dom} d = \operatorname{dom} e$, and

(ii) for every natural number i such that $i \in \operatorname{dom} d$ holds $e(i) = d(i) \bmod n$.

Then $\sum d \bmod n = \sum e \bmod n$. The theorem is a consequence of (14). Proof: Define $\mathcal{P}[\text{finite 0-sequence of } \mathbb{Z}] \equiv$ for every finite 0-sequence e of $\mathbb{Z}$ such that $\operatorname{dom} \$_1 = \operatorname{dom} e$ and for every natural number i such that $i \in \operatorname{dom} \$_1$ holds $e(i) = \$_1(i) \bmod n$ holds $\sum \$_1 \bmod n = \sum e \bmod n$. For every finite 0-sequence p of $\mathbb{Z}$ and for every element l of $\mathbb{Z}$ such that $\mathcal{P}[p]$ holds $\mathcal{P}[p \frown \langle l \rangle]$ by [2, (44), (13)], [25, (33)]. $\mathcal{P}[\langle\rangle_{\mathbb{Z}}]$ by [25, (15)]. For every finite 0-sequence p of $\mathbb{Z}$, $\mathcal{P}[p]$ from [18, Sch. 2]. $\square$

(18) Let us consider finite 0-sequences f, g of $\mathbb{N}$ and an integer i. Suppose

(i) $\operatorname{dom} f = \operatorname{dom} g$, and

(ii) for every element n such that $n \in \operatorname{dom} f$ holds $f(n) = i \cdot g(n)$.

Then $\sum f = i \cdot \sum g$.

(19) If $b > 1$, then $n = b \cdot \operatorname{value}(\operatorname{mid}(\operatorname{digits}(n, b), 2, \operatorname{len} \operatorname{digits}(n, b)), b) + (\operatorname{digits}(n, b))(0)$. The theorem is a consequence of (2), (18), and (3).

Let us consider natural numbers n, k. Now we state the propositions:

(20) If $k = 10^{2 \cdot n} - 1$, then $11 \mid k$.

(21) If $k = 10^{2 \cdot n + 1} + 1$, then $11 \mid k$.

Now we state the propositions:

(22) 7 and 10 are relatively prime.

(23) 29 is prime.

(24) 31 is prime.

(25) 41 is prime.

(26) 47 is prime.

(27) 53 is prime.

(28) 59 is prime.

(29) 61 is prime.

(30) 67 is prime.

(31) 71 is prime.

(32) 73 is prime.

(33) 79 is prime.

(34) 89 is prime.

(35) 97 is prime.

(36) 101 is prime.

3. Divisibility Criteria for Primes up to 101

Let us consider a prime natural number p and natural numbers n, f, b. Now we state the propositions:

(37) Suppose there exists a natural number k such that $b \cdot f + 1 = p \cdot k$ and $b > 1$ and p and b are relatively prime. Then $p \mid n$ if and only if $p \mid \mathrm{value}(\mathrm{mid}(\mathrm{digits}(n, b), 2, \mathrm{len}\,\mathrm{digits}(n, b)), b) - f \cdot (\mathrm{digits}(n, b))(0)$.

(38) Suppose there exists a natural number k such that $b \cdot f - 1 = p \cdot k$ and $b > 1$ and p and b are relatively prime. Then $p \mid n$ if and only if $p \mid \mathrm{value}(\mathrm{mid}(\mathrm{digits}(n, b), 2, \mathrm{len}\,\mathrm{digits}(n, b)), b) + f \cdot (\mathrm{digits}(n, b))(0)$.

Now we state the propositions:

(39) Divisibility rule–Divisibility by 7:
$7 \mid n$ if and only if $7 \mid \mathrm{value}(\mathrm{mid}(\mathrm{digits}(n, 10), 2, \mathrm{len}\,\mathrm{digits}(n, 10)), 10) - 2 \cdot (\mathrm{digits}(n, 10))(0)$. The theorem is a consequence of (37) and (22).

