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FORMALIZED MATHEMATICS
Vol. 22, No. 1, Pages 1–10, 2014
DOI: 10.2478/forma-2014-0001

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Definition of Flat Poset and Existence Theorems for Recursive Call

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Summary. This text includes the definition and basic notions of product of posets, chain-complete and flat posets, flattening operation, and the existence theorems of recursive call using the flattening operator. First part of the article, devoted to product and flat posets has a purely mathematical quality. Definition 3 allows to construct a flat poset from arbitrary non-empty set [12] in order to provide formal apparatus which eanbles to work with recursive calls within the Mizar langauge. To achieve this we extensively use technical Mizar functors like `BaseFunc` or `RecFunc`. The remaining part builds the background for information engineering approach for lists, namely recursive call for posets [21]. We formalized some facts from Chapter 8 of this book as an introduction to the next two sections where we concentrate on binary product of posets rather than on a more general case.

MSC: 06A11 68N30 03B35

Keywords: flat posets; recursive calls for posets; flattening operator

MML identifier: `POSET_2`, version: `8.1.02 5.22.1199`

The notation and terminology used in this paper have been introduced in the following articles: [1], [17], [11], [6], [7], [8], [2], [13], [19], [14], [4], [9], [15], [22], [23], [20], [5], [16], and [10].

[1]My work was supported by JSPS KAKENHI 22300285.

ISSN 1426–2630(Print), 1898-9934(Online)

1. Preliminaries from Poset Theory

From now on a, Z_1, Z_2, Z_3 denote sets, x, y, z denote objects, and k denotes a natural number.

Now we state the propositions:

(1) Let us consider a lower-bounded non empty poset P and an element p of P. If $p \leqslant$ the carrier of P, then $p = \bot_P$.

(2) Let us consider a chain-complete non empty poset P, a non empty chain L of P, and an element p of P. If $p \in L$, then $p \leqslant \sup L$.

(3) Let us consider a chain-complete non empty poset P, a non empty chain L of P, and an element p_1 of P. Suppose an element p of P. If $p \in L$, then $p \leqslant p_1$. Then $\sup L \leqslant p_1$.

2. On the Product of Posets

Now we state the proposition:

(4) Let us consider non empty relational structures P, Q and an object x. Then x is an element of $P \times Q$ if and only if there exists an element p of P and there exists an element q of Q such that $x = \langle p, q \rangle$.

Let P, Q be non empty posets and L be a non empty chain of $P \times Q$. The functors: $\pi_1(L)$ and $\pi_2(L)$ yield non empty chains of P. Let P, Q_1, Q_2 be non empty posets, f_1 be a monotone function from P into Q_1, and f_2 be a monotone function from P into Q_2. One can verify that $\langle f_1, f_2 \rangle$ is monotone as a function from P into $Q_1 \times Q_2$.

Let P, Q be chain-complete non empty posets. Observe that $P \times Q$ is chain-complete.

Now we state the proposition:

(5) Let us consider chain-complete non empty posets P, Q and a non empty chain L of $P \times Q$. Then $\sup L = \langle \sup \pi_1(L), \sup \pi_2(L) \rangle$.

Let P, Q_1, Q_2 be strict chain-complete non empty posets, f_1 be a continuous function from P into Q_1, and f_2 be a continuous function from P into Q_2. Note that $\langle f_1, f_2 \rangle$ is continuous as a function from P into $Q_1 \times Q_2$.

3. Definition of Flat Poset and Poset Flattening

Let I_3 be a relational structure. We say that I_3 is flat if and only if

(Def. 1) There exists an element a of I_3 such that for every elements x, y of I_3, $x \leqslant y$ iff $x = a$ or $x = y$.

One can verify that every non empty relational structure which is discrete is also reflexive and every discrete non empty relational structure which is trivial is also flat and there exists a poset which is strict, non empty, and flat and every relational structure which is flat is also reflexive transitive and antisymmetric and every non empty poset which is flat is also lower-bounded.

In the sequel S denotes a relational structure, P, Q denote non empty flat posets, p, p_1, p_2 denote elements of P, and K denotes a non empty chain of P.

Now we state the proposition:

(6) Let us consider a non empty flat poset P and a non empty chain K of P. Then there exists an element a of P such that $K = \{a\}$ or $K = \{\perp_P, a\}$.

Let us consider a function f from P into Q. Now we state the propositions:

(7) There exists an element a of P such that $K = \{a\}$ and $f^\circ K = \{f(a)\}$ or $K = \{\perp_P, a\}$ and $f^\circ K = \{f(\perp_P), f(a)\}$. The theorem is a consequence of (6).

(8) If $f(\perp_P) = \perp_Q$, then f is monotone.

Now we state the proposition:

(9) If $K = \{\perp_P, p\}$, then $\sup K = p$.

One can verify that there exists a poset which is strict, non empty, flat, and chain-complete and every poset which is non empty and flat is also chain-complete.

Now we state the proposition:

(10) Let us consider strict non empty chain-complete flat posets P, Q and a function f from P into Q. If $f(\perp_P) = \perp_Q$, then f is continuous. Proof: For every non empty chain K of P, $f(\sup K) \leqslant \sup(f^\circ K)$ by [15, (1)], (7), [5, (39)], (9). □

4. Primaries for Existence Theorems of Recursive Call Using Flattening

In the sequel X, Y denote non empty sets.

Let X be a non empty set. The functor $\operatorname{FlatRelat} X$ yielding a relation between $\operatorname{succ} X$ and $\operatorname{succ} X$ is defined by the term

(Def. 2) $(\{\langle X, X\rangle\} \cup \{X\} \times X) \cup \mathrm{id}_X$.

Now we state the proposition:

(11) Let us consider elements x, y of $\operatorname{succ} X$. Then $\langle x, y\rangle \in \operatorname{FlatRelat} X$ if and only if $x = X$ or $x = y$.

Let X be a non empty set. The functor $\operatorname{FlatPoset} X$ yielding a strict non empty chain-complete flat poset is defined by the term

(Def. 3) $\langle \operatorname{succ} X, \operatorname{FlatRelat} X\rangle$.

Now we state the propositions:

(12) Let us consider elements x, y of $\operatorname{FlatPoset} X$. Then $x \leqslant y$ if and only if $x = X$ or $x = y$.

(13) X is an element of $\operatorname{FlatPoset} X$.

Let us consider X. Let us observe that $\perp_{\operatorname{FlatPoset} X}$ reduces to X.

Let x be an object, X, Y be non empty sets, and f be a function from X into Y. The functor $\operatorname{Flatten}(f, x)$ yielding a set is defined by the term

(Def. 4) $\begin{cases} f(x), & \textbf{if } x \in X, \\ Y, & \textbf{otherwise}. \end{cases}$

The functor $\operatorname{Flatten}(f)$ yielding a function from $\operatorname{FlatPoset} X$ into $\operatorname{FlatPoset} Y$ is defined by

(Def. 5) (i) $it(X) = Y$, and

(ii) for every element x of $\operatorname{FlatPoset} X$ such that $x \neq X$ holds $it(x) = f(x)$.

Let us observe that $\operatorname{Flatten}(f)$ is continuous.

Now we state the proposition:

(14) Let us consider a function f from X into Y. If $x \in X$, then $(\operatorname{Flatten}(f))(x) \in Y$.

Let us consider X and Y. The functor $\operatorname{FlatConF}(X, Y)$ yielding a strict chain-complete non empty poset is defined by the term

(Def. 6) $\operatorname{ConPoset}(\operatorname{FlatPoset} X, \operatorname{FlatPoset} Y)$.

Let L be a flat poset. One can verify that every chain of L is finite and there exists a lattice which is non empty, flat, and lower-bounded.

Now we state the propositions:

(15) Let us consider a non empty lattice L, an element x of L, and an x-chain A of x. Then $\overline{\overline{A}} = 1$. Proof: For every element z of L such that $z \in A$ holds $z \in \{x\}$ by [19, (2)]. □

(16) Let us consider a non empty flat lower-bounded lattice L, an element x of L, and a $\perp_L$-chain A of x. Then $\overline{\overline{A}} \leqslant 2$. The theorem is a consequence of (6) and (15).

(17) Let us consider a finite lower-bounded antisymmetric non empty lattice L. Then L is flat if and only if for every element x of L, $\operatorname{height} x \leqslant 2$. Proof: There exists an element a of L such that for every elements x, y of L, $x \leqslant y$ iff $x = a$ or $x = y$ by [5, (44)], [13, (2), (6)], [3, (13)]. □

5. EXISTENCE THEOREM OF RECURSIVE CALL FOR SINGLE-EQUATION

From now on D denotes a subset of X, I denotes a function from X into Y, J denotes a function from $X \times Y$ into Y, and E denotes a function from X into X.

Let X be a non empty set, D be a subset of X, and E be a function from X into X. We say that E is well founded with minimal set D if and only if

(Def. 7) There exists a function l from X into $\mathbb{N}$ such that for every element x of X, if $l(x) \leqslant 0$, then $x \in D$ and if $x \notin D$, then $l(E(x)) < l(x)$.

Let X, Y be non empty sets. Let I be a function from X into Y, J be a function from $X \times Y$ into Y, and x, y be objects. The functor $\mathrm{BaseFunc01}(x, y, I, J, D)$ yielding a set is defined by the term

(Def. 8) $$\begin{cases} I(x), & \textbf{if } x \in D, \\ J(\langle x, y \rangle), & \textbf{if } x \notin D \text{ and } x \in X \text{ and } y \in Y, \\ Y, & \textbf{otherwise}. \end{cases}$$

Let E be a function from X into X and h be an object. Assume h is a continuous function from $\mathrm{FlatPoset}\, X$ into $\mathrm{FlatPoset}\, Y$.

The functor $\mathrm{RecFunc01}(h, E, I, J, D)$ yielding a continuous function from $\mathrm{FlatPoset}\, X$ into $\mathrm{FlatPoset}\, Y$ is defined by

(Def. 9) Let us consider an element x of $\mathrm{FlatPoset}\, X$ and a continuous function f from $\mathrm{FlatPoset}\, X$ into $\mathrm{FlatPoset}\, Y$. Suppose $h = f$. Then $it(x) = \mathrm{BaseFunc01}(x, f((\mathrm{Flatten}(E))(x)), I, J, D)$.

Now we state the propositions:

(18) There exists a continuous function W from $\mathrm{FlatConF}(X, Y)$ into $\mathrm{FlatConF}(X, Y)$ such that for every element f of $\mathrm{ConFuncs}(\mathrm{FlatPoset}\, X, \mathrm{FlatPoset}\, Y)$, $W(f) = \mathrm{RecFunc01}(f, E, I, J, D)$. PROOF: Set $F_1 = \mathrm{FlatPoset}\, X$. Set $F_2 = \mathrm{FlatPoset}\, Y$. Set $F_3 = \mathrm{FlatConF}(X, Y)$. Set $C_1 = \mathrm{ConFuncs}(F_1, F_2)$. Define $\mathcal{H}(\text{object}) = \mathrm{RecFunc01}(\$_1, E, I, J, D)$. For every continuous function h from F_1 into F_2, $h \in C_1$ by [7, (8)]. For every set h such that $h \in C_1$ holds h is a continuous function from F_1 into F_2. There exists a function W from C_1 into C_1 such that for every object f such that $f \in C_1$ holds $W(f) = \mathcal{H}(f)$ from [7, Sch. 2]. Consider I_3 being a function from C_1 into C_1 such that for every object f such that $f \in C_1$ holds $I_3(f) = \mathcal{H}(f)$. I_3 is a continuous function from F_3 into F_3 by [7, (5)], (12), [24, (9)], [15, (1), (11)]. $\square$

(19) There exists a set f such that

 (i) $f \in \mathrm{ConFuncs}(\mathrm{FlatPoset}\, X, \mathrm{FlatPoset}\, Y)$, and

 (ii) $f = \mathrm{RecFunc01}(f, E, I, J, D)$.

The theorem is a consequence of (18).

Let us assume that E is well founded with minimal set D. Now we state the propositions:

(20) There exists a continuous function f from FlatPoset X into FlatPoset Y such that for every element x of X, $f(x) \in Y$ and $f(x) = \mathrm{BaseFunc01}(x, f(E(x)), I, J, D)$. PROOF: Consider f being a set such that $f \in$ ConFuncs (FlatPoset X, FlatPoset Y) and $f = \mathrm{RecFunc01}(f, E, I, J, D)$. Consider l being a function from X into $\mathbb{N}$ such that for every element x_0 of X, if $l(x_0) \leqslant 0$, then $x_0 \in D$ and if $x_0 \notin D$, then $l(E(x_0)) < l(x_0)$. Define $\mathcal{P}$[natural number] $\equiv$ for every element x_0 of X such that $l(x_0) \leqslant \$_1$ holds $f(x_0) \in Y$ and $f(x_0) = \mathrm{BaseFunc01}(x_0, f(E(x_0)), I, J, D)$. $\mathcal{P}[0]$ by [7, (5)]. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [7, (5)], [3, (13)]. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. For every element x of X, $f(x) \in Y$ and $f(x) = \mathrm{BaseFunc01}(x, f(E(x)), I, J, D)$. $\square$

(21) There exists a function f from X into Y such that for every element x of X, if $x \in D$, then $f(x) = I(x)$ and if $x \notin D$, then $f(x) = J(\langle x, f(E(x))\rangle)$.

Now we state the proposition:

(22) Let us consider functions f_1, f_2 from X into Y. Suppose
   (i) E is well founded with minimal set D, and
   (ii) for every element x of X, if $x \in D$, then $f_1(x) = I(x)$ and if $x \notin D$, then $f_1(x) = J(\langle x, f_1(E(x))\rangle)$, and
   (iii) for every element x of X, if $x \in D$, then $f_2(x) = I(x)$ and if $x \notin D$, then $f_2(x) = J(\langle x, f_2(E(x))\rangle)$.

Then $f_1 = f_2$. PROOF: Consider l being a function from X into $\mathbb{N}$ such that for every element x of X, if $l(x) \leqslant 0$, then $x \in D$ and if $x \notin D$, then $l(E(x)) < l(x)$. Define $\mathcal{P}$[natural number] $\equiv$ for every element x of X such that $l(x) \leqslant \$_1$ holds $f_1(x) = f_2(x)$. $\mathcal{P}[0]$. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [3, (13)]. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. For every element x of X, $f_1(x) = f_2(x)$. $\square$

6. Existence Theorem of Recursive Calls for 2-equations

From now on D denotes a subset of X, I, I_1, I_2 denote functions from X into Y, J, J_1, J_2 denote functions from $X \times Y \times Y$ into Y, and E_1, E_2 denote functions from X into X.

Let X be a non empty set, D be a subset of X, and E_1, E_2 be functions from X into X. We say that (E_1,E_2) is well founded with minimal set D if and only if

(Def. 10) There exists a function l from X into $\mathbb{N}$ such that for every element x of X, if $l(x) \leqslant 0$, then $x \in D$ and if $x \notin D$, then $l(E_1(x)) < l(x)$ and $l(E_2(x)) < l(x)$.

Let X, Y be non empty sets. Let I be a function from X into Y, J be a function from $X \times Y \times Y$ into Y, and x, y_1, y_2 be objects. The functor BaseFunc02(x, y_1, y_2, I, J, D) yielding a set is defined by the term

$$\text{(Def. 11)} \quad \begin{cases} I(x), & \textbf{if } x \in D, \\ J(\langle x, y_1, y_2 \rangle), & \textbf{if } x \notin D \text{ and } x \in X \text{ and } \mathrm{y}_1, y_2 \in Y, \\ Y, & \textbf{otherwise}. \end{cases}$$

Let E_1, E_2 be functions from X into X and h_1, h_2 be objects. Assume h_1 is a continuous function from FlatPoset X into FlatPoset Y and h_2 is a continuous function from FlatPoset X into FlatPoset Y. The functor RecFunc02$(h_1, h_2, E_1, E_2, I, J, D)$ yielding a continuous function from FlatPoset X into FlatPoset Y is defined by

(Def. 12) Let us consider an element x of FlatPoset X and continuous functions f_1, f_2 from FlatPoset X into FlatPoset Y. Suppose

(i) $h_1 = f_1$, and

(ii) $h_2 = f_2$.

Then $it(x) =$ BaseFunc02$(x, f_1((\mathrm{Flatten}(E_1))(x)), f_2((\mathrm{Flatten}(E_2))(x)), I, J, D)$.

Now we state the propositions:

(23) There exists a continuous function W from FlatConF$(X, Y) \times$ FlatConF(X, Y) into FlatConF(X, Y) such that for every set f such that $f \in$ ConFuncs(FlatPoset X, FlatPoset Y)$\times$ConFuncs(FlatPoset X, FlatPoset Y) holds $W(f) =$ RecFunc02$(f_{\mathbf{1}}, f_{\mathbf{2}}, E_1, E_2, I, J, D)$. Proof: Set $F_1 =$ FlatPoset X. Set $F_2 =$ FlatPoset Y. Set $F_3 =$ FlatConF(X, Y). Set $C_1 =$ ConFuncs(F_1, F_2). Set $F_4 = F_3 \times F_3$. Set $C_2 = C_1 \times C_1$. Define $\mathcal{H}$(object) $=$ RecFunc02$(\$_{1\mathbf{1}}, \$_{1\mathbf{2}}, E_1, E_2, I, J, D)$. For every continuous function h from F_1 into F_2, $h \in C_1$ by [7, (8)]. For every set h such that $h \in C_1$ holds h is a continuous function from F_1 into F_2. For every element h of F_4, there exist continuous functions h_1, h_2 from F_1 into F_2 such that $h = \langle h_1, h_2 \rangle$. There exists a function W from C_2 into C_1 such that for every object f such that $f \in C_2$ holds $W(f) = \mathcal{H}(f)$ from [7, Sch. 2]. Consider I_3 being a function from C_2 into C_1 such that for every object f such that $f \in C_2$ holds $I_3(f) = \mathcal{H}(f)$. I_3 is a continuous function from F_4 into F_3 by [7, (5)], [16, (12)], (12), [24, (9)]. □

(24) There exist sets f, g such that

(i) f, $g \in$ ConFuncs(FlatPoset X, FlatPoset Y), and

(ii) $f =$ RecFunc02$(f, g, E_1, E_2, I_1, J_1, D)$, and

(iii) $g =$ RecFunc02$(f, g, E_1, E_2, I_2, J_2, D)$.

The theorem is a consequence of (23) and (4).

Let us assume that (E_1,E_2) is well founded with minimal set D. Now we state the propositions:

(25) There exist continuous functions f, g from FlatPoset X into FlatPoset Y such that for every element x of X, $f(x) \in Y$ and $f(x) = \mathrm{BaseFunc02}(x, f(E_1(x)), g(E_2(x)), I_1, J_1, D)$ and $g(x) \in Y$ and $g(x) = \mathrm{BaseFunc02}(x, f(E_1(x)), g(E_2(x)), I_2, J_2, D)$. PROOF: Consider f, g being sets such that f, $g \in \mathrm{ConFuncs}(\mathrm{FlatPoset}\,X, \mathrm{FlatPoset}\,Y)$ and $f = \mathrm{RecFunc02}(f, g, E_1, E_2, I_1, J_1, D)$ and $g = \mathrm{RecFunc02}(f, g, E_1, E_2, I_2, J_2, D)$. Consider l being a function from X into $\mathbb{N}$ such that for every element x_0 of X, if $l(x_0) \leqslant 0$, then $x_0 \in D$ and if $x_0 \notin D$, then $l(E_1(x_0)) < l(x_0)$ and $l(E_2(x_0)) < l(x_0)$. Define $\mathcal{P}$[natural number] $\equiv$ for every elements x_1, x_2 of X such that $l(x_1) \leqslant \$_1$ and $l(x_2) \leqslant \$_1$ holds $f(x_1) \in Y$ and $f(x_1) = \mathrm{BaseFunc02}(x_1, f(E_1(x_1)), g(E_2(x_1)), I_1, J_1, D)$ and $g(x_2) \in Y$ and $g(x_2) = \mathrm{BaseFunc02}(x_2, f(E_1(x_2)), g(E_2(x_2)), I_2, J_2, D)$. $\mathcal{P}[0]$ by [7, (5)]. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [7, (5)], [3, (13)], [18, (69)]. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. For every elements x_1, x_2 of X, $f(x_1) \in Y$ and $f(x_1) = \mathrm{BaseFunc02}(x_1, f(E_1(x_1)), g(E_2(x_1)), I_1, J_1, D)$ and $g(x_2) \in Y$ and $g(x_2) = \mathrm{BaseFunc02}(x_2, f(E_1(x_2)), g(E_2(x_2)), I_2, J_2, D)$ by [3, (11)]. $\square$

(26) There exist functions f, g from X into Y such that for every element x of X, if $x \in D$, then $f(x) = I_1(x)$ and $g(x) = I_2(x)$ and if $x \notin D$, then $f(x) = J_1(\langle x, f(E_1(x)), g(E_2(x))\rangle)$ and $g(x) = J_2(\langle x, f(E_1(x)), g(E_2(x))\rangle)$.

Now we state the propositions:

(27) Let us consider functions f_1, g_1, f_2, g_2 from X into Y. Suppose

(i) (E_1,E_2) is well founded with minimal set D, and

(ii) for every element x of X, if $x \in D$, then $f_1(x) = I_1(x)$ and $g_1(x) = I_2(x)$ and if $x \notin D$, then $f_1(x) = J_1(\langle x, f_1(E_1(x)), g_1(E_2(x))\rangle)$ and $g_1(x) = J_2(\langle x, f_1(E_1(x)), g_1(E_2(x))\rangle)$, and

(iii) for every element x of X, if $x \in D$, then $f_2(x) = I_1(x)$ and $g_2(x) = I_2(x)$ and if $x \notin D$, then $f_2(x) = J_1(\langle x, f_2(E_1(x)), g_2(E_2(x))\rangle)$ and $g_2(x) = J_2(\langle x, f_2(E_1(x)), g_2(E_2(x))\rangle)$.

Then

(iv) $f_1 = f_2$, and

(v) $g_1 = g_2$.

PROOF: Consider l being a function from X into $\mathbb{N}$ such that for every element x of X, if $l(x) \leqslant 0$, then $x \in D$ and if $x \notin D$, then $l(E_1(x)) < l(x)$ and $l(E_2(x)) < l(x)$. Define $\mathcal{P}$[natural number] $\equiv$ for every element x of X such that $l(x) \leqslant \$_1$ holds $f_1(x) = f_2(x)$ and $g_1(x) = g_2(x)$. $\mathcal{P}[0]$. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [3, (13)]. For every natural number

k, $\mathcal{P}[k]$ from [3, Sch. 2]. For every element x of X, $f_1(x) = f_2(x)$ and $g_1(x) = g_2(x)$. □

(28) Suppose (E_1,E_2) is well founded with minimal set D. Then there exists a function f from X into Y such that for every element x of X, if $x \in D$, then $f(x) = I(x)$ and if $x \notin D$, then $f(x) = J(\langle x, f(E_1(x)), f(E_2(x))\rangle)$. The theorem is a consequence of (26).

(29) Let us consider functions f_1, f_2 from X into Y. Suppose

(i) (E_1,E_2) is well founded with minimal set D, and

(ii) for every element x of X, if $x \in D$, then $f_1(x) = I(x)$ and if $x \notin D$, then $f_1(x) = J(\langle x, f_1(E_1(x)), f_1(E_2(x))\rangle)$, and

(iii) for every element x of X, if $x \in D$, then $f_2(x) = I(x)$ and if $x \notin D$, then $f_2(x) = J(\langle x, f_2(E_1(x)), f_2(E_2(x))\rangle)$.

Then $f_1 = f_2$. Proof: Consider l being a function from X into $\mathbb{N}$ such that for every element x of X, if $l(x) \leqslant 0$, then $x \in D$ and if $x \notin D$, then $l(E_1(x)) < l(x)$ and $l(E_2(x)) < l(x)$. Define $\mathcal{P}$[natural number] $\equiv$ for every element x of X such that $l(x) \leqslant \$_1$ holds $f_1(x) = f_2(x)$. $\mathcal{P}[0]$. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [3, (13)]. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. For every element x of X, $f_1(x) = f_2(x)$. □

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Received February 11, 2014

FORMALIZED MATHEMATICS
Vol. 22, No. 1, Pages 11–19, 2014
DOI: 10.2478/forma-2014-0002

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Tietze Extension Theorem for n-dimensional Spaces[1]

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Summary. In this article we prove the Tietze extension theorem for an arbitrary convex compact subset of $\mathcal{E}^n$ with a non-empty interior. This theorem states that, if T is a normal topological space, X is a closed subset of T, and A is a convex compact subset of $\mathcal{E}^n$ with a non-empty interior, then a continuous function $f : X \to A$ can be extended to a continuous function $g : T \to \mathcal{E}^n$. Additionally we show that a subset A is replaceable by an arbitrary subset of a topological space that is homeomorphic with a convex compact subset of $\mathcal{E}^n$ with a non-empty interior. This article is based on [20]; [23] and [22] can also serve as reference books.

MSC: 54A05 03B35

Keywords: Tietze extension; hypercube

MML identifier: TIETZE_2, version: 8.1.02 5.22.1199

The notation and terminology used in this paper have been introduced in the following articles: [8], [36], [24], [30], [1], [15], [21], [16], [25], [6], [9], [17], [37], [10], [11], [3], [34], [5], [12], [26], [33], [35], [41], [42], [13], [40], [19], [31], [28], [43], [18], [44], [29], and [14].

1. Closed Hypercube

From now on n, m, i denote natural numbers, p, q denote points of $\mathcal{E}^n_{\mathrm{T}}$, r, s denote real numbers, and R denotes a real-valued finite sequence.

Note that every finite sequence which is empty is also non-negative yielding.

[1]The paper has been financed by the resources of the Polish National Science Centre granted by decision no DEC-2012/07/N/ST6/02147.

ISSN 1426–2630(Print), 1898-9934(Online)

Let n be a non zero natural number, X be a set, and F be an element of ((the carrier of $\mathbb{R}^{\mathbf{1}})^X)^n$. Let us note that the functor $\prod^* F$ yields a function from X into $\mathcal{E}^n_{\mathrm{T}}$. Now we state the proposition:

(1) Let us consider sets X, Y, a function yielding function F, and objects x, y. Suppose

(i) F is (Y^X)-valued, or

(ii) $y \in \operatorname{dom} \prod^* F$.

Then $F(x)(y) = (\prod^* F)(y)(x)$.

Let us consider n, p, and r. The functor OpenHypercube(p, r) yielding an open subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined by

(Def. 1) There exists a point e of $\mathcal{E}^n$ such that

(i) $p = e$, and

(ii) $it =$ OpenHypercube(e, r).

Now we state the propositions:

(2) If $q \in$ OpenHypercube(p, r) and $s \in\,]p(i) - r, p(i) + r[$, then $q +\cdot (i, s) \in$ OpenHypercube(p, r). Proof: Consider e being a point of $\mathcal{E}^n$ such that $p = e$ and OpenHypercube$(p, r) =$ OpenHypercube(e, r). Set $I =$ Intervals(e, r). Set $q_3 = q +\cdot (i, s)$. For every object x such that $x \in \operatorname{dom} I$ holds $q_3(x) \in I(x)$ by [2, (9)], [7, (31), (32)]. □

(3) If $i \in \operatorname{Seg} n$, then $(\mathrm{PROJ}(n, i))^\circ($OpenHypercube$(p, r)) = \,]p(i) - r, p(i) + r[$. The theorem is a consequence of (2).

(4) $q \in$ OpenHypercube(p, r) if and only if for every i such that $i \in \operatorname{Seg} n$ holds $q(i) \in\,]p(i) - r, p(i) + r[$. The theorem is a consequence of (3).

Let us consider n, p, and R. The functor ClosedHypercube(p, R) yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined by

(Def. 2) $q \in it$ if and only if for every i such that $i \in \operatorname{Seg} n$ holds $q(i) \in [p(i) - R(i), p(i) + R(i)]$.

Now we state the propositions:

(5) If there exists i such that $i \in \operatorname{Seg} n \cap \operatorname{dom} R$ and $R(i) < 0$, then ClosedHypercube(p, R) is empty.

