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Morley's Trisector Theorem

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Summary. Morley's trisector theorem states that "The points of intersection of the adjacent trisectors of the angles of any triangle are the vertices of an equilateral triangle" [10].

There are many proofs of Morley's trisector theorem [12, 16, 9, 13, 8, 20, 3, 18]. We follow the proof given by A. Letac in [15].

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The notation and terminology used in this paper have been introduced in the following articles: [1], [11], [7], [14], [19], [2], [4], [23], [5], [24], [21], [22], and [6].

1. Preliminaries

From now on A, B, C, D, E, F, G denote points of $\mathcal{E}^2_{\rm T}$.

Now we state the propositions:

(1) $\measuredangle(A, B, A) = 0$.

(2) $0 \leqslant \measuredangle(A, B, C) < 2 \cdot \pi$.

(3) (i) $0 \leqslant \measuredangle(A, B, C) < \pi$, or

(ii) $\measuredangle(A, B, C) = \pi$, or

(iii) $\pi < \measuredangle(A, B, C) < 2 \cdot \pi$.

The theorem is a consequence of (2).

(4) $|F - E|^{\mathbf{2}} = |A - E|^{\mathbf{2}} + |A - F|^{\mathbf{2}} - 2 \cdot |A - E| \cdot |A - F| \cdot \cos \measuredangle(E, A, F)$.

(5) If A, B, C are mutually different and $0 < \measuredangle(A, B, C) < \pi$, then $0 < \measuredangle(B, C, A) < \pi$ and $0 < \measuredangle(C, A, B) < \pi$.

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(6) Suppose A, B, C are mutually different and $\measuredangle(A, B, C) = 0$. Then

(i) $\measuredangle(B, C, A) = 0$ and $\measuredangle(C, A, B) = \pi$, or

(ii) $\measuredangle(B, C, A) = \pi$ and $\measuredangle(C, A, B) = 0$ and $\measuredangle(A, B, C) + \measuredangle(B, C, A) + \measuredangle(C, A, B) = \pi$.

(7) Suppose A, B, C are mutually different and $\measuredangle(A, B, C) = \pi$. Then

(i) $\measuredangle(B, C, A) = 0$, and

(ii) $\measuredangle(C, A, B) = 0$, and

(iii) $\measuredangle(A, B, C) + \measuredangle(B, C, A) + \measuredangle(C, A, B) = \pi$.

(8) If A, B, C are mutually different and $\measuredangle(A, B, C) > \pi$, then $\measuredangle(A, B, C) + \measuredangle(B, C, A) + \measuredangle(C, A, B) = 5 \cdot \pi$.

Let us assume that $\measuredangle(C, B, A) < \pi$. Now we state the propositions:

(9) $0 \leqslant$ area of $\triangle(A, B, C)$. The theorem is a consequence of (2).

(10) $0 \leqslant \varnothing_{\circlearrowleft}(A, B, C)$. The theorem is a consequence of (9).

2. Morley's Theorem

Now we state the propositions:

(11) Suppose A, F, C form a triangle and $\measuredangle(C, F, A) < \pi$ and $\measuredangle(A, C, F) = \measuredangle(A, C, B)/3$ and $\measuredangle(F, A, C) = \measuredangle(B, A, C)/3$ and $(\measuredangle(A, C, B)/3) + (\measuredangle(B, A, C)/3) + (\measuredangle(C, B, A)/3) = \pi/3$.
Then $|A - F| \cdot \sin((\pi/3) - (\measuredangle(C, B, A)/3)) = |A - C| \cdot \sin(\measuredangle(A, C, B)/3)$.

(12) Suppose A, B, C form a triangle and A, F, C form a triangle and $\measuredangle(C, F, A) < \pi$ and $\measuredangle(A, C, F) = \measuredangle(A, C, B)/3$ and $\measuredangle(F, A, C) = \measuredangle(B, A, C)/3$ and $(\measuredangle(A, C, B)/3) + (\measuredangle(B, A, C)/3) + (\measuredangle(C, B, A)/3) = \pi/3$ and $\sin((\pi/3) - (\measuredangle(C, B, A)/3)) \neq 0$. Then $|A - F| = 4 \cdot \varnothing_{\circlearrowleft}(A, B, C) \cdot \sin(\measuredangle(C, B, A)/3) \cdot \sin((\pi/3) + (\measuredangle(C, B, A)/3)) \cdot \sin(\measuredangle(A, C, B)/3)$. The theorem is a consequence of (11).