(40) $7 \mid n$ if and only if $7 \mid \mathrm{value}((\mathrm{digits}(n, 10))_{\downharpoonright 1}, 10) - 2 \cdot (\mathrm{digits}(n, 10))(0)$. The theorem is a consequence of (3) and (39).

(41) $11 \mid n$ if and only if $11 \mid \mathrm{value}(\mathrm{mid}(\mathrm{digits}(n, 10), 2, \mathrm{len}\,\mathrm{digits}(n, 10)), 10) - (\mathrm{digits}(n, 10))(0)$. The theorem is a consequence of (37).

(42) $11 \mid n$ if and only if $11 \mid \mathrm{value}((\mathrm{digits}(n, 10))_{\downharpoonright 1}, 10) - (\mathrm{digits}(n, 10))(0)$. The theorem is a consequence of (3) and (41).

Now we state the proposition:

(43) Divisibility rule–Divisibility by 11:
$11 \mid n$ if and only if $11 \mid \sum$ the $\mathbb{N}_{\mathrm{even}}$-subsequence of $\mathrm{digits}(n, 10) - \sum$ the $\mathbb{N}_{\mathrm{odd}}$-subsequence of $\mathrm{digits}(n, 10)$. The theorem is a consequence of (10), (7), (8), (11), (6), (12), (13), (20), (16), (21), and (14). Proof: Set $d = \mathrm{digits}(n, 10)$. Consider p being a finite 0-sequence of $\mathbb{N}$ such that $\mathrm{dom}\, p = \mathrm{dom}\, d$ and for every natural number i such that $i \in \mathrm{dom}\, p$ holds $p(i) = d(i) \cdot 10^i$ and $\mathrm{value}(d, 10) = \sum p$. Set $p_3 =$ the $\mathbb{N}_{\mathrm{even}}$-subsequence of p. Set $p_2 =$ the $\mathbb{N}_{\mathrm{odd}}$-subsequence of p. Set $d_2 =$ the $\mathbb{N}_{\mathrm{even}}$-subsequence of d. Set $d_3 =$ the $\mathbb{N}_{\mathrm{odd}}$-subsequence of d. For every natural number i such that $i \in \mathrm{dom}\, d_2$ holds $d_2(i) = d(2 \cdot i)$ by [8, (11), (12)]. For every natural number i such that $i \in \mathrm{dom}\, p_3$ holds $p_3(i) = d_2(i) \cdot 10^{2 \cdot i}$ by [8, (11), (12)]. For every natural number i such that $i \in \mathrm{dom}\, d_3$ holds $d_3(i) = d(2 \cdot i + 1)$ by [8, (11), (12)]. For every natural number i such that $i \in \mathrm{dom}\, p_2$ holds $p_2(i) = d_3(i) \cdot 10^{2 \cdot i + 1}$ by [8, (11), (12)]. Define $\mathcal{E}[\mathrm{set}, \mathrm{set}] \equiv \$_2 = p_3(\$_1) - d_2(\$_1)$. For every natural number k such that $k \in \mathbb{Z}_{\mathrm{dom}\, p_3}$ there exists an element x of $\mathbb{Z}$ such that $\mathcal{E}[k, x]$. Consider p_1 being a finite 0-sequence of $\mathbb{Z}$ such that $\mathrm{dom}\, p_1 = \mathbb{Z}_{\mathrm{dom}\, p_3}$ and for every natural number k such that $k \in \mathbb{Z}_{\mathrm{dom}\, p_3}$ holds $\mathcal{E}[k, p_1(k)]$ from [20, Sch. 5]. For every natural number i such that $i \in \mathrm{dom}\, p_3$ holds $p_3(i) = +_{\mathbb{Z}}(p_1(i), d_2(i))$. Define $\mathcal{O}[\mathrm{set}, \mathrm{set}] \equiv \$_2 = p_2(\$_1) + d_3(\$_1)$. Consider p_4 being a finite 0-sequence of

$\mathbb{N}$ such that $\operatorname{dom} p_4 = \mathbb{Z}_{\operatorname{dom} p_2}$ and for every natural number k such that $k \in \mathbb{Z}_{\operatorname{dom} p_2}$ holds $\mathcal{O}[k, p_4(k)]$ from [20, Sch. 5]. Set $m = (-1) \cdot d_3$. For every natural number i such that $i \in \operatorname{dom} p_2$ holds $p_2(i) = +_{\mathbb{Z}}(p_4(i), m(i))$. If $11 \mid n$, then $11 \mid \sum d_2 - \sum d_3$ by [19, (5)], [23, (62)]. If $11 \mid \sum d_2 - \sum d_3$, then $11 \mid n$ by [23, (62)], [19, (5)]. □

Now we state the propositions:

(44) DIVISIBILITY RULE–DIVISIBILITY BY 13:
$13 \mid n$ if and only if $13 \mid \text{value}(\text{mid}(\text{digits}(n, 10), 2, \text{len}\,\text{digits}(n, 10)), 10) + 4 \cdot (\text{digits}(n, 10))(0)$. The theorem is a consequence of (38).

(45) $13 \mid n$ if and only if $13 \mid \text{value}((\text{digits}(n, 10))_{\lfloor 1}, 10) + 4 \cdot (\text{digits}(n, 10))(0)$. The theorem is a consequence of (3) and (44).

(46) $17 \mid n$ if and only if $17 \mid \text{value}(\text{mid}(\text{digits}(n, 10), 2, \text{len}\,\text{digits}(n, 10)), 10) - 5 \cdot (\text{digits}(n, 10))(0)$. The theorem is a consequence of (37).

(47) $17 \mid n$ if and only if $17 \mid \text{value}((\text{digits}(n, 10))_{\lfloor 1}, 10) - 5 \cdot (\text{digits}(n, 10))(0)$. The theorem is a consequence of (3) and (46).

(48) $19 \mid n$ if and only if $19 \mid \text{value}(\text{mid}(\text{digits}(n, 10), 2, \text{len}\,\text{digits}(n, 10)), 10) + 2 \cdot (\text{digits}(n, 10))(0)$. The theorem is a consequence of (38).

(49) $19 \mid n$ if and only if $19 \mid \text{value}((\text{digits}(n, 10))_{\lfloor 1}, 10) + 2 \cdot (\text{digits}(n, 10))(0)$. The theorem is a consequence of (3) and (48).

(50) $23 \mid n$ if and only if $23 \mid \text{value}(\text{mid}(\text{digits}(n, 10), 2, \text{len}\,\text{digits}(n, 10)), 10) + 7 \cdot (\text{digits}(n, 10))(0)$. The theorem is a consequence of (38).

(51) $23 \mid n$ if and only if $23 \mid \text{value}((\text{digits}(n, 10))_{\lfloor 1}, 10) + 7 \cdot (\text{digits}(n, 10))(0)$. The theorem is a consequence of (3) and (50).

(52) $29 \mid n$ if and only if $29 \mid \text{value}(\text{mid}(\text{digits}(n, 10), 2, \text{len}\,\text{digits}(n, 10)), 10) + 3 \cdot (\text{digits}(n, 10))(0)$. The theorem is a consequence of (23) and (38).

(53) $29 \mid n$ if and only if $29 \mid \text{value}((\text{digits}(n, 10))_{\lfloor 1}, 10) + 3 \cdot (\text{digits}(n, 10))(0)$. The theorem is a consequence of (3) and (52).

(54) $31 \mid n$ if and only if $31 \mid \text{value}(\text{mid}(\text{digits}(n, 10), 2, \text{len}\,\text{digits}(n, 10)), 10) - 3 \cdot (\text{digits}(n, 10))(0)$. The theorem is a consequence of (24) and (37).

(55) $31 \mid n$ if and only if $31 \mid \text{value}((\text{digits}(n, 10))_{\lfloor 1}, 10) - 3 \cdot (\text{digits}(n, 10))(0)$. The theorem is a consequence of (3) and (54).

(56) $37 \mid n$ if and only if $37 \mid \text{value}(\text{mid}(\text{digits}(n, 10), 2, \text{len}\,\text{digits}(n, 10)), 10) - 11 \cdot (\text{digits}(n, 10))(0)$. The theorem is a consequence of (37).

(57) $37 \mid n$ if and only if $37 \mid \text{value}((\text{digits}(n, 10))_{\lfloor 1}, 10) - 11 \cdot (\text{digits}(n, 10))(0)$. The theorem is a consequence of (3) and (56).

(58) $41 \mid n$ if and only if $41 \mid \text{value}(\text{mid}(\text{digits}(n, 10), 2, \text{len}\,\text{digits}(n, 10)), 10) - 4 \cdot (\text{digits}(n, 10))(0)$. The theorem is a consequence of (25) and (37).

(59) $41 \mid n$ if and only if $41 \mid \text{value}((\text{digits}(n, 10))_{\lfloor 1}, 10) - 4 \cdot (\text{digits}(n, 10))(0)$. The theorem is a consequence of (3) and (58).

(60) $43 \mid n$ if and only if $43 \mid \mathrm{value}(\mathrm{mid}(\mathrm{digits}(n,10),2,\mathrm{len}\,\mathrm{digits}(n,10)),10)+13\cdot(\mathrm{digits}(n,10))(0)$. The theorem is a consequence of (38).

(61) $43 \mid n$ if and only if $43 \mid \mathrm{value}((\mathrm{digits}(n,10))_{\lfloor 1},10)+13\cdot(\mathrm{digits}(n,10))(0)$. The theorem is a consequence of (3) and (60).

(62) $47 \mid n$ if and only if $47 \mid \mathrm{value}(\mathrm{mid}(\mathrm{digits}(n,10),2,\mathrm{len}\,\mathrm{digits}(n,10)),10)-14\cdot(\mathrm{digits}(n,10))(0)$. The theorem is a consequence of (26) and (37).

(63) $47 \mid n$ if and only if $47 \mid \mathrm{value}((\mathrm{digits}(n,10))_{\lfloor 1},10)-14\cdot(\mathrm{digits}(n,10))(0)$. The theorem is a consequence of (3) and (62).

(64) $53 \mid n$ if and only if $53 \mid \mathrm{value}(\mathrm{mid}(\mathrm{digits}(n,10),2,\mathrm{len}\,\mathrm{digits}(n,10)),10)+16\cdot(\mathrm{digits}(n,10))(0)$. The theorem is a consequence of (27) and (38).

(65) $53 \mid n$ if and only if $53 \mid \mathrm{value}((\mathrm{digits}(n,10))_{\lfloor 1},10)+16\cdot(\mathrm{digits}(n,10))(0)$. The theorem is a consequence of (3) and (64).

(66) $59 \mid n$ if and only if $59 \mid \mathrm{value}(\mathrm{mid}(\mathrm{digits}(n,10),2,\mathrm{len}\,\mathrm{digits}(n,10)),10)+6\cdot(\mathrm{digits}(n,10))(0)$. The theorem is a consequence of (28) and (38).

(67) $59 \mid n$ if and only if $59 \mid \mathrm{value}((\mathrm{digits}(n,10))_{\lfloor 1},10)+6\cdot(\mathrm{digits}(n,10))(0)$. The theorem is a consequence of (3) and (66).

(68) $61 \mid n$ if and only if $61 \mid \mathrm{value}(\mathrm{mid}(\mathrm{digits}(n,10),2,\mathrm{len}\,\mathrm{digits}(n,10)),10)-6\cdot(\mathrm{digits}(n,10))(0)$. The theorem is a consequence of (29) and (37).

(69) $61 \mid n$ if and only if $61 \mid \mathrm{value}((\mathrm{digits}(n,10))_{\lfloor 1},10)-6\cdot(\mathrm{digits}(n,10))(0)$. The theorem is a consequence of (3) and (68).

(70) $67 \mid n$ if and only if $67 \mid \mathrm{value}(\mathrm{mid}(\mathrm{digits}(n,10),2,\mathrm{len}\,\mathrm{digits}(n,10)),10)-20\cdot(\mathrm{digits}(n,10))(0)$. The theorem is a consequence of (30) and (37).

(71) $67 \mid n$ if and only if $67 \mid \mathrm{value}((\mathrm{digits}(n,10))_{\lfloor 1},10)-20\cdot(\mathrm{digits}(n,10))(0)$. The theorem is a consequence of (3) and (70).

(72) $71 \mid n$ if and only if $71 \mid \mathrm{value}(\mathrm{mid}(\mathrm{digits}(n,10),2,\mathrm{len}\,\mathrm{digits}(n,10)),10)-7\cdot(\mathrm{digits}(n,10))(0)$. The theorem is a consequence of (31) and (37).

(73) $71 \mid n$ if and only if $71 \mid \mathrm{value}((\mathrm{digits}(n,10))_{\lfloor 1},10)-7\cdot(\mathrm{digits}(n,10))(0)$. The theorem is a consequence of (3) and (72).

(74) $73 \mid n$ if and only if $73 \mid \mathrm{value}(\mathrm{mid}(\mathrm{digits}(n,10),2,\mathrm{len}\,\mathrm{digits}(n,10)),10)+22\cdot(\mathrm{digits}(n,10))(0)$. The theorem is a consequence of (32) and (38).

(75) $73 \mid n$ if and only if $73 \mid \mathrm{value}((\mathrm{digits}(n,10))_{\lfloor 1},10)+22\cdot(\mathrm{digits}(n,10))(0)$. The theorem is a consequence of (3) and (74).

(76) $79 \mid n$ if and only if $79 \mid \mathrm{value}(\mathrm{mid}(\mathrm{digits}(n,10),2,\mathrm{len}\,\mathrm{digits}(n,10)),10)+8\cdot(\mathrm{digits}(n,10))(0)$. The theorem is a consequence of (33) and (38).

(77) $79 \mid n$ if and only if $79 \mid \mathrm{value}((\mathrm{digits}(n,10))_{\lfloor 1},10)+8\cdot(\mathrm{digits}(n,10))(0)$. The theorem is a consequence of (3) and (76).

(78) $83 \mid n$ if and only if $83 \mid \mathrm{value}(\mathrm{mid}(\mathrm{digits}(n,10),2,\mathrm{len}\,\mathrm{digits}(n,10)),10)+25\cdot(\mathrm{digits}(n,10))(0)$. The theorem is a consequence of (38).

(79) $83 \mid n$ if and only if $83 \mid \text{value}((\text{digits}(n,10))_{\downarrow 1},10)+25\cdot(\text{digits}(n,10))(0)$. The theorem is a consequence of (3) and (78).

(80) $89 \mid n$ if and only if $89 \mid \text{value}(\text{mid}(\text{digits}(n,10),2,\text{len}\,\text{digits}(n,10)),10)+9\cdot(\text{digits}(n,10))(0)$. The theorem is a consequence of (34) and (38).

(81) $89 \mid n$ if and only if $89 \mid \text{value}((\text{digits}(n,10))_{\downarrow 1},10)+9\cdot(\text{digits}(n,10))(0)$. The theorem is a consequence of (3) and (80).

(82) $97 \mid n$ if and only if $97 \mid \text{value}(\text{mid}(\text{digits}(n,10),2,\text{len}\,\text{digits}(n,10)),10)-29\cdot(\text{digits}(n,10))(0)$. The theorem is a consequence of (35) and (37).

(83) $97 \mid n$ if and only if $97 \mid \text{value}((\text{digits}(n,10))_{\downarrow 1},10)-29\cdot(\text{digits}(n,10))(0)$. The theorem is a consequence of (3) and (82).

(84) $101 \mid n$ if and only if $101 \mid \text{value}(\text{mid}(\text{digits}(n,10),2,\text{len}\,\text{digits}(n,10)),10)-10\cdot(\text{digits}(n,10))(0)$. The theorem is a consequence of (36) and (37).

(85) $101 \mid n$ if and only if $101 \mid \text{value}((\text{digits}(n,10))_{\downarrow 1},10)-10\cdot(\text{digits}(n,10))(0)$. The theorem is a consequence of (3) and (84).

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