(6) If for every i such that $i \in \operatorname{Seg} n \cap \operatorname{dom} R$ holds $R(i) \geqslant 0$, then $p \in$ ClosedHypercube(p, R).

Let us consider n and p. Let R be a non-negative yielding real-valued finite sequence. One can check that ClosedHypercube(p, R) is non empty.

Let us consider R. Let us observe that ClosedHypercube(p, R) is convex and compact.

Now we state the propositions:

(7) If $i \in \operatorname{Seg} n$ and $q \in \text{ClosedHypercube}(p, R)$ and $r \in [p(i) - R(i), p(i) + R(i)]$, then $q +\cdot (i, r) \in \text{ClosedHypercube}(p, R)$. PROOF: Set $p_4 = q +\cdot (i, r)$. For every natural number j such that $j \in \operatorname{Seg} n$ holds $p_4(j) \in [p(j) - R(j), p(j) + R(j)]$ by [7, (32), (31)]. □

(8) Suppose $i \in \operatorname{Seg} n$ and ClosedHypercube(p, R) is not empty. Then $(\text{PROJ}(n, i))^\circ(\text{ClosedHypercube}(p, R)) = [p(i) - R(i), p(i) + R(i)]$. The theorem is a consequence of (5), (7), and (6).

(9) If $n \leqslant \operatorname{len} R$ and $r \leqslant \inf \operatorname{rng} R$, then $\text{OpenHypercube}(p, r) \subseteq \text{ClosedHypercube}(p, R)$.

(10) $q \in \operatorname{Fr} \text{ClosedHypercube}(p, R)$ if and only if $q \in \text{ClosedHypercube}(p, R)$ and there exists i such that $i \in \operatorname{Seg} n$ and $q(i) = p(i) - R(i)$ or $q(i) = p(i) + R(i)$. PROOF: Set $T_4 = \mathcal{E}^n_{\mathrm{T}}$. If $q \in \operatorname{Fr} \text{ClosedHypercube}(p, R)$, then $q \in \text{ClosedHypercube}(p, R)$ and there exists i such that $i \in \operatorname{Seg} n$ and $q(i) = p(i) - R(i)$ or $q(i) = p(i) + R(i)$ by [16, (22)], [32, (105)], [14, (33)], [6, (3)]. For every subset S of T_4 such that S is open and $q \in S$ holds ClosedHypercube(p, R) meets S and $(\text{ClosedHypercube}(p, R))^{\mathrm{c}}$ meets S by [16, (67)], [43, (23)], [38, (5)], [31, (13)]. □

(11) If $r \geqslant 0$, then $p \in \text{ClosedHypercube}(p, n \mapsto r)$.

(12) If $r > 0$, then $\operatorname{Int} \text{ClosedHypercube}(p, n \mapsto r) = \text{OpenHypercube}(p, r)$. PROOF: Set $O = \text{OpenHypercube}(p, r)$. Set $C = \text{ClosedHypercube}(p, n \mapsto r)$. Set $T_4 = \mathcal{E}^n_{\mathrm{T}}$. Set $R = n \mapsto r$. Consider e being a point of $\mathcal{E}^n$ such that $p = e$ and $\text{OpenHypercube}(p, r) = \text{OpenHypercube}(e, r)$. $\operatorname{Int} C \subseteq O$ by [43, (39)], [9, (57)], (10), [39, (29)]. Reconsider $q = x$ as a point of T_4. For every i such that $i \in \operatorname{Seg} n$ holds $q(i) \in [p(i) - R(i), p(i) + R(i)]$ by [9, (57)], (3). Consider i such that $i \in \operatorname{Seg} n$ and $q(i) = p(i) - R(i)$ or $q(i) = p(i) + R(i)$. $(\text{PROJ}(n, i))^\circ O =]e(i) - r, e(i) + r[$. □

(13) $\text{OpenHypercube}(p, r) \subseteq \text{ClosedHypercube}(p, n \mapsto r)$.

(14) If $r < s$, then $\text{ClosedHypercube}(p, n \mapsto r) \subseteq \text{OpenHypercube}(p, s)$. The theorem is a consequence of (4).

Let us consider n and p. Let r be a positive real number. Let us note that ClosedHypercube$(p, n \mapsto r)$ is non boundary.

2. PROPERTIES OF THE PRODUCT OF CLOSED HYPERCUBE

From now on T_1, T_2, S_1, S_2 denote non empty topological spaces, t_1 denotes a point of T_1, t_2 denotes a point of T_2, p_2, q_2 denote points of $\mathcal{E}^n_{\mathrm{T}}$, and p_1, q_1 denote points of $\mathcal{E}^m_{\mathrm{T}}$.

Now we state the propositions:

(15) Let us consider a function f from T_1 into T_2 and a function g from S_1 into S_2. Suppose

(i) f is a homeomorphism, and

(ii) g is a homeomorphism.

Then $f \times g$ is a homeomorphism.

(16) Suppose $r > 0$ and $s > 0$. Then there exists a function h from $(\mathcal{E}^n_{\rm T} \restriction \mathrm{ClosedHypercube}(p_2, n \mapsto r)) \times (\mathcal{E}^m_{\rm T} \restriction \mathrm{ClosedHypercube}(p_1, m \mapsto s))$ into $\mathcal{E}^{n+m}_{\rm T} \restriction \mathrm{ClosedHypercube}(0_{\mathcal{E}^{n+m}_{\rm T}}, (n+m) \mapsto 1)$ such that

(i) h is a homeomorphism, and

(ii) $h^\circ(\mathrm{OpenHypercube}(p_2, r) \times \mathrm{OpenHypercube}(p_1, s)) = \mathrm{OpenHypercube}(0_{\mathcal{E}^{n+m}_{\rm T}}, 1)$.

Proof: Set $T_6 = \mathcal{E}^n_{\rm T}$. Set $T_5 = \mathcal{E}^m_{\rm T}$. Set $n_1 = n+m$. Set $T_7 = \mathcal{E}^{n_1}_{\rm T}$. Set $R_2 = \mathrm{ClosedHypercube}(0_{T_6}, n \mapsto 1)$. Set $R_4 = \mathrm{ClosedHypercube}(p_2, n \mapsto r)$. Set $R_5 = \mathrm{ClosedHypercube}(p_1, m \mapsto s)$. Set $R_1 = \mathrm{ClosedHypercube}(0_{T_5}, m \mapsto 1)$. Set $R_3 = \mathrm{ClosedHypercube}(0_{T_7}, n_1 \mapsto 1)$. Reconsider $R_{10} = R_5$, $R_6 = R_1$ as a non empty subset of T_5. Consider h_3 being a function from $T_5 \restriction R_{10}$ into $T_5 \restriction R_6$ such that h_3 is a homeomorphism and $h_3{}^\circ(\mathrm{Fr}\, R_{10}) = \mathrm{Fr}\, R_6$. Reconsider $R_9 = R_4$, $R_7 = R_2$ as a non empty subset of T_6. Consider h_4 being a function from $T_6 \restriction R_9$ into $T_6 \restriction R_7$ such that h_4 is a homeomorphism and $h_4{}^\circ(\mathrm{Fr}\, R_9) = \mathrm{Fr}\, R_7$. Set $O_8 = \mathrm{OpenHypercube}(p_2, r)$. Set $O_9 = \mathrm{OpenHypercube}(p_1, s)$. Set $O_6 = \mathrm{OpenHypercube}(0_{T_7}, 1)$. $\mathrm{Int}\, R_{10} = O_9$. Set $O_5 = \mathrm{OpenHypercube}(0_{T_6}, 1)$. Set $O_7 = \mathrm{OpenHypercube}(0_{T_5}, 1)$. Reconsider $R_8 = R_3$ as a non empty subset of T_7. Consider f being a function from $T_6 \times T_5$ into T_7 such that f is a homeomorphism and for every element f_5 of T_6 and for every element f_6 of T_5, $f(f_5, f_6) = f_5 \frown f_6$. $f^\circ(R_7 \times R_6) \subseteq R_8$ by [14, (87)], [9, (57)], [6, (25)]. $R_8 \subseteq f^\circ(R_7 \times R_6)$ by [9, (23)], [27, (17)], [4, (11)], [6, (5)]. Set $h_5 = h_4 \times h_3$. h_5 is a homeomorphism. $\mathrm{Int}\, R_7 = O_5$. Reconsider $f_1 = f \restriction (R_7 \times R_6)$ as a function from $(T_6 \restriction R_7) \times (T_5 \restriction R_6)$ into $T_7 \restriction R_8$. Reconsider $h = f_1 \cdot h_5$ as a function from $(T_6 \restriction R_4) \times (T_5 \restriction R_5)$ into $T_7 \restriction R_3$. $\mathrm{Int}\, R_6 = O_7$. $\mathrm{Int}\, R_9 = O_8$. $h^\circ(O_8 \times O_9) \subseteq O_6$ by [14, (87)], [10, (12)], [43, (40)], [10, (49)]. Reconsider $p_3 = y$ as a point of T_7. Consider p, q being finite sequences of elements of $\mathbb{R}$ such that $\mathrm{len}\, p = n$ and $\mathrm{len}\, q = m$ and $p_3 = p \frown q$. $q \in O_7$. $q \in R_6$. Consider x_2 being an object such that $x_2 \in \mathrm{dom}\, h_3$ and $h_3(x_2) = q$. $p \in O_5$. $p \in R_7$. Consider x_1 being an object such that $x_1 \in \mathrm{dom}\, h_4$ and $h_4(x_1) = p$. □

(17) Suppose $r > 0$ and $s > 0$. Let us consider a function f from T_1 into $\mathcal{E}^n_{\rm T} \restriction \mathrm{ClosedHypercube}(p_2, n \mapsto r)$ and a function g from T_2 into $\mathcal{E}^m_{\rm T} \restriction \mathrm{ClosedHypercube}(p_1, m \mapsto s)$. Suppose

(i) f is a homeomorphism, and

(ii) g is a homeomorphism.

Then there exists a function h from $T_1 \times T_2$ into

$\mathcal{E}^{n+m}_{\mathrm{T}} \restriction \mathrm{ClosedHypercube}(0_{\mathcal{E}^{n+m}_{\mathrm{T}}}, (n+m) \mapsto 1)$ such that

(iii) h is a homeomorphism, and

(iv) for every t_1 and t_2, $f(t_1) \in \mathrm{OpenHypercube}(p_2, r)$ and $g(t_2) \in \mathrm{OpenHypercube}(p_1, s)$ iff $h(t_1, t_2) \in \mathrm{OpenHypercube}(0_{\mathcal{E}^{n+m}_{\mathrm{T}}}, 1)$.

Proof: Set $n_1 = n + m$. Set $T_6 = \mathcal{E}^n_{\mathrm{T}}$. Set $T_5 = \mathcal{E}^m_{\mathrm{T}}$. Set $T_7 = \mathcal{E}^{n_1}_{\mathrm{T}}$. Set $R_7 = n \mapsto r$. Set $R_6 = m \mapsto s$. Set $R_8 = n_1 \mapsto 1$. Set $R_4 = \mathrm{ClosedHypercube}(p_2, R_7)$. Set $R_5 = \mathrm{ClosedHypercube}(p_1, R_6)$. Set $C_2 = \mathrm{ClosedHypercube}(0_{T_7}, R_8)$. Reconsider $R_{10} = R_5$ as a non empty subset of T_5. Reconsider $R_9 = R_4$ as a non empty subset of T_6. Set $O_8 = \mathrm{OpenHypercube}(p_2, r)$. Set $O_9 = \mathrm{OpenHypercube}(p_1, s)$. Set $O = \mathrm{OpenHypercube}(0_{T_7}, 1)$. Consider h being a function from $(T_6 \restriction R_9) \times (T_5 \restriction R_{10})$ into $T_7 \restriction C_2$ such that h is a homeomorphism and $h^{\circ}(O_8 \times O_9) = O$. Reconsider $G = g$ as a function from T_2 into $T_5 \restriction R_{10}$. Reconsider $F = f$ as a function from T_1 into $T_6 \restriction R_9$. Reconsider $f_4 = h \cdot (F \times G)$ as a function from $T_1 \times T_2$ into $T_7 \restriction C_2$. $F \times G$ is a homeomorphism. $O_9 \subseteq R_{10}$. $O_8 \subseteq R_9$. If $f(t_1) \in O_8$ and $g(t_2) \in O_9$, then $f_4(t_1, t_2) \in O$ by [14, (87)], [10, (12)]. Consider x_3 being an object such that $x_3 \in \mathrm{dom}\, h$ and $x_3 \in O_8 \times O_9$ and $h(x_3) = h(\langle f(t_1), g(t_2) \rangle)$. $\square$

Let us consider n. One can check that there exists a subset of $\mathcal{E}^n_{\mathrm{T}}$ which is non boundary, convex, and compact.

Now we state the propositions:

(18) Let us consider a non boundary convex compact subset A of $\mathcal{E}^n_{\mathrm{T}}$, a non boundary convex compact subset B of $\mathcal{E}^m_{\mathrm{T}}$, a non boundary convex compact subset C of $\mathcal{E}^{n+m}_{\mathrm{T}}$, a function f from T_1 into $\mathcal{E}^n_{\mathrm{T}} \restriction A$, and a function g from T_2 into $\mathcal{E}^m_{\mathrm{T}} \restriction B$. Suppose

(i) f is a homeomorphism, and

(ii) g is a homeomorphism.

Then there exists a function h from $T_1 \times T_2$ into $\mathcal{E}^{n+m}_{\mathrm{T}} \restriction C$ such that

(iii) h is a homeomorphism, and

(iv) for every t_1 and t_2, $f(t_1) \in \mathrm{Int}\, A$ and $g(t_2) \in \mathrm{Int}\, B$ iff $h(t_1, t_2) \in \mathrm{Int}\, C$.

Proof: Set $T_6 = \mathcal{E}^n_{\mathrm{T}}$. Set $T_5 = \mathcal{E}^m_{\mathrm{T}}$. Set $n_1 = n+m$. Set $T_7 = \mathcal{E}^{n_1}_{\mathrm{T}}$. Set $R_7 = \mathrm{ClosedHypercube}(0_{T_6}, n \mapsto 1)$. Set $R_6 = \mathrm{ClosedHypercube}(0_{T_5}, m \mapsto 1)$. Set $R_8 = \mathrm{ClosedHypercube}(0_{T_7}, n_1 \mapsto 1)$. Consider g_1 being a function from $T_5 \restriction B$ into $T_5 \restriction R_6$ such that g_1 is a homeomorphism and $g_1{}^{\circ}(\mathrm{Fr}\, B) = \mathrm{Fr}\, R_6$. Reconsider $g_2 = g_1 \cdot g$ as a function from T_2 into $T_5 \restriction R_6$. Consider f_7 being a function from $T_6 \restriction A$ into $T_6 \restriction R_7$ such that f_7 is a homeomorphism and $f_7{}^{\circ}(\mathrm{Fr}\, A) = \mathrm{Fr}\, R_7$. Reconsider $f_8 = f_7 \cdot f$ as a function from T_1 into $T_6 \restriction R_7$. Set $O_3 = \mathrm{OpenHypercube}(0_{T_6}, 1)$. Set $O_2 = \mathrm{OpenHypercube}(0_{T_5}, 1)$. Set $O_4 = \mathrm{OpenHypercube}(0_{T_7}, 1)$. Consider H

being a function from $T_7 \restriction R_8$ into $T_7 \restriction C$ such that H is a homeomorphism and $H^{\circ}(\operatorname{Fr} R_8) = \operatorname{Fr} C$. $\operatorname{Int} R_6 = O_2$. Consider P being a function from $T_1 \times T_2$ into $T_7 \restriction R_8$ such that P is a homeomorphism and for every t_1 and t_2, $f_8(t_1) \in O_3$ and $g_2(t_2) \in O_2$ iff $P(t_1, t_2) \in O_4$. Reconsider $H_1 = H \cdot P$ as a function from $T_1 \times T_2$ into $T_7 \restriction C$. $\operatorname{Int} R_8 = O_4$. If $f(t_1) \in \operatorname{Int} A$ and $g(t_2) \in \operatorname{Int} B$, then $H_1(t_1, t_2) \in \operatorname{Int} C$ by [10, (11), (12)], (12). $P(\langle t_1, t_2 \rangle) \in \operatorname{Int} R_8$. $P(t_1, t_2) \in O_4$. $\operatorname{Int} R_7 = O_3$. $f(t_1) \in \operatorname{Int} A$ by [43, (40)]. □

(19) Let us consider a point p_2 of $\mathcal{E}^n_{\mathrm{T}}$, a point p_1 of $\mathcal{E}^m_{\mathrm{T}}$, r, and s. Suppose

(i) $r > 0$, and

(ii) $s > 0$.

Then there exists a function h from $\operatorname{Tdisk}(p_2, r) \times \operatorname{Tdisk}(p_1, s)$ into $\operatorname{Tdisk}(0_{\mathcal{E}^{n+m}_{\mathrm{T}}}, 1)$ such that

(iii) h is a homeomorphism, and

(iv) $h^{\circ}(\operatorname{Ball}(p_2, r) \times \operatorname{Ball}(p_1, s)) = \operatorname{Ball}(0_{\mathcal{E}^{n+m}_{\mathrm{T}}}, 1)$.

Proof: Set $T_6 = \mathcal{E}^n_{\mathrm{T}}$. Set $T_5 = \mathcal{E}^m_{\mathrm{T}}$. Set $n_1 = n + m$. Set $T_7 = \mathcal{E}^{n_1}_{\mathrm{T}}$. Reconsider $C_4 = \overline{\operatorname{Ball}}(p_2, r)$ as a non empty subset of T_6. Reconsider $C_3 = \overline{\operatorname{Ball}}(p_1, s)$ as a non empty subset of T_5. Reconsider $C_5 = \overline{\operatorname{Ball}}(0_{T_7}, 1)$ as a non empty subset of T_7. Set $R_7 = \operatorname{ClosedHypercube}(0_{T_6}, n \mapsto 1)$. Set $R_6 = \operatorname{ClosedHypercube}(0_{T_5}, m \mapsto 1)$. Consider f_7 being a function from $T_6 \restriction C_4$ into $T_6 \restriction R_7$ such that f_7 is a homeomorphism and $f_7{}^{\circ}(\operatorname{Fr} C_4) = \operatorname{Fr} R_7$. Consider g_1 being a function from $T_5 \restriction C_3$ into $T_5 \restriction R_6$ such that g_1 is a homeomorphism and $g_1{}^{\circ}(\operatorname{Fr} C_3) = \operatorname{Fr} R_6$. Consider P being a function from $\operatorname{Tdisk}(p_2, r) \times \operatorname{Tdisk}(p_1, s)$ into $\operatorname{Tdisk}(0_{T_7}, 1)$ such that P is a homeomorphism and for every point t_1 of $T_6 \restriction C_4$ and for every point t_2 of $T_5 \restriction C_3$, $f_7(t_1) \in \operatorname{Int} R_7$ and $g_1(t_2) \in \operatorname{Int} R_6$ iff $P(t_1, t_2) \in \operatorname{Int} C_5$. $P^{\circ}(\operatorname{Ball}(p_2, r) \times \operatorname{Ball}(p_1, s)) \subseteq \operatorname{Ball}(0_{T_7}, 1)$ by [30, (3)], [43, (40)]. Consider x being an object such that $x \in \operatorname{dom} P$ and $P(x) = y$. Consider y_1, y_2 being objects such that $y_1 \in C_4$ and $y_2 \in C_3$ and $x = \langle y_1, y_2 \rangle$. □

(20) Suppose $r > 0$ and $s > 0$ and T_1 and $\mathcal{E}^n_{\mathrm{T}} \restriction \operatorname{Ball}(p_2, r)$ are homeomorphic and T_2 and $\mathcal{E}^m_{\mathrm{T}} \restriction \operatorname{Ball}(p_1, s)$ are homeomorphic. Then $T_1 \times T_2$ and $\mathcal{E}^{n+m}_{\mathrm{T}} \restriction \operatorname{Ball}(0_{\mathcal{E}^{n+m}_{\mathrm{T}}}, 1)$ are homeomorphic.

3. Tietze Extension Theorem

In the sequel T, S denote topological spaces, A denotes a closed subset of T, and B denotes a subset of S.

Now we state the propositions:

(21) Let us consider a non zero natural number n and an element F of $((\text{the carrier of } \mathbb{R}^{\mathbf{1}})^{\alpha})^n$. Suppose If $i \in \operatorname{dom} F$, then for every function

h from T into $\mathbb{R}^{\mathbf{1}}$ such that $h = F(i)$ holds h is continuous. Then $\prod^* F$ is continuous, where α is the carrier of T. PROOF: Set $T_4 = \mathcal{E}^n_{\mathrm{T}}$. Set $F_1 = \prod^* F$. For every subset Y of T_4 such that Y is open holds $F_1{}^{-1}(Y)$ is open by [16, (67)], [11, (2)], (1), [19, (17)]. □

(22) Suppose T is normal. Let us consider a function f from $T \restriction A$ into $\mathcal{E}^n_{\mathrm{T}} \restriction \mathrm{ClosedHypercube}(0_{\mathcal{E}^n_{\mathrm{T}}}, n \mapsto 1)$. Suppose f is continuous. Then there exists a function g from T into $\mathcal{E}^n_{\mathrm{T}} \restriction \mathrm{ClosedHypercube}(0_{\mathcal{E}^n_{\mathrm{T}}}, n \mapsto 1)$ such that

(i) g is continuous, and

(ii) $g \restriction A = f$.

The theorem is a consequence of (8), (1), and (21).

(23) Suppose T is normal. Let us consider a subset X of $\mathcal{E}^n_{\mathrm{T}}$. Suppose X is compact, non boundary, and convex. Let us consider a function f from $T \restriction A$ into $\mathcal{E}^n_{\mathrm{T}} \restriction X$. Suppose f is continuous. Then there exists a function g from T into $\mathcal{E}^n_{\mathrm{T}} \restriction X$ such that

(i) g is continuous, and

(ii) $g \restriction A = f$.

The theorem is a consequence of (22).

Now we state the proposition:

(24) THE FIRST IMPLICATION OF TIETZE EXTENSION THEOREM FOR n-DIMENSIONAL SPACES:
Suppose T is normal. Let us consider a subset X of $\mathcal{E}^n_{\mathrm{T}}$. Suppose

(i) X is compact, non boundary, and convex, and

(ii) B and X are homeomorphic.

Let us consider a function f from $T \restriction A$ into $S \restriction B$. Suppose f is continuous. Then there exists a function g from T into $S \restriction B$ such that

(iii) g is continuous, and

(iv) $g \restriction A = f$.

The theorem is a consequence of (23).

Now we state the proposition:

(25) THE SECOND IMPLICATION OF TIETZE EXTENSION THEOREM FOR n-DIMENSIONAL SPACES:
Let us consider a non empty topological space T and n. Suppose

(i) $n \geqslant 1$, and

(ii) for every topological space S and for every non empty closed subset A of T and for every subset B of S such that there exists a subset X of $\mathcal{E}^n_{\mathrm{T}}$ such that X is compact, non boundary, and convex and B and

> X are homeomorphic for every function f from $T \restriction A$ into $S \restriction B$ such that f is continuous there exists a function g from T into $S \restriction B$ such that g is continuous and $g \restriction A = f$.

Then T is normal. PROOF: Set $C_1 = [-1, 1]_{\mathrm{T}}$. For every non empty closed subset A of T and for every continuous function f from $T \restriction A$ into C_1, there exists a continuous function g from T into $[-1, 1]_{\mathrm{T}}$ such that $g \restriction A = f$ by [19, (18), (17)], [11, (2)], [33, (26)]. □

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Received February 11, 2014

FORMALIZED MATHEMATICS
Vol. 22, No. 1, Pages 21–28, 2014
DOI: 10.2478/forma-2014-0003

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Brouwer Invariance of Domain Theorem[1]

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Summary. In this article we focus on a special case of the Brouwer invariance of domain theorem. Let us A, B be a subsets of $\mathcal{E}^n$, and $f : A \to B$ be a homeomorphic. We prove that, if A is closed then f transform the boundary of A to the boundary of B; and if B is closed then f transform the interior of A to the interior of B. These two cases are sufficient to prove the topological invariance of dimension, which is used to prove basic properties of the n-dimensional manifolds, and also to prove basic properties of the boundary and the interior of manifolds, e.g. the boundary of an n-dimension manifold with boundary is an $(n-1)$-dimension manifold. This article is based on [18]; [21] and [20] can also serve as reference books.

MSC: 54A05 03B35

Keywords: continuous transformations; topological dimension

MML identifier: BROUWER3, version: 8.1.02 5.22.1199

The notation and terminology used in this paper have been introduced in the following articles: [27], [1], [14], [4], [6], [15], [37], [7], [8], [40], [31], [34], [38], [2], [3], [9], [5], [33], [13], [44], [45], [10], [42], [43], [35], [17], [28], [29], [25], [46], [16], [47], [26], [30], [32], and [12].

1. Preliminaries

From now on x, X denote sets, n, m, i denote natural numbers, p, q denote points of $\mathcal{E}^n_{\mathrm{T}}$, A, B denote subsets of $\mathcal{E}^n_{\mathrm{T}}$, and r, s denote real numbers.

Let us consider X and n. One can verify that every function from X into $\mathcal{E}^n_{\mathrm{T}}$ is finite sequence-yielding.

[1]The paper has been financed by the resources of the Polish National Science Centre granted by decision no DEC-2012/07/N/ST6/02147.

ISSN 1426–2630(Print), 1898-9934(Online)

Let us consider m. Let f be a function from X into $\mathcal{E}^n_{\mathrm{T}}$ and g be a function from X into $\mathcal{E}^m_{\mathrm{T}}$. Let us observe that the functor $f \frown g$ yields a function from X into $\mathcal{E}^{n+m}_{\mathrm{T}}$. Let T be a topological space. Let f be a continuous function from T into $\mathcal{E}^n_{\mathrm{T}}$ and g be a continuous function from T into $\mathcal{E}^m_{\mathrm{T}}$. Note that $f \frown g$ is continuous as a function from T into $\mathcal{E}^{n+m}_{\mathrm{T}}$.

Let f be a real-valued function. The functor $|[f]|$ yielding a function is defined by

(Def. 1) (i) $\operatorname{dom} it = \operatorname{dom} f$, and

(ii) for every object x such that $x \in \operatorname{dom} it$ holds $it(x) = |[f(x)]|$.

One can verify that $|[f]|$ is (the carrier of $\mathcal{E}^1_{\mathrm{T}}$)-valued.

Let us consider X. Let Y be a non empty real-membered set and f be a function from X into Y. One can verify that the functor $|[f]|$ yields a function from X into $\mathcal{E}^1_{\mathrm{T}}$. Let T be a non empty topological space and f be a continuous function from T into $\mathbb{R}^{\mathbf{1}}$. Note that $|[f]|$ is continuous as a function from T into $\mathcal{E}^1_{\mathrm{T}}$.

Let f be a continuous real map of T. Observe that $|[f]|$ is continuous as a function from T into $\mathcal{E}^1_{\mathrm{T}}$.

2. A Distribution of Sphere

In the sequel N denotes a non zero natural number and u, t denote points of $\mathcal{E}^{N+1}_{\mathrm{T}}$.

Now we state the propositions:

(1) Let us consider an element F of ((the carrier of $\mathbb{R}^{\mathbf{1}})^{\alpha})^N$. Suppose If $i \in \operatorname{dom} F$, then $F(i) = \mathrm{PROJ}(N+1, i)$. Then

(i) for every t, $(\prod^* F)(t) = t{\upharpoonright}N$, and

(ii) for every subsets S_3, S_2 of $\mathcal{E}^{N+1}_{\mathrm{T}}$ such that $S_3 = \{u : u(N+1) \geqslant 0$ and $|u| = 1\}$ and $S_2 = \{t : t(N+1) \leqslant 0$ and $|t| = 1\}$ holds $(\prod^* F)^{\circ}S_3 = \overline{\mathrm{Ball}}(0_{\mathcal{E}^N_{\mathrm{T}}}, 1)$ and $(\prod^* F)^{\circ}S_2 = \overline{\mathrm{Ball}}(0_{\mathcal{E}^N_{\mathrm{T}}}, 1)$ and $(\prod^* F)^{\circ}(S_3 \cap S_2) = \mathrm{Sphere}(0_{\mathcal{E}^N_{\mathrm{T}}}, 1)$ and for every function H from $\mathcal{E}^{N+1}_{\mathrm{T}}{\upharpoonright}S_3$ into $\mathrm{Tdisk}(0_{\mathcal{E}^N_{\mathrm{T}}}, 1)$ such that $H = \prod^* F{\upharpoonright}S_3$ holds H is a homeomorphism and for every function H from $\mathcal{E}^{N+1}_{\mathrm{T}}{\upharpoonright}S_2$ into $\mathrm{Tdisk}(0_{\mathcal{E}^N_{\mathrm{T}}}, 1)$ such that $H = \prod^* F{\upharpoonright}S_2$ holds H is a homeomorphism,

where α is the carrier of $\mathcal{E}^{N+1}_{\mathrm{T}}$. Proof: Set $N_2 = N+1$. Set $T_{10} = \mathcal{E}^{N_2}_{\mathrm{T}}$. Set $T_4 = \mathcal{E}^N_{\mathrm{T}}$. Set $N_3 = N\,\mathrm{NormF}$. Set $N_4 = N_3 \cdot N_3$. Reconsider $O = 1$ as an element of $\mathbb{N}$. Set $T_3 = \mathrm{Tdisk}(0_{\mathcal{E}^N_{\mathrm{T}}}, 1)$. Reconsider $m_2 = -N_4$ as a function from T_4 into $\mathbb{R}^{\mathbf{1}}$. Reconsider $m_1 = 1 + m_2$ as a function from T_4 into $\mathbb{R}^{\mathbf{1}}$. Set $F_1 = \prod^* F$. For every t, $(\prod^* F)(t) = t{\upharpoonright}N$ by [2, (13)], [41, (25)], [4, (1)]. $\overline{\mathrm{Ball}}(0_{T_4}, 1) \subseteq F_1{}^{\circ}S_3$ by [14, (22)], [28, (11)], [6, (16)], [11, (145)]. $\overline{\text{Ball}}(0_{T_4}, 1) \subseteq F_1{}^\circ S_2$ by [14, (22)], [28, (11)], [6, (16)], [11, (145)]. $\text{Sphere}(0_{T_4}, 1) \subseteq F_1{}^\circ(S_2 \cap S_3)$ by [14, (22)], [28, (12)], [6, (16), (92)]. $F_1{}^\circ S_3 \subseteq \overline{\text{Ball}}(0_{T_4}, 1)$ by [14, (22)], [4, (59)], [24, (17)], [19, (10)]. $F_1{}^\circ S_2 \subseteq \overline{\text{Ball}}(0_{T_4}, 1)$ by [14, (22)], [4, (59)], [24, (17)], [19, (10)]. $F_1{}^\circ(S_2 \cap S_3) \subseteq \text{Sphere}(0_{T_4}, 1)$ by [14, (22)], [4, (59)], [24, (17)], [19, (10)]. For every function H from $\mathcal{E}_{\mathrm{T}}^{N+1} \restriction S_3$ into $\text{Tdisk}(0_{\mathcal{E}_{\mathrm{T}}^N}, 1)$ such that $H = \prod^* F \restriction S_3$ holds H is a homeomorphism by [24, (17)], [17, (17)], [2, (11)], [25, (13)]. For every objects x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} H$ and $H(x_1) = H(x_2)$ holds $x_1 = x_2$ by [14, (22)], [19, (10)], [7, (47)], [39, (40)]. Set $T_3 = \text{Tdisk}(0_{T_4}, 1)$. Set $M = m_1 \restriction T_3$. Reconsider $M_1 = M$ as a continuous function from T_3 into $\mathbb{R}$. Reconsider $M_2 = -\sqrt{M_1}$ as a function from T_3 into $\mathbb{R}$. For every point p of T_4 such that $p \in$ the carrier of T_3 holds $M_1(p) = 1 - |p| \cdot |p|$ by [7, (49)]. Reconsider $S_1 = |[M_2]|$ as a continuous function from T_3 into $\mathcal{E}_{\mathrm{T}}^1$. Reconsider $I_3 = \text{id}_{T_3}$ as a continuous function from T_3 into T_4. Reconsider $I_4 = I_3 {}^\frown S_1$ as a continuous function from T_3 into $\mathcal{E}_{\mathrm{T}}^{N+O}$. For every objects y, x, $y \in \operatorname{rng} H$ and $x = I_4(y)$ iff $x \in \operatorname{dom} H$ and $y = H(x)$ by [7, (17)], [11, (145), (144), (55)]. For every subset P of $T_{10} \restriction S_2$, P is open iff $H^\circ P$ is open by [4, (1)], [2, (13)], [25, (57)]. □

(2) Let us consider subsets S_3, S_2 of $\mathcal{E}_{\mathrm{T}}^n$. Suppose

(i) $S_3 = \{s$, where s is a point of $\mathcal{E}_{\mathrm{T}}^n : s(n) \geqslant 0$ and $|s| = 1\}$, and

(ii) $S_2 = \{t$, where t is a point of $\mathcal{E}_{\mathrm{T}}^n : t(n) \leqslant 0$ and $|t| = 1\}$.

Then

(iii) S_3 is closed, and

(iv) S_2 is closed.

(3) Let us consider a metrizable topological space T_2. Suppose T_2 is finite-ind and second-countable. Let us consider a closed subset F of T_2. Suppose $\operatorname{ind} F^{\mathrm{c}} \leqslant n$. Let us consider a continuous function f from $T_2 \restriction F$ into $\text{TopUnitCircle}(n + 1)$. Then there exists a continuous function g from T_2 into $\text{TopUnitCircle}(n + 1)$ such that $g \restriction F = f$. PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every metrizable topological space T_2 such that T_2 is finite-ind and second-countable for every closed subset F of T_2 such that $\operatorname{ind} F^{\mathrm{c}} \leqslant \$_1$ for every continuous function f from $T_2 \restriction F$ into $\text{TopUnitCircle}(\$_1 + 1)$, there exists a function g from T_2 into TopUnitCircle $(\$_1 + 1)$ such that g is continuous and $g \restriction F = f$. For every n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n + 1]$ by (2), [29, (9)], [42, (13)], [44, (121)]. $\mathcal{P}[(0$ **qua** natural number$)]$ by [44, (143), (135)], [29, (9)], [14, (70)]. For every n, $\mathcal{P}[n]$ from [2, Sch. 2]. □

(4) Suppose $p \notin A$ and $r > 0$. Then there exists a function h from $\mathcal{E}_{\mathrm{T}}^n \restriction A$ into $\mathcal{E}_{\mathrm{T}}^n \restriction \text{Sphere}(p, r)$ such that

(i) h is continuous, and

(ii) $h{\restriction}\operatorname{Sphere}(p,r) = \mathrm{id}_{A\cap\operatorname{Sphere}(p,r)}$.

(5) If $r + |p - q| \leqslant s$, then $\operatorname{Ball}(p,r) \subseteq \operatorname{Ball}(q,s)$.

(6) If A is not boundary, then $\operatorname{ind} A = n$.

Now we state the proposition:

(7) THE SMALL INDUCTIVE DIMENSION OF THE SPHERE:
If $r > 0$, then $\operatorname{ind}\operatorname{Sphere}(p,r) = n - 1$. PROOF: If $\operatorname{ind} A \leqslant i$ and $\operatorname{ind} B \leqslant i$ and A is closed, then $\operatorname{ind}(A \cup B) \leqslant i$ by [33, (31)], [23, (93)], [35, (22)], [36, (5)]. □

3. A Characterization of Open Sets in Euclidean Space in Terms of Continuous Transformations

Now we state the propositions:

(8) Suppose $n > 0$ and $p \in A$ and for every r such that $r > 0$ there exists an open subset U of $\mathcal{E}^n_{\mathrm{T}}{\restriction}A$ such that $p \in U$ and $U \subseteq \operatorname{Ball}(p,r)$ and for every function f from $\mathcal{E}^n_{\mathrm{T}}{\restriction}(A \setminus U)$ into TopUnitCircle n such that f is continuous there exists a function h from $\mathcal{E}^n_{\mathrm{T}}{\restriction}A$ into TopUnitCircle n such that h is continuous and $h{\restriction}(A \setminus U) = f$. Then $p \in \operatorname{Fr} A$. PROOF: Set $T_7 = \mathcal{E}^n_{\mathrm{T}}$. Set $c_1 =$ the carrier of T_7. Set $S = \operatorname{Sphere}(0_{T_7}, 1)$. Set $T_9 =$ TopUnitCircle n. Reconsider $c = c_1 \setminus \{0_{T_7}\}$ as a non empty open subset of T_7. Set $n_3 = n$ NormF. Set $T_8 = T_7{\restriction}c$. Set $G = \operatorname{transl}(p, T_7)$. Reconsider $I = \overset{T_8}{\hookrightarrow}$ as a continuous function from T_8 into T_7. $0 \notin \operatorname{rng}(n_3{\restriction}T_8)$ by [44, (57)], [14, (22)], [7, (47)], [14, (8), (70)]. Reconsider $n_2 = n_3{\restriction}T_8$ as a non-empty continuous function from T_8 into $\mathbb{R}^{\mathbf{1}}$. Reconsider $b = I/n_2$ as a function from T_8 into T_7. Set $E_1 = \mathcal{E}^n$. Set $T_2 = E_{1\mathrm{top}}$. Reconsider $e = p$ as a point of E_1. Reconsider $I_1 = \operatorname{Int} A$ as a subset of T_2. Consider r being a real number such that $r > 0$ and $\operatorname{Ball}(e,r) \subseteq I_1$. Set $r_2 = \frac{r}{2}$. Consider U being an open subset of $T_7{\restriction}A$ such that $p \in U$ and $U \subseteq \operatorname{Ball}(p, r_2)$ and for every function f from $T_7{\restriction}(A \setminus U)$ into T_9 such that f is continuous there exists a function h from $T_7{\restriction}A$ into T_9 such that h is continuous and $h{\restriction}(A \setminus U) = f$. Reconsider $S_4 = \operatorname{Sphere}(p, r_2)$ as a non empty subset of T_7. Consider a being an object such that $a \in S_4$. Reconsider $C_2 = \overline{\operatorname{Ball}}(p, r_2)$ as a non empty subset of T_7. Reconsider $s_2 = S_4$ as a non empty subset of $T_7{\restriction}C_2$. Reconsider $A_1 = A \setminus U$ as a non empty subset of T_7. Set $T_1 = T_7{\restriction}A_1$. Set $t = \operatorname{transl}(-p, T_7)$. Set $T = t{\restriction}T_1$. $\operatorname{rng} T \subseteq c$ by [7, (47)], [42, (21)]. Reconsider $T_1 1 = T$ as a continuous function from T_1 into T_8. For every point p of T_7 such that $p \in c$ holds $b(p) = \frac{1}{|p|} \cdot p$ and $|\frac{1}{|p|} \cdot p| = 1$ by [22, (84)], [7, (49)], [26, (72)], [12, (56)]. $\operatorname{rng} b \subseteq S$ by [42, (13)]. Reconsider $B = b$ as a function from T_8 into T_9. Set $m = r_2 \bullet T_7$. Set $M = m{\restriction}T_9$. Reconsider $M = m{\restriction}T_9$ as a continuous function from T_9 into T_7. Reconsider $c_2 = C_2$ as a subset of $T_7{\restriction}A$. Consider h being a function from $T_7{\restriction}A$ into T_9 such

that h is continuous and $h{\restriction}(A \setminus U) = B \cdot T_1 1$. Reconsider $G_2 = G \cdot (M \cdot h)$ as a continuous function from $T_7{\restriction}A$ into T_7. $\operatorname{rng} G_2 \subseteq S_4$ by [7, (12), (11), (47)], [42, (28), (15)]. Reconsider $g_2 = G_2$ as a function from $T_7{\restriction}A$ into $T_7{\restriction}S_4$. Reconsider $g_1 = g_2{\restriction}((T_7{\restriction}A){\restriction}c_2)$ as a continuous function from $T_7{\restriction}C_2$ into $(T_7{\restriction}C_2){\restriction}s_2$. For every point w of $T_7{\restriction}C_2$ such that $w \in S_4$ holds $g_1(w) = w$ by [7, (11), (12)], [44, (61)], [7, (47)]. □

(9) Suppose $p \in \operatorname{Fr} A$ and A is closed. Suppose $r > 0$. Then there exists an open subset U of $\mathcal{E}^n_{\mathrm{T}}{\restriction}A$ such that

  (i) $p \in U$, and

  (ii) $U \subseteq \operatorname{Ball}(p, r)$, and

  (iii) for every function f from $\mathcal{E}^n_{\mathrm{T}}{\restriction}(A \setminus U)$ into TopUnitCircle n such that f is continuous there exists a function h from $\mathcal{E}^n_{\mathrm{T}}{\restriction}A$ into TopUnitCircle n such that h is continuous and $h{\restriction}(A \setminus U) = f$.

Proof: $n > 0$ by [14, (77), (22)], [12, (33)]. Set $r_3 = \frac{r}{3}$. Set $r_2 = 2 \cdot r_3$. Set $B = \operatorname{Ball}(p, r_3)$. Consider x being an object such that $x \in A^{\mathrm{c}}$ and $x \in B$. Set $u = \operatorname{Ball}(x, r_2)$. $u \subseteq \operatorname{Ball}(p, r)$. □

4. Brouwer Invariance of Domain Theorem – Special Case

Let us consider a function h from $\mathcal{E}^n_{\mathrm{T}}{\restriction}A$ into $\mathcal{E}^n_{\mathrm{T}}{\restriction}B$. Now we state the propositions:

(10) If A is closed and $p \in \operatorname{Fr} A$, then if h is a homeomorphism, then $h(p) \in \operatorname{Fr} B$. The theorem is a consequence of (9) and (8).

(11) If B is closed and $p \in \operatorname{Int} A$, then if h is a homeomorphism, then $h(p) \in \operatorname{Int} B$. The theorem is a consequence of (8) and (9).

(12) Suppose A is closed and B is closed. Then if h is a homeomorphism, then $h^{\circ}(\operatorname{Int} A) = \operatorname{Int} B$ and $h^{\circ}(\operatorname{Fr} A) = \operatorname{Fr} B$. Proof: $h^{\circ}(\operatorname{Int} A) = \operatorname{Int} B$ by (11), (10), [46, (39)]. □

5. Topological Invariance of Dimension – An Introduction to Manifolds

Now we state the proposition:

(13) Suppose $r > 0$. Let us consider a subset U of $\operatorname{Tdisk}(p, r)$. Suppose U is open and non empty. Let us consider a subset A of $\mathcal{E}^n_{\mathrm{T}}$. If $A = U$, then $\operatorname{Int} A$ is not empty.

Let us consider a non empty topological space T, subsets A, B of T, r, s, a point p_1 of $\mathcal{E}^n_{\mathrm{T}}$, and a point p_2 of $\mathcal{E}^m_{\mathrm{T}}$.

Let us assume that $r > 0$ and $s > 0$. Now we state the propositions:

(14) Suppose $T \restriction A$ and $\mathrm{Tdisk}(p_1, r)$ are homeomorphic and $T \restriction B$ and $\mathrm{Tdisk}(p_2, s)$ are homeomorphic and $\mathrm{Int}\, A$ meets $\mathrm{Int}\, B$. Then $n = m$. The theorem is a consequence of (13) and (6).

(15) Suppose $T \restriction A$ and $\mathcal{E}^n_{\mathrm{T}} \restriction \mathrm{Ball}(p_1, r)$ are homeomorphic and $T \restriction B$ and $\mathrm{Tdisk}(p_2, s)$ are homeomorphic and $\mathrm{Int}\, A$ meets $\mathrm{Int}\, B$. Then $n = m$. The theorem is a consequence of (13) and (6).

Now we state the propositions:

(16) (i) $(\mathrm{transl}(p, \mathcal{E}^n_{\mathrm{T}}))^\circ(\mathrm{Ball}(q, r)) = \mathrm{Ball}(q + p, r)$, and

(ii) $(\mathrm{transl}(p, \mathcal{E}^n_{\mathrm{T}}))^\circ(\overline{\mathrm{Ball}}(q, r)) = \overline{\mathrm{Ball}}(q + p, r)$, and

(iii) $(\mathrm{transl}(p, \mathcal{E}^n_{\mathrm{T}}))^\circ(\mathrm{Sphere}(q, r)) = \mathrm{Sphere}((q + p), r)$.

Proof: Set $T_5 = \mathcal{E}^n_{\mathrm{T}}$. Set $T = \mathrm{transl}(p, T_5)$. $T^\circ(\mathrm{Ball}(q, r)) = \mathrm{Ball}(q + p, r)$ by [28, (7)], [42, (27)]. $T^\circ(\overline{\mathrm{Ball}}(q, r)) = \overline{\mathrm{Ball}}(q + p, r)$ by [28, (8)], [42, (27)]. $T^\circ(\mathrm{Sphere}(q, r)) \subseteq \mathrm{Sphere}((q + p), r)$ by [28, (9)]. □

(17) Suppose $s > 0$. Then

(i) $(s \bullet \mathcal{E}^n_{\mathrm{T}})^\circ(\mathrm{Ball}(p, r)) = \mathrm{Ball}(s \cdot p, r \cdot s)$, and

(ii) $(s \bullet \mathcal{E}^n_{\mathrm{T}})^\circ(\overline{\mathrm{Ball}}(p, r)) = \overline{\mathrm{Ball}}(s \cdot p, r \cdot s)$, and

(iii) $(s \bullet \mathcal{E}^n_{\mathrm{T}})^\circ(\mathrm{Sphere}(p, r)) = \mathrm{Sphere}((s \cdot p), (r \cdot s))$.

Proof: Set $T_5 = \mathcal{E}^n_{\mathrm{T}}$. Set $M = s \bullet T_5$. $M^\circ(\mathrm{Ball}(p, r)) = \mathrm{Ball}(s \cdot p, r \cdot s)$ by [42, (34)], [14, (11)], [28, (7)]. $M^\circ(\overline{\mathrm{Ball}}(p, r)) = \overline{\mathrm{Ball}}(s \cdot p, r \cdot s)$ by [42, (34)], [14, (11)], [28, (8)]. $M^\circ(\mathrm{Sphere}(p, r)) \subseteq \mathrm{Sphere}((s \cdot p), (r \cdot s))$ by [42, (34)], [14, (11)], [28, (9)]. □

(18) Let us consider a rotation homogeneous additive function f from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$. Suppose f is onto. Then

(i) $f^\circ(\mathrm{Ball}(p, r)) = \mathrm{Ball}(f(p), r)$, and

(ii) $f^\circ(\overline{\mathrm{Ball}}(p, r)) = \overline{\mathrm{Ball}}(f(p), r)$, and

(iii) $f^\circ(\mathrm{Sphere}(p, r)) = \mathrm{Sphere}((f(p)), r)$.

Proof: $f^\circ(\mathrm{Ball}(p, r)) = \mathrm{Ball}(f(p), r)$ by [28, (7)]. $f^\circ(\overline{\mathrm{Ball}}(p, r)) = \overline{\mathrm{Ball}}(f(p), r)$ by [28, (8)]. $f^\circ(\mathrm{Sphere}(p, r)) \subseteq \mathrm{Sphere}((f(p)), r)$ by [28, (9)]. Consider x being an object such that $x \in \mathrm{dom}\, f$ and $f(x) = y$. □

(19) Let us consider points p, q of $\mathcal{E}^{n+1}_{\mathrm{T}}$, r, and s. Suppose

(i) $s \leqslant r \leqslant |p - q|$, and

(ii) $s < |p - q| < s + r$.

Then there exists a function h from $\mathcal{E}^{n+1}_{\mathrm{T}} \restriction (\mathrm{Sphere}(p, r) \cap \overline{\mathrm{Ball}}(q, s))$ into $\mathrm{Tdisk}(0_{\mathcal{E}^n_{\mathrm{T}}}, 1)$ such that

(iii) h is a homeomorphism, and

(iv) $h^\circ(\mathrm{Sphere}(p, r) \cap \mathrm{Sphere}(q, s)) = \mathrm{Sphere}(0_{\mathcal{E}^n_{\mathrm{T}}}, 1)$.

Proof: Set $n_1 = n + 1$. Set $T_6 = \mathcal{E}^{n_1}_{\mathrm{T}}$. Set $y = \frac{1}{r} \cdot (q - p)$. Set $Y = \langle \underbrace{0, \ldots, 0}_{n_1} \rangle +\cdot (n_1, |y|)$. There exists a homogeneous additive rotation function R from T_6 into T_6 such that R is a homeomorphism and $R(y) = Y$ by [34, (40), (41)]. Consider R being a homogeneous additive rotation function from T_6 into T_6 such that R is a homeomorphism and $R(y) = Y$. $s > 0$. □

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Received February 11, 2014

FORMALIZED MATHEMATICS
Vol. 22, No. 1, Pages 29–35, 2014
DOI: 10.2478/forma-2014-0004

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The Formalization of Decision-Free Petri Net

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Summary. In this article we formalize the definition of Decision-Free Petri Net (DFPN) presented in [19]. Then we formalize the concept of directed path and directed circuit nets in Petri nets to prove properties of DFPN. We also present the definition of firing transitions and transition sequences with natural numbers marking that always check whether transition is enabled or not and after firing it only removes the available tokens (i.e., it does not remove from zero number of tokens). At the end of this article, we show that the total number of tokens in a circuit of decision-free Petri net always remains the same after firing any sequences of the transition.

MSC: 68Q60 68Q85 03B35

Keywords: specification and verification of discrete systems; Petri net;

MML identifier: PETRI_DF, version: 8.1.02 5.22.1199

The notation and terminology used in this paper have been introduced in the following articles: [1], [4], [17], [14], [8], [5], [6], [15], [12], [3], [9], [10], [20], [11], [13], [18], and [7].

1. Preliminaries

From now on N denotes a place/transition net structure, P denotes a Petri net, and i denotes a natural number.

[1]This work was supported in part by JSPS Kakenhi grant number 2230028502.
[2]This work was supported in part by the University of Bialystok grant BST225 *Database of mathematical texts checked by computer*.

ISSN 1426–2630(Print), 1898-9934(Online)

Now we state the propositions:

(1) Let us consider natural numbers x, y and a finite sequence f. Suppose

(i) $f_{\restriction 1}$ is one-to-one, and

(ii) $1 < x \leqslant \operatorname{len} f$, and

(iii) $1 < y \leqslant \operatorname{len} f$, and

(iv) $f(x) = f(y)$.

Then $x = y$.

(2) Let us consider a non empty set D and a non empty finite sequence f of elements of D. If f is circular, then $f(1) = f(\operatorname{len} f)$.

Let D be a non empty set and a, b be elements of D. Let us observe that $\langle a, b, a\rangle$ is circular as a finite sequence of elements of D.

Now we state the proposition:

(3) Let us consider objects a, b. If $a \neq b$, then $\langle a, b, a\rangle$ is almost one-to-one.

Let X be a set, Y be a non empty set, P_1 be a finite subset of X, and M_1 be a function from X into Y.

An enumeration of M_1 and P_1 is a finite sequence of elements of Y and is defined by

(Def. 1) (i) $\operatorname{len} it = \operatorname{len}$ the enumeration of P_1 and for every i such that $i \in \operatorname{dom} it$ holds $it(i) = M_1$(the enumeration of $P_1(i)$), **if** P_1 is not empty,

(ii) $it = \varepsilon_Y$, **otherwise**.

The functor PN_0 yielding a Petri net is defined by the term

(Def. 2) $\langle \{0\}, \{1\}, \Omega_{\{1\}}(\{0\}), \Omega_{\{0\}}(\{1\})\rangle$.

Let us consider N. We introduce the places and transitions of N as a synonym of Elements(N).

Let us consider P. Let us note that the places and transitions of P is non empty.

In the sequel f_1 denotes a finite sequence of elements of the places and transitions of P.

Let us consider P and f_1. The functors: the places of f_1 and the transitions of f_1 yielding finite subsets of P are defined by terms,

(Def. 3) $\{p$, where p is a place of $P : p \in \operatorname{rng} f_1\}$,

(Def. 4) $\{t$, where t is a transition of $P : t \in \operatorname{rng} f_1\}$,

respectively.

2. The Number of Tokens in a Circuit

Let us consider N. The markings of N yielding a non empty set of functions from the carrier of N to $\mathbb{N}$ is defined by the term

(Def. 5) $\mathbb{N}^{\alpha}$, where α is the carrier of N.

A marking of N is an element of the markings of N. Let P_1 be a finite subset of N and M_1 be a marking of N. The number of tokens of P_1 and M_1 yielding an element of $\mathbb{N}$ is defined by the term

(Def. 6) $\sum$ the enumeration of M_1 and P_1.

3. Decision-Free Petri Net Concept and Properties of Circuits in Petri Nets

Let I be a Petri net. We say that I is decision-free-like if and only if

(Def. 7) Let us consider a place s of I. Then

- (i) there exists a transition t of I such that $\langle t, s\rangle \in$ the T-S arcs of I, and
- (ii) for every transitions t_1, t_2 of I such that $\langle t_1, s\rangle$, $\langle t_2, s\rangle \in$ the T-S arcs of I holds $t_1 = t_2$, and
- (iii) there exists a transition t of I such that $\langle s, t\rangle \in$ the S-T arcs of I, and
- (iv) for every transitions t_1, t_2 of I such that $\langle s, t_1\rangle$, $\langle s, t_2\rangle \in$ the S-T arcs of I holds $t_1 = t_2$.

Let us consider P. Let I be a finite sequence of elements of the places and transitions of P. We say that I is directed path if and only if

(Def. 8)
- (i) $\operatorname{len} I \geqslant 3$, and
- (ii) $\operatorname{len} I \bmod 2 = 1$, and
- (iii) for every i such that $i \bmod 2 = 1$ and $i + 1 < \operatorname{len} I$ holds $\langle I(i), I(i+1)\rangle \in$ the S-T arcs of P and $\langle I(i+1), I(i+2)\rangle \in$ the T-S arcs of P, and
- (iv) $I(\operatorname{len} I) \in$ the carrier of P.

Now we state the proposition:

(4) Let us consider a finite sequence f_1 of elements of the places and transitions of PN_0. Suppose $f_1 = \langle 0, 1, 0\rangle$. Then f_1 is directed path. Proof: f_1 is directed path by [2, (13)], [4, (45)]. □

Let us consider P. Observe that every finite sequence of elements of the places and transitions of P which is directed path is also non empty.

Let I be a Petri net. We say that I has directed path if and only if

(Def. 9) There exists a finite sequence f_1 of elements of the places and transitions of I such that f_1 is directed path.

Let us consider P. We say that P has directed circuit if and only if

(Def. 10) There exists f_1 such that f_1 is directed path, circular, and almost one-to-one.

One can verify that PN_0 is decision-free-like and Petri-like and has directed circuit and there exists a Petri net which is Petri-like and decision-free-like and has directed circuit and every Petri net which has directed circuit has also directed path and there exists a Petri net which has directed path.

Let D_1 be a Petri net with directed path. Let us note that there exists a finite sequence of elements of the places and transitions of D_1 which is directed path.

From now on D_1 denotes a Petri net with directed path and d denotes a directed path finite sequence of elements of the places and transitions of D_1.

Now we state the propositions:

(5) $\langle d(1), d(2) \rangle \in$ the S-T arcs of D_1.

(6) $\langle d(\mathrm{len}\, d - 1), d(\mathrm{len}\, d) \rangle \in$ the T-S arcs of D_1.

From now on D_1 denotes a Petri-like Petri net with directed path and d denotes a directed path finite sequence of elements of the places and transitions of D_1.

Now we state the proposition:

(7) If $d(i) \in$ the places of d and $i \in \mathrm{dom}\, d$, then $i \bmod 2 = 1$. PROOF: Consider p being a place of D_1 such that $p = d(i)$ and $p \in \mathrm{rng}\, d$. $i \bmod 2 = 1$ by [2, (21)], [16, (25)], [7, (87)]. □

Let us assume that $d(i) \in$ the transitions of d and $i \in \mathrm{dom}\, d$. Now we state the propositions:

(8) $i \bmod 2 = 0$. PROOF: $\langle d(\mathrm{len}\, d - 1), d(\mathrm{len}\, d) \rangle \in$ the T-S arcs of D_1. Consider t being a transition of D_1 such that $t = d(i)$ and $t \in \mathrm{rng}\, d$. $i \neq \mathrm{len}\, d$ by [7, (87)]. $i + 1 \neq \mathrm{len}\, d$ by [7, (87)], [2, (11)], [16, (25)], [5, (3)]. □

(9) (i) $\langle d(i - 1), d(i) \rangle \in$ the S-T arcs of D_1, and

(ii) $\langle d(i), d(i + 1) \rangle \in$ the T-S arcs of D_1.

PROOF: $\langle d(\mathrm{len}\, d - 1), d(\mathrm{len}\, d) \rangle \in$ the T-S arcs of D_1. Consider t being a transition of D_1 such that $t = d(i)$ and $t \in \mathrm{rng}\, d$. $i \neq \mathrm{len}\, d$ by [7, (87)]. □

Now we state the proposition:

(10) Suppose $d(i) \in$ the places of d and $1 < i < \mathrm{len}\, d$. Then

(i) $\langle d(i - 2), d(i - 1) \rangle \in$ the S-T arcs of D_1, and

(ii) $\langle d(i - 1), d(i) \rangle \in$ the T-S arcs of D_1, and

(iii) $\langle d(i), d(i + 1) \rangle \in$ the S-T arcs of D_1, and

(iv) $\langle d(i+1), d(i+2)\rangle \in$ the T-S arcs of D_1, and

(v) $3 \leqslant i$.

PROOF: $i \bmod 2 = 1$. $\langle d(\operatorname{len} d - 1), d(\operatorname{len} d)\rangle \in$ the T-S arcs of D_1. $\langle d(1), d(2)\rangle \in$ the S-T arcs of D_1. Consider p being a place of D_1 such that $p = d(i)$ and $p \in \operatorname{rng} d$. $i + 1 \neq \operatorname{len} d$ by [7, (87)]. $2 \neq i$ by [7, (87)]. □

4. Firable and Firing Conditions for Transitions and Transition Sequences with Natural Marking

From now on M_1 denotes a marking of P, t denotes a transition of P, and Q, Q_1 denote finite sequences of elements of the carrier' of P.

Let us consider P, M_1, and t. We say that t is firable at M_1 if and only if

(Def. 11) Let us consider a natural number m. If $m \in M_1{}^{\circ}({}^{*}\{t\})$, then $m > 0$.

The functor Firing(t, M_1) yielding a marking of P is defined by

(Def. 12) (i) for every place s of P, if $s \in {}^{*}\{t\}$ and $s \notin \overline{\{t\}}$, then $it(s) = M_1(s) - 1$ and if $s \in \overline{\{t\}}$ and $s \notin {}^{*}\{t\}$, then $it(s) = M_1(s) + 1$ and if $s \in {}^{*}\{t\}$ and $s \in \overline{\{t\}}$ or $s \notin {}^{*}\{t\}$ and $s \notin \overline{\{t\}}$, then $it(s) = M_1(s)$, **if** t is firable at M_1,

(ii) $it = M_1$, **otherwise**.

Let us consider Q. We say that Q is firable at M_1 if and only if

(Def. 13) (i) $Q = \emptyset$, or

(ii) there exists a finite sequence M of elements of the markings of P such that $\operatorname{len} Q = \operatorname{len} M$ and Q_1 is firable at M_1 and $M_1 = \text{Firing}(Q_1, M_1)$ and for every i such that $i < \operatorname{len} Q$ and $i > 0$ holds Q_{i+1} is firable at M_i and $M_{i+1} = \text{Firing}(Q_{i+1}, M_i)$.

The functor Firing(Q, M_1) yielding a marking of P is defined by

(Def. 14) (i) $it = M_1$, **if** $Q = \emptyset$,

(ii) there exists a finite sequence M of elements of the markings of P such that $\operatorname{len} Q = \operatorname{len} M$ and $it = M_{\operatorname{len} M}$ and $M_1 = \text{Firing}(Q_1, M_1)$ and for every i such that $i < \operatorname{len} Q$ and $i > 0$ holds $M_{i+1} = \text{Firing}(Q_{i+1}, M_i)$, **otherwise**.

Now we state the propositions:

(11) Firing$(t, M_1) = $ Firing$(\langle t\rangle, M_1)$.

(12) t is firable at M_1 if and only if $\langle t\rangle$ is firable at M_1.

(13) Firing$(Q \frown Q_1, M_1) = $ Firing$(Q_1, \text{Firing}(Q, M_1))$.

(14) If $Q \frown Q_1$ is firable at M_1, then Q_1 is firable at Firing(Q, M_1) and Q is firable at M_1.

5. The Theorem Stating that the Number of Tokens in a Circuit Remains the Same After any Firing Sequences

Now we state the proposition:

(15) Let us consider a Petri-like decision-free-like Petri net D_1 with directed path, a directed path finite sequence d of elements of the places and transitions of D_1, and a transition t of D_1. Suppose

(i) d is circular, and

(ii) there exists a place p_1 of D_1 such that $p_1 \in$ the places of d and $\langle p_1, t\rangle \in$ the S-T arcs of D_1 or $\langle t, p_1\rangle \in$ the T-S arcs of D_1.

Then $t \in$ the transitions of d. The theorem is a consequence of (7), (5), (6), and (2).

A decision-free Petri net is a Petri-like decision-free-like Petri net with directed circuit. Let D_1 be a Petri net with directed circuit. Observe that there exists a finite sequence of elements of the places and transitions of D_1 which is directed path, circular, and almost one-to-one.

A circuit of places and transitions of D_1 is a directed path circular almost one-to-one finite sequence of elements of the places and transitions of D_1. Now we state the propositions:

(16) Let us consider a decision-free Petri net D_1, a circuit d of places and transitions of D_1, a marking M_1 of D_1, and a transition t of D_1. Then the number of tokens of the places of d and $M_1 =$ the number of tokens of the places of d and $\mathrm{Firing}(t, M_1)$. The theorem is a consequence of (6), (5), (8), (2), (9), (1), (10), and (15).

(17) Let us consider a decision-free Petri net D_1, a circuit d of places and transitions of D_1, a marking M_1 of D_1, and a finite sequence Q of elements of the carrier' of D_1. Then the number of tokens of the places of d and $M_1 =$ the number of tokens of the places of d and $\mathrm{Firing}(Q, M_1)$. The theorem is a consequence of (16).

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Received March 31, 2014

FORMALIZED MATHEMATICS
Vol. 22, No. 1, Pages 37–56, 2014
DOI: 10.2478/forma-2014-0005

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Abstract Reduction Systems and Idea of Knuth-Bendix Completion Algorithm

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Summary. Educational content for abstract reduction systems concerning reduction, convertibility, normal forms, divergence and convergence, Church-Rosser property, term rewriting systems, and the idea of the Knuth-Bendix Completion Algorithm. The theory is based on [1].

MSC: 68Q42 03B35

Keywords: abstract reduction systems; Knuth-Bendix algorithm

MML identifier: ABSRED_0, version: 8.1.02 5.22.1199

The notation and terminology used in this paper have been introduced in the following articles: [2], [17], [16], [7], [9], [20], [14], [18], [10], [11], [8], [22], [3], [4], [12], [5], [23], [24], [6], [21], [15], and [13].

1. Reduction and Convertibility

We consider ARS's which extend 1-sorted structures and are systems

$$\langle \text{a carrier}, \text{a reduction} \rangle$$

where the carrier is a set, the reduction is a binary relation on the carrier.

Let A be a non empty set and r be a binary relation on A. Observe that $\langle A, r \rangle$ is non empty and there exists an ARS which is non empty and strict.

Let X be an ARS and x, y be elements of X. We say that $x \to y$ if and only if

(Def. 1) $\langle x, y \rangle \in$ the reduction of X.

We introduce $y \leftarrow x$ as a synonym of $x \to y$.

We say that $x \to_{01} y$ if and only if

ISSN 1426–2630(Print), 1898-9934(Online)

(Def. 2) (i) $x = y$, or

(ii) $x \to y$.

One can verify that the predicate is reflexive. We say that $x \to_* y$ if and only if

(Def. 3) The reduction of X reduces x to y.

Let us observe that the predicate is reflexive.

From now on X denotes an ARS and a, b, c, u, v, w, x, y, z denote elements of X.

Now we state the propositions:

(1) If $a \to b$, then X is not empty.

(2) If $x \to y$, then $x \to_* y$.

(3) If $x \to_* y \to_* z$, then $x \to_* z$.

The scheme *Star* deals with an ARS $\mathcal{X}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 1) For every elements x, y of $\mathcal{X}$ such that $x \to_* y$ and $\mathcal{P}[x]$ holds $\mathcal{P}[y]$

provided

- for every elements x, y of $\mathcal{X}$ such that $x \to y$ and $\mathcal{P}[x]$ holds $\mathcal{P}[y]$.

The scheme *Star1* deals with an ARS $\mathcal{X}$ and a unary predicate $\mathcal{P}$ and elements a, b of $\mathcal{X}$ and states that

(Sch. 2) $\mathcal{P}[b]$

provided

- $a \to_* b$ and
- $\mathcal{P}[a]$ and
- for every elements x, y of $\mathcal{X}$ such that $x \to y$ and $\mathcal{P}[x]$ holds $\mathcal{P}[y]$.

The scheme *StarBack* deals with an ARS $\mathcal{X}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 3) For every elements x, y of $\mathcal{X}$ such that $x \to_* y$ and $\mathcal{P}[y]$ holds $\mathcal{P}[x]$

provided

- for every elements x, y of $\mathcal{X}$ such that $x \to y$ and $\mathcal{P}[y]$ holds $\mathcal{P}[x]$.

The scheme *StarBack1* deals with an ARS $\mathcal{X}$ and a unary predicate $\mathcal{P}$ and elements a, b of $\mathcal{X}$ and states that

(Sch. 4) $\mathcal{P}[a]$

provided

- $a \to_* b$ and
- $\mathcal{P}[b]$ and

- for every elements x, y of $\mathcal{X}$ such that $x \to y$ and $\mathcal{P}[y]$ holds $\mathcal{P}[x]$.

Let X be an ARS and x, y be elements of X. We say that $x \to_+ y$ if and only if

(Def. 4) There exists an element z of X such that $x \to z \to_* y$.

Now we state the proposition:

(4) $x \to_+ y$ if and only if there exists z such that $x \to_* z \to y$. Proof: If $x \to_+ y$, then there exists z such that $x \to_* z \to y$. Define $\mathcal{P}$[element of X] $\equiv$ there exists u such that $\$_1 \to u \to_* y$. For every y and z such that $y \to z$ and $\mathcal{P}[z]$ holds $\mathcal{P}[y]$. For every y and z such that $y \to_* z$ and $\mathcal{P}[z]$ holds $\mathcal{P}[y]$ from *StarBack*. □

Let us consider X, x, and y. We introduce $y \leftarrow_{01} x$ as a synonym of $x \to_{01} y$ and $y \leftarrow_* x$ as a synonym of $x \to_* y$ and $y \leftarrow_+ x$ as a synonym of $x \to_+ y$.

We say that $x \leftrightarrow y$ if and only if

(Def. 5) (i) $x \to y$, or

(ii) $x \leftarrow y$.

One can check that the predicate is symmetric.

Now we state the proposition:

(5) $x \leftrightarrow y$ if and only if $\langle x, y\rangle \in$ (the reduction of X) $\cup$ (the reduction of X)$^\smile$.

Let us consider X, x, and y. We say that $x \leftrightarrow_{01} y$ if and only if

(Def. 6) (i) $x = y$, or

(ii) $x \leftrightarrow y$.

Observe that the predicate is reflexive and symmetric. We say that $x \leftrightarrow_* y$ if and only if

(Def. 7) x and y are convertible w.r.t. the reduction of X.

One can check that the predicate is reflexive and symmetric.

Now we state the propositions:

(6) If $x \leftrightarrow y$, then $x \leftrightarrow_* y$.

(7) If $x \leftrightarrow_* y \leftrightarrow_* z$, then $x \leftrightarrow_* z$.

The scheme *Star2* deals with an ARS $\mathcal{X}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 5) For every elements x, y of $\mathcal{X}$ such that $x \leftrightarrow_* y$ and $\mathcal{P}[x]$ holds $\mathcal{P}[y]$

provided

- for every elements x, y of $\mathcal{X}$ such that $x \leftrightarrow y$ and $\mathcal{P}[x]$ holds $\mathcal{P}[y]$.

The scheme *Star2A* deals with an ARS $\mathcal{X}$ and a unary predicate $\mathcal{P}$ and elements a, b of $\mathcal{X}$ and states that

(Sch. 6) $\mathcal{P}[b]$

provided

- $a \leftrightarrow_* b$ and
- $\mathcal{P}[a]$ and
- for every elements x, y of $\mathcal{X}$ such that $x \leftrightarrow y$ and $\mathcal{P}[x]$ holds $\mathcal{P}[y]$.

Let us consider X, x, and y. We say that $x \leftrightarrow_+ y$ if and only if

(Def. 8) There exists z such that $x \leftrightarrow z \leftrightarrow_* y$.

One can check that the predicate is symmetric.

Now we state the propositions:

(8) $x \leftrightarrow_+ y$ if and only if there exists z such that $x \leftrightarrow_* z \leftrightarrow y$.

(9) If $x \rightarrow_{01} y$, then $x \rightarrow_* y$.

(10) If $x \rightarrow_+ y$, then $x \rightarrow_* y$. The theorem is a consequence of (2) and (3).

(11) If $x \rightarrow y$, then $x \rightarrow_+ y$.

(12) If $x \rightarrow y \rightarrow z$, then $x \rightarrow_* z$. The theorem is a consequence of (2) and (3).

(13) If $x \rightarrow y \rightarrow_{01} z$, then $x \rightarrow_* z$. The theorem is a consequence of (2), (9), and (3).

(14) If $x \rightarrow y \rightarrow_* z$, then $x \rightarrow_* z$. The theorem is a consequence of (2) and (3).

(15) If $x \rightarrow y \rightarrow_+ z$, then $x \rightarrow_* z$. The theorem is a consequence of (2), (10), and (3).

(16) If $x \rightarrow_{01} y \rightarrow z$, then $x \rightarrow_* z$. The theorem is a consequence of (9), (2), and (3).

(17) If $x \rightarrow_{01} y \rightarrow_{01} z$, then $x \rightarrow_* z$. The theorem is a consequence of (9) and (3).

(18) If $x \rightarrow_{01} y \rightarrow_* z$, then $x \rightarrow_* z$. The theorem is a consequence of (9) and (3).

(19) If $x \rightarrow_{01} y \rightarrow_+ z$, then $x \rightarrow_* z$. The theorem is a consequence of (9), (10), and (3).

(20) If $x \rightarrow_* y \rightarrow z$, then $x \rightarrow_* z$. The theorem is a consequence of (2) and (3).

(21) If $x \rightarrow_* y \rightarrow_{01} z$, then $x \rightarrow_* z$. The theorem is a consequence of (9) and (3).

(22) If $x \rightarrow_* y \rightarrow_+ z$, then $x \rightarrow_* z$. The theorem is a consequence of (10) and (3).

(23) If $x \rightarrow_+ y \rightarrow z$, then $x \rightarrow_* z$. The theorem is a consequence of (10), (2), and (3).

(24) If $x \to_{+} y \to_{01} z$, then $x \to_{*} z$. The theorem is a consequence of (10), (9), and (3).

(25) If $x \to_{+} y \to_{+} z$, then $x \to_{*} z$. The theorem is a consequence of (10) and (3).

(26) If $x \to y \to z$, then $x \to_{+} z$.

(27) If $x \to y \to_{01} z$, then $x \to_{+} z$.

(28) If $x \to y \to_{+} z$, then $x \to_{+} z$.

(29) If $x \to_{01} y \to z$, then $x \to_{+} z$.

(30) If $x \to_{01} y \to_{+} z$, then $x \to_{+} z$. The theorem is a consequence of (4) and (18).

(31) If $x \to_{*} y \to_{+} z$, then $x \to_{+} z$. The theorem is a consequence of (4) and (3).

(32) If $x \to_{+} y \to z$, then $x \to_{+} z$.

(33) If $x \to_{+} y \to_{01} z$, then $x \to_{+} z$.

(34) If $x \to_{+} y \to_{*} z$, then $x \to_{+} z$.

(35) If $x \to_{+} y \to_{+} z$, then $x \to_{+} z$.

(36) If $x \leftrightarrow_{01} y$, then $x \leftrightarrow_{*} y$.

(37) If $x \leftrightarrow_{+} y$, then $x \leftrightarrow_{*} y$. The theorem is a consequence of (6) and (7).

(38) If $x \leftrightarrow y$, then $x \leftrightarrow_{+} y$.

(39) If $x \leftrightarrow y \leftrightarrow z$, then $x \leftrightarrow_{*} z$. The theorem is a consequence of (6) and (7).

(40) If $x \leftrightarrow y \leftrightarrow_{01} z$, then $x \leftrightarrow_{*} z$. The theorem is a consequence of (6), (36), and (7).

(41) If $x \leftrightarrow_{01} y \leftrightarrow z$, then $x \leftrightarrow_{*} z$.

(42) If $x \leftrightarrow y \leftrightarrow_{*} z$, then $x \leftrightarrow_{*} z$. The theorem is a consequence of (6) and (7).

(43) If $x \leftrightarrow_{*} y \leftrightarrow z$, then $x \leftrightarrow_{*} z$.

(44) If $x \leftrightarrow y \leftrightarrow_{+} z$, then $x \leftrightarrow_{*} z$. The theorem is a consequence of (6), (37), and (7).

(45) If $x \leftrightarrow_{+} y \leftrightarrow z$, then $x \leftrightarrow_{*} z$.

(46) If $x \leftrightarrow_{01} y \leftrightarrow_{01} z$, then $x \leftrightarrow_{*} z$. The theorem is a consequence of (36) and (7).

(47) If $x \leftrightarrow_{01} y \leftrightarrow_{*} z$, then $x \leftrightarrow_{*} z$. The theorem is a consequence of (36) and (7).

(48) If $x \leftrightarrow_{*} y \leftrightarrow_{01} z$, then $x \leftrightarrow_{*} z$.

(49) If $x \leftrightarrow_{01} y \leftrightarrow_{+} z$, then $x \leftrightarrow_{*} z$. The theorem is a consequence of (36), (37), and (7).

(50) If $x \leftrightarrow_{+} y \leftrightarrow_{01} z$, then $x \leftrightarrow_{*} z$.

(51) If $x \leftrightarrow_* y \leftrightarrow_+ z$, then $x \leftrightarrow_* z$. The theorem is a consequence of (37) and (7).

(52) If $x \leftrightarrow_+ y \leftrightarrow_+ z$, then $x \leftrightarrow_* z$. The theorem is a consequence of (37) and (7).

(53) If $x \leftrightarrow y \leftrightarrow z$, then $x \leftrightarrow_+ z$.

(54) If $x \leftrightarrow y \leftrightarrow_{01} z$, then $x \leftrightarrow_+ z$.

(55) If $x \leftrightarrow y \leftrightarrow_+ z$, then $x \leftrightarrow_+ z$.

(56) If $x \leftrightarrow_{01} y \leftrightarrow_+ z$, then $x \leftrightarrow_+ z$. The theorem is a consequence of (8) and (47).

(57) If $x \leftrightarrow_* y \leftrightarrow_+ z$, then $x \leftrightarrow_+ z$. The theorem is a consequence of (8) and (7).

(58) If $x \leftrightarrow_+ y \leftrightarrow_+ z$, then $x \leftrightarrow_+ z$.

(59) If $x \leftrightarrow_{01} y$, then $x \leftarrow y$ or $x = y$ or $x \rightarrow y$.

(60) If $x \leftarrow y$ or $x = y$ or $x \rightarrow y$, then $x \leftrightarrow_{01} y$.

(61) If $x \leftrightarrow_{01} y$, then $x \leftarrow_{01} y$ or $x \rightarrow y$.

(62) If $x \leftarrow_{01} y$ or $x \rightarrow y$, then $x \leftrightarrow_{01} y$.

Let us assume that $x \leftrightarrow_{01} y$. Now we state the propositions:

(63) (i) $x \leftarrow_{01} y$, or

(ii) $x \rightarrow_+ y$.

(64) (i) $x \leftarrow_{01} y$, or

(ii) $x \leftrightarrow y$.

Now we state the propositions:

(65) If $x \leftarrow_{01} y$ or $x \leftrightarrow y$, then $x \leftrightarrow_{01} y$.

(66) If $x \leftrightarrow_* y \rightarrow z$, then $x \leftrightarrow_+ z$.

(67) If $x \leftrightarrow_+ y \rightarrow z$, then $x \leftrightarrow_+ z$. The theorem is a consequence of (37).

Let us assume that $x \leftrightarrow_{01} y$. Now we state the propositions:

(68) (i) $x \leftarrow_{01} y$, or

(ii) $x \rightarrow y$.

(69) (i) $x \leftarrow_{01} y$, or

(ii) $x \rightarrow_+ y$.

Now we state the propositions:

(70) If $x \leftarrow_{01} y$ or $x \rightarrow y$, then $x \leftrightarrow_{01} y$.

(71) If $x \leftarrow_{01} y$ or $x \leftrightarrow y$, then $x \leftrightarrow_{01} y$.

(72) If $x \leftrightarrow_{01} y$, then $x \leftarrow_{01} y$ or $x \leftrightarrow y$.

(73) If $x \leftrightarrow_+ y \rightarrow z$, then $x \leftrightarrow_+ z$. The theorem is a consequence of (37).

(74) If $x \leftrightarrow_* y \rightarrow z$, then $x \leftrightarrow_+ z$.

(75) If $x \leftrightarrow_{01} y \rightarrow z$, then $x \leftrightarrow_+ z$. The theorem is a consequence of (36).

(76) If $x \leftrightarrow_+ y \rightarrow_{01} z$, then $x \leftrightarrow_+ z$. The theorem is a consequence of (70) and (56).

(77) If $x \leftrightarrow y \rightarrow_{01} z$, then $x \leftrightarrow_+ z$. The theorem is a consequence of (70), (38), and (56).

(78) If $x \rightarrow y \rightarrow z \rightarrow u$, then $x \rightarrow_+ u$.

(79) If $x \rightarrow y \rightarrow_{01} z \rightarrow u$, then $x \rightarrow_+ u$.

(80) If $x \rightarrow y \rightarrow_* z \rightarrow u$, then $x \rightarrow_+ u$.

(81) If $x \rightarrow y \rightarrow_+ z \rightarrow u$, then $x \rightarrow_+ u$. The theorem is a consequence of (15) and (4).

(82) If $x \rightarrow_* y$, then $x \leftrightarrow_* y$. Proof: Define $\mathcal{P}[\text{element of } X] \equiv x \leftrightarrow_* \$_1$. For every y and z such that $y \rightarrow z$ and $\mathcal{P}[y]$ holds $\mathcal{P}[z]$. $\mathcal{P}[y]$ from *Star1*. □

(83) Suppose for every x and y such that $x \rightarrow z$ and $x \rightarrow y$ holds $y \rightarrow z$. If $x \rightarrow z$ and $x \rightarrow_* y$, then $y \rightarrow z$. Proof: Define $\mathcal{P}[\text{element of } X] \equiv \$_1 \rightarrow z$. For every u and v such that $u \rightarrow_* v$ and $\mathcal{P}[u]$ holds $\mathcal{P}[v]$ from *Star*. □

(84) If for every x and y such that $x \rightarrow y$ holds $y \rightarrow x$, then for every x and y such that $x \leftrightarrow_* y$ holds $x \rightarrow_* y$. Proof: Define $\mathcal{P}[\text{element of } X] \equiv x \rightarrow_* \$_1$. For every u and v such that $u \leftrightarrow v$ and $\mathcal{P}[u]$ holds $\mathcal{P}[v]$. For every u and v such that $u \leftrightarrow_* v$ and $\mathcal{P}[u]$ holds $\mathcal{P}[v]$ from *Star2*. □

(85) If $x \rightarrow_* y$, then $x = y$ or $x \rightarrow_+ y$. Proof: Define $\mathcal{P}[\text{element of } X] \equiv x = \$_1$ or $x \rightarrow_+ \$_1$. For every y and z such that $y \rightarrow z$ and $\mathcal{P}[y]$ holds $\mathcal{P}[z]$. $\mathcal{P}[y]$ from *Star1*. □

(86) If for every x, y, and z such that $x \rightarrow y \rightarrow z$ holds $x \rightarrow z$, then for every x and y such that $x \rightarrow_+ y$ holds $x \rightarrow y$. Proof: Consider z such that $x \rightarrow z$ and $z \rightarrow_* y$. Define $\mathcal{P}[\text{element of } X] \equiv x \rightarrow \$_1$. $\mathcal{P}[y]$ from *Star1*. □

2. Examples of an Abstract Reduction System

The scheme *ARSex* deals with a non empty set $\mathcal{A}$ and a binary predicate $\mathcal{R}$ and states that

(Sch. 7) There exists a strict non empty ARS X such that the carrier of $X = \mathcal{A}$ and for every elements x, y of X, $x \rightarrow y$ iff $\mathcal{R}[x, y]$.

The functors: ARS_{01} and ARS_{02} yielding strict ARS's are defined by conditions,

(Def. 9) (i) the carrier of $\text{ARS}_{01} = \{0, 1\}$, and

(ii) the reduction of $\text{ARS}_{01} = \{0\} \times \{0, 1\}$,

(Def. 10) (i) the carrier of $\text{ARS}_{02} = \{0, 1, 2\}$, and

(ii) the reduction of $\text{ARS}_{02} = \{0\} \times \{0, 1, 2\}$,

respectively. One can check that ARS_{01} is non empty and ARS_{02} is non empty.

From now on i, j, k denote elements of ARS_{01}.

Now we state the propositions:

(87) Let us consider a set s. Then s is an element of ARS_{01} if and only if $s = 0$ or $s = 1$.

(88) $i \to j$ if and only if $i = 0$. The theorem is a consequence of (87).

In the sequel l, m, n denote elements of ARS_{02}.

Now we state the propositions:

(89) Let us consider a set s. Then s is an element of ARS_{02} if and only if $s = 0$ or $s = 1$ or $s = 2$.

(90) $m \to n$ if and only if $m = 0$. The theorem is a consequence of (89).

3. Normal Forms

Let us consider X and x. We say that x is a normal form if and only if

(Def. 11) There exists no y such that $x \to y$.

Now we state the proposition:

(91) x is a normal form if and only if x is a normal form w.r.t. the reduction of X. Proof: If x is a normal form, then x is a normal form w.r.t. the reduction of X by [13, (87)]. □

Let us consider X, x, and y. We say that x is a normal form of y if and only if

(Def. 12) (i) x is a normal form, and

(ii) $y \to_* x$.

Now we state the proposition:

(92) x is a normal form of y if and only if x is a normal form of y w.r.t. the reduction of X. The theorem is a consequence of (91).

Let us consider X and x. We say that x is normalizable if and only if

(Def. 13) There exists y such that y is a normal form of x.

Now we state the proposition:

(93) x is normalizable if and only if x has a normal form w.r.t. the reduction of X. The theorem is a consequence of (92).

Let us consider X. We say that X is WN if and only if

(Def. 14) x is normalizable.

We say that X is SN if and only if

(Def. 15) Let us consider a function f from $\mathbb{N}$ into the carrier of X. Then there exists a natural number i such that $f(i) \not\to f(i+1)$.

We say that X is UN* if and only if

(Def. 16) If y is a normal form of x and z is a normal form of x, then $y = z$.

We say that X is UN if and only if

(Def. 17) If x is a normal form and y is a normal form and $x \leftrightarrow_* y$, then $x = y$.

We say that X is NF if and only if

(Def. 18) If x is a normal form and $x \leftrightarrow_* y$, then $y \rightarrow_* x$.

Now we state the propositions:

(94) X is WN if and only if the reduction of X is weakly-normalizing. The theorem is a consequence of (93).

(95) If X is SN, then the reduction of X is strongly-normalizing.

(96) If X is not empty and the reduction of X is strongly-normalizing, then X is SN.

From now on A denotes a set.

Now we state the proposition:

(97) X is SN if and only if there exists no A and there exists z such that $z \in A$ and for every x such that $x \in A$ there exists y such that $y \in A$ and $x \rightarrow y$.

The scheme *notSN* deals with an ARS $\mathcal{X}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 8) $\mathcal{X}$ is not SN

provided

- there exists an element x of $\mathcal{X}$ such that $\mathcal{P}[x]$ and
- for every element x of $\mathcal{X}$ such that $\mathcal{P}[x]$ there exists an element y of $\mathcal{X}$ such that $\mathcal{P}[y]$ and $x \rightarrow y$.

Now we state the propositions:

(98) X is UN if and only if the reduction of X has unique normal form property. PROOF: Set $R =$ the reduction of X. If X is UN, then R has unique normal form property by (91), [6, (28), (31)]. x is a normal form w.r.t. R and y is a normal form w.r.t. R and x and y are convertible w.r.t. R. $\square$

(99) X is NF if and only if the reduction of X has normal form property. PROOF: Set $R =$ the reduction of X. If X is NF, then R has normal form property by (91), [6, (28), (31), (12)]. $\square$

Let us consider X and x. Assume there exists y such that y is a normal form of x and for every y and z such that y is a normal form of x and z is a normal form of x holds $y = z$. The functor nf x yielding an element of X is defined by

(Def. 19) *it* is a normal form of x.

Now we state the propositions:

(100) Suppose there exists y such that y is a normal form of x and for every y and z such that y is a normal form of x and z is a normal form of x holds

$y = z$. Then $\mathrm{nf}\, x = \mathrm{nf}_\alpha(x)$, where α is the reduction of X. The theorem is a consequence of (92).

(101) If x is a normal form and $x \to_* y$, then $x = y$. The theorem is a consequence of (85).

(102) If x is a normal form, then x is a normal form of x.

(103) If x is a normal form and $y \to x$, then x is a normal form of y.

(104) If x is a normal form and $y \to_{01} x$, then x is a normal form of y.

(105) If x is a normal form and $y \to_+ x$, then x is a normal form of y.

(106) If x is a normal form of y and y is a normal form of x, then $x = y$.

(107) If x is a normal form of y and $z \to y$, then x is a normal form of z.

(108) If x is a normal form of y and $z \to_* y$, then x is a normal form of z.

(109) If x is a normal form of y and $z \to_* x$, then x is a normal form of z.

Let us consider X. One can check that every element of X which is a normal form is also normalizable.

Now we state the propositions:

(110) If x is normalizable and $y \to x$, then y is normalizable.

(111) X is WN if and only if for every x, there exists y such that y is a normal form of x.

(112) If for every x, x is a normal form, then X is WN. The theorem is a consequence of (102).

One can verify that every ARS which is SN is also WN.

Now we state the propositions:

(113) If $x \neq y$ and for every a and b, $a \to b$ iff $a = x$, then y is a normal form and x is normalizable. The theorem is a consequence of (2).

(114) There exists X such that

(i) X is WN, and

(ii) X is not SN.

PROOF: Define $\mathcal{R}[\text{set}, \text{set}] \equiv \$_1 = 0$. Consider X being a strict non empty ARS such that the carrier of $X = \{0, 1\}$ and for every elements x, y of X, $x \to y$ iff $\mathcal{R}[x, y]$ from *ARSex*. X is WN. □

One can verify that every ARS which is NF is also UN* and every ARS which is NF is also UN and every ARS which is UN is also UN*.

Now we state the proposition:

(115) If X is WN and UN* and x is a normal form and $x \leftrightarrow_* y$, then $y \to_* x$.
PROOF: Define $\mathcal{P}[\text{element of } X] \equiv \$_1 \to_* x$. For every y and z such that $y \leftrightarrow z$ and $\mathcal{P}[y]$ holds $\mathcal{P}[z]$. For every y and z such that $y \leftrightarrow_* z$ and $\mathcal{P}[y]$ holds $\mathcal{P}[z]$ from *Star2*. □

Observe that every ARS which is WN and UN* is also NF and every ARS which is WN and UN* is also UN.

Now we state the propositions:

(116) If y is a normal form of x and z is a normal form of x and $y \neq z$, then $x \rightarrow_{+} y$. The theorem is a consequence of (85) and (101).

(117) If X is WN and UN*, then $\mathrm{nf}\, x$ is a normal form of x.

(118) If X is WN and UN* and y is a normal form of x, then $y = \mathrm{nf}\, x$.

Let us assume that X is WN and UN*. Now we state the propositions:

(119) $\mathrm{nf}\, x$ is a normal form. The theorem is a consequence of (117).

(120) $\mathrm{nf}\, \mathrm{nf}\, x = \mathrm{nf}\, x$. The theorem is a consequence of (119), (102), and (118).

Now we state the propositions:

(121) If X is WN and UN* and $x \rightarrow_{*} y$, then $\mathrm{nf}\, x = \mathrm{nf}\, y$. The theorem is a consequence of (117), (108), and (118).

(122) If X is WN and UN* and $x \leftrightarrow_{*} y$, then $\mathrm{nf}\, x = \mathrm{nf}\, y$. PROOF: Define $\mathcal{P}[\text{element of } X] \equiv \mathrm{nf}\, x = \mathrm{nf}\, \$_1$. For every z and u such that $z \leftrightarrow u$ and $\mathcal{P}[z]$ holds $\mathcal{P}[u]$. $\mathcal{P}[y]$ from *Star2A*. $\square$

(123) If X is WN and UN* and $\mathrm{nf}\, x = \mathrm{nf}\, y$, then $x \leftrightarrow_{*} y$. The theorem is a consequence of (117), (82), and (7).

4. DIVERGENCE AND CONVERGENCE

Let us consider X, x, and y. We say that $x \swarrow^{*}\searrow y$ if and only if

(Def. 20) There exists z such that $x \leftarrow_{*} z \rightarrow_{*} y$.

Observe that the predicate is symmetric and reflexive. We say that $x \searrow_{*}\swarrow y$ if and only if

(Def. 21) There exists z such that $x \rightarrow_{*} z \leftarrow_{*} y$.

One can check that the predicate is symmetric and reflexive. We say that $x \swarrow^{01}\searrow y$ if and only if

(Def. 22) There exists z such that $x \leftarrow_{01} z \rightarrow_{01} y$.

Observe that the predicate is symmetric and reflexive. We say that $x \searrow_{01}\swarrow y$ if and only if

(Def. 23) There exists z such that $x \rightarrow_{01} z \leftarrow_{01} y$.

One can check that the predicate is symmetric and reflexive.

Now we state the propositions:

(124) $x \swarrow^{*}\searrow y$ if and only if x and y are divergent w.r.t. the reduction of X.

(125) $x \searrow_{*}\swarrow y$ if and only if x and y are convergent w.r.t. the reduction of X.

(126) $x \swarrow^{01}\searrow y$ if and only if x and y are divergent at most in 1 step w.r.t. the reduction of X.

(127) $x \searrow_{01}\swarrow y$ if and only if x and y are convergent at most in 1 step w.r.t. the reduction of X.

Let us consider X. We say that X is DIAMOND if and only if

(Def. 24) If $x \swarrow^{01}\searrow y$, then $x \searrow_{01}\swarrow y$.

We say that X is CONF if and only if

(Def. 25) If $x \swarrow^{*}\searrow y$, then $x \searrow_{*}\swarrow y$.

We say that X is CR if and only if

(Def. 26) If $x \leftrightarrow_{*} y$, then $x \searrow_{*}\swarrow y$.

We say that X is WCR if and only if

(Def. 27) If $x \swarrow^{01}\searrow y$, then $x \searrow_{*}\swarrow y$.

We say that X is COMP if and only if

(Def. 28) X is SN and CONF.

The scheme *isCR* deals with a non empty ARS $\mathcal{X}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{X}$ and states that

(Sch. 9) $\mathcal{X}$ is CR

provided

- for every element x of $\mathcal{X}$, $x \rightarrow_{*} \mathcal{F}(x)$ and
- for every elements x, y of $\mathcal{X}$ such that $x \leftrightarrow_{*} y$ holds $\mathcal{F}(x) = \mathcal{F}(y)$.

The scheme *isCOMP* deals with a non empty ARS $\mathcal{X}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{X}$ and states that

(Sch. 10) $\mathcal{X}$ is COMP

provided

- $\mathcal{X}$ is SN and
- for every element x of $\mathcal{X}$, $x \rightarrow_{*} \mathcal{F}(x)$ and
- for every elements x, y of $\mathcal{X}$ such that $x \leftrightarrow_{*} y$ holds $\mathcal{F}(x) = \mathcal{F}(y)$.

Now we state the propositions:

(128) If $x \swarrow^{01}\searrow y$, then $x \swarrow^{*}\searrow y$. The theorem is a consequence of (9).

(129) If $x \searrow_{01}\swarrow y$, then $x \searrow_{*}\swarrow y$. The theorem is a consequence of (9).

Let us assume that $x \rightarrow y$. Now we state the propositions:

(130) $x \swarrow^{01}\searrow y$.

(131) $x \searrow_{01}\swarrow y$.

Let us assume that $x \rightarrow_{01} y$. Now we state the propositions:

(132) $x \swarrow^{01}\searrow y$.

(133) $x \searrow_{01}\swarrow y$.

Let us assume that $x \leftrightarrow y$. Now we state the propositions:

(134) $x \swarrow^{01}\searrow y$.

(135) $x \searrow_{01}\swarrow y$.

Let us assume that $x \leftrightarrow_{01} y$. Now we state the propositions:

(136) $x \swarrow^{01}\searrow y$. The theorem is a consequence of (59).

(137) $x \searrow_{01}\swarrow y$. The theorem is a consequence of (59).

Now we state the proposition:

(138) If $x \to y$, then $x \searrow_{*}\swarrow y$.

Let us assume that $x \to_{*} y$. Now we state the propositions:

(139) $x \searrow_{*}\swarrow y$.

(140) $x \swarrow^{*}\searrow y$.

Let us assume that $x \to_{+} y$. Now we state the propositions:

(141) $x \searrow_{*}\swarrow y$. The theorem is a consequence of (10).

(142) $x \swarrow^{*}\searrow y$. The theorem is a consequence of (10).

Now we state the propositions:

(143) If $x \to y$ and $x \to z$, then $y \swarrow^{01}\searrow z$.

(144) If $x \to y$ and $z \to y$, then $x \searrow_{01}\swarrow z$.

(145) If $x \searrow_{*}\swarrow z \leftarrow y$, then $x \searrow_{*}\swarrow y$. The theorem is a consequence of (14).

(146) If $x \searrow_{*}\swarrow z \leftarrow_{01} y$, then $x \searrow_{*}\swarrow y$. The theorem is a consequence of (18).

(147) If $x \searrow_{*}\swarrow z \leftarrow_{*} y$, then $x \searrow_{*}\swarrow y$. The theorem is a consequence of (3).

(148) If $x \swarrow^{*}\searrow y$, then $x \leftrightarrow_{*} y$. The theorem is a consequence of (82) and (7).

(149) If $x \searrow_{*}\swarrow y$, then $x \leftrightarrow_{*} y$. The theorem is a consequence of (82) and (7).

5. Church-Rosser Property

Now we state the propositions:

(150) X is DIAMOND if and only if the reduction of X is subcommutative. PROOF: Set $R =$ the reduction of X. If X is DIAMOND, then R is subcommutative by [23, (15)], (127). □

(151) X is CONF if and only if the reduction of X is confluent. PROOF: Set $R =$ the reduction of X. If X is CONF, then R is confluent by [6, (37), (32)], (124), (125). x and y are divergent w.r.t. R. □

(152) X is CR if and only if the reduction of X has Church-Rosser property. PROOF: Set $R =$ the reduction of X. If X is CR, then R has Church-Rosser property by [6, (32)], (125), [6, (38)]. □

(153) X is WCR if and only if the reduction of X is locally-confluent. PROOF: Set $R =$ the reduction of X. If X is WCR, then R is locally-confluent by [23, (15)], (125). □

(154) Let us consider a non empty ARS X. Then X is COMP if and only if the reduction of X is complete. The theorem is a consequence of (151), (95), and (96).

(155) If X is DIAMOND and $x \leftarrow_* z \rightarrow_{01} y$, then there exists u such that $x \rightarrow_{01} u \leftarrow_* y$. Proof: Define $\mathcal{P}[\text{element of } X] \equiv$ there exists u such that $\$_1 \rightarrow_{01} u \leftarrow_* y$. For every u and v such that $u \rightarrow v$ and $\mathcal{P}[u]$ holds $\mathcal{P}[v]$. For every u and v such that $u \rightarrow_* v$ and $\mathcal{P}[u]$ holds $\mathcal{P}[v]$ from *Star*. $\square$

(156) If X is DIAMOND and $x \leftarrow_{01} y \rightarrow_* z$, then there exists u such that $x \rightarrow_* u \leftarrow_{01} z$. The theorem is a consequence of (155).

One can verify that every ARS which is DIAMOND is also CONF and every ARS which is DIAMOND is also CR and every ARS which is CR is also WCR and every ARS which is CR is also CONF and every ARS which is CONF is also CR.

Now we state the proposition:

(157) If X is non CONF and WN, then there exists x and there exists y and there exists z such that y is a normal form of x and z is a normal form of x and $y \neq z$. The theorem is a consequence of (108).

Newman Lemma: Every ARS which is SN and WCR is also CR and every ARS which is CR is also NF and every ARS which is WN and UN is also CR and every ARS which is SN and CR is also COMP and every ARS which is COMP is also CR WCR NF UN UN* and WN.

Now we state the proposition:

(158) If X is COMP, then for every x and y such that $x \leftrightarrow_* y$ holds $\mathrm{nf}\, x = \mathrm{nf}\, y$.

Observe that every ARS which is WN and UN* is also CR and every ARS which is SN and UN* is also COMP.

6. Term Rewriting Systems

We consider TRS structures which extend ARS's and universal algebra structures and are systems

$$\langle \text{a carrier}, \text{a characteristic}, \text{a reduction} \rangle$$

where the carrier is a set, the characteristic is a finite sequence of operational functions of the carrier, the reduction is a binary relation on the carrier.

One can verify that there exists a TRS structure which is non empty, non-empty, and strict.

Let S be a non empty universal algebra structure. We say that S is group-like if and only if

(Def. 29) (i) $\mathrm{Seg}\, 3 \subseteq \mathrm{dom}(\text{the characteristic of } S)$, and

(ii) for every non empty homogeneous partial function f from (the carrier of S)* to the carrier of S, if $f =$ (the characteristic of S)(1), then arity $f = 0$ and if $f =$ (the characteristic of S)(2), then arity $f = 1$ and if $f =$ (the characteristic of S)(3), then arity $f = 2$.

Now we state the propositions:

(159) Let us consider a non empty set X and non empty homogeneous partial functions f_1, f_2, f_3 from X^* to X. Suppose

(i) arity $f_1 = 0$, and

(ii) arity $f_2 = 1$, and

(iii) arity $f_3 = 2$.

Let us consider a non empty universal algebra structure S. Suppose

(iv) the carrier of $S = X$, and

(v) $\langle f_1, f_2, f_3 \rangle \subseteq$ the characteristic of S.

Then S is group-like.

(160) Let us consider a non empty set X, non empty quasi total homogeneous partial functions f_1, f_2, f_3 from X^* to X, and a non empty universal algebra structure S. Suppose

(i) the carrier of $S = X$, and

(ii) $\langle f_1, f_2, f_3 \rangle =$ the characteristic of S.

Then S is quasi total and partial. PROOF: S is quasi total by [7, (89)], [19, (1)], [7, (45)]. □

Let S be a non empty non-empty universal algebra structure, o be an operation of S, and a be an element of dom o. Let us note that the functor $o(a)$ yields an element of S. One can check that every operation of S is non empty.

Note that every element of dom o is relation-like and function-like.

Let S be a partial non empty non-empty universal algebra structure. Let us observe that every operation of S is homogeneous.

Let S be a quasi total non empty non-empty universal algebra structure. One can check that every operation of S is quasi total.

Now we state the propositions:

(161) Let us consider a non empty non-empty universal algebra structure S. Suppose S is group-like. Then

(i) 1 is an operation symbol of S, and

(ii) 2 is an operation symbol of S, and

(iii) 3 is an operation symbol of S.

(162) Let us consider a partial non empty non-empty universal algebra structure S. Suppose S is group-like. Then

(i) arity Den(1($\in$ dom(the characteristic of S)), S) $= 0$, and

(ii) arity Den(2($\in$ dom(the characteristic of S)), S) $= 1$, and

(iii) arity Den(3($\in$ dom(the characteristic of S)), S) $= 2$.

The theorem is a consequence of (161).

Let S be a non empty non-empty TRS structure. We say that S is invariant if and only if

(Def. 30) Let us consider an operation symbol o of S, elements a, b of dom Den(o, S), and a natural number i. Suppose $i \in \operatorname{dom} a$. Let us consider elements x, y of S. Suppose

(i) $x = a(i)$, and

(ii) $b = a +\cdot (i, y)$, and

(iii) $x \to y$.

Then $(\mathrm{Den}(o, S))(a) \to (\mathrm{Den}(o, S))(b)$.

We say that S is compatible if and only if

(Def. 31) Let us consider an operation symbol o of S and elements a, b of dom Den(o, S). Suppose a natural number i. Suppose $i \in \operatorname{dom} a$. Let us consider elements x, y of S. If $x = a(i)$ and $y = b(i)$, then $x \to y$. Then (Den(o, S))(a) $\to_*$ (Den(o, S))(b).

Now we state the proposition:

(163) Let us consider a natural number n, a non empty set X, and an element x of X. Then there exists a non empty homogeneous quasi total partial function f from X^* to X such that

(i) arity $f = n$, and

(ii) $f = X^n \longmapsto x$.

PROOF: Set $f = X^n \longmapsto x$. f is quasi total by [9, (132), (133)]. f is homogeneous by [9, (132)]. $\square$

Let X be a non empty set, O be a finite sequence of operational functions of X, and r be a binary relation on X. Observe that $\langle X, O, r \rangle$ is non empty.

Let O be a non empty non-empty finite sequence of operational functions of X. Let us note that $\langle X, O, r \rangle$ is non-empty.

Let x be an element of X. The functor TotalTRS(X, x) yielding a non empty non-empty strict TRS structure is defined by

(Def. 32) (i) the carrier of *it* $= X$, and

(ii) the characteristic of *it* $= \langle X^0 \longmapsto x, X^1 \longmapsto x, X^2 \longmapsto x \rangle$, and

(iii) the reduction of *it* $= \nabla_X$.

One can verify that TotalTRS(X, x) is quasi total partial group-like and invariant and there exists a non empty non-empty TRS structure which is strict, quasi total, partial, group-like, and invariant.

Let S be a group-like quasi total partial non empty non-empty TRS structure. The functor 1_S yielding an element of S is defined by the term

(Def. 33) (Den(1($\in$ dom(the characteristic of S)), S))($\emptyset$).

Let a be an element of S. The functor a^{-1} yielding an element of S is defined by the term

(Def. 34) (Den(2($\in$ dom(the characteristic of S)), S))($\langle a \rangle$).

Let b be an element of S. The functor $a \cdot b$ yielding an element of S is defined by the term

(Def. 35) (Den(3($\in$ dom(the characteristic of S)), S))($\langle a, b \rangle$).

In the sequel S denotes a group-like quasi total partial invariant non empty non-empty TRS structure and a, b, c denote elements of S.

Let us assume that $a \to b$. Now we state the propositions:

(164) $a^{-1} \to b^{-1}$. The theorem is a consequence of (162).

(165) $a \cdot c \to b \cdot c$. The theorem is a consequence of (162).

(166) $c \cdot a \to c \cdot b$. The theorem is a consequence of (162).

7. Idea of Knuth-Bendix Algorithm

In the sequel S denotes a group-like quasi total partial non empty non-empty TRS structure and a, b, c denote elements of S.

Let us consider S. We say that S is (R1) if and only if

(Def. 36) $1_S \cdot a \to a$.

We say that S is (R2) if and only if

(Def. 37) $a^{-1} \cdot a \to 1_S$.

We say that S is (R3) if and only if

(Def. 38) $(a \cdot b) \cdot c \to a \cdot (b \cdot c)$.

We say that S is (R4) if and only if

(Def. 39) $a^{-1} \cdot (a \cdot b) \to b$.

We say that S is (R5) if and only if

(Def. 40) $(1_S)^{-1} \cdot a \to a$.

We say that S is (R6) if and only if

(Def. 41) $(a^{-1})^{-1} \cdot 1_S \to a$.

We say that S is (R7) if and only if

(Def. 42) $(a^{-1})^{-1} \cdot b \to a \cdot b$.

We say that S is (R8) if and only if

(Def. 43) $a \cdot 1_S \to a$.

We say that S is (R9) if and only if

(Def. 44) $(a^{-1})^{-1} \to a$.

We say that S is (R10) if and only if

(Def. 45) $(1_S)^{-1} \to 1_S$.

We say that S is (R11) if and only if

(Def. 46) $a \cdot a^{-1} \to 1_S$.

We say that S is (R12) if and only if

(Def. 47) $a \cdot (a^{-1} \cdot b) \to b$.

We say that S is (R13) if and only if

(Def. 48) $a \cdot (b \cdot (a \cdot b)^{-1}) \to 1_S$.

We say that S is (R14) if and only if

(Def. 49) $a \cdot (b \cdot a)^{-1} \to b^{-1}$.

We say that S is (R15) if and only if

(Def. 50) $(a \cdot b)^{-1} \to b^{-1} \cdot a^{-1}$.

In the sequel S denotes a group-like quasi total partial invariant non empty non-empty TRS structure and a, b, c denote elements of S.

Now we state the propositions:

(167) If S is (R1), (R2), and (R3), then $a^{-1} \cdot (a \cdot b) \swarrow^{*}\searrow b$. The theorem is a consequence of (2), (165), and (3).

(168) If S is (R1) and (R4), then $(1_S)^{-1} \cdot a \swarrow^{*}\searrow a$. The theorem is a consequence of (2) and (166).

(169) If S is (R2) and (R4), then $(a^{-1})^{-1} \cdot 1_S \swarrow^{*}\searrow a$. The theorem is a consequence of (2) and (166).

(170) If S is (R1), (R3), and (R6), then $(a^{-1})^{-1} \cdot b \swarrow^{*}\searrow a \cdot b$. The theorem is a consequence of (2), (166), (3), and (165).

(171) If S is (R6) and (R7), then $a \cdot 1_S \swarrow^{*}\searrow a$. The theorem is a consequence of (2).

(172) If S is (R6) and (R8), then $(a^{-1})^{-1} \swarrow^{*}\searrow a$. The theorem is a consequence of (2).

(173) If S is (R5) and (R8), then $(1_S)^{-1} \swarrow^{*}\searrow 1_S$. The theorem is a consequence of (2).

(174) If S is (R2) and (R9), then $a \cdot a^{-1} \swarrow^{*}\searrow 1_S$. The theorem is a consequence of (2) and (165).

(175) If S is (R1), (R3), and (R11), then $a \cdot (a^{-1} \cdot b) \swarrow^{*}\searrow b$. The theorem is a consequence of (2), (165), and (12).

(176) If S is (R3) and (R11), then $a \cdot (b \cdot (a \cdot b)^{-1}) \swarrow^{*}\searrow 1_S$. The theorem is a consequence of (2).

(177) If S is (R4), (R8), and (R13), then $a \cdot (b \cdot a)^{-1} \swarrow^{*}\searrow b^{-1}$. The theorem is a consequence of (2), (166), and (12).

(178) If S is (R4) and (R14), then $(a \cdot b)^{-1} \swarrow^{*}\searrow b^{-1} \cdot a^{-1}$. The theorem is a consequence of (2) and (166).

(179) If S is (R1) and (R10), then $(1_S)^{-1} \cdot a \to_* a$. The theorem is a consequence of (165) and (12).

(180) If S is (R8) and (R9), then $(a^{-1})^{-1} \cdot 1_S \to_* a$. The theorem is a consequence of (12).

(181) If S is (R9), then $(a^{-1})^{-1} \cdot b \to_* a \cdot b$. The theorem is a consequence of (2) and (165).

(182) If S is (R11) and (R14), then $a \cdot (b \cdot (a \cdot b)^{-1}) \to_* 1_S$. The theorem is a consequence of (166) and (12).

(183) If S is (R12) and (R15), then $a \cdot (b \cdot a)^{-1} \to_* b^{-1}$. The theorem is a consequence of (166) and (12).

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Received March 31, 2014

FORMALIZED MATHEMATICS
Vol. 22, No. 1, Pages 57–68, 2014
DOI: 10.2478/forma-2014-0006

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Double Series and Sums[1]

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Summary. In this paper the author constructs several properties for double series and its convergence. The notions of convergence of double sequence have already been introduced in our previous paper [18]. In section 1 we introduce double series and their convergence. Then we show the relationship between Pringsheim-type convergence and iterated convergence. In section 2 we study double series having non-negative terms. As a result, we have equality of three type sums of non-negative double sequence. In section 3 we show that if a non-negative sequence is summable, then the squence of rearrangement of terms is summable and it has the same sums. In the last section two basic relations between double sequences and matrices are introduced.

MSC: 40A05 40B05 03B35

Keywords: double series

MML identifier: DBLSEQ_2, version: 8.1.03 5.23.1204

The notation and terminology used in this paper have been introduced in the following articles: [7], [1], [2], [18], [6], [9], [16], [11], [12], [23], [25], [30], [17], [3], [4], [13], [21], [20], [28], [29], [14], [22], [24], [27], and [15].

1. Double Series and their Convergence

From now on R_1, R_2, R_3 denote functions from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$.

Let f be a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Let us note that f is non-negative yielding if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let us consider natural numbers i, j. Then $f(i, j) \geqslant 0$.

Now we state the propositions:

(1) Suppose R_1 is non-decreasing. Then

[1]This work was supported by JSPS KAKENHI 23500029.

ISSN 1426–2630(Print), 1898-9934(Online)

(i) for every element m of $\mathbb{N}$, curry(R_1, m) is non-decreasing, and

(ii) for every element n of $\mathbb{N}$, curry$'(R_1, n)$ is non-decreasing.

(2) If R_1 is non-decreasing and convergent in the first coordinate, then the lim in the first coordinate of R_1 is non-decreasing.

(3) If R_1 is non-decreasing and convergent in the second coordinate, then the lim in the second coordinate of R_1 is non-decreasing.

(4) If R_1 is non-decreasing and p-convergent, then for every natural numbers n, m, $R_1(n, m) \leqslant$ P-lim R_1.

(5) (i) $\mathrm{dom}(R_2 + R_3) = \mathbb{N} \times \mathbb{N}$, and

(ii) $\mathrm{dom}(R_2 - R_3) = \mathbb{N} \times \mathbb{N}$, and

(iii) for every natural numbers n, m, $(R_2 + R_3)(n, m) = R_2(n, m) + R_3(n, m)$, and

(iv) for every natural numbers n, m, $(R_2 - R_3)(n, m) = R_2(n, m) - R_3(n, m)$.

(6) Let us consider non empty sets C, D, E and a function f from $C \times D$ into E. Then there exists a function g from $D \times C$ into E such that for every element d of D for every element c of C, $g(d, c) = f(c, d)$. PROOF: Define $\mathcal{F}$(element of D, element of C) $= f(\$_2, \$_1)$. Consider I being a function from $D \times C$ into E such that for every element d of D and for every element c of C, $I(d, c) = \mathcal{F}(d, c)$ from [5, Sch. 2]. □

Let C, D, E be non empty sets and f be a function from $C \times D$ into E. The functor f^{T} yielding a function from $D \times C$ into E is defined by

(Def. 2) Let us consider an element d of D and an element c of C. Then $it(d, c) = f(c, d)$.

Now we state the proposition:

(7) Let us consider non empty sets C, D, E and a function f from $C \times D$ into E. Then $f = (f^{\mathrm{T}})^{\mathrm{T}}$.

The scheme *RecEx2D1* deals with a non empty set $\mathcal{C}$ and a non empty set $\mathcal{D}$ and a function $\mathcal{H}$ from $\mathcal{C}$ into $\mathcal{D}$ and a ternary functor $\mathcal{F}$ yielding an element of $\mathcal{D}$ and states that

(Sch. 1) There exists a function g from $\mathcal{C} \times \mathbb{N}$ into $\mathcal{D}$ such that for every element a of $\mathcal{C}$, $g(a, 0) = \mathcal{H}(a)$ and for every natural number n, $g(a, n+1) = \mathcal{F}(g(a, n), a, n)$.

The scheme *RecEx2D1R* deals with a non empty set $\mathcal{C}$ and a function $\mathcal{H}$ from $\mathcal{C}$ into $\mathbb{R}$ and a ternary functor $\mathcal{F}$ yielding a real number and states that

(Sch. 2) There exists a function g from $\mathcal{C} \times \mathbb{N}$ into $\mathbb{R}$ such that for every element a of $\mathcal{C}$, $g(a, 0) = \mathcal{H}(a)$ and for every natural number n, $g(a, n+1) = \mathcal{F}(g(a, n), a, n)$.

The scheme *RecEx2D2* deals with a non empty set $\mathcal{C}$ and a non empty set $\mathcal{D}$ and a function $\mathcal{H}$ from $\mathcal{C}$ into $\mathcal{D}$ and a ternary functor $\mathcal{F}$ yielding an element of $\mathcal{D}$ and states that

(Sch. 3) There exists a function g from $\mathbb{N} \times \mathcal{C}$ into $\mathcal{D}$ such that for every element a of $\mathcal{C}$, $g(0, a) = \mathcal{H}(a)$ and for every natural number n, $g(n + 1, a) = \mathcal{F}(g(n, a), a, n)$.

The scheme *RecEx2D2R* deals with a non empty set $\mathcal{C}$ and a function $\mathcal{H}$ from $\mathcal{C}$ into $\mathbb{R}$ and a ternary functor $\mathcal{F}$ yielding a real number and states that

(Sch. 4) There exists a function g from $\mathbb{N} \times \mathcal{C}$ into $\mathbb{R}$ such that for every element a of $\mathcal{C}$, $g(0, a) = \mathcal{H}(a)$ and for every natural number n, $g(n + 1, a) = \mathcal{F}(g(n, a), a, n)$.

Let R_1 be a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. The partial sums in the second coordinate of R_1 yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ is defined by

(Def. 3) Let us consider natural numbers n, m. Then

(i) $it(n, 0) = R_1(n, 0)$, and

(ii) $it(n, m + 1) = it(n, m) + R_1(n, m + 1)$.

The partial sums in the first coordinate of R_1 yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ is defined by

(Def. 4) Let us consider natural numbers n, m. Then

(i) $it(0, m) = R_1(0, m)$, and

(ii) $it(n + 1, m) = it(n, m) + R_1(n + 1, m)$.

Now we state the propositions:

(8) (i) the partial sums in the second coordinate of $R_2 + R_3$ = (the partial sums in the second coordinate of R_2)+(the partial sums in the second coordinate of R_3), and

(ii) the partial sums in the first coordinate of $R_2 + R_3$ = (the partial sums in the first coordinate of R_2) + (the partial sums in the first coordinate of R_3).

The theorem is a consequence of (5).

(9) Let us consider natural numbers n, m. Then

(i) (the partial sums in the second coordinate of R_1)(n, m) = (the partial sums in the first coordinate of ${R_1}^{\mathrm{T}}$)(m, n), and

(ii) (the partial sums in the first coordinate of R_1)(n, m) = (the partial sums in the second coordinate of ${R_1}^{\mathrm{T}}$)(m, n).

(10) (i) the partial sums in the second coordinate of R_1 = (the partial sums in the first coordinate of ${R_1}^{\mathrm{T}}$)$^{\mathrm{T}}$, and

(ii) the partial sums in the second coordinate of ${R_1}^{\mathrm{T}}$ = (the partial sums in the first coordinate of R_1)$^{\mathrm{T}}$, and

(iii) (the partial sums in the second coordinate of R_1)$^{\rm T}$ = the partial sums in the first coordinate of ${R_1}^{\rm T}$, and

(iv) (the partial sums in the second coordinate of ${R_1}^{\rm T}$)$^{\rm T}$ = the partial sums in the first coordinate of R_1.

The theorem is a consequence of (9).

Let R_1 be a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. The functor $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}$ yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ is defined by the term

(Def. 5) The partial sums in the second coordinate of the partial sums in the first coordinate of R_1.

Now we state the propositions:

(11) Let us consider natural numbers n, m. Then

(i) $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(n+1, m)$ = (the partial sums in the second coordinate of R_1)$(n+1, m) + (\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(n, m)$, and

(ii) (the partial sums in the first coordinate of the partial sums in the second coordinate of R_1)$(n, m+1)$ = (the partial sums in the first coordinate of R_1)$(n, m+1)$ + (the partial sums in the first coordinate of the partial sums in the second coordinate of R_1)(n, m).

Proof: Set $R_4 = (\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}$. Set C_5 = the partial sums in the first coordinate of the partial sums in the second coordinate of R_1. Set R_5 = the partial sums in the first coordinate of R_1. Set C_6 = the partial sums in the second coordinate of R_1. Define $\mathcal{P}[\text{natural number}] \equiv R_4(n+1, \$_1) = C_6(n+1, \$_1) + R_4(n, \$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. Define $\mathcal{Q}[\text{natural number}] \equiv C_5(\$_1, m+1) = R_5(\$_1, m+1) + C_5(\$_1, m)$. For every natural number k such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k+1]$. For every natural number k, $\mathcal{Q}[k]$ from [3, Sch. 2]. $\square$

(12) $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}$ = the partial sums in the first coordinate of the partial sums in the second coordinate of R_1.

Let us consider natural numbers n, m. Now we state the propositions:

(13) $R_1(n+1, m+1) = (\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(n+1, m+1) - (\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}$ $(n, m+1) - (\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(n+1, m) + (\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(n, m)$.

(14) $R_1(n+1, m+1)$ = (the partial sums in the first coordinate of the partial sums in the second coordinate of R_1)$(n+1, m+1)$ − (the partial sums in the first coordinate of the partial sums in the second coordinate of R_1)$(n+1, m)$ − (the partial sums in the first coordinate of the partial sums in the second coordinate of R_1)$(n, m+1)$ + (the partial sums in the first coordinate of the partial sums in the second coordinate of R_1)(n, m).

Now we state the propositions:

(15) If $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}$ is p-convergent, then R_1 is p-convergent and P-lim R_1

$= 0$. Proof: For every real number e such that $0 < e$ there exists a natural number N such that for every natural numbers n, m such that $n \geqslant N$ and $m \geqslant N$ holds $|R_1(n, m) - 0| < e$ by [3, (13), (20)], (13), [8, (57)]. $\square$

(16) $(\sum_{\alpha=0}^{\kappa}(R_2 + R_3)(\alpha))_{\kappa \in \mathbb{N}} = (\sum_{\alpha=0}^{\kappa} R_2(\alpha))_{\kappa \in \mathbb{N}} + (\sum_{\alpha=0}^{\kappa} R_3(\alpha))_{\kappa \in \mathbb{N}}$. The theorem is a consequence of (8).

(17) Suppose $(\sum_{\alpha=0}^{\kappa} R_2(\alpha))_{\kappa \in \mathbb{N}}$ is p-convergent and $(\sum_{\alpha=0}^{\kappa} R_3(\alpha))_{\kappa \in \mathbb{N}}$ is p-convergent. Then $(\sum_{\alpha=0}^{\kappa}(R_2 + R_3)(\alpha))_{\kappa \in \mathbb{N}}$ is p-convergent. The theorem is a consequence of (16).

(18) Let us consider elements m, n of $\mathbb{N}$. Then

   (i) (the partial sums in the first coordinate of $R_1)(m, n) = (\sum_{\alpha=0}^{\kappa}(\mathrm{curry}'(R_1, n))(\alpha))_{\kappa \in \mathbb{N}}(m)$, and

   (ii) (the partial sums in the second coordinate of $R_1)(m, n) = (\sum_{\alpha=0}^{\kappa}(\mathrm{curry}(R_1, m))(\alpha))_{\kappa \in \mathbb{N}}(n)$.

   Proof: Define $\mathcal{P}$[natural number] $\equiv$ (the partial sums in the first coordinate of $R_1)(\$_1, n) = (\sum_{\alpha=0}^{\kappa}(\mathrm{curry}'(R_1, n))(\alpha))_{\kappa \in \mathbb{N}}(\$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. Define $\mathcal{Q}$[natural number] $\equiv$ (the partial sums in the second coordinate of $R_1)(m, \$_1) = (\sum_{\alpha=0}^{\kappa}(\mathrm{curry}(R_1, m))(\alpha))_{\kappa \in \mathbb{N}}(\$_1)$. For every natural number k such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k + 1]$. For every natural number k, $\mathcal{Q}[k]$ from [3, Sch. 2]. $\square$

(19) (i) $\mathrm{curry}((\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}, 0) = \mathrm{curry}$(the partial sums in the second coordinate of $R_1, 0)$, and

   (ii) $\mathrm{curry}'((\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}, 0) = \mathrm{curry}'$(the partial sums in the first coordinate of $R_1, 0)$.

   The theorem is a consequence of (12).

(20) Let us consider non empty sets C, D, functions F_1, F_2 from $C \times D$ into $\mathbb{R}$, and an element c of C. Then $\mathrm{curry}(F_1 + F_2, c) = \mathrm{curry}(F_1, c) + \mathrm{curry}(F_2, c)$.

(21) Let us consider non empty sets C, D, functions F_1, F_2 from $C \times D$ into $\mathbb{R}$, and an element d of D. Then $\mathrm{curry}'(F_1 + F_2, d) = \mathrm{curry}'(F_1, d) + \mathrm{curry}'(F_2, d)$.

(22) $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the first coordinate if and only if the partial sums in the first coordinate of R_1 is convergent in the first coordinate. The theorem is a consequence of (19), (12), and (11).

(23) $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the second coordinate if and only if the partial sums in the second coordinate of R_1 is convergent in the second coordinate. The theorem is a consequence of (19), (12), and (11).

Let us consider a natural number k. Now we state the propositions:

(24) Suppose $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the first coordinate. Then (the lim in the first coordinate of $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}})(k + 1) =$ (the lim

in the first coordinate of $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}})(k)$ + (the lim in the first coordinate of the partial sums in the first coordinate of $R_1)(k+1)$. The theorem is a consequence of (22).

(25) Suppose $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}$ is convergent in the second coordinate. Then (the lim in the second coordinate of $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}})(k+1)$ = (the lim in the second coordinate of $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}})(k)$ + (the lim in the second coordinate of the partial sums in the second coordinate of $R_1)(k+1)$. The theorem is a consequence of (23) and (12).

Now we state the propositions:

(26) Suppose $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}$ is convergent in the first coordinate. Then the lim in the first coordinate of $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}$(the lim in the first coordinate of the partial sums in the first coordinate of $R_1)(\alpha))_{\kappa\in\mathbb{N}}$. The theorem is a consequence of (19) and (24).

(27) Suppose $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}$ is convergent in the second coordinate. Then the lim in the second coordinate of $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}$(the lim in the second coordinate of the partial sums in the second coordinate of $R_1)(\alpha))_{\kappa\in\mathbb{N}}$. The theorem is a consequence of (19) and (25).

2. Double Series of Non-Negative Double Sequence

Let us assume that R_1 is non-negative yielding. Now we state the propositions:

(28) (i) the partial sums in the second coordinate of R_1 is non-negative yielding, and

(ii) the partial sums in the first coordinate of R_1 is non-negative yielding.

(29) $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}$ is non-decreasing. The theorem is a consequence of (11) and (28).

(30) $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}$ is p-convergent if and only if $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}$ is lower bounded and upper bounded. The theorem is a consequence of (29).

Let us consider natural numbers i, j. Now we state the propositions:

(31) Suppose for every natural numbers n, m, $R_2(n,m) \leqslant R_3(n,m)$. Then

(i) (the partial sums in the first coordinate of $R_2)(i,j) \leqslant$ (the partial sums in the first coordinate of $R_3)(i,j)$, and

(ii) (the partial sums in the second coordinate of $R_2)(i,j) \leqslant$ (the partial sums in the second coordinate of $R_3)(i,j)$.

Proof: Set R_4 = the partial sums in the first coordinate of R_2. Set R_5 = the partial sums in the first coordinate of R_3. Set C_1 = the partial sums in the second coordinate of R_2. Set C_2 = the partial sums in the second coordinate of R_3. Define $\mathcal{R}$[natural number] $\equiv R_4(\$_1, j) \leqslant R_5(\$_1, j)$. For every natural number k such that $\mathcal{R}[k]$ holds $\mathcal{R}[k+1]$. For every natural number k, $\mathcal{R}[k]$ from [3, Sch. 2]. Define $\mathcal{C}[\text{natural number}] \equiv C_1(i, \$_1) \leqslant C_2(i, \$_1)$. For every natural number k such that $\mathcal{C}[k]$ holds $\mathcal{C}[k+1]$. For every natural number k, $\mathcal{C}[k]$ from [3, Sch. 2]. □

(32) Suppose R_2 is non-negative yielding and for every natural numbers n, m, $R_2(n,m) \leqslant R_3(n,m)$. Then $(\sum_{\alpha=0}^{\kappa} R_2(\alpha))_{\kappa \in \mathbb{N}}(i,j) \leqslant (\sum_{\alpha=0}^{\kappa} R_3(\alpha))_{\kappa \in \mathbb{N}}(i, j)$. Proof: Set $R_4 = (\sum_{\alpha=0}^{\kappa} R_2(\alpha))_{\kappa \in \mathbb{N}}$. Set $R_5 = (\sum_{\alpha=0}^{\kappa} R_3(\alpha))_{\kappa \in \mathbb{N}}$. Define $\mathcal{P}[\text{natural number}] \equiv R_4(i, \$_1) \leqslant R_5(i, \$_1)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. □

Now we state the propositions:

(33) Suppose R_2 is non-negative yielding and for every natural numbers n, m, $R_2(n,m) \leqslant R_3(n,m)$ and $(\sum_{\alpha=0}^{\kappa} R_3(\alpha))_{\kappa \in \mathbb{N}}$ is p-convergent. Then $(\sum_{\alpha=0}^{\kappa} R_2(\alpha))_{\kappa \in \mathbb{N}}$ is p-convergent. The theorem is a consequence of (29) and (32).

(34) Let us consider a sequence r_1 of real numbers and a natural number m. Suppose r_1 is non-negative. Then $r_1(m) \leqslant (\sum_{\alpha=0}^{\kappa} r_1(\alpha))_{\kappa \in \mathbb{N}}(m)$. Proof: Define $\mathcal{P}[\text{natural number}] \equiv r_1(\$_1) \leqslant (\sum_{\alpha=0}^{\kappa} r_1(\alpha))_{\kappa \in \mathbb{N}}(\$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [19, (34)]. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. □

Let us assume that R_1 is non-negative yielding. Now we state the propositions:

(35) Let us consider natural numbers m, n. Then

(i) $R_1(m,n) \leqslant$ (the partial sums in the first coordinate of R_1)(m,n), and

(ii) $R_1(m,n) \leqslant$ (the partial sums in the second coordinate of R_1)(m,n).

The theorem is a consequence of (34) and (18).

(36) (i) for every natural numbers i_1, i_2, j such that $i_1 \leqslant i_2$ holds (the partial sums in the first coordinate of R_1)$(i_1, j) \leqslant$ (the partial sums in the first coordinate of R_1)(i_2, j), and

(ii) for every natural numbers i, j_1, j_2 such that $j_1 \leqslant j_2$ holds (the partial sums in the second coordinate of R_1)$(i, j_1) \leqslant$ (the partial sums in the second coordinate of R_1)(i, j_2).

(37) (i) for every natural numbers i_1, i_2, j such that $i_1 \leqslant i_2$ holds $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(i_1, j) \leqslant (\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(i_2, j)$, and

(ii) for every natural numbers i, j_1, j_2 such that $j_1 \leqslant j_2$ holds $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(i, j_1) \leqslant (\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(i, j_2)$.

The theorem is a consequence of (36).

(38) Let us consider natural numbers i_1, i_2, j_1, j_2. Suppose

(i) $i_1 \leqslant i_2$, and

(ii) $j_1 \leqslant j_2$.

Then $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(i_1, j_1) \leqslant (\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(i_2, j_2)$. The theorem is a consequence of (37).

(39) Let us consider an element k of $\mathbb{N}$. Then

(i) curry$'$(the partial sums in the first coordinate of R_1, k) is non-decreasing, and

(ii) curry(the partial sums in the second coordinate of R_1, k) is non-decreasing, and

(iii) curry$'$(the partial sums in the first coordinate of R_1, k) is non-negative, and

(iv) curry(the partial sums in the second coordinate of R_1, k) is non-negative, and

(v) curry$'$(the partial sums in the second coordinate of R_1, k) is non-negative, and

(vi) curry(the partial sums in the first coordinate of R_1, k) is non-negative.

The theorem is a consequence of (18) and (34).

Let us assume that R_1 is non-negative yielding and $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}$ is p-convergent. Now we state the propositions:

(40) (i) the partial sums in the first coordinate of R_1 is convergent in the first coordinate, and

(ii) the partial sums in the second coordinate of R_1 is convergent in the second coordinate.

The theorem is a consequence of (39), (18), (34), and (29).

(41) $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the first coordinate and convergent in the second coordinate. The theorem is a consequence of (40), (22), and (23).

(42) (i) the lim in the first coordinate of the partial sums in the first coordinate of R_1 is summable, and

(ii) the lim in the second coordinate of the partial sums in the second coordinate of R_1 is summable.

The theorem is a consequence of (41), (26), and (27).

(43) (i) P-lim$(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}} = \sum$(the lim in the first coordinate of the partial sums in the first coordinate of R_1), and

(ii) P-lim$(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}} = \sum$(the lim in the second coordinate of the partial sums in the second coordinate of R_1).

The theorem is a consequence of (41), (26), and (27).

3. Summability for Rearrangements of Non-Negative Real Sequence

Now we state the propositions:

(44) Let us consider sequences s_1, s_2 of real numbers. Suppose

(i) s_1 is non-negative, and

(ii) s_1 and s_2 are fiberwise equipotent.

Then s_2 is non-negative.

(45) Let us consider a non empty set X, a sequence s of X, and a natural number n. Then $\operatorname{dom}\operatorname{Shift}(s{\restriction}\mathbb{Z}_n, 1) = \operatorname{Seg} n$.

Let X be a non empty set, s be a sequence of X, and n be a natural number. Note that $\operatorname{Shift}(s{\restriction}\mathbb{Z}_n, 1)$ is finite sequence-like.

Now we state the propositions:

(46) Let us consider a non empty set X, a sequence s of X, and a natural number n. Then $\operatorname{Shift}(s{\restriction}\mathbb{Z}_n, 1)$ is a finite sequence of elements of X.

(47) Let us consider a non empty set X, a sequence s of X, and natural numbers n, m. Suppose $m+1 \in \operatorname{dom}\operatorname{Shift}(s{\restriction}\mathbb{Z}_n, 1)$. Then $(\operatorname{Shift}(s{\restriction}\mathbb{Z}_n, 1))(m+1) = s(m)$.

(48) Let us consider a non empty set X and a sequence s of X. Then

(i) $\operatorname{Shift}(s{\restriction}\mathbb{Z}_0, 1) = \emptyset$, and

(ii) $\operatorname{Shift}(s{\restriction}\mathbb{Z}_1, 1) = \langle s(0)\rangle$, and

(iii) $\operatorname{Shift}(s{\restriction}\mathbb{Z}_2, 1) = \langle s(0), s(1)\rangle$, and

(iv) for every natural number n, $\operatorname{Shift}(s{\restriction}\mathbb{Z}_{n+1}, 1) = \operatorname{Shift}(s{\restriction}\mathbb{Z}_n, 1)^\frown\langle s(n)\rangle$.

The theorem is a consequence of (45) and (47).

(49) Let us consider a sequence s of real numbers and a natural number n. Then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \sum \operatorname{Shift}(s{\restriction}\mathbb{Z}_{n+1}, 1)$. Proof: Define $\mathcal{P}$[natural number] $\equiv (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(\$_1) = \sum \operatorname{Shift}(s{\restriction}\mathbb{Z}_{\$_1+1}, 1)$. $\operatorname{Shift}(s{\restriction}\mathbb{Z}_{0+1}, 1) = \langle s(0)\rangle$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by (48), [14, (74)]. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. □

(50) Let us consider a non empty set X, sequences s_1, s_2 of X, and a natural number n. Suppose s_1 and s_2 are fiberwise equipotent. Then there exists a natural number m and there exists a subset f_2 of $\operatorname{Shift}(s_2{\restriction}\mathbb{Z}_m, 1)$ such that $\operatorname{Shift}(s_1{\restriction}\mathbb{Z}_{n+1}, 1)$ and f_2 are fiberwise equipotent. Proof: Consider P being a permutation of $\operatorname{dom} s_1$ such that $s_1 = s_2 \cdot P$. Define $\mathcal{F}(\text{set}) = P(\$_1) + 1$. Define $\mathcal{G}[\text{set}] \equiv \$_1$ is a natural number. $\{\mathcal{F}(i)$, where i is a natural number : $i \leqslant n$ and $\mathcal{G}[i]\}$ is finite from [6, Sch. 6]. Reconsider $D = \{\mathcal{F}(i)$, where i is a natural number : $i \leqslant n$ and $\mathcal{G}[i]\}$ as a finite set. Set $f_2 = \{\langle j+1, s_2(j)\rangle$, where j is a natural number : $j+1 \in D\}$. Define $\mathcal{G}[\text{object}, \text{object}] \equiv$ there exists a natural number i such that $\$_1 = i + 1$

and $\$_2 = P(i) + 1$. For every object x such that $x \in \mathrm{Seg}(n + 1)$ there exists an object y such that $\mathcal{G}[x, y]$ by [6, (1)], [3, (21)]. Consider G being a function such that $\mathrm{dom}\, G = \mathrm{Seg}(n + 1)$ and for every object x such that $x \in \mathrm{Seg}(n + 1)$ holds $\mathcal{G}[x, G(x)]$ from [11, Sch. 2]. $\mathrm{dom}\, G = \mathrm{dom}\,\mathrm{Shift}(s_1 \restriction \mathbb{Z}_{n+1}, 1)$. $\mathrm{dom}(f_2 \cdot G) = \mathrm{dom}\,\mathrm{Shift}(s_1 \restriction \mathbb{Z}_{n+1}, 1)$. For every object x such that $x \in \mathrm{dom}\,\mathrm{Shift}(s_1 \restriction \mathbb{Z}_{n+1}, 1)$ holds $(\mathrm{Shift}(s_1 \restriction \mathbb{Z}_{n+1}, 1))(x) = (f_2 \cdot G)(x)$ by (45), [6, (1)], [3, (21)], (47). □

(51) Let us consider a non empty set X, a finite sequence f_1 of elements of X, and a subset f_3 of f_1. Then $\mathrm{Seq}\, f_3$ and f_3 are fiberwise equipotent.

(52) Let us consider sequences s_1, s_2 of real numbers and a natural number n. Suppose

(i) s_1 and s_2 are fiberwise equipotent, and

(ii) s_1 is non-negative.

Then there exists a natural number m such that $(\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa \in \mathbb{N}}(n) \leqslant (\sum_{\alpha=0}^{\kappa} s_2(\alpha))_{\kappa \in \mathbb{N}}(m)$. The theorem is a consequence of (44), (50), (46), (51), (47), (49), and (48).

(53) Let us consider sequences s_1, s_2 of real numbers. Suppose

(i) s_1 and s_2 are fiberwise equipotent, and

(ii) s_1 is non-negative and summable.

Then

(iii) s_2 is summable, and

(iv) $\sum s_1 = \sum s_2$.

The theorem is a consequence of (44) and (52).

4. Basic Relations between Double Sequences and Matrices

Now we state the propositions:

(54) Let us consider a non empty set D, a function R_1 from $\mathbb{N} \times \mathbb{N}$ into D, and natural numbers n, m. Then there exists a matrix M over D of dimension $n+1 \times m+1$ such that for every natural numbers i, j such that $i \leqslant n$ and $j \leqslant m$ holds $R_1(i, j) = M_{i+1,j+1}$. Proof: Define $\mathcal{P}$[natural number, natural number, object] $\equiv$ there exist natural numbers i_1, j_1 such that $i_1 = \$_1 - 1$ and $j_1 = \$_2 - 1$ and $\$_3 = R_1(i_1, j_1)$. Consider M being a matrix over D of dimension $n + 1 \times m + 1$ such that for every natural numbers i, j such that $\langle i, j \rangle \in$ the indices of M holds $\mathcal{P}[i, j, M_{i,j}]$. □

(55) Let us consider natural numbers n, m, a function R_1 from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$, and a matrix M over $\mathbb{R}$ of dimension $n + 1 \times m + 1$. Suppose natural numbers i, j. If $i \leqslant n$ and $j \leqslant m$, then $R_1(i, j) = M_{i+1,j+1}$. Then

$(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}(n,m) = \mathrm{SumAll}\, M$. Proof: For every natural number i such that $i \leqslant n$ holds (the partial sums in the second coordinate of $R_1)(i,m) = (\mathrm{LineSum}\, M)(i+1)$ by [3, (11)], [6, (1), (59)], [26, (112)]. Define $\mathcal{G}$[natural number] $\equiv$ if $\$_1 \leqslant n$, then (the partial sums in the first coordinate of the partial sums in the second coordinate of $R_1)(\$_1, m) = \sum(\mathrm{LineSum}\, M\restriction(\$_1+1))$. For every natural number k such that $\mathcal{G}[k]$ holds $\mathcal{G}[k+1]$ by [3, (11)], [30, (20)], [6, (59)], [10, (21)]. For every natural number k, $\mathcal{G}[k]$ from [3, Sch. 2]. □

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Received March 31, 2014

FORMALIZED MATHEMATICS
Vol. 22, No. 1, Pages 69–77, 2014
DOI: 10.2478/forma-2014-0007

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Dual Spaces and Hahn-Banach Theorem[1]

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Summary. In this article, we deal with dual spaces and the Hahn-Banach Theorem. At the first, we defined dual spaces of real linear spaces and proved related basic properties. Next, we defined dual spaces of real normed spaces. We formed the definitions based on dual spaces of real linear spaces. In addition, we proved properties of the norm about elements of dual spaces. For the proof we referred to descriptions in the article [21]. Finally, applying theorems of the second section, we proved the Hahn-Banach extension theorem in real normed spaces. We have used extensively used [17].

MSC: 46A22 46E15 03B35

Keywords: dual space; Hahn-Banach extension

MML identifier: DUALSP01, version: 8.1.03 5.23.1204

The notation and terminology used in this paper have been introduced in the following articles: [5], [16], [23], [18], [6], [7], [17], [15], [21], [24], [1], [2], [20], [3], [8], [4], [28], [25], [26], [10], [22], [12], [13], [27], [14], and [9].

1. Dual Spaces of Real Linear Spaces

From now on V denotes a non empty real linear space.

Let X be a real linear space. The functor MultFReal$*\, X$ yielding a function from (the carrier of $\mathbb{R}_{\mathrm{F}}$) $\times$ (the carrier of X) into the carrier of X is defined by the term

(Def. 1) The external multiplication of X.

[1]This work was supported by JSPS KAKENHI 22300285, 23500029.

ISSN 1426–2630(Print), 1898-9934(Online)

Now we state the proposition:

(1) Let us consider a real linear space X. Then $\langle$the carrier of X, the addition of X, the zero of X, MultFReal$*\, X\rangle$ is a vector space over $\mathbb{R}_{\rm F}$.

Let X be a real linear space. The functor RLSp2RVSp X yielding a vector space over $\mathbb{R}_{\rm F}$ is defined by the term

(Def. 2) $\langle$the carrier of X, the addition of X, the zero of X, MultFReal$*\, X\rangle$.

Let X be a vector space structure over $\mathbb{R}_{\rm F}$. The functor MultReal$*\, X$ yielding a function from $\mathbb{R} \times$ (the carrier of X) into the carrier of X is defined by the term

(Def. 3) The left multiplication of X.

Now we state the proposition:

(2) Let us consider a vector space X over $\mathbb{R}_{\rm F}$. Then $\langle$the carrier of X, the zero of X, the addition of X, MultReal$*\, X\rangle$ is a real linear space.

Let X be a vector space over $\mathbb{R}_{\rm F}$. The functor RVSp2RLSp X yielding a real linear space is defined by the term

(Def. 4) $\langle$the carrier of X, the zero of X, the addition of X, MultReal$*\, X\rangle$.

Now we state the propositions:

(3) Let us consider a real linear space X, elements v, w of X, and elements v_1, w_1 of RLSp2RVSp X. If $v = v_1$ and $w = w_1$, then $v + w = v_1 + w_1$ and $v - w = v_1 - w_1$.

(4) Let us consider a vector space X over $\mathbb{R}_{\rm F}$, elements v, w of X, and elements v_1, w_1 of RVSp2RLSp X. If $v = v_1$ and $w = w_1$, then $v + w = v_1 + w_1$ and $v - w = v_1 - w_1$.

Let V be a non empty real linear space. The functor $\overline{V}$ yielding a strict non empty real linear space is defined by

(Def. 5) There exists a non empty vector space X over $\mathbb{R}_{\rm F}$ such that

(i) $X =$ RLSp2RVSp V, and

(ii) $it =$ RVSp2RLSp $\overline{X}$.

Now we state the proposition:

(5) Let us consider an object x. Then $x \in$ the carrier of $\overline{V}$ if and only if x is a linear functional in V.

Let V be a non empty real linear space. One can check that $\overline{V}$ is constituted functions.

Let f be an element of $\overline{V}$ and v be a vector of V. Note that the functor $f(v)$ yields an element of $\mathbb{R}$. Now we state the propositions:

(6) Let us consider a non empty real linear space V and vectors f, g, h of $\overline{V}$. Then $h = f + g$ if and only if for every vector x of V, $h(x) = f(x) + g(x)$.

(7) Let us consider a non empty real linear space V, vectors f, h of $\overline{V}$, and a real number a. Then $h = a \cdot f$ if and only if for every vector x of V, $h(x) = a \cdot f(x)$.

(8) Let us consider a non empty real linear space V. Then $0_{\overline{V}} =$ (the carrier of V) $\longmapsto 0$.

(9) Let us consider a real linear space X. Then (the carrier of X) $\longmapsto 0$ is a linear functional in X. PROOF: Set $f =$ (the carrier of X) $\longmapsto 0$. f is additive by [23, (7)]. f is homogeneous by [23, (7)]. $\square$

Let X be a real linear space. The linear functionals of X yielding a subset of $\mathbb{R}_{\mathbb{R}}^{(\text{the carrier of } X)}$ is defined by

(Def. 6) Let us consider an object x. Then $x \in it$ if and only if x is a linear functional in X.

Let X be a real normed space. One can verify that the linear functionals of X is non empty.

Let X be a real linear space. One can verify that the linear functionals of X is non empty and functional.

Let us consider a real linear space X. Now we state the propositions:

(10) The linear functionals of X is linearly closed. PROOF: Set $W =$ the linear functionals of X. For every vectors v, u of $\mathbb{R}_{\mathbb{R}}^{\alpha}$ such that $v, u \in$ the linear functionals of X holds $v + u \in$ the linear functionals of X, where α is the carrier of X by [7, (66)], [18, (1)]. For every real number a and for every vector v of $\mathbb{R}_{\mathbb{R}}^{\alpha}$ such that $v \in W$ holds $a \cdot v \in W$, where α is the carrier of X by [7, (66)], [18, (4)]. $\square$

(11) $\langle$the linear functionals of X, Zero(the linear functionals of $X, \mathbb{R}_{\mathbb{R}}^{\alpha}$), Add (the linear functionals of $X, \mathbb{R}_{\mathbb{R}}^{\alpha}$), Mult(the linear functionals of $X, \mathbb{R}_{\mathbb{R}}^{\alpha}$)$\rangle$ is a subspace of $\mathbb{R}_{\mathbb{R}}^{\alpha}$, where α is the carrier of X.

Let X be a real linear space. Note that $\langle$the linear functionals of X, Zero (the linear functionals of $X, \mathbb{R}_{\mathbb{R}}^{(\text{the carrier of } X)}$), Add(the linear functionals of $X, \mathbb{R}_{\mathbb{R}}^{(\text{the carrier of } X)}$), Mult(the linear functionals of $X, \mathbb{R}_{\mathbb{R}}^{(\text{the carrier of } X)}$)$\rangle$ is Abelian add-associative right zeroed right complementable scalar distributive vector distributive scalar associative and scalar unital.

The functor $\overline{X}$ yielding a strict real linear space is defined by the term

(Def. 7) $\langle$the linear functionals of X, Zero(the linear functionals of $X, \mathbb{R}_{\mathbb{R}}^{\alpha}$), Add (the linear functionals of $X, \mathbb{R}_{\mathbb{R}}^{\alpha}$), Mult(the linear functionals of $X, \mathbb{R}_{\mathbb{R}}^{\alpha}$)$\rangle$, where α is the carrier of X.

Observe that $\overline{X}$ is constituted functions.

Let f be an element of $\overline{X}$ and v be a vector of X. One can verify that the functor $f(v)$ yields an element of $\mathbb{R}$. Now we state the propositions:

(12) Let us consider a real linear space X and vectors f, g, h of $\overline{X}$. Then $h = f + g$ if and only if for every vector x of X, $h(x) = f(x) + g(x)$. The

theorem is a consequence of (10).

(13) Let us consider a real linear space X, vectors f, h of $\overline{X}$, and a real number a. Then $h = a \cdot f$ if and only if for every vector x of X, $h(x) = a \cdot f(x)$. The theorem is a consequence of (10).

(14) Let us consider a real linear space X. Then $0_{\overline{X}} = (\text{the carrier of } X) \longmapsto 0$. The theorem is a consequence of (10).

2. Dual Spaces of Real Normed Spaces

In the sequel S denotes a sequence of real numbers, k, n, m, m_1 denote natural numbers, and g, h, r, x denote real numbers.

Let S be a sequence of real numbers and x be a real number. The functor $S - x$ yielding a sequence of real numbers is defined by

(Def. 8) $it(n) = S(n) - x$.

Now we state the proposition:

(15) If S is convergent, then $S - x$ is convergent and $\lim(S - x) = \lim S - x$.

Let X be a real normed space and I be a functional in X. We say that I is Lipschitzian if and only if

(Def. 9) There exists a real number K such that

(i) $0 \leqslant K$, and

(ii) for every vector x of X, $|I(x)| \leqslant K \cdot \|x\|$.

Now we state the proposition:

(16) Let us consider a real normed space X and a functional f in X. If for every vector x of X, $f(x) = 0$, then f is Lipschitzian.

Let X be a real normed space. One can check that there exists a linear functional in X which is Lipschitzian.

The bounded linear functionals X yielding a subset of $\overline{X}$ is defined by

(Def. 10) Let us consider a set x. Then $x \in it$ if and only if x is a Lipschitzian linear functional in X.

One can check that the bounded linear functionals X is non empty.

Let us consider a real normed space X. Now we state the propositions:

(17) The bounded linear functionals X is linearly closed. Proof: Set $W =$ the bounded linear functionals X. For every vectors v, u of $\overline{X}$ such that $v, u \in W$ holds $v + u \in W$ by [5, (56)], (12). For every real number a and for every vector v of $\overline{X}$ such that $v \in W$ holds $a \cdot v \in W$ by [5, (46), (65)], (13). □

(18) $\langle$the bounded linear functionals X, Zero(the bounded linear functionals $X, \overline{X}$), Add(the bounded linear functionals $X, \overline{X}$), Mult(the bounded linear functionals $X, \overline{X}$)$\rangle$ is a subspace of $\overline{X}$.

Let X be a real normed space. Let us observe that $\langle$the bounded linear functionals X, Zero(the bounded linear functionals $X, \overline{X}$), Add(the bounded linear functionals $X, \overline{X}$), Mult(the bounded linear functionals $X, \overline{X}$)$\rangle$ is Abelian add-associative right zeroed right complementable vector distributive scalar distributive scalar associative and scalar unital.

The $\mathbb{R}$-vector space of bounded linear functionals of X yielding a strict real linear space is defined by the term

(Def. 11) $\langle$the bounded linear functionals X, Zero(the bounded linear functionals $X, \overline{X}$), Add(the bounded linear functionals $X, \overline{X}$), Mult(the bounded linear functionals $X, \overline{X}$)$\rangle$.

One can check that every element of the $\mathbb{R}$-vector space of bounded linear functionals of X is function-like and relation-like.

Let f be an element of the $\mathbb{R}$-vector space of bounded linear functionals of X and v be a vector of X. Note that the functor $f(v)$ yields an element of $\mathbb{R}$. Now we state the propositions:

(19) Let us consider a real normed space X and vectors f, g, h of the $\mathbb{R}$-vector space of bounded linear functionals of X. Then $h = f + g$ if and only if for every vector x of X, $h(x) = f(x) + g(x)$. The theorem is a consequence of (17) and (12).

(20) Let us consider a real normed space X, vectors f, h of the $\mathbb{R}$-vector space of bounded linear functionals of X, and a real number a. Then $h = a \cdot f$ if and only if for every vector x of X, $h(x) = a \cdot f(x)$. The theorem is a consequence of (17) and (13).

(21) Let us consider a real normed space X. Then $0_\alpha =$ (the carrier of X) $\longmapsto 0$, where α is the $\mathbb{R}$-vector space of bounded linear functionals of X. The theorem is a consequence of (14) and (17).

Let X be a real normed space and f be an object.

The functor Bound2Lipschitz(f, X) yielding a Lipschitzian linear functional in X is defined by the term

(Def. 12) $f(\in$ the bounded linear functionals $X)$.

Let u be a linear functional in X. The functor PreNorms(u) yielding a non empty subset of $\mathbb{R}$ is defined by the term

(Def. 13) $\{|u(t)|$, where t is a vector of $X : \|t\| \leqslant 1\}$.

Let g be a Lipschitzian linear functional in X. Observe that PreNorms(g) is upper bounded.

Now we state the proposition:

(22) Let us consider a real normed space X and a linear functional g in X. Then g is Lipschitzian if and only if PreNorms(g) is upper bounded.

Let X be a real normed space. The bounded linear functionals norm X yielding a function from the bounded linear functionals X into $\mathbb{R}$ is defined by

(Def. 14) Let us consider an object x. Suppose $x \in$ the bounded linear functionals X. Then $it(x) = \sup \mathrm{PreNorms}(\mathrm{Bound2Lipschitz}(x, X))$.

Let us consider a real normed space X and a Lipschitzian linear functional f in X. Now we state the propositions:

(23) $\mathrm{Bound2Lipschitz}(f, X) = f$.

(24) (The bounded linear functionals norm $X)(f) = \sup \mathrm{PreNorms}(f)$. The theorem is a consequence of (23).

Let X be a real normed space. The functor DualSp X yielding a non empty normed structure is defined by the term

(Def. 15) $\langle$the bounded linear functionals X, Zero(the bounded linear functionals $X, \overline{X}$), Add(the bounded linear functionals $X, \overline{X}$), Mult(the bounded linear functionals $X, \overline{X}$), the bounded linear functionals norm $X\rangle$.

Now we state the propositions:

(25) Let us consider a real normed space X. Then (the carrier of X) $\longmapsto 0 = 0_{\mathrm{DualSp}\, X}$. The theorem is a consequence of (21).

(26) Let us consider a real normed space X, a point f of DualSp X, and a Lipschitzian linear functional g in X. Suppose $g = f$. Let us consider a vector t of X. Then $|g(t)| \leqslant \|f\| \cdot \|t\|$. The theorem is a consequence of (24).

(27) Let us consider a real normed space X and a point f of DualSp X. Then $0 \leqslant \|f\|$. The theorem is a consequence of (24).

(28) Let us consider real normed spaces X, Y and a point f of DualSp X. If $f = 0_{\mathrm{DualSp}\, X}$, then $0 = \|f\|$. PROOF: $\|f\| = 0$ by [23, (45)], [13, (45)], (25), [23, (7)]. □

Let X be a real normed space. Note that every element of DualSp X is function-like and relation-like.

Let f be an element of DualSp X and v be a vector of X. Let us note that the functor $f(v)$ yields an element of $\mathbb{R}$. Now we state the propositions:

(29) Let us consider a real normed space X and points f, g, h of DualSp X. Then $h = f + g$ if and only if for every vector x of X, $h(x) = f(x) + g(x)$. The theorem is a consequence of (19).

(30) Let us consider a real normed space X, points f, h of DualSp X, and a real number a. Then $h = a \cdot f$ if and only if for every vector x of X, $h(x) = a \cdot f(x)$. The theorem is a consequence of (20).

(31) Let us consider a real normed space X, points f, g of DualSp X, and a real number a. Then

(i) $\|f\| = 0$ iff $f = 0_{\mathrm{DualSp}\, X}$, and

(ii) $\|a \cdot f\| = |a| \cdot \|f\|$, and

(iii) $\|f + g\| \leqslant \|f\| + \|g\|$.

Proof: $\|f+g\| \leqslant \|f\| + \|g\|$ by [13, (45)], (27), [5, (56)], (26). $\|a \cdot f\| = |a| \cdot \|f\|$ by (27), (26), [5, (65), (46)]. □

Let X be a real normed space. Note that DualSp X is reflexive discernible and real normed space-like.

Now we state the proposition:

(32) Let us consider a real normed space X. Then DualSp X is a real normed space.

Let X be a real normed space. Let us note that DualSp X is reflexive discernible real normed space-like vector distributive scalar distributive scalar associative scalar unital Abelian add-associative right zeroed and right complementable.

Now we state the proposition:

(33) Let us consider a real normed space X and points f, g, h of DualSp X. Then $h = f - g$ if and only if for every vector x of X, $h(x) = f(x) - g(x)$. The theorem is a consequence of (29).

Let X be a real normed space, s be a sequence of DualSp X, and n be a natural number. Let us note that the functor $s(n)$ yields an element of DualSp X. Now we state the propositions:

(34) Let us consider a real normed space X and a sequence s_1 of DualSp X. If s_1 is Cauchy sequence by norm, then s_1 is convergent. Proof: Define $\mathcal{P}[\text{set}, \text{set}] \equiv$ there exists a sequence x_1 of $\mathbb{R}$ such that for every natural number n, $x_1(n) = (\text{Bound2Lipschitz}(vseq(n), X))(\$_1)$ and x_1 is convergent and $\$_2 = \lim x_1$. For every element x of X, there exists an element y of $\mathbb{R}$ such that $\mathcal{P}[x, y]$ by (23), (33), (26), [5, (44)]. Consider f being a function from the carrier of X into $\mathbb{R}$ such that for every element x of X, $\mathcal{P}[x, f(x)]$ from [7, Sch. 3]. Reconsider $t_1 = f$ as a function from the carrier of X into $\mathbb{R}$. t_1 is Lipschitzian by [13, (14)], [11, (12)], (23), (26). For every real number e such that $e > 0$ there exists a natural number k such that for every natural number n such that $n \geqslant k$ for every vector x of X, $|(\text{Bound2Lipschitz}(vseq(n), X))(x) - t_1(x)| \leqslant e \cdot \|x\|$ by [22, (8)], (23), (33), (26). Reconsider $t_2 = t_1$ as a point of DualSp X. For every real number e such that $e > 0$ there exists a natural number k such that for every natural number n such that $n \geqslant k$ holds $\|vseq(n) - t_2\| \leqslant e$ by (23), (33), [13, (45)], (24). For every real number e such that $e > 0$ there exists a natural number m such that for every natural number n such that $n \geqslant m$ holds $\|vseq(n) - t_2\| < e$. □

(35) Let us consider a real normed space X. Then DualSp X is a real Banach space. The theorem is a consequence of (34).

Let X be a real normed space. One can verify that DualSp X is complete.

3. Hahn-Banach Extension Theorem

Let V be a real normed space.

A subreal normal space of V is a real normed space and is defined by

(Def. 16) (i) the carrier of $it \subseteq$ the carrier of V, and

(ii) $0_{it} = 0_V$, and

(iii) the addition of $it =$ (the addition of V) $\restriction$ (the carrier of it), and

(iv) the external multiplication of $it =$ (the external multiplication of V)$\restriction$($\mathbb{R} \times$ (the carrier of it)), and

(v) the normed of $it =$ (the normed of V)$\restriction$(the carrier of it).

(36) Let us consider a real normed space V, a subreal normal space X of V, a Lipschitzian linear functional f in X, and a point F of DualSp X. Suppose $f = F$. Then there exists a Lipschitzian linear functional g in V and there exists a point G of DualSp V such that $g = G$ and $g\restriction$(the carrier of X) $= f$ and $\|G\| = \|F\|$. Proof: Reconsider $X_0 = X$ as a real linear space. Reconsider $f_3 = f$ as a linear functional in X_0. Define $\mathcal{F}$(element of the carrier of V) $= \|F\| \cdot \|\$_1\|$. Consider q being a function from the carrier of V into $\mathbb{R}$ such that for every element v of the carrier of V, $q(v) = \mathcal{F}(v)$ from [7, Sch. 8]. q is a Banach functional in V. For every vector x of X_0 and for every vector v of V such that $x = v$ holds $f_3(x) \leqslant q(v)$ by [19, (4)], (26), [6, (49)]. Consider g being a linear functional in V such that $g\restriction$(the carrier of X_0) $= f_3$ and for every vector x of V, $g(x) \leqslant q(x)$. For every vector x of V, $|g(x)| \leqslant \|F\| \cdot \|x\|$ by [26, (16)], [20, (2)], [19, (5)]. (The bounded linear functionals norm V)$(g) \leqslant$ (the bounded linear functionals norm X)(f). (The bounded linear functionals norm X)$(f) \leqslant \sup \mathrm{PreNorms}(g)$. (The bounded linear functionals norm X)$(f) \leqslant$ (the bounded linear functionals norm V)(g). □

(37) Hahn-Banach extension theorem (real normed spaces):
Let us consider a real normed space V, a subreal normal space X of V, a Lipschitzian linear functional f in X, and a point F of DualSp X. Suppose

(i) $f = F$, and

(ii) for every vector x of X and for every vector v of V such that $x = v$ holds $f(x) \leqslant \|v\|$.

Then there exists a Lipschitzian linear functional g in V and there exists a point G of DualSp V such that $g = G$ and $g\restriction$(the carrier of X) $= f$ and for every vector x of V, $g(x) \leqslant \|x\|$ and $\|G\| = \|F\|$. Proof: Consider g being a Lipschitzian linear functional in V, G being a point of DualSp V such that $g = G$ and $g\restriction$(the carrier of X) $= f$ and $\|G\| = \|F\|$. $\|G\| \leqslant 1$. For every vector x of V, $g(x) \leqslant \|x\|$ by [19, (4)], (26). □

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Received March 31, 2014

FORMALIZED MATHEMATICS
Vol. 22, No. 1, Pages 79–84, 2014
DOI: 10.2478/forma-2014-0008

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Semiring of Sets

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Summary. Schmets [22] has developed a measure theory from a generalized notion of a semiring of sets. Goguadze [15] has introduced another generalized notion of semiring of sets and proved that all known properties that semiring have according to the old definitions are preserved. We show that this two notions are almost equivalent. We note that Patriota [20] has defined this quasi-semiring. We propose the formalization of some properties developed by the authors.

MSC: 28A05 03E02 03E30 03B35

Keywords: sets; set partitions; distributive lattice

MML identifier: SRINGS_1, version: 8.1.03 5.23.1204

The notation and terminology used in this paper have been introduced in the following articles: [1], [3], [4], [21], [6], [12], [24], [8], [9], [25], [13], [23], [11], [5], [17], [18], [27], [28], [19], [26], [14], [16], and [10].

1. Preliminaries

From now on X denotes a set and S denotes a family of subsets of X.

Now we state the proposition:

(1) Let us consider sets X, Y. Then $(X \cup Y) \setminus (Y \setminus X) = X$.

Let us consider X and S. Let S_1, S_2 be finite subsets of S. Let us note that $S_1 \Cap S_2$ is finite.

Now we state the proposition:

(2) Let us consider a family S of subsets of X and an element A of S. Then $\{x$, where x is an element of $S : x \in \bigcup(\mathrm{PARTITIONS}(A) \cap \mathrm{Fin}\, S)\} = \bigcup(\mathrm{PARTITIONS}(A) \cap \mathrm{Fin}\, S)$.

Let us consider X and S. Note that $\bigcup(\mathrm{PARTITIONS}(\emptyset) \cap \mathrm{Fin}\, S)$ is empty. Note that 2^X_* has empty element. Now we state the proposition:

ISSN 1426–2630(Print), 1898-9934(Online)

(3) Let us consider a set X. Suppose X is $\cap$-closed and $\cup$-closed. Then X is a ring of sets.

2. The Existence of Partitions

Let X be a set and S be a family of subsets of X. We say that S is $\cap_{fp}$-closed if and only if

(Def. 1) Let us consider elements S_1, S_2 of S. Suppose $S_1 \cap S_2$ is not empty. Then there exists a finite subset x of S such that x is a partition of $S_1 \cap S_2$.

Let us observe that 2_*^X is $\cap_{fp}$-closed.

Observe that there exists a family of subsets of X which is $\cap_{fp}$-closed.

One can verify that every family of subsets of X which is $\cap$-closed is also $\cap_{fp}$-closed.

Now we state the propositions:

(4) Let us consider a non empty set A, a $\cap_{fp}$-closed family S of subsets of X, and partitions P_1, P_2 of A. Suppose

(i) P_1 is a finite subset of S, and

(ii) P_2 is a finite subset of S.

Then there exists a partition P of A such that

(iii) P is a finite subset of S, and

(iv) $P \Subset P_1 \wedge P_2$.

Proof: Define $\mathcal{F}[\text{object}, \text{object}] \equiv \$_1 \in P_1 \wedge P_2$ and $\$_2$ is a finite subset of S and there exists a set A such that $A = \$_1$ and $\$_2$ is a partition of A. Set $F_1 = \{y$, where y is a finite subset of S : there exists a set t such that $t \in P_1 \wedge P_2$ and y is a partition of $t\}$. $F_1 \subseteq 2^{2^x}$ by [10, (67)]. For every object u such that $u \in P_1 \wedge P_2$ there exists an object v such that $v \in F_1$ and $\mathcal{F}[u, v]$. Consider f being a function such that $\operatorname{dom} f = P_1 \wedge P_2$ and $\operatorname{rng} f \subseteq F_1$ and for every object x such that $x \in P_1 \wedge P_2$ holds $\mathcal{F}[x, f(x)]$ from [8, Sch. 6]. $\bigcup f$ is a finite subset of S by [2, (88)]. $\bigcup f$ is a partition of x by [10, (77), (81), (74)]. $\bigcup f \Subset P_1 \wedge P_2$. $\square$

(5) Let us consider a $\cap_{fp}$-closed family S of subsets of X and finite subsets A, B of S. Suppose

(i) A is mutually-disjoint, and

(ii) B is mutually-disjoint.

Then there exists a finite subset P of S such that P is a partition of $\bigcup A \cap \bigcup B$.

(6) Let us consider a $\cap_{fp}$-closed family S of subsets of X and a finite subset W of S. Then there exists a finite subset P of S such that P is a partition of $\bigcap W$.

(7) Let us consider a $\cap_{fp}$-closed family S of subsets of X. Then $\{\bigcup x$, where x is a finite subset of $S : x$ is mutually-disjoint$\}$ is $\cap$-closed. The theorem is a consequence of (5).

Let X be a set and S be a family of subsets of X. We say that S is $\setminus_{fp}$-closed if and only if

(Def. 2) Let us consider elements S_1, S_2 of S. Suppose $S_1 \setminus S_2$ is not empty. Then there exists a finite subset x of S such that x is a partition of $S_1 \setminus S_2$.

Let us note that 2^X_* is $\setminus_{fp}$-closed.

Note that there exists a family of subsets of X which is $\setminus_{fp}$-closed.

Observe that every family of subsets of X which is diff-closed is also $\setminus_{fp}$-closed. Now we state the proposition:

(8) Let us consider a $\setminus_{fp}$-closed family S of subsets of X, an element S_1 of S, and a finite subset T of S. Then there exists a finite subset P of S such that P is a partition of $S_1 \setminus \bigcup T$. Proof: Consider p_0 being a finite sequence such that $T = \operatorname{rng} p_0$. Define $\mathcal{P}[\text{finite sequence}] \equiv$ there exists a finite subset p_1 of S such that p_1 is a partition of $S_1 \setminus \bigcup \operatorname{rng} \$_1$. For every finite sequence p of elements of S and for every element x of S such that $\mathcal{P}[p]$ holds $\mathcal{P}[p \frown \langle x \rangle]$ by [6, (31)], [10, (78)], [6, (38)], [12, (8), (7)]. $\mathcal{P}[\varepsilon_S]$ by [26, (1)], [21, (45)], [26, (41)], [21, (39)]. For every finite sequence p of elements of S, $\mathcal{P}[p]$ from [7, Sch. 2]. □

3. Partitions in a Difference of Sets

Let X be a set and S be a family of subsets of X. We say that S is $\setminus^{\subseteq}_{fp}$-closed if and only if

(Def. 3) Let us consider elements S_1, S_2 of S. Suppose $S_2 \subseteq S_1$. Then there exists a finite subset x of S such that x is a partition of $S_1 \setminus S_2$.

Now we state the proposition:

(9) Let us consider a family S of subsets of X. Suppose S is $\setminus_{fp}$-closed. Then S is $\setminus^{\subseteq}_{fp}$-closed.

Let us consider X. Note that every family of subsets of X which is $\setminus_{fp}$-closed is also $\setminus^{\subseteq}_{fp}$-closed.

Observe that 2^X_* is $\setminus^{\subseteq}_{fp}$-closed. Observe that there exists a family of subsets of X which is $\setminus^{\subseteq}_{fp}$-closed, $\setminus_{fp}$-closed, and $\cap_{fp}$-closed and has empty element.

Now we state the propositions:

(10) Let us consider a $\setminus_{fp}$-closed family S of subsets of X. Then $\{\bigcup x$, where x is a finite subset of $S : x$ is mutually-disjoint$\}$ is diff-closed. Proof: Set $Y = \{\bigcup x$, where x is a finite subset of $S : x$ is mutually-disjoint$\}$. For every sets A, B such that $A, B \in Y$ holds $A \setminus B \in Y$ by [6, (52)], (8), (5), [12, (8), (7)]. □

(11) Let us consider a $\cap_{fp}$-closed $\setminus^{\subseteq}_{fp}$-closed family S of subsets of X, an element A of S, and a finite subset Q of S. Suppose

(i) $\bigcup Q \subseteq A$, and

(ii) Q is a partition of $\bigcup Q$.

Then there exists a finite subset R of S such that

(iii) $\bigcup R$ misses $\bigcup Q$, and

(iv) $Q \cup R$ is a partition of A.

(12) Every $\setminus^{\subseteq}_{fp}$-closed $\cap_{fp}$-closed family of subsets of X is $\setminus_{fp}$-closed. Proof: For every elements S_1, S_2 of S such that $S_1 \setminus S_2$ is not empty there exists a finite subset P_0 of S such that P_0 is a partition of $S_1 \setminus S_2$ by (11), [10, (77), (81)]. □

Let X be a set. Let us observe that every $\cap_{fp}$-closed family of subsets of X which is $\setminus^{\subseteq}_{fp}$-closed is also $\setminus_{fp}$-closed. Now we state the propositions:

(13) Let us consider a $\cap_{fp}$-closed $\setminus^{\subseteq}_{fp}$-closed family S of subsets of X and finite subsets W, T of S. Then there exists a finite subset P of S such that P is a partition of $\bigcap W \setminus \bigcup T$.

(14) Let us consider a $\cap_{fp}$-closed $\setminus^{\subseteq}_{fp}$-closed family S of subsets of X and a finite subset W of S. Then there exists a finite subset P of S such that

(i) P is a partition of $\bigcup W$, and

(ii) for every element Y of W, $Y = \bigcup\{s$, where s is an element of $S : s \in P$ and $s \subseteq Y\}$.

(15) Let us consider a $\cap_{fp}$-closed $\setminus^{\subseteq}_{fp}$-closed family S of subsets of X and a function W from $\mathbb{N}^+$ into S. Then there exists a countable subset P of S such that

(i) P is a partition of $\bigcup W$, and

(ii) for every positive natural number n, $\bigcup(W \restriction \mathrm{Seg}\, n) = \bigcup\{s$, where s is an element of $S : s \in P$ and $s \subseteq \bigcup(W \restriction \mathrm{Seg}\, n)\}$.

The theorem is a consequence of (8).

4. Countable Covers

Let X be a set and S be a family of subsets of X. We say that S has countable cover if and only if

(Def. 4) There exists a countable subset X_1 of S such that X_1 is a cover of X.

Let us consider X. One can check that 2^X_* has countable cover.

One can check that there exists a family of subsets of X which is $\setminus^{\subseteq}_{fp}$-closed, $\setminus_{fp}$-closed, and $\cap_{fp}$-closed and has empty element and countable cover.

Now we state the proposition:

(16) Let us consider a $\cap_{fp}$-closed $\setminus_{fp}^{\subseteq}$-closed family S of subsets of X. Suppose S has countable cover. Then there exists a countable subset P of S such that P is a partition of X. The theorem is a consequence of (15).

5. Semiring of Sets

Let X be a set. A semiring of sets of X is a $\cap_{fp}$-closed $\setminus_{fp}^{\subseteq}$-closed family of subsets of X with empty element.

Let us consider a $\cap_{fp}$-closed family S of subsets of X and an element A of S. Now we state the propositions:

(17) $\{x$, where x is an element of $S : x \in \bigcup(\mathrm{PARTITIONS}(A) \cap \mathrm{Fin}\, S)\}$ is a $\cap_{fp}$-closed family of subsets of A. The theorem is a consequence of (4).

(18) $\{x$, where x is an element of $S : x \in \bigcup(\mathrm{PARTITIONS}(A) \cap \mathrm{Fin}\, S)\}$ is a $\setminus_{fp}^{\subseteq}$-closed family of subsets of A. The theorem is a consequence of (4).

(19) $\bigcup(\mathrm{PARTITIONS}(A) \cap \mathrm{Fin}\, S)$ is $\cap_{fp}$-closed $\setminus_{fp}$-closed family of subsets of A and has non empty elements. The theorem is a consequence of (2), (17), and (18).

(20) $\{\emptyset\} \cup \bigcup(\mathrm{PARTITIONS}(A) \cap \mathrm{Fin}\, S)$ is a semiring of sets of A. PROOF: Set $A_1 = \bigcup(\mathrm{PARTITIONS}(A) \cap \mathrm{Fin}\, S)$. Set $B = \bigcup(\mathrm{PARTITIONS}(A) \cap \mathrm{Fin}\, S) \cup \{\emptyset\}$. A_1 is a $\cap_{fp}$-closed $\setminus_{fp}$-closed family of subsets of A. $B \subseteq 2^A$. B is $\cap_{fp}$-closed. B is $\setminus_{fp}$-closed by (19), [21, (39)]. $\square$

6. A Ring of Sets

Let us consider a $\cap_{fp}$-closed $\setminus_{fp}$-closed family S of subsets of X. Now we state the propositions:

(21) $\{\bigcup x$, where x is a finite subset of $S : x$ is mutually-disjoint$\}$ is $\cup$-closed. The theorem is a consequence of (14).

(22) $\{\bigcup x$, where x is a finite subset of $S : x$ is mutually-disjoint$\}$ is a ring of sets. The theorem is a consequence of (7), (21), and (3).

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Received March 31, 2014

FORMALIZED MATHEMATICS
Vol. 22, No. 1, Pages 85–88, 2014
DOI: 10.2478/forma-2014-0009

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Semiring of Sets: Examples

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Summary. This article proposes the formalization of some examples of semiring of sets proposed by Goguadze [8] and Schmets [13].

MSC: 28A05 03E02 03E30 03B35

Keywords: semiring of sets

MML identifier: SRINGS_2, version: 8.1.03 5.23.1204

The notation and terminology used in this paper have been introduced in the following articles: [2], [14], [7], [17], [15], [5], [16], [9], [12], [19], [10], [18], and [6].

1. Preliminaries

From now on X denotes a set and S denotes a family of subsets of X.

Now we state the propositions:

(1) Let us consider sets X_1, X_2, a family S_1 of subsets of X_1, and a family S_2 of subsets of X_2. Then $\{a \times b$, where a is an element of S_1, b is an element of $S_2 : a \in S_1$ and $b \in S_2\} = \{s$, where s is a subset of $X_1 \times X_2 :$ there exist sets a, b such that $a \in S_1$ and $b \in S_2$ and $s = a \times b\}$. Proof: $\{a \times b$, where a is an element of S_1, b is an element of $S_2 : a \in S_1$ and $b \in S_2\} \subseteq \{s$, where s is a subset of $X_1 \times X_2 :$ there exist sets a, b such that $a \in S_1$ and $b \in S_2$ and $s = a \times b\}$ by [6, (96)]. $\square$

(2) Let us consider sets X_1, X_2, a non empty family S_1 of subsets of X_1, and a non empty family S_2 of subsets of X_2. Then $\{s$, where s is a subset of $X_1 \times X_2 :$ there exist sets x_1, x_2 such that $x_1 \in S_1$ and $x_2 \in S_2$ and $s = x_1 \times x_2\} =$ the set of all $x_1 \times x_2$ where x_1 is an element of S_1, x_2 is an element of S_2.

(3) Let us consider sets X_1, X_2, a family S_1 of subsets of X_1, and a family S_2 of subsets of X_2. Suppose

ISSN 1426–2630(Print), 1898-9934(Online)

(i) S_1 is $\cap$-closed, and

(ii) S_2 is $\cap$-closed.

Then $\{s$, where s is a subset of $X_1 \times X_2$: there exist sets x_1, x_2 such that $x_1 \in S_1$ and $x_2 \in S_2$ and $s = x_1 \times x_2\}$ is $\cap$-closed. Proof: Set $Y = \{s$, where s is a subset of $X_1 \times X_2$: there exist sets x_1, x_2 such that $x_1 \in S_1$ and $x_2 \in S_2$ and $s = x_1 \times x_2\}$. Y is $\cap$-closed by [6, (100)]. $\square$

Let X be a set. Note that every σ-field of subsets of X is $\cap_{fp}$-closed and $\setminus_{fp}^{\subseteq}$-closed and has countable cover and empty element.

2. Ordinary Examples of Semirings of Sets

Now we state the proposition:

(4) Every σ-field of subsets of X is a semiring of sets of X.

Let X be a set. Note that 2^X is $\cap_{fp}$-closed and $\setminus_{fp}^{\subseteq}$-closed and has countable cover and empty element as a family of subsets of X.

Now we state the proposition:

(5) 2^X is a semiring of sets of X.

Let us consider X. Note that $\operatorname{Fin} X$ is $\cap_{fp}$-closed and $\setminus_{fp}^{\subseteq}$-closed and has empty element as a family of subsets of X.

Let D be a denumerable set. Observe that $\operatorname{Fin} D$ has countable cover as a family of subsets of D.

Now we state the propositions:

(6) $\operatorname{Fin} X$ is a semiring of sets of X.

(7) Let us consider sets X_1, X_2, a semiring S_1 of sets of X_1, and a semiring S_2 of sets of X_2. Then $\{s$, where s is a subset of $X_1 \times X_2$: there exist sets x_1, x_2 such that $x_1 \in S_1$ and $x_2 \in S_2$ and $s = x_1 \times x_2\}$ is a semiring of sets of $X_1 \times X_2$. Proof: Set $Y = \{s$, where s is a subset of $X_1 \times X_2$: there exist sets x_1, x_2 such that $x_1 \in S_1$ and $x_2 \in S_2$ and $s = x_1 \times x_2\}$. Y has empty element. Y is $\cap_{fp}$-closed by [6, (100)], [4, (8)], [1, (10)]. Y is $\setminus_{fp}$-closed by [1, (10)], [11, (39)], [4, (8)], [11, (45)]. $\square$

(8) Let us consider non empty sets X_1, X_2, a family S_1 of subsets of X_1 with countable cover, a family S_2 of subsets of X_2 with countable cover, and a family S of subsets of $X_1 \times X_2$. Suppose $S = \{s$, where s is a subset of $X_1 \times X_2$: there exist sets x_1, x_2 such that $x_1 \in S_1$ and $x_2 \in S_2$ and $s = x_1 \times x_2\}$. Then S has countable cover. Proof: There exists a countable subset U of S such that $\bigcup U = X_1 \times X_2$ and U is a subset of S by [6, (2), (77)], [2, (95)], [3, (7)]. $\square$

Let us consider a family S of subsets of $\mathbb{R}$. Now we state the propositions:

(9) Suppose $S = \{]a, b]$, where a, b are real numbers : $a \leqslant b\}$. Then

(i) S is $\cap$-closed, and

(ii) S is $\setminus_{fp}$-closed and has empty element, and

(iii) S has countable cover.

(10) Suppose $S = \{s$, where s is a subset of $\mathbb{R}$: s is left open interval$\}$. Then

(i) S is $\cap$-closed, and

(ii) S is $\setminus_{fp}$-closed and has empty element, and

(iii) S has countable cover.

Proof: S is $\cap$-closed. S has empty element. S is $\setminus_{fp}$-closed by [11, (39)], [6, (75)]. □

3. Numerical Example

The functor sring_8^4 yielding a family of subsets of $\{1, 2, 3, 4\}$ is defined by the term

(Def. 1) $\{\{1, 2, 3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1\}, (\{2\}), (\{3\}), (\{4\}), (\emptyset)\}$.

One can verify that sring_8^4 has empty element and sring_8^4 is $\cap_{fp}$-closed and non $\cap$-closed and sring_8^4 is $\setminus_{fp}$-closed.

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Received March 31, 2014

FORMALIZED MATHEMATICS
Vol. 22, No. 1, Pages 89–97, 2014
DOI: 10.2478/forma-2014-0010

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Topological Interpretation of Rough Sets

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Summary. Rough sets, developed by Pawlak, are an important model of incomplete or partially known information. In this article, which is essentially a continuation of [11], we characterize rough sets in terms of topological closure and interior, as the approximations have the properties of the Kuratowski operators. We decided to merge topological spaces with tolerance approximation spaces. As a testbed for our developed approach, we restated the results of Isomichi [13] (formalized in Mizar in [14]) and about fourteen sets of Kuratowski [17] (encoded with the help of Mizar adjectives and clusters' registrations in [1]) in terms of rough approximations. The upper bounds which were 14 and 7 in the original paper of Kuratowski, in our case are six and three, respectively.

It turns out that within the classification given by Isomichi, 1^{st} class subsets are precisely crisp sets, 2^{nd} class subsets are proper rough sets, and there are no 3^{rd} class subsets in topological spaces generated by approximations. Also the important results about these spaces is that they are extremally disconnected [15], hence lattices of their domains are Boolean.

Furthermore, we develop the theory of abstract spaces equipped with maps possessing characteristic properties of rough approximations which enables us to freely use the notions from the theory of rough sets and topological spaces formalized in the Mizar Mathematical Library [10].

MSC: 54H10 68T37 03B35

Keywords: rough sets; rough approximations; Kuratowski closure-complement problem; topological spaces

MML identifier: ROUGHS_4, version: 8.1.03 5.23.1204

The notation and terminology used in this paper have been introduced in the following articles: [2], [22], [4], [9], [24], [20], [21], [5], [6], [14], [1], [25], [3], [7], [19], [27], [11], [12], [18], [26], [15], [28], [16], and [8].

ISSN 1426–2630(Print), 1898-9934(Online)

1. Preliminaries

Now we state the proposition:

(1) Let us consider a set T and a family F of subsets of T. Then $F = \{B$, where B is a subset of $T : B \in F\}$.

Let f be a function and A be a set. We say that A is f-closed if and only if

(Def. 1) $A = f(A)$.

Let X be a set and F be a family of subsets of X. One can check that F is $\cap$-closed if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let us consider subsets a, b of X. If a, $b \in F$, then $a \cap b \in F$.

We say that F is union-closed if and only if

(Def. 3) Let us consider a family a of subsets of X. If $a \subseteq F$, then $\bigcup a \in F$.

We say that F is topology-like if and only if

(Def. 4) (i) $\emptyset$, $X \in F$, and

(ii) F is union-closed and $\cap$-closed.

Let us observe that there exists a family of subsets of X which is topology-like.

2. Ordinary Properties of Maps

Let X be a set and f be a function from 2^X into 2^X. We say that f is extensive if and only if

(Def. 5) Let us consider a subset A of X. Then $A \subseteq f(A)$.

We say that f is intensive if and only if

(Def. 6) Let us consider a subset A of X. Then $f(A) \subseteq A$.

We say that f is idempotent if and only if

(Def. 7) Let us consider a subset A of X. Then $f(f(A)) = f(A)$.

We say that f is $\subseteq$-monotone if and only if

(Def. 8) Let us consider subsets A, B of X. If $A \subseteq B$, then $f(A) \subseteq f(B)$.

We say that f preserves $\cup$ if and only if

(Def. 9) Let us consider subsets A, B of X. Then $f(A \cup B) = f(A) \cup f(B)$.

We say that f preserves $\cap$ if and only if

(Def. 10) Let us consider subsets A, B of X. Then $f(A \cap B) = f(A) \cap f(B)$.

Let O be a function from 2^X into 2^X. We say that O is a preclosure if and only if

(Def. 11) O is extensive and preserves $\cup$ and empty set.

We say that O is closure if and only if

(Def. 12) O is extensive and idempotent and preserves $\cup$ and empty set.

We say that O is a preinterior if and only if

(Def. 13) O is intensive and preserves $\cap$ and universe.

We say that O is an interior if and only if

(Def. 14) O is intensive and idempotent and preserves $\cap$ and universe.

Let us observe that every function from 2^X into 2^X which preserves $\cup$ is also $\subseteq$-monotone and every function from 2^X into 2^X which preserves $\cap$ is also $\subseteq$-monotone.

One can verify that id_{2^X} is closure as a function from 2^X into 2^X and id_{2^X} is an interior as a function from 2^X into 2^X.

One can check that there exists a function from 2^X into 2^X which is closure and interior.

Observe that every function from 2^X into 2^X which is closure is also a preclosure.

3. Structural Part

Let T be a 1-sorted structure.

A map of T is a function from $2^{(\text{the carrier of } T)}$ into $2^{(\text{the carrier of } T)}$. We consider 1stOpStrs which extend 1-sorted structures and are systems

$$\langle \text{a carrier}, \text{a FirstOp} \rangle$$

where the carrier is a set, the FirstOp is a function from $2^{(\text{the carrier})}$ into $2^{(\text{the carrier})}$.

We consider 2ndOpStrs which extend 1-sorted structures and are systems

$$\langle \text{a carrier}, \text{a SecondOp} \rangle$$

where the carrier is a set, the SecondOp is a function from $2^{(\text{the carrier})}$ into $2^{(\text{the carrier})}$.

We consider TwoOpStructs which extend 1stOpStrs and 2ndOpStrs and are systems

$$\langle \text{a carrier}, \text{a FirstOp}, \text{a SecondOp} \rangle$$

where the carrier is a set, the FirstOp and the SecondOp are functions from $2^{(\text{the carrier})}$ into $2^{(\text{the carrier})}$.

Let X be a 1stOpStr. We say that X has closure if and only if

(Def. 15) The FirstOp of X is closure.

We say that X has preclosure if and only if

(Def. 16) The FirstOp of X is a preclosure.

Let T be a topological space. Let us observe that $\mathrm{ClMap}\, T$ is closure and $\mathrm{IntMap}\, T$ is an interior and there exists a 1stOpStr which is non empty and has closure and every 1stOpStr which has closure has also preclosure.

Let X be a 1stOpStr and A be a subset of X. We say that A is op-closed if and only if

(Def. 17) $A = (\text{the FirstOp of } X)(A)$.

We say that X has op-closed subsets if and only if

(Def. 18) There exists a subset A of X such that A is op-closed.

One can check that there exists a 1stOpStr which has op-closed subsets.

Let X be 1stOpStr with op-closed subsets. One can check that there exists a subset of X which is op-closed.

Let X be a 2ndOpStr and A be a subset of X. We say that A is op-open if and only if

(Def. 19) $A = (\text{the SecondOp of } X)(A)$.

We say that X has op-open subsets if and only if

(Def. 20) There exists a subset A of X such that A is op-open.

Let us observe that there exists a 2ndOpStr which has op-open subsets.

Let X be 2ndOpStr with op-open subsets. Let us observe that there exists a subset of X which is op-open.

Let X be a 2ndOpStr. We say that X has interior if and only if

(Def. 21) The SecondOp of X is an interior.

We say that X has preinterior if and only if

(Def. 22) The SecondOp of X is a preinterior.

Note that there exists a TwoOpStruct which has closure and interior.

4. Merging with Topologies

We consider 1TopStructs which extend 1stOpStrs and topological structures and are systems

$$\langle \text{a carrier}, \text{a FirstOp}, \text{a topology} \rangle$$

where the carrier is a set, the FirstOp is a function from $2^{(\text{the carrier})}$ into $2^{(\text{the carrier})}$, the topology is a family of subsets of the carrier.

We consider 2TopStructs which extend 2ndOpStrs and topological structures and are systems

$$\langle \text{a carrier}, \text{a SecondOp}, \text{a topology} \rangle$$

where the carrier is a set, the SecondOp is a function from $2^{(\text{the carrier})}$ into $2^{(\text{the carrier})}$, the topology is a family of subsets of the carrier.

Let us observe that there exists a 1TopStruct which is non empty and strict and there exists a 2TopStruct which is non empty and strict.

Let T be a 1TopStruct. We say that T has properly defined topology if and only if

(Def. 23) Let us consider an object x. Then $x \in$ the topology of T if and only if there exists a subset S of T such that $S^{\mathrm{c}} = x$ and S is op-closed.

Let T be a 2TopStruct. We say that T has properly defined Topology if and only if

(Def. 24) Let us consider an object x. Then $x \in$ the topology of T if and only if there exists a subset S of T such that $S = x$ and S is op-open.

One can verify that there exists a 1TopStruct which has closure and properly defined topology and there exists a 2TopStruct which has interior and properly defined Topology.

(2) Let us consider 1TopStruct T with properly defined topology and a subset A of T. Then A is op-closed if and only if A is closed. Proof: If A is op-closed, then A is closed by [28, (3)]. If A is closed, then A is op-closed by [28, (3)]. □

Observe that every 1TopStruct with properly defined topology which has preclosure is also topological space-like.

(3) Let us consider 2TopStruct T with properly defined Topology and a subset A of T. Then A is op-open if and only if A is open.

Note that every 2TopStruct with properly defined Topology which has preinterior is also topological space-like.

(4) Let us consider 1TopStruct T with closure properly defined topology and a subset A of T. Then (the FirstOp of T)$(A) = \overline{A}$. Proof: Set $f =$ the FirstOp of T. Consider F being a family of subsets of T such that for every subset C of T, $C \in F$ iff C is closed and $A \subseteq C$ and $\overline{A} = \bigcap F$. $\overline{A} \subseteq f(A)$ by (2), [18, (3)]. Define $\mathcal{P}$[subset of T] $\equiv \$_1 \in F$. Set $G = \{f(B)$, where B is a subset of $T : B \in F\}$. Define $\mathcal{T} = 2^{(\text{the carrier of } T)}$. Define $\mathcal{F}$(element of $\mathcal{T}$) $= f(\$_1)$. Define $\mathcal{G}$(element of $\mathcal{T}$) $= \$_1$. For every element B of $\mathcal{T}$ such that $\mathcal{P}[B]$ holds $\mathcal{F}(B) = \mathcal{G}(B)$. $\{\mathcal{F}(B)$, where B is an element of $\mathcal{T} : \mathcal{P}[B]\} = \{\mathcal{G}(B)$, where B is an element of $\mathcal{T} : \mathcal{P}[B]\}$ from [23, Sch. 6]. $F = G$. For every set Z such that $Z \in G$ holds $f(A) \subseteq Z$. □

5. Introducing Rough Sets

Let R be a tolerance space. Let us note that LAp(R) is a preinterior and UAp(R) is a preclosure.

Let R be an approximation space. Observe that LAp(R) is an interior and UAp(R) is closure.

Let X be a set and f be a function from 2^X into 2^X. The functor GenTop f yielding a family of subsets of X is defined by

(Def. 25) Let us consider an object x. Then $x \in it$ if and only if there exists a subset S of X such that $S = x$ and S is f-closed.

Now we state the proposition:

(5) Let us consider a set X and a function f from 2^X into 2^X. If f is a preinterior, then $\operatorname{GenTop} f$ is topology-like. PROOF: Set $F = \operatorname{GenTop} f$. There exists a subset S of X such that $S = X$ and S is f-closed. There exists a subset S of X such that $S = \emptyset$ and S is f-closed. F is $\cap$-closed. For every family a of subsets of X such that $a \subseteq F$ holds $\bigcup a \in F$ by [8, (74), (76)]. $\square$

Let C be a set, I be a binary relation on C, and f be a topology-like family of subsets of C. Observe that $\langle C, I, f\rangle$ is topological space-like and there exists a FR-structure which is topological space-like and non empty and has equivalence relation.

6. On Sequential Closure and Frechet Spaces

Let T be a non empty topological space. The functor $\mathrm{Cl}_{\mathrm{Seq}}\, T$ yielding a map of T is defined by

(Def. 26) Let us consider a subset A of T. Then $it(A) = \mathrm{Cl}_{\mathrm{Seq}}\, A$.

One can verify that $\mathrm{Cl}_{\mathrm{Seq}}\, T$ is a preclosure and there exists a non empty topological space which is Frechet.

Let T be a Frechet non empty topological space. Note that $\mathrm{Cl}_{\mathrm{Seq}}\, T$ is closure.

7. Connections between Closures and Approximations

Let T be a non empty FR-structure. We say that T is Natural if and only if

(Def. 27) Let us consider a subset x of T. Then $x \in$ the topology of T if and only if x is $(\mathrm{LAp}(T))$-closed.

We say that T is naturally generated if and only if

(Def. 28) The topology of $T = \operatorname{GenTop} \mathrm{LAp}(T)$.

Now we state the proposition:

(6) Let us consider a non empty FR-structure T. Suppose T is naturally generated. Let us consider a subset A of T. Then A is open if and only if $\mathrm{LAp}(A) = A$.

Let us consider a non empty FR-structure T and a non empty relational structure R.

Let us assume that the relational structure of $T =$ the relational structure of R. Now we state the propositions:

(7) $\mathrm{LAp}(T) = \mathrm{LAp}(R)$.

(8) $\mathrm{UAp}(T) = \mathrm{UAp}(R)$.

One can verify that there exists a non empty FR-structure which is Natural and topological space-like and has equivalence relation and every non empty FR-structure with equivalence relation which is naturally generated is also topological space-like and there exists a non empty FR-structure which is naturally generated and topological space-like and has equivalence relation.

Let T be a naturally generated non empty FR-structure with equivalence relation and A be a subset of T. One can check that $\mathrm{LAp}(A)$ is open.

Let us consider a naturally generated non empty FR-structure T with equivalence relation and a subset A of T. Now we state the propositions:

(9) $\mathrm{LAp}(A) = \mathrm{Int}\, A$. Proof: $\mathrm{Int}\, A \subseteq \mathrm{LAp}(A)$ by [28, (22), (23)], [11, (24)]. □

(10) A is closed if and only if $\mathrm{UAp}(A) = A$. Proof: If A is closed, then $\mathrm{UAp}(A) = A$ by (6), [11, (28)]. □

Let T be a naturally generated non empty FR-structure with equivalence relation and A be a subset of T. One can check that $\mathrm{UAp}(A)$ is closed.

Let us consider a naturally generated non empty FR-structure T with equivalence relation and a subset A of T. Now we state the propositions:

(11) $\mathrm{UAp}(A) = \overline{A}$. Proof: $\mathrm{UAp}(A) \subseteq \overline{A}$ by (10), [11, (25)], [19, (15)]. □

(12) $\mathrm{BndAp}(A) = \mathrm{Fr}\, A$. The theorem is a consequence of (11) and (9).

Let T be a naturally generated non empty FR-structure with equivalence relation and A be a subset of T. We identify $\mathrm{LAp}(A)$ with $\mathrm{Int}\, A$. We identify $\mathrm{UAp}(A)$ with $\overline{A}$. We identify $\mathrm{Int}\, A$ with $\mathrm{LAp}(A)$. We identify $\overline{A}$ with $\mathrm{UAp}(A)$. We identify $\mathrm{Fr}\, A$ with $\mathrm{BndAp}(A)$. We identify $\mathrm{BndAp}(A)$ with $\mathrm{Fr}\, A$.

8. Isomichi Results Reuse

Let us consider a naturally generated non empty FR-structure T with equivalence relation and a subset A of T. Now we state the propositions:

(13) A is 1^{st} class if and only if $\mathrm{LAp}(\mathrm{UAp}(A)) \subseteq \mathrm{UAp}(\mathrm{LAp}(A))$.

(14) A is 1^{st} class if and only if $\mathrm{UAp}(A) \subseteq \mathrm{LAp}(A)$.

(15) A is 1^{st} class if and only if A is exact. Proof: If A is 1^{st} class, then A is exact by [11, (14)], (14), [11, (13), (12)]. □

Let T be a naturally generated non empty FR-structure with equivalence relation. Note that every subset of T which is 1^{st} class is also exact and every subset of T which is exact is also 1^{st} class.

Let us consider a naturally generated non empty FR-structure T with equivalence relation and a subset A of T. Now we state the propositions:

(16) A is 2^{nd} class if and only if $\mathrm{LAp}(A) \subset \mathrm{UAp}(A)$.

(17) A is 2^{nd} class if and only if A is rough. Proof: $\mathrm{LAp}(A) \neq \mathrm{UAp}(A)$ by [11, (13), (12)]. □

Let T be a naturally generated non empty FR-structure with equivalence relation. Note that every subset of T which is 2^{nd} class is also rough and every subset of T which is rough is also 2^{nd} class.

Now we state the propositions:

(18) Let us consider a naturally generated non empty FR-structure T with equivalence relation and a subset A of T. Then $\overline{\mathrm{Int}\,A}$ and $\overline{A}$ are $\subseteq$-comparable.

(19) Let us consider a naturally generated non empty FR-structure T with equivalence relation and a subset A of T. Then A is not 3^{rd} class.

Let T be a topological space.

Observe that every naturally generated non empty FR-structure with equivalence relation is without 3rd class subsets and there exists a topological space which is without 3rd class subsets.

Let T be a topological space and A be a 1^{st} class subset of T. One can verify that Border A is empty.

Let T be a naturally generated non empty FR-structure with equivalence relation and A be a subset of T. Note that $\overline{A}$ is open and Int A is closed and every naturally generated non empty FR-structure with equivalence relation is extremally disconnected.

9. Reexamination of Kuratowski's 14 Sets for Approximation Spaces

Let us consider a naturally generated non empty FR-structure T with equivalence relation and a subset A of T. Now we state the propositions:

(20) $\mathrm{Kurat7Set}(A) = \{A, \overline{A}, \mathrm{Int}\,A\}$.

(21) $\overline{\overline{\mathrm{Kurat7Set}(A)}} \leqslant 3$. The theorem is a consequence of (20).

(22) $\mathrm{Kurat14Set}(A) = \{A, \mathrm{UAp}(A), (\mathrm{UAp}(A))^{\mathrm{c}}, A^{\mathrm{c}}, (\mathrm{LAp}(A))^{\mathrm{c}}, \mathrm{LAp}(A)\}$.

(23) $\overline{\overline{\mathrm{Kurat14Set}(A)}} \leqslant 6$. The theorem is a consequence of (22).

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Received March 31, 2014