(13) Suppose C, A, B form a triangle and A, F, C form a triangle and F, A, E form a triangle and E, A, B form a triangle and $\measuredangle(B, A, E) = \measuredangle(B, A, C)/3$ and $\measuredangle(F, A, C) = \measuredangle(B, A, C)/3$. Then $\measuredangle(E, A, F) = \measuredangle(B, A, C)/3$. Proof: $\measuredangle(E, A, F) \neq 4 \cdot \pi + (\measuredangle(B, A, C)/3)$ by [17, (5)], (2), [7, (30)]. $\measuredangle(E, A, F) \neq 2 \cdot \pi + (\measuredangle(B, A, C)/3)$ by (2), [7, (30)]. □

(14) Suppose C, A, B form a triangle and $\measuredangle(A, C, B) < \pi$ and A, F, C form a triangle and F, A, E form a triangle and E, A, B form a triangle and $\measuredangle(B, A, E) = \measuredangle(B, A, C)/3$ and $\measuredangle(F, A, C) = \measuredangle(B, A, C)/3$. Then $(\pi/3) + (\measuredangle(A, C, B)/3) + ((\pi/3) + (\measuredangle(C, B, A)/3)) + \measuredangle(E, A, F) = \pi$. The theorem is a consequence of (13).

(15) If A, C, B form a triangle, then $\sin((\pi/3) - (\measuredangle(A,C,B)/3)) \neq 0$. The theorem is a consequence of (2).

(16) Suppose A, B, C form a triangle and A, B, E form a triangle and $\measuredangle(E,B,A) = \measuredangle(C,B,A)/3$ and $\measuredangle(B,A,E) = \measuredangle(B,A,C)/3$ and A, F, C form a triangle and $\measuredangle(A,C,F) = \measuredangle(A,C,B)/3$ and $\measuredangle(F,A,C) = \measuredangle(B,A,C)/3$ and $\measuredangle(A,C,B) < \pi$. Then $|F - E| = 4 \cdot \varnothing_{\triangleleft}(A,B,C) \cdot \sin(\measuredangle(A,C,B)/3) \cdot \sin(\measuredangle(C,B,A)/3) \cdot \sin(\measuredangle(B,A,C)/3)$.

PROOF: $\sin((\pi/3) - (\measuredangle(A,C,B)/3)) \neq 0$. $\sin((\pi/3) - (\measuredangle(C,B,A)/3)) \neq 0$. $0 < \measuredangle(A,C,B)$. $\measuredangle(C,B,A) < \pi$. $0 < \measuredangle(A,C,B) < \pi$ and A, C, B are mutually different. $\measuredangle(B,A,C) < \pi$. $0 < \measuredangle(B,A,E) < \pi$. $\measuredangle(A,E,B) < \pi$. $0 < \measuredangle(F,A,C) < \pi$. $\measuredangle(C,F,A) < \pi$. F, A, E form a triangle by [19, (4)], (5), [17, (5)], [7, (31)]. $|A - F| = \varnothing_{\triangleleft}(A,B,C) \cdot 4 \cdot \sin(\measuredangle(C,B,A)/3) \cdot \sin((\pi/3) + (\measuredangle(C,B,A)/3)) \cdot \sin(\measuredangle(A,C,B)/3)$. $(\pi/3) + (\measuredangle(A,C,B)/3) + ((\pi/3) + (\measuredangle(C,B,A)/3)) + \measuredangle(E,A,F) = \pi$. $|F - E|^{\mathbf{2}} = |A - E|^{\mathbf{2}} + |A - F|^{\mathbf{2}} - 2 \cdot |A - E| \cdot |A - F| \cdot \cos\measuredangle(E,A,F)$. □

(17) Suppose A, B, C form a triangle and $\measuredangle(E,B,A) = \measuredangle(C,B,A)/3$ and $\measuredangle(B,A,E) = \measuredangle(B,A,C)/3$. Then A, B, E form a triangle. The theorem is a consequence of (1) and (2).

(18) Suppose A, B, C form a triangle and $\measuredangle(A,C,F) = \measuredangle(A,C,B)/3$ and $\measuredangle(F,A,C) = \measuredangle(B,A,C)/3$. Then A, F, C form a triangle. The theorem is a consequence of (1) and (2).

(19) Suppose A, B, C form a triangle and $\measuredangle(C,B,G) = \measuredangle(C,B,A)/3$ and $\measuredangle(G,C,B) = \measuredangle(A,C,B)/3$. Then C, G, B form a triangle. The theorem is a consequence of (1) and (2).

Let us assume that A, B, C form a triangle and $\measuredangle(A,C,B) < \pi$ and $\measuredangle(E,B,A) = \measuredangle(C,B,A)/3$ and $\measuredangle(B,A,E) = \measuredangle(B,A,C)/3$ and $\measuredangle(A,C,F) = \measuredangle(A,C,B)/3$ and $\measuredangle(F,A,C) = \measuredangle(B,A,C)/3$ and $\measuredangle(C,B,G) = \measuredangle(C,B,A)/3$ and $\measuredangle(G,C,B) = \measuredangle(A,C,B)/3$. Now we state the propositions:

(20) (i) $|F - E| = 4 \cdot \varnothing_{\triangleleft}(A,B,C) \cdot \sin(\measuredangle(A,C,B)/3) \cdot \sin(\measuredangle(C,B,A)/3) \cdot \sin(\measuredangle(B,A,C)/3)$, and

(ii) $|G - F| = 4 \cdot \varnothing_{\triangleleft}(C,A,B) \cdot \sin(\measuredangle(C,B,A)/3) \cdot \sin(\measuredangle(B,A,C)/3) \cdot \sin(\measuredangle(A,C,B)/3)$, and

(iii) $|E - G| = 4 \cdot \varnothing_{\triangleleft}(B,C,A) \cdot \sin(\measuredangle(B,A,C)/3) \cdot \sin(\measuredangle(A,C,B)/3) \cdot \sin(\measuredangle(C,B,A)/3)$.

The theorem is a consequence of (17), (18), (19), (2), (5), and (16).

(21) (i) $|F - E| = |G - F|$, and

(ii) $|F - E| = |E - G|$, and

(iii) $|G - F| = |E - G|$.

The theorem is a consequence of (20).

(22) Morley's trisector theorem:
Suppose A, B, C form a triangle and $\measuredangle(A, B, C) < \pi$ and $\measuredangle(E, C, A) = \measuredangle(B, C, A)/3$ and $\measuredangle(C, A, E) = \measuredangle(C, A, B)/3$ and $\measuredangle(A, B, F) = \measuredangle(A, B, C)/3$ and $\measuredangle(F, A, B) = \measuredangle(C, A, B)/3$ and $\measuredangle(B, C, G) = \measuredangle(B, C, A)/3$ and $\measuredangle(G, B, C) = \measuredangle(A, B, C)/3$. Then

(i) $|F - E| = |G - F|$, and

(ii) $|F - E| = |E - G|$, and

(iii) $|G - F| = |E - G|$.

The theorem is a consequence of (21).

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[3] Alexander Bogomolny. Morley's miracle from interactive mathematics miscellany and puzzles. *Cut the Knot*, 2015.
[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**): 55–65, 1990.
[5] Czesław Byliński. Introduction to real linear topological spaces. *Formalized Mathematics*, 13(**1**):99–107, 2005.
[6] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[7] Roland Coghetto. Some facts about trigonometry and Euclidean geometry. *Formalized Mathematics*, 22(**4**):313–319, 2014. doi:10.2478/forma-2014-0031.
[8] Alain Connes. A new proof of Morley's theorem. *Publications Mathématiques de l'IHÉS*, 88:43–46, 1998.
[9] John Conway. On Morley's trisector theorem. *The Mathematical Intelligencer*, 36(3):3, 2014. ISSN 0343-6993. doi:10.1007/s00283-014-9463-3.
[10] H.S.M. Coxeter and S.L. Greitzer. *Geometry Revisited.* The Mathematical Association of America (Inc.), 1967.
[11] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[12] Cesare Donolato. A vector-based proof of Morley's trisector theorem. In *Forum Geometricorum*, volume 13, pages 233–235, 2013.
[13] O.A.S. Karamzadeh. Is John Conway's proof of Morley's theorem the simplest and free of A Deus Ex Machina ? *The Mathematical Intelligencer*, 36(3):4–7, 2014. ISSN 0343-6993. doi:10.1007/s00283-014-9481-1.
[14] Akihiro Kubo and Yatsuka Nakamura. Angle and triangle in Euclidean topological space. *Formalized Mathematics*, 11(**3**):281–287, 2003.
[15] A. Letac. Solutions (Morley's triangle). Problem N 490. *Sphinx: revue mensuelle des questions récréatives*, 9, 1939.
[16] Eli Maor and Eugen Jost. *Beautiful geometry.* