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Pseudo-Canonical Formulae are Classical

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Summary. An original result about Hilbert Positive Propositional Calculus introduced in [11] is proven. That is, it is shown that the pseudo-canonical formulae of that calculus (and hence also the canonical ones, see [17]) are a subset of the classical tautologies.

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The notation and terminology used in this paper have been introduced in the following articles: [13], [1], [14], [10], [9], [15], [3], [4], [5], [6], [11], [16], [17], [2], [7], [18], [20], [22], [21], [12], [19], and [8].

1. Preliminaries about Injectivity, Involutiveness, Fixed Points

From now on a, b, c, x, y, z, A, B, C, X, Y denote sets, f, g denote functions, V denotes a SetValuation, P denotes a permutation of V, p, q, r, s denote elements of HP-WFF, and n denotes an element of $\mathbb{N}$.

Let us consider X and Y. Let f be a relation between X and Y. Note that $\mathrm{id}_X \cdot f$ reduces to f and $f \cdot \mathrm{id}_Y$ reduces to f.

Now we state the proposition:

(1) Let us consider one-to-one functions f, g. If $f^{-1} = g^{-1}$, then $f = g$.

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One can verify that there exists a function which is involutive.

Let us consider A. Let us observe that there exists a permutation of A which is involutive.

Now we state the propositions:

(2) Let us consider an involutive function f. Suppose $\operatorname{rng} f \subseteq \operatorname{dom} f$. Then $f \cdot f = \mathrm{id}_{\operatorname{dom} f}$.

(3) Let us consider a function f. If $f \cdot f = \mathrm{id}_{\operatorname{dom} f}$, then f is involutive.

(4) Let us consider an involutive function f from A into A. Then $f \cdot f = \mathrm{id}_A$. The theorem is a consequence of (2).

(5) Let us consider a function f from A into A. If $f \cdot f = \mathrm{id}_A$, then f is involutive. The theorem is a consequence of (3).

Observe that every function which is involutive is also one-to-one.

Let us consider A. Let f be an involutive permutation of A. One can verify that f^{-1} is involutive.

Let n be a non zero natural number. Observe that $[0 \longmapsto n, n \longmapsto 0]$ is without fixpoints.

Let z be a zero natural number. Note that fixpoints$[z \longmapsto n, n \longmapsto z]$ is empty.

Let X be a non empty set. Observe that there exists a permutation of X which is non empty and involutive.

Let us consider A and B. Let f be an involutive function from A into A and g be an involutive function from B into B. Observe that $f \times g$ is involutive.

Let A, B be non empty sets, f be an involutive permutation of A, and g be an involutive permutation of B. Observe that $f \Rightarrow g$ is involutive.

2. Facts about Perm's Fixed Points

Now we state the propositions:

(6) If x is a fixpoint of $\mathrm{Perm}(P, q)$, then $\mathrm{SetVal}(V, p) \longmapsto x$ is a fixpoint of $\mathrm{Perm}(P, p \Rightarrow q)$.

(7) If $\mathrm{Perm}(P, q)$ has fixpoints, then $\mathrm{Perm}(P, p \Rightarrow q)$ has fixpoints. The theorem is a consequence of (6).

(8) If $\mathrm{Perm}(P, p)$ has fixpoints and $\mathrm{Perm}(P, q)$ is without fixpoints, then $\mathrm{Perm}(P, p \Rightarrow q)$ is without fixpoints.

3. Axiom of Choice in Functional Form via the Fraenkel Operator

Let X be a set. The functor ChoiceOn X yielding a set is defined by the term

(Def. 1) $\{\langle x, \text{the element of } x\rangle, \text{ where } x \text{ is an element of } X \setminus \{\emptyset\} : x \in X \setminus \{\emptyset\}\}$.

One can check that ChoiceOn X is relation-like and function-like.

Let us consider f. The functor FieldCover f yielding a set is defined by the term

(Def. 2) $\{\{x, f(x)\}$, where x is an element of $\operatorname{dom} f : x \in \operatorname{dom} f\}$.

The functor SomePoints f yielding a set is defined by the term

(Def. 3) field $f \setminus \operatorname{rng}$ ChoiceOn FieldCover f.

The functor OtherPoints f yielding a set is defined by the term

(Def. 4) (field $f \setminus$ fixpoints $f) \setminus$ SomePoints f.

Let us consider g. Let us observe that OtherPoints $g \cap$ SomePoints g is empty.

4. Building a Suitable Set Valuation and a Suitable Permutation of It

Let us consider x. The functor ToHilb(x) yielding a set is defined by the term

(Def. 5) $(\mathrm{id}_1 {+\cdot} (1 \times \emptyset^x) \cdot (\emptyset^x \times \{1\})) {+\cdot} (\{1\} \times \emptyset^x) \cdot (\emptyset^x \times \{0\})$.

Note that ToHilb(x) is function-like and relation-like.

Now we state the propositions:

(9) If $x \neq \emptyset$, then ToHilb$(x) = \mathrm{id}_1$.

(10) ToHilb$(\emptyset) = [0 \longmapsto 1, 1 \longmapsto 0]$.

Let v be a function. The functor ToHilbPerm(v) yielding a set is defined by the term

(Def. 6) the set of all $\langle n$, ToHilb$(v(n))\rangle$ where n is an element of $\mathbb{N}$.

The functor ToHilbVal(v) yielding a set is defined by the term

(Def. 7) the set of all $\langle n$, dom ToHilb$(v(n))\rangle$ where n is an element of $\mathbb{N}$.

One can check that ToHilbVal(v) is function-like and relation-like and ToHilb-Perm (v) is function-like and relation-like and ToHilbVal(v) is $\mathbb{N}$-defined and ToHilbVal(v) is total and ToHilbPerm(v) is $\mathbb{N}$-defined and ToHilbPerm(v) is total.

One can verify that ToHilbVal(v) is non-empty.

Let us consider x. Let us note that ToHilb(x) is symmetric.

Let v be a function. Observe that the functor ToHilbPerm(v) yields a permutation of ToHilbVal(v).

A set valuation is a many sorted set indexed by $\mathbb{N}$. From now on v denotes a set valuation.

Let us consider p and v. Note that Perm(ToHilbPerm$(v), p)$ is involutive.

5. Classical Semantics via SetVal_0, an Extension of SetVal

Let V be a set valuation. The functor $\mathrm{SetVal}_0\,V$ yielding a many sorted set indexed by HP-WFF is defined by

(Def. 8) (i) $it(\mathrm{VERUM}) = 1$, and

(ii) for every n, $it(\mathrm{prop}\,n) = V(n)$, and

(iii) for every p and q, $it(p \wedge q) = it(p) \times it(q)$ and $it(p \Rightarrow q) = (it(q))^{it(p)}$.

Let us consider v and p. The functor $\mathrm{SetVal}_0(v, p)$ yielding a set is defined by the term

(Def. 9) $(\mathrm{SetVal}_0\,v)(p)$.

We say that p is classical if and only if

(Def. 10) $\mathrm{SetVal}_0(v, p) \neq \emptyset$.

One can check that every element of HP-WFF which is pseudo-canonical is also classical.

Let us consider v. Let p be a classical element of HP-WFF. Note that $\mathrm{SetVal}_0(v, p)$ is non empty.

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