

Categorical Pullbacks

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Summary. The main purpose of this article is to introduce the categorical concept of pullback in Mizar. In the first part of this article we redefine hom-sets, monomorphisms, epimorphisms and isomorphisms [7] within a free-object category [1] and it is shown there that ordinal numbers can be considered as categories. Then the pullback is introduced in terms of its universal property and the Pullback Lemma is formalized [15]. In the last part of the article we formalize the pullback of functors [14] and it is also shown that it is not possible to write an equivalent definition in the context of the previous Mizar formalization of category theory [8].

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The notation and terminology used in this paper have been introduced in the following articles: [2], [8], [17], [18], [6], [13], [9], [10], [3], [11], [20], [21], [16], [19], [4], [5], and [12].

1. PRELIMINARIES

One can verify that every set which is ordinal is also non pair.

Let \mathcal{C} be an empty category structure. Let us note that $\text{Mor } \mathcal{C}$ is empty.

Let \mathcal{C} be a non empty category structure. Note that $\text{Mor } \mathcal{C}$ is non empty.

Let \mathcal{C} be an empty category structure with identities. Let us note that $\text{Ob } \mathcal{C}$ is empty.

Let \mathcal{C} be a non empty category structure with identities. Observe that $\text{Ob } \mathcal{C}$ is non empty.

Let \mathcal{C} be category structure with identities and a be an object of \mathcal{C} . One can check that id_a is identity.

Now we state the propositions:

- (1) Let us consider a category structure \mathcal{C} , and a morphism f of \mathcal{C} . Suppose \mathcal{C} is not empty. Then $f \in$ the carrier of \mathcal{C} .
- (2) Let us consider category structure \mathcal{C} with identities, and an object a of \mathcal{C} . Suppose \mathcal{C} is not empty. Then $a \in$ the carrier of \mathcal{C} .
- (3) Let us consider a composable category structure \mathcal{C} , and morphisms f_1, f_2, f_3 of \mathcal{C} . Suppose $f_1 \triangleright f_2$ and $f_2 \triangleright f_3$ and f_2 is identity. Then $f_1 \triangleright f_3$.
- (4) Let us consider a composable category structure \mathcal{C} with identities, and morphisms f_1, f_2 of \mathcal{C} . Suppose $f_1 \triangleright f_2$. Then
 - (i) $\text{dom}(f_1 \circ f_2) = \text{dom } f_2$, and
 - (ii) $\text{cod}(f_1 \circ f_2) = \text{cod } f_1$.
- (5) Let us consider a non empty, composable category structure \mathcal{C} with identities, and morphisms f_1, f_2 of \mathcal{C} . Then $f_1 \triangleright f_2$ if and only if $\text{dom } f_1 = \text{cod } f_2$.
- (6) Let us consider a composable category structure \mathcal{C} with identities, and a morphism f of \mathcal{C} . If f is identity, then $\text{dom } f = f$ and $\text{cod } f = f$.
- (7) Let us consider a composable category structure \mathcal{C} with identities, and morphisms f_1, f_2 of \mathcal{C} . Suppose $f_1 \triangleright f_2$ and f_1 is identity and f_2 is identity. Then $f_1 = f_2$.

Let us consider a non empty, composable category structure \mathcal{C} with identities and morphisms f_1, f_2 of \mathcal{C} . Now we state the propositions:

- (8) If $\text{dom } f_1 = f_2$, then $f_1 \triangleright f_2$ and $f_1 \circ f_2 = f_1$.
- (9) If $f_1 = \text{cod } f_2$, then $f_1 \triangleright f_2$ and $f_1 \circ f_2 = f_2$.

Now we state the propositions:

- (10) Let us consider categories $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$, a functor \mathcal{F} from \mathcal{C}_1 to \mathcal{C}_2 , a functor \mathcal{G} from \mathcal{C}_2 to \mathcal{C}_3 , and a functor \mathcal{H} from \mathcal{C}_3 to \mathcal{C}_4 . Suppose \mathcal{F} is covariant and \mathcal{G} is covariant and \mathcal{H} is covariant. Then $\mathcal{H} \circ (\mathcal{G} \circ \mathcal{F}) = (\mathcal{H} \circ \mathcal{G}) \circ \mathcal{F}$.
- (11) Let us consider categories \mathcal{C}, \mathcal{D} , and a functor \mathcal{F} from \mathcal{C} to \mathcal{D} . Suppose \mathcal{F} is covariant. Then
 - (i) $\mathcal{F} \circ \text{id}_{\mathcal{C}} = \mathcal{F}$, and
 - (ii) $\text{id}_{\mathcal{D}} \circ \mathcal{F} = \mathcal{F}$.
- (12) Let us consider composable category structures \mathcal{C}, \mathcal{D} with identities. Then $\mathcal{C} \cong \mathcal{D}$ if and only if there exists a functor \mathcal{F} from \mathcal{C} to \mathcal{D} such that \mathcal{F} is covariant and bijective. The theorem is a consequence of (5).

- (13) Let us consider empty category structures \mathcal{C} , \mathcal{D} with identities. Then $\mathcal{C} \cong \mathcal{D}$.

Let us consider category structures \mathcal{C} , \mathcal{D} with identities. Now we state the propositions:

- (14) Suppose $\mathcal{C} \cong \mathcal{D}$. Then

- (i) $\overline{\text{Mor } \mathcal{C}} = \overline{\text{Mor } \mathcal{D}}$, and
- (ii) $\overline{\text{Ob } \mathcal{C}} = \overline{\text{Ob } \mathcal{D}}$.

- (15) If $\mathcal{C} \cong \mathcal{D}$ and \mathcal{C} is empty, then \mathcal{D} is empty. The theorem is a consequence of (14).

2. HOM-SETS

Let \mathcal{C} be a category structure and a, b be objects of \mathcal{C} . The functor $\text{hom}(a, b)$ yielding a subset of $\text{Mor } \mathcal{C}$ is defined by the term

- (Def. 1) $\{f, \text{ where } f \text{ is a morphism of } \mathcal{C} : \text{there exist morphisms } f_1, f_2 \text{ of } \mathcal{C} \text{ such that } a = f_1 \text{ and } b = f_2 \text{ and } f \triangleright f_1 \text{ and } f_2 \triangleright f\}$.

Let \mathcal{C} be a non empty, composable category structure with identities. Observe that the functor $\text{hom}(a, b)$ yields a subset of $\text{Mor } \mathcal{C}$ and is defined by the term

- (Def. 2) $\{f, \text{ where } f \text{ is a morphism of } \mathcal{C} : \text{dom } f = a \text{ and } \text{cod } f = b\}$.

Let \mathcal{C} be a category structure. Assume $\text{hom}(a, b) \neq \emptyset$.

A morphism from a to b is a morphism of \mathcal{C} and is defined by

- (Def. 3) $it \in \text{hom}(a, b)$.

Let \mathcal{C} be category structure with identities and a be an object of \mathcal{C} . Assume $\text{hom}(a, a) \neq \emptyset$. Observe that the functor id_a yields a morphism from a to a . Let \mathcal{C} be a non empty category structure with identities. Note that $\text{hom}(a, a)$ is non empty.

Let \mathcal{C} be a composable category structure with identities, a, b, c be objects of \mathcal{C} , f be a morphism from a to b , and g be a morphism from b to c . Assume $\text{hom}(a, b) \neq \emptyset$ and $\text{hom}(b, c) \neq \emptyset$. The functor $g \cdot f$ yielding a morphism from a to c is defined by the term

- (Def. 4) $g \circ f$.

Now we state the propositions:

- (16) Let us consider a category structure \mathcal{C} , objects a, b of \mathcal{C} , and a morphism f from a to b . Suppose $\text{hom}(a, b) \neq \emptyset$. Then there exist morphisms f_1, f_2 of \mathcal{C} such that

- (i) $a = f_1$, and

- (ii) $b = f_2$, and
- (iii) $f \triangleright f_1$, and
- (iv) $f_2 \triangleright f$.

- (17) Let us consider a composable category structure \mathcal{C} with identities, objects a, b, c of \mathcal{C} , a morphism f_1 from a to b , and a morphism f_2 from b to c . Suppose $\text{hom}(a, b) \neq \emptyset$ and $\text{hom}(b, c) \neq \emptyset$. Then $f_2 \triangleright f_1$. The theorem is a consequence of (16) and (3).
- (18) Let us consider a composable category structure \mathcal{C} with identities, objects a, b of \mathcal{C} , and a morphism f from a to b . Suppose $\text{hom}(a, b) \neq \emptyset$. Then
- (i) $f \cdot \text{id}_a = f$, and
 - (ii) $\text{id}_b \cdot f = f$.

The theorem is a consequence of (17).

- (19) Let us consider a non empty, composable category structure \mathcal{C} with identities, and a morphism f of \mathcal{C} . Then $f \in \text{hom}(\text{dom } f, \text{cod } f)$.
- (20) Let us consider a non empty, composable category structure \mathcal{C} with identities, objects a, b of \mathcal{C} , and a morphism f of \mathcal{C} . Then $f \in \text{hom}(a, b)$ if and only if $\text{dom } f = a$ and $\text{cod } f = b$.
- (21) Let us consider a non empty, composable category structure \mathcal{C} with identities, and an object a of \mathcal{C} . Then $a \in \text{hom}(a, a)$. The theorem is a consequence of (6).
- (22) Let us consider a composable category structure \mathcal{C} with identities, and objects a, b, c of \mathcal{C} . Suppose $\text{hom}(a, b) \neq \emptyset$ and $\text{hom}(b, c) \neq \emptyset$. Then $\text{hom}(a, c) \neq \emptyset$. The theorem is a consequence of (16) and (3).
- (23) Let us consider a category \mathcal{C} , objects a, b, c, d of \mathcal{C} , a morphism f_1 from a to b , a morphism f_2 from b to c , and a morphism f_3 from c to d . Suppose $\text{hom}(a, b) \neq \emptyset$ and $\text{hom}(b, c) \neq \emptyset$ and $\text{hom}(c, d) \neq \emptyset$. Then $f_3 \cdot (f_2 \cdot f_1) = (f_3 \cdot f_2) \cdot f_1$. The theorem is a consequence of (22) and (17).
- (24) Let us consider a composable category structure \mathcal{C} with identities, objects a, b, c of \mathcal{C} , a morphism f_1 from a to b , and a morphism f_2 from b to c . Suppose $\text{hom}(a, b) \neq \emptyset$ and $\text{hom}(b, c) \neq \emptyset$. Then
- (i) if f_1 is identity, then $f_2 \cdot f_1 = f_2$, and
 - (ii) if f_2 is identity, then $f_2 \cdot f_1 = f_1$.

PROOF: $f_2 \triangleright f_1$. If f_1 is identity, then $f_2 \cdot f_1 = f_2$ by [17, (22), (23)]. \square

3. MONOMORPHISMS, EPIMORPHISMS AND ISOMORPHISMS

Let \mathcal{C} be a composable category structure with identities, a, b be objects of \mathcal{C} , and f be a morphism from a to b . We say that f is monomorphic if and only if

(Def. 5) $\text{hom}(a, b) \neq \emptyset$ and for every object c of \mathcal{C} such that $\text{hom}(c, a) \neq \emptyset$ for every morphisms g_1, g_2 from c to a such that $f \cdot g_1 = f \cdot g_2$ holds $g_1 = g_2$.

We say that f is epimorphic if and only if

(Def. 6) $\text{hom}(a, b) \neq \emptyset$ and for every object c of \mathcal{C} such that $\text{hom}(b, c) \neq \emptyset$ for every morphisms g_1, g_2 from b to c such that $g_1 \cdot f = g_2 \cdot f$ holds $g_1 = g_2$.

Now we state the proposition:

(25) Let us consider a composable category structure \mathcal{C} with identities, objects a, b of \mathcal{C} , and a morphism f_1 from a to b . Suppose $\text{hom}(a, b) \neq \emptyset$ and f_1 is identity. Then f_1 is monomorphic. The theorem is a consequence of (24).

Let us consider a category \mathcal{C} , objects a, b, c of \mathcal{C} , a morphism f_1 from a to b , and a morphism f_2 from b to c . Now we state the propositions:

(26) If f_1 is monomorphic and f_2 is monomorphic, then $f_2 \cdot f_1$ is monomorphic. The theorem is a consequence of (22) and (23).

(27) If $f_2 \cdot f_1$ is monomorphic and $\text{hom}(a, b) \neq \emptyset$ and $\text{hom}(b, c) \neq \emptyset$, then f_1 is monomorphic. The theorem is a consequence of (23).

Let \mathcal{C} be a composable category structure with identities, a, b be objects of \mathcal{C} , and f be a morphism from a to b . We say that f is a section if and only if

(Def. 7) $\text{hom}(a, b) \neq \emptyset$ and $\text{hom}(b, a) \neq \emptyset$ and there exists a morphism g from b to a such that $g \cdot f = \text{id-}a$.

We say that f is a retraction if and only if

(Def. 8) $\text{hom}(a, b) \neq \emptyset$ and $\text{hom}(b, a) \neq \emptyset$ and there exists a morphism g from b to a such that $f \cdot g = \text{id-}b$.

Now we state the propositions:

(28) Let us consider a category \mathcal{C} , objects a, b of \mathcal{C} , and a morphism f from a to b . If f is a section, then f is monomorphic. The theorem is a consequence of (23) and (18).

(29) Let us consider a composable category structure \mathcal{C} with identities, objects a, b of \mathcal{C} , and a morphism f_1 from a to b . Suppose $\text{hom}(a, b) \neq \emptyset$ and f_1 is identity. Then f_1 is epimorphic. The theorem is a consequence of (24).

Let us consider a category \mathcal{C} , objects a, b, c of \mathcal{C} , a morphism f_1 from a to b , and a morphism f_2 from b to c . Now we state the propositions:

- (30) If f_1 is epimorphic and f_2 is epimorphic, then $f_2 \cdot f_1$ is epimorphic. The theorem is a consequence of (22) and (23).
- (31) If $f_2 \cdot f_1$ is epimorphic and $\text{hom}(a, b) \neq \emptyset$ and $\text{hom}(b, c) \neq \emptyset$, then f_2 is epimorphic. The theorem is a consequence of (23).
- (32) Let us consider a category \mathcal{C} , objects a, b of \mathcal{C} , and a morphism f from a to b . If f is a retraction, then f is epimorphic. The theorem is a consequence of (23) and (18).

Let \mathcal{C} be a composable category structure with identities, a, b be objects of \mathcal{C} , and f be a morphism from a to b . We say that f is isomorphism if and only if

- (Def. 9) $\text{hom}(a, b) \neq \emptyset$ and $\text{hom}(b, a) \neq \emptyset$ and there exists a morphism g from b to a such that $g \cdot f = \text{id-}a$ and $f \cdot g = \text{id-}b$.

We say that a and b are isomorphic if and only if

- (Def. 10) there exists a morphism f from a to b such that f is isomorphism.

Note that a and b are isomorphic if and only if the condition (Def. 11) is satisfied.

- (Def. 11) $\text{hom}(a, b) \neq \emptyset$ and $\text{hom}(b, a) \neq \emptyset$ and there exists a morphism f from a to b and there exists a morphism g from b to a such that $g \cdot f = \text{id-}a$ and $f \cdot g = \text{id-}b$.

Now we state the proposition:

- (33) Let us consider a category \mathcal{C} , objects a, b of \mathcal{C} , and a morphism f from a to b . If f is isomorphism, then f is monomorphic and epimorphic. The theorem is a consequence of (28) and (32).

4. ORDINAL NUMBERS AS CATEGORIES

Let \mathcal{C} be a category structure. We say that \mathcal{C} is a preorder if and only if

- (Def. 12) for every objects a, b of \mathcal{C} and for every morphisms f_1, f_2 of \mathcal{C} such that $f_1, f_2 \in \text{hom}(a, b)$ holds $f_1 = f_2$.

Observe that every category structure which is empty is also a preorder and there exists a category structure which is strict and preorder and every composable category structure with identities which is a preorder is also associative.

Let \mathcal{C} be category structure with identities. The functor $\text{RelOb } \mathcal{C}$ yielding a binary relation on $\text{Ob } \mathcal{C}$ is defined by the term

- (Def. 13) $\{\langle a, b \rangle, \text{ where } a, b \text{ are objects of } \mathcal{C} : \text{ there exists a morphism } f \text{ of } \mathcal{C} \text{ such that } f \in \text{hom}(a, b)\}$.

Let \mathcal{C} be an empty category structure with identities. Let us note that $\text{RelOb } \mathcal{C}$ is empty.

Now we state the propositions:

- (34) Let us consider a composable category structure \mathcal{C} with identities. Then
- (i) $\text{dom RelOb } \mathcal{C} = \text{Ob } \mathcal{C}$, and
 - (ii) $\text{rng RelOb } \mathcal{C} = \text{Ob } \mathcal{C}$.

The theorem is a consequence of (6) and (19).

- (35) Let us consider composable category structures $\mathcal{C}_1, \mathcal{C}_2$ with identities. Suppose $\mathcal{C}_1 \cong \mathcal{C}_2$. Then $\text{RelOb } \mathcal{C}_1$ and $\text{RelOb } \mathcal{C}_2$ are isomorphic. The theorem is a consequence of (15), (34), and (20).

Let \mathcal{C} be a non empty, composable category structure with identities. One can verify that $\text{RelOb } \mathcal{C}$ is non empty.

Now we state the propositions:

- (36) Let us consider preorder, composable category structure \mathcal{C} with identities. Suppose \mathcal{C} is not empty. Then there exists a function \mathcal{F} from \mathcal{C} into $\text{RelOb } \mathcal{C}$ such that
- (i) \mathcal{F} is bijective, and
 - (ii) for every morphism f of \mathcal{C} , $\mathcal{F}(f) = \langle \text{dom } f, \text{cod } f \rangle$.

PROOF: Reconsider $\mathcal{C}_1 = \mathcal{C}$ as a non empty, composable category structure with identities. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ for every morphism f of \mathcal{C}_1 such that $\$1 = f$ holds $\$2 = \langle \text{dom } f, \text{cod } f \rangle$. For every element x of the carrier of \mathcal{C}_1 , there exists an element y of $\text{RelOb } \mathcal{C}_1$ such that $\mathcal{P}[x, y]$. Consider \mathcal{F} being a function from the carrier of \mathcal{C}_1 into $\text{RelOb } \mathcal{C}_1$ such that for every element x of the carrier of \mathcal{C}_1 , $\mathcal{P}[x, \mathcal{F}(x)]$ from [10, Sch. 3]. For every object y such that $y \in \text{RelOb } \mathcal{C}$ holds $y \in \text{rng } \mathcal{F}$ by (20), [9, (3)]. For every objects x_1, x_2 such that $x_1, x_2 \in \text{dom } \mathcal{F}$ and $\mathcal{F}(x_1) = \mathcal{F}(x_2)$ holds $x_1 = x_2$. \square

- (37) Let us consider an ordinal number O . Then there exists a strict, a pre-order category \mathcal{C} such that
- (i) $\text{Ob } \mathcal{C} = O$, and
 - (ii) for every objects o_1, o_2 of \mathcal{C} such that $o_1 \in o_2$ holds $\text{hom}(o_1, o_2) = \{\langle o_1, o_2 \rangle\}$, and
 - (iii) $\text{RelOb } \mathcal{C} = \subseteq_O$, and
 - (iv) $\text{Mor } \mathcal{C} = O \cup \{\langle o_1, o_2 \rangle, \text{ where } o_1, o_2 \text{ are elements of } O : o_1 \in o_2\}$.

The theorem is a consequence of (6), (20), and (21).

Let O be an ordinal number and \mathcal{C} be a composable category structure with identities. We say that \mathcal{C} is O -ordered if and only if

- (Def. 14) $\text{RelOb } \mathcal{C}$ and \subseteq_O are isomorphic.

Let O be a non empty, ordinal number. Let us observe that every composable category structure with identities which is O -ordered is also non empty.

Let O be an ordinal number. Note that there exists a composable category structure with identities which is strict, O -ordered, and preorder.

Let O be an empty, ordinal number. Let us observe that every composable category structure with identities which is O -ordered is also empty.

Now we state the proposition:

- (38) Let us consider ordinal numbers O_1, O_2 , a O_1 -ordered, a preorder category \mathcal{C}_1 , and a O_2 -ordered, a preorder category \mathcal{C}_2 . Then $O_1 = O_2$ if and only if $\mathcal{C}_1 \cong \mathcal{C}_2$.

PROOF: If $O_1 = O_2$, then $\mathcal{C}_1 \cong \mathcal{C}_2$ by (13), [4, (39), (41)], (36). If $\mathcal{C}_1 \cong \mathcal{C}_2$, then $O_1 = O_2$ by (35), [4, (42), (40)], [5, (10)]. \square

Let O be an ordinal number. The functor \mathbf{O} yielding a strict, O -ordered, a preorder category is defined by the term

(Def. 15) the strict, O -ordered, a preorder category.

Now we state the proposition:

- (39) There exists a morphism f of $\mathbf{2}$ such that

- (i) f is not identity, and
- (ii) $\text{Ob } \mathbf{2} = \{\text{dom } f, \text{cod } f\}$, and
- (iii) $\text{Mor } \mathbf{2} = \{\text{dom } f, \text{cod } f, f\}$, and
- (iv) $\text{dom } f, \text{cod } f, f$ are mutually different.

PROOF: Consider \mathcal{C} being a strict, a preorder category such that $\text{Ob } \mathcal{C} = 2$ and for every objects o_1, o_2 of \mathcal{C} such that $o_1 \in o_2$ holds $\text{hom}(o_1, o_2) = \{\langle o_1, o_2 \rangle\}$ and $\text{RelOb } \mathcal{C} = \subseteq_2$ and $\text{Mor } \mathcal{C} = 2 \cup \{\langle o_1, o_2 \rangle\}$, where o_1, o_2 are elements of $2 : o_1 \in o_2$. $\mathcal{C} \cong \mathbf{2}$. Consider \mathcal{F} being a functor from \mathcal{C} to $\mathbf{2}$, \mathcal{G} being a functor from $\mathbf{2}$ to \mathcal{C} such that \mathcal{F} is covariant and \mathcal{G} is covariant and $\mathcal{G} \circ \mathcal{F} = \text{id}_{\mathcal{C}}$ and $\mathcal{F} \circ \mathcal{G} = \text{id}_{\mathbf{2}}$. Reconsider $g = \langle 0, 1 \rangle$ as a morphism of \mathcal{C} . g is not identity by [17, (22)]. Set $f = \mathcal{F}(g)$. f is not identity by [9, (18)], [17, (34)]. $\overline{\text{Ob } \mathbf{2}} = \overline{2}$. Consider x, y being objects such that $x \neq y$ and $\text{Ob } \mathbf{2} = \{x, y\}$. $\text{dom } f \neq \text{cod } f$. For every object x , $x \in \text{Mor } \mathbf{2}$ iff $x \in \{\text{dom } f, \text{cod } f, f\}$ by [17, (22)], [9, (18)], [17, (34)], [2, (50), (49)]. \square

Let \mathcal{C} be a non empty category and f be a morphism of \mathcal{C} . The functor \mathcal{M}_f yielding a covariant functor from $\mathbf{2}$ to \mathcal{C} is defined by

(Def. 16) for every morphism g of $\mathbf{2}$ such that g is not identity holds $it(g) = f$.

Now we state the proposition:

- (40) Let us consider a non empty category \mathcal{C} , and a morphism f of \mathcal{C} . Suppose f is identity. Let us consider a morphism g of $\mathbf{2}$. Then $(\mathcal{M}_f)(g) = f$. The theorem is a consequence of (39) and (6).

5. PULLBACKS

Let \mathcal{C} be a category, c, c_1, c_2, d be objects of \mathcal{C} , and f_1 be a morphism from c_1 to c . Assume $\text{hom}(c_1, c) \neq \emptyset$. Let f_2 be a morphism from c_2 to c . Assume $\text{hom}(c_2, c) \neq \emptyset$. Let p_1 be a morphism from d to c_1 . Assume $\text{hom}(d, c_1) \neq \emptyset$. Let p_2 be a morphism from d to c_2 . Assume $\text{hom}(d, c_2) \neq \emptyset$. We say that $\langle d, p_1, p_2 \rangle$ is a pullback of f_1, f_2 if and only if

(Def. 17) $f_1 \cdot p_1 = f_2 \cdot p_2$ and for every object d_1 of \mathcal{C} and for every morphism g_1 from d_1 to c_1 and for every morphism g_2 from d_1 to c_2 such that $\text{hom}(d_1, c_1) \neq \emptyset$ and $\text{hom}(d_1, c_2) \neq \emptyset$ and $f_1 \cdot g_1 = f_2 \cdot g_2$ holds $\text{hom}(d_1, d) \neq \emptyset$ and there exists a morphism h from d_1 to d such that $p_1 \cdot h = g_1$ and $p_2 \cdot h = g_2$ and for every morphism h_1 from d_1 to d such that $p_1 \cdot h_1 = g_1$ and $p_2 \cdot h_1 = g_2$ holds $h = h_1$.

Now we state the proposition:

(41) Let us consider a category \mathcal{C} , objects c, c_1, c_2, d, e of \mathcal{C} , a morphism f_1 from c_1 to c , a morphism f_2 from c_2 to c , a morphism p_1 from d to c_1 , a morphism p_2 from d to c_2 , a morphism q_1 from e to c_1 , and a morphism q_2 from e to c_2 . Suppose $\text{hom}(c_1, c) \neq \emptyset$ and $\text{hom}(c_2, c) \neq \emptyset$ and $\text{hom}(d, c_1) \neq \emptyset$ and $\text{hom}(d, c_2) \neq \emptyset$ and $\text{hom}(e, c_1) \neq \emptyset$ and $\text{hom}(e, c_2) \neq \emptyset$ and $\langle d, p_1, p_2 \rangle$ is a pullback of f_1, f_2 and $\langle e, q_1, q_2 \rangle$ is a pullback of f_1, f_2 . Then d and e are isomorphic. The theorem is a consequence of (23) and (18).

Let us consider a category \mathcal{C} , objects c, c_1, c_2, d of \mathcal{C} , a morphism f_1 from c_1 to c , a morphism f_2 from c_2 to c , a morphism p_1 from d to c_1 , and a morphism p_2 from d to c_2 . Now we state the propositions:

(42) Suppose $\text{hom}(c_1, c) \neq \emptyset$ and $\text{hom}(c_2, c) \neq \emptyset$ and $\text{hom}(d, c_1) \neq \emptyset$ and $\text{hom}(d, c_2) \neq \emptyset$ and $\langle d, p_1, p_2 \rangle$ is a pullback of f_1, f_2 .

Then $\langle d, p_2, p_1 \rangle$ is a pullback of f_2, f_1 .

(43) Suppose $\text{hom}(c_1, c) \neq \emptyset$ and $\text{hom}(c_2, c) \neq \emptyset$ and $\text{hom}(d, c_1) \neq \emptyset$ and $\text{hom}(d, c_2) \neq \emptyset$ and $\langle d, p_1, p_2 \rangle$ is a pullback of f_1, f_2 and f_1 is monomorphic. Then p_2 is monomorphic. The theorem is a consequence of (22) and (23).

(44) Suppose $\text{hom}(c_1, c) \neq \emptyset$ and $\text{hom}(c_2, c) \neq \emptyset$ and $\text{hom}(d, c_1) \neq \emptyset$ and $\text{hom}(d, c_2) \neq \emptyset$ and $\langle d, p_1, p_2 \rangle$ is a pullback of f_1, f_2 and f_1 is isomorphism. Then p_2 is isomorphism. The theorem is a consequence of (22), (23), and (18).

(45) Let us consider a category \mathcal{C} , objects $c_1, c_1, c_2, c_3, c_4, c_5, c_6$ of \mathcal{C} , a morphism f_1 from c_1 to c_2 , a morphism f_2 from c_2 to c_3 , a morphism f_3 from c_1 to c_4 , a morphism f_4 from c_2 to c_5 , a morphism f_5 from

c_3 to c_6 , a morphism f_6 from c_4 to c_5 , and a morphism f_7 from c_5 to c_6 . Suppose $\text{hom}(c_1, c_2) \neq \emptyset$ and $\text{hom}(c_2, c_3) \neq \emptyset$ and $\text{hom}(c_1, c_4) \neq \emptyset$ and $\text{hom}(c_2, c_5) \neq \emptyset$ and $\text{hom}(c_3, c_6) \neq \emptyset$ and $\text{hom}(c_4, c_5) \neq \emptyset$ and $\text{hom}(c_5, c_6) \neq \emptyset$ and $\langle c_2, f_2, f_4 \rangle$ is a pullback of f_5, f_7 . Then $\langle c_1, f_1, f_3 \rangle$ is a pullback of f_4, f_6 if and only if $\langle c_1, f_2 \cdot f_1, f_3 \rangle$ is a pullback of $f_5, f_7 \cdot f_6$ and $f_4 \cdot f_1 = f_6 \cdot f_3$. The theorem is a consequence of (22) and (23).

6. PULLBACKS OF FUNCTORS

Let \mathcal{C}, \mathcal{D} be categories and \mathcal{F} be a functor from \mathcal{C} to \mathcal{D} . We say that \mathcal{F} is monomorphic if and only if

(Def. 18) \mathcal{F} is covariant and for every category \mathcal{B} and for every functors $\mathcal{G}_1, \mathcal{G}_2$ from \mathcal{B} to \mathcal{C} such that \mathcal{G}_1 is covariant and \mathcal{G}_2 is covariant and $\mathcal{F} \circ \mathcal{G}_1 = \mathcal{F} \circ \mathcal{G}_2$ holds $\mathcal{G}_1 = \mathcal{G}_2$.

We say that \mathcal{F} is isomorphism if and only if

(Def. 19) \mathcal{F} is covariant and there exists a functor \mathcal{G} from \mathcal{D} to \mathcal{C} such that \mathcal{G} is covariant and $\mathcal{G} \circ \mathcal{F} = \text{id}_{\mathcal{C}}$ and $\mathcal{F} \circ \mathcal{G} = \text{id}_{\mathcal{D}}$.

Let $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2, \mathcal{D}$ be categories and \mathcal{F}_1 be a functor from \mathcal{C}_1 to \mathcal{C} . Assume \mathcal{F}_1 is covariant. Let \mathcal{F}_2 be a functor from \mathcal{C}_2 to \mathcal{C} . Assume \mathcal{F}_2 is covariant. Let \mathcal{P}_1 be a functor from \mathcal{D} to \mathcal{C}_1 . Assume \mathcal{P}_1 is covariant. Let \mathcal{P}_2 be a functor from \mathcal{D} to \mathcal{C}_2 . Assume \mathcal{P}_2 is covariant. We say that $\langle \mathcal{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$ if and only if

(Def. 20) $\mathcal{F}_1 \circ \mathcal{P}_1 = \mathcal{F}_2 \circ \mathcal{P}_2$ and for every category \mathcal{D}_1 and for every functor \mathcal{G}_1 from \mathcal{D}_1 to \mathcal{C}_1 and for every functor \mathcal{G}_2 from \mathcal{D}_1 to \mathcal{C}_2 such that \mathcal{G}_1 is covariant and \mathcal{G}_2 is covariant and $\mathcal{F}_1 \circ \mathcal{G}_1 = \mathcal{F}_2 \circ \mathcal{G}_2$ there exists a functor \mathcal{H} from \mathcal{D}_1 to \mathcal{D} such that \mathcal{H} is covariant and $\mathcal{P}_1 \circ \mathcal{H} = \mathcal{G}_1$ and $\mathcal{P}_2 \circ \mathcal{H} = \mathcal{G}_2$ and for every functor \mathcal{H}_1 from \mathcal{D}_1 to \mathcal{D} such that \mathcal{H}_1 is covariant and $\mathcal{P}_1 \circ \mathcal{H}_1 = \mathcal{G}_1$ and $\mathcal{P}_2 \circ \mathcal{H}_1 = \mathcal{G}_2$ holds $\mathcal{H} = \mathcal{H}_1$.

Now we state the proposition:

(46) Let us consider categories $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2, \mathcal{D}, \mathcal{E}$, a functor \mathcal{F}_1 from \mathcal{C}_1 to \mathcal{C} , a functor \mathcal{F}_2 from \mathcal{C}_2 to \mathcal{C} , a functor \mathcal{P}_1 from \mathcal{D} to \mathcal{C}_1 , a functor \mathcal{P}_2 from \mathcal{D} to \mathcal{C}_2 , a functor \mathcal{Q}_1 from \mathcal{E} to \mathcal{C}_1 , and a functor \mathcal{Q}_2 from \mathcal{E} to \mathcal{C}_2 . Suppose \mathcal{F}_1 is covariant and \mathcal{F}_2 is covariant and \mathcal{P}_1 is covariant and \mathcal{P}_2 is covariant and \mathcal{Q}_1 is covariant and \mathcal{Q}_2 is covariant and $\langle \mathcal{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$ and $\langle \mathcal{E}, \mathcal{Q}_1, \mathcal{Q}_2 \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$. Then $\mathcal{D} \cong \mathcal{E}$.

PROOF: There exists a functor \mathcal{F}_8 from \mathcal{D} to \mathcal{E} and there exists a functor \mathcal{G}_3 from \mathcal{E} to \mathcal{D} such that \mathcal{F}_8 is covariant and \mathcal{G}_3 is covariant and $\mathcal{G}_3 \circ \mathcal{F}_8 = \text{id}_{\mathcal{D}}$ and $\mathcal{F}_8 \circ \mathcal{G}_3 = \text{id}_{\mathcal{E}}$ by (10), (11), [17, (35)]. \square

Let us consider categories \mathcal{C} , \mathcal{C}_1 , \mathcal{C}_2 , \mathcal{D} , a functor \mathcal{F}_1 from \mathcal{C}_1 to \mathcal{C} , a functor \mathcal{F}_2 from \mathcal{C}_2 to \mathcal{C} , a functor \mathcal{P}_1 from \mathcal{D} to \mathcal{C}_1 , and a functor \mathcal{P}_2 from \mathcal{D} to \mathcal{C}_2 . Now we state the propositions:

(47) Suppose \mathcal{F}_1 is covariant and \mathcal{F}_2 is covariant and \mathcal{P}_1 is covariant and \mathcal{P}_2 is covariant and $\langle \mathcal{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$.
Then $\langle \mathcal{D}, \mathcal{P}_2, \mathcal{P}_1 \rangle$ is a pullback of $\mathcal{F}_2, \mathcal{F}_1$.

(48) Suppose \mathcal{F}_1 is covariant and \mathcal{F}_2 is covariant and \mathcal{P}_1 is covariant and \mathcal{P}_2 is covariant and $\langle \mathcal{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$ and \mathcal{F}_1 is monomorphic. Then \mathcal{P}_2 is monomorphic.

PROOF: For every category \mathcal{D}_1 and for every functors $\mathcal{Q}_1, \mathcal{Q}_2$ from \mathcal{D}_1 to \mathcal{D} such that \mathcal{Q}_1 is covariant and \mathcal{Q}_2 is covariant and $\mathcal{P}_2 \circ \mathcal{Q}_1 = \mathcal{P}_2 \circ \mathcal{Q}_2$ holds $\mathcal{Q}_1 = \mathcal{Q}_2$ by [17, (35)], (10). \square

(49) Suppose \mathcal{F}_1 is covariant and \mathcal{F}_2 is covariant and \mathcal{P}_1 is covariant and \mathcal{P}_2 is covariant and $\langle \mathcal{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$ and \mathcal{F}_1 is isomorphism. Then \mathcal{P}_2 is isomorphism. The theorem is a consequence of (10) and (11).

(50) Let us consider categories $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6$, a functor \mathcal{F}_1 from \mathcal{C}_1 to \mathcal{C}_2 , a functor \mathcal{F}_2 from \mathcal{C}_2 to \mathcal{C}_3 , a functor \mathcal{F}_3 from \mathcal{C}_1 to \mathcal{C}_4 , a functor \mathcal{F}_4 from \mathcal{C}_2 to \mathcal{C}_5 , a functor \mathcal{F}_5 from \mathcal{C}_3 to \mathcal{C}_6 , a functor \mathcal{F}_6 from \mathcal{C}_4 to \mathcal{C}_5 , and a functor \mathcal{F}_7 from \mathcal{C}_5 to \mathcal{C}_6 . Suppose \mathcal{F}_1 is covariant and \mathcal{F}_2 is covariant and \mathcal{F}_3 is covariant and \mathcal{F}_4 is covariant and \mathcal{F}_5 is covariant and \mathcal{F}_6 is covariant and \mathcal{F}_7 is covariant and $\langle \mathcal{C}_2, \mathcal{F}_2, \mathcal{F}_4 \rangle$ is a pullback of $\mathcal{F}_5, \mathcal{F}_7$. Then $\langle \mathcal{C}_1, \mathcal{F}_1, \mathcal{F}_3 \rangle$ is a pullback of $\mathcal{F}_4, \mathcal{F}_6$ if and only if $\langle \mathcal{C}_1, \mathcal{F}_2 \circ \mathcal{F}_1, \mathcal{F}_3 \rangle$ is a pullback of $\mathcal{F}_5, \mathcal{F}_7 \circ \mathcal{F}_6$ and $\mathcal{F}_4 \circ \mathcal{F}_1 = \mathcal{F}_6 \circ \mathcal{F}_3$.

PROOF: For every category \mathcal{D}_1 and for every functor \mathcal{G}_1 from \mathcal{D}_1 to \mathcal{C}_2 and for every functor \mathcal{G}_2 from \mathcal{D}_1 to \mathcal{C}_4 such that \mathcal{G}_1 is covariant and \mathcal{G}_2 is covariant and $\mathcal{F}_4 \circ \mathcal{G}_1 = \mathcal{F}_6 \circ \mathcal{G}_2$ there exists a functor \mathcal{H} from \mathcal{D}_1 to \mathcal{C}_1 such that \mathcal{H} is covariant and $\mathcal{F}_1 \circ \mathcal{H} = \mathcal{G}_1$ and $\mathcal{F}_3 \circ \mathcal{H} = \mathcal{G}_2$ and for every functor \mathcal{H}_1 from \mathcal{D}_1 to \mathcal{C}_1 such that \mathcal{H}_1 is covariant and $\mathcal{F}_1 \circ \mathcal{H}_1 = \mathcal{G}_1$ and $\mathcal{F}_3 \circ \mathcal{H}_1 = \mathcal{G}_2$ holds $\mathcal{H} = \mathcal{H}_1$ by [17, (35)], (10). \square

(51) Let us consider categories \mathcal{C} , \mathcal{C}_1 , \mathcal{C}_2 , a functor \mathcal{F}_1 from \mathcal{C}_1 to \mathcal{C} , and a functor \mathcal{F}_2 from \mathcal{C}_2 to \mathcal{C} . Suppose \mathcal{F}_1 is covariant and \mathcal{F}_2 is covariant. Then there exists a strict category \mathcal{D} and there exists a functor \mathcal{P}_1 from \mathcal{D} to \mathcal{C}_1 and there exists a functor \mathcal{P}_2 from \mathcal{D} to \mathcal{C}_2 such that the carrier of $\mathcal{D} = \{ \langle f_1, f_2 \rangle, \text{ where } f_1 \text{ is a morphism of } \mathcal{C}_1, f_2 \text{ is a morphism of } \mathcal{C}_2 : f_1 \in \text{the carrier of } \mathcal{C}_1 \text{ and } f_2 \in \text{the carrier of } \mathcal{C}_2 \text{ and } \mathcal{F}_1(f_1) = \mathcal{F}_2(f_2) \}$ and the composition of $\mathcal{D} = \{ \langle \langle f_1, f_2 \rangle, f_3 \rangle, \text{ where } f_1, f_2, f_3 \text{ are morphisms of } \mathcal{D} : f_1, f_2, f_3 \in \text{the carrier of } \mathcal{D} \text{ and for every morphisms } f_{11}, f_{12}, f_{13} \text{ of } \mathcal{C}_1 \text{ and for every morphisms } f_{21}, f_{22}, f_{23} \text{ of } \mathcal{C}_2 \text{ such that } f_1 = \langle f_{11}, f_{21} \rangle \text{ and } f_2 = \langle f_{12}, f_{22} \rangle \text{ and } f_3 = \langle f_{13}, f_{23} \rangle \text{ holds}$

$f_{11} \triangleright f_{12}$ and $f_{21} \triangleright f_{22}$ and $f_{13} = f_{11} \circ f_{12}$ and $f_{23} = f_{21} \circ f_{22}$ and \mathcal{P}_1 is covariant and \mathcal{P}_2 is covariant and $\langle \mathcal{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$.

PROOF: Reconsider $c_7 = \{\langle f_1, f_2 \rangle\}$, where f_1 is a morphism of \mathcal{C}_1, f_2 is a morphism of $\mathcal{C}_2 : f_1 \in$ the carrier of \mathcal{C}_1 and $f_2 \in$ the carrier of \mathcal{C}_2 and $\mathcal{F}_1(f_1) = \mathcal{F}_2(f_2)$ as a set. Set $c_8 = \{\langle \langle x_1, x_2 \rangle, x_3 \rangle\}$, where x_1, x_2, x_3 are elements of $c_7 : x_1, x_2, x_3 \in c_7$ and for every morphisms f_{11}, f_{12}, f_{13} of \mathcal{C}_1 and for every morphisms f_{21}, f_{22}, f_{23} of \mathcal{C}_2 such that $x_1 = \langle f_{11}, f_{21} \rangle$ and $x_2 = \langle f_{12}, f_{22} \rangle$ and $x_3 = \langle f_{13}, f_{23} \rangle$ holds $f_{11} \triangleright f_{12}$ and $f_{21} \triangleright f_{22}$ and $f_{13} = f_{11} \circ f_{12}$ and $f_{23} = f_{21} \circ f_{22}$. For every object x such that $x \in c_8$ holds $x \in (c_7 \times c_7) \times c_7$. For every objects x, y_1, y_2 such that $\langle x, y_1 \rangle, \langle x, y_2 \rangle \in c_8$ holds $y_1 = y_2$. Set $\mathcal{D} = \langle c_7, c_8 \rangle$. For every morphisms g_1, g_2 of \mathcal{D} such that $g_1 \triangleright g_2$ there exist morphisms f_{11}, f_{12}, f_{13} of \mathcal{C}_1 and there exist morphisms f_{21}, f_{22}, f_{23} of \mathcal{C}_2 such that $g_1 = \langle f_{11}, f_{21} \rangle$ and $g_2 = \langle f_{12}, f_{22} \rangle$ and $\mathcal{F}_1(f_{11}) = \mathcal{F}_2(f_{21})$ and $\mathcal{F}_1(f_{12}) = \mathcal{F}_2(f_{22})$ and $f_{11} \triangleright f_{12}$ and $f_{21} \triangleright f_{22}$ and $f_{13} = f_{11} \circ f_{12}$ and $f_{23} = f_{21} \circ f_{22}$ and $g_1 \circ g_2 = \langle f_{13}, f_{23} \rangle$ by (1), [17, (1)], [9, (1)]. For every morphisms g_1, g_2 of \mathcal{D} such that there exist morphisms f_{11}, f_{12} of \mathcal{C}_1 and there exist morphisms f_{21}, f_{22} of \mathcal{C}_2 such that $g_1 = \langle f_{11}, f_{21} \rangle$ and $g_2 = \langle f_{12}, f_{22} \rangle$ and $\mathcal{F}_1(f_{11}) = \mathcal{F}_2(f_{21})$ and $\mathcal{F}_1(f_{12}) = \mathcal{F}_2(f_{22})$ and $f_{11} \triangleright f_{12}$ and $f_{21} \triangleright f_{22}$ holds $g_1 \triangleright g_2$ by (1), [17, (1)]. For every morphisms g, g_1, g_2 of \mathcal{D} such that $g_1 \triangleright g_2$ holds $g_1 \circ g_2 \triangleright g$ iff $g_2 \triangleright g$. For every morphisms g, g_1, g_2 of \mathcal{D} such that $g_1 \triangleright g_2$ holds $g \triangleright g_1 \circ g_2$ iff $g \triangleright g_1$. For every morphism g_1 of \mathcal{D} such that $g_1 \in$ the carrier of \mathcal{D} there exists a morphism g of \mathcal{D} such that $g \triangleright g_1$ and g is left identity by (2), [17, (31), (32)]. For every morphism g_1 of \mathcal{D} such that $g_1 \in$ the carrier of \mathcal{D} there exists a morphism g of \mathcal{D} such that $g_1 \triangleright g$ and g is right identity by (2), [17, (31), (32)]. For every morphisms g_1, g_2, g_3 of \mathcal{D} such that $g_1 \triangleright g_2$ and $g_2 \triangleright g_3$ and $g_1 \circ g_2 \triangleright g_3$ and $g_1 \triangleright g_2 \circ g_3$ holds $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$. For every object $x, x \in c_8$ iff $x \in \{\langle \langle f_1, f_2 \rangle, f_3 \rangle\}$, where f_1, f_2, f_3 are morphisms of $\mathcal{D} : f_1, f_2, f_3 \in$ the carrier of \mathcal{D} and for every morphisms f_{11}, f_{12}, f_{13} of \mathcal{C}_1 and for every morphisms f_{21}, f_{22}, f_{23} of \mathcal{C}_2 such that $f_1 = \langle f_{11}, f_{21} \rangle$ and $f_2 = \langle f_{12}, f_{22} \rangle$ and $f_3 = \langle f_{13}, f_{23} \rangle$ holds $f_{11} \triangleright f_{12}$ and $f_{21} \triangleright f_{22}$ and $f_{13} = f_{11} \circ f_{12}$ and $f_{23} = f_{21} \circ f_{22}$. There exists a functor \mathcal{P}_1 from \mathcal{D} to \mathcal{C}_1 and there exists a functor \mathcal{P}_2 from \mathcal{D} to \mathcal{C}_2 such that \mathcal{P}_1 is covariant and \mathcal{P}_2 is covariant and $\mathcal{F}_1 \circ \mathcal{P}_1 = \mathcal{F}_2 \circ \mathcal{P}_2$ and for every category \mathcal{D}_1 and for every functor \mathcal{G}_1 from \mathcal{D}_1 to \mathcal{C}_1 and for every functor \mathcal{G}_2 from \mathcal{D}_1 to \mathcal{C}_2 such that \mathcal{G}_1 is covariant and \mathcal{G}_2 is covariant and $\mathcal{F}_1 \circ \mathcal{G}_1 = \mathcal{F}_2 \circ \mathcal{G}_2$ there exists a functor \mathcal{H} from \mathcal{D}_1 to \mathcal{D} such that \mathcal{H} is covariant and $\mathcal{P}_1 \circ \mathcal{H} = \mathcal{G}_1$ and $\mathcal{P}_2 \circ \mathcal{H} = \mathcal{G}_2$ and for every functor \mathcal{H}_1 from \mathcal{D}_1 to \mathcal{D} such that \mathcal{H}_1 is covariant and $\mathcal{P}_1 \circ \mathcal{H}_1 = \mathcal{G}_1$ and $\mathcal{P}_2 \circ \mathcal{H}_1 = \mathcal{G}_2$ holds $\mathcal{H} = \mathcal{H}_1$ by [17, (31)], [9, (13)], (1), [17, (32), (34)]. Consider \mathcal{P}_1

being a functor from \mathcal{D} to \mathcal{C}_1 , \mathcal{P}_2 being a functor from \mathcal{D} to \mathcal{C}_2 such that \mathcal{P}_1 is covariant and \mathcal{P}_2 is covariant and $\mathcal{F}_1 \circ \mathcal{P}_1 = \mathcal{F}_2 \circ \mathcal{P}_2$ and for every category \mathcal{D}_1 and for every functor \mathcal{G}_1 from \mathcal{D}_1 to \mathcal{C}_1 and for every functor \mathcal{G}_2 from \mathcal{D}_1 to \mathcal{C}_2 such that \mathcal{G}_1 is covariant and \mathcal{G}_2 is covariant and $\mathcal{F}_1 \circ \mathcal{G}_1 = \mathcal{F}_2 \circ \mathcal{G}_2$ there exists a functor \mathcal{H} from \mathcal{D}_1 to \mathcal{D} such that \mathcal{H} is covariant and $\mathcal{P}_1 \circ \mathcal{H} = \mathcal{G}_1$ and $\mathcal{P}_2 \circ \mathcal{H} = \mathcal{G}_2$ and for every functor \mathcal{H}_1 from \mathcal{D}_1 to \mathcal{D} such that \mathcal{H}_1 is covariant and $\mathcal{P}_1 \circ \mathcal{H}_1 = \mathcal{G}_1$ and $\mathcal{P}_2 \circ \mathcal{H}_1 = \mathcal{G}_2$ holds $\mathcal{H} = \mathcal{H}_1$. \square

Let \mathcal{C} , \mathcal{C}_1 , \mathcal{C}_2 be categories and \mathcal{F}_1 be a functor from \mathcal{C}_1 to \mathcal{C} . Assume \mathcal{F}_1 is covariant. Let \mathcal{F}_2 be a functor from \mathcal{C}_2 to \mathcal{C} . Assume \mathcal{F}_2 is covariant.

A pullback of \mathcal{F}_1 , \mathcal{F}_2 is a triple object and is defined by

(Def. 21) there exists a strict category \mathcal{D} and there exists a functor \mathcal{P}_1 from \mathcal{D} to \mathcal{C}_1 and there exists a functor \mathcal{P}_2 from \mathcal{D} to \mathcal{C}_2 such that $it = \langle \mathcal{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ and \mathcal{P}_1 is covariant and \mathcal{P}_2 is covariant and $\langle \mathcal{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$.

Assume \mathcal{F}_1 is covariant. Assume \mathcal{F}_2 is covariant. The functor $[[\mathcal{F}_1, \mathcal{F}_2]]$ yielding a strict category is defined by the term

(Def. 22) the pullback of $\mathcal{F}_1, \mathcal{F}_{21,3}$.

Assume \mathcal{F}_1 is covariant. Assume \mathcal{F}_2 is covariant. The functor $\pi_1(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ yielding a functor from $[[\mathcal{F}_1, \mathcal{F}_2]]$ to \mathcal{C}_1 is defined by the term

(Def. 23) the pullback of $\mathcal{F}_1, \mathcal{F}_{22,3}$.

The functor $\pi_2(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ yielding a functor from $[[\mathcal{F}_1, \mathcal{F}_2]]$ to \mathcal{C}_2 is defined by the term

(Def. 24) the pullback of $\mathcal{F}_1, \mathcal{F}_{23,3}$.

Let us consider categories \mathcal{C} , \mathcal{C}_1 , \mathcal{C}_2 , a functor \mathcal{F}_1 from \mathcal{C}_1 to \mathcal{C} , and a functor \mathcal{F}_2 from \mathcal{C}_2 to \mathcal{C} . Let us assume that \mathcal{F}_1 is covariant and \mathcal{F}_2 is covariant. Now we state the propositions:

(52) (i) $\pi_1(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ is covariant, and

(ii) $\pi_2(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ is covariant, and

(iii) $\langle [[\mathcal{F}_1, \mathcal{F}_2]], \pi_1(\mathcal{F}_1 \boxtimes \mathcal{F}_2), \pi_2(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$.

(53) $[[\mathcal{F}_1, \mathcal{F}_2]] \cong [[\mathcal{F}_2, \mathcal{F}_1]]$. The theorem is a consequence of (52), (47), and (46).

(54) There exist object-categories \mathcal{C} , \mathcal{C}_1 , \mathcal{C}_2 and there exists a functor \mathcal{F}_1 from \mathcal{C}_1 to \mathcal{C} and there exists a functor \mathcal{F}_2 from \mathcal{C}_2 to \mathcal{C} such that there exists no object-category \mathcal{D} and there exists a functor \mathcal{P}_1 from \mathcal{D} to \mathcal{C}_1 and there exists a functor \mathcal{P}_2 from \mathcal{D} to \mathcal{C}_2 such that $\mathcal{F}_1 \cdot \mathcal{P}_1 = \mathcal{F}_2 \cdot \mathcal{P}_2$ and for every object-category \mathcal{D}_1 and for every functor \mathcal{G}_1 from \mathcal{D}_1 to \mathcal{C}_1 and for every functor \mathcal{G}_2 from \mathcal{D}_1 to \mathcal{C}_2 such that $\mathcal{F}_1 \cdot \mathcal{G}_1 = \mathcal{F}_2 \cdot \mathcal{G}_2$ there exists a functor \mathcal{H} from \mathcal{D}_1 to \mathcal{D} such that $\mathcal{P}_1 \cdot \mathcal{H} = \mathcal{G}_1$ and $\mathcal{P}_2 \cdot \mathcal{H} = \mathcal{G}_2$ and for

every functor \mathcal{H}_1 from \mathcal{D}_1 to \mathcal{D} such that $\mathcal{P}_1 \cdot \mathcal{H}_1 = \mathcal{G}_1$ and $\mathcal{P}_2 \cdot \mathcal{H}_1 = \mathcal{G}_2$ holds $\mathcal{H} = \mathcal{H}_1$. The theorem is a consequence of (39) and (40).

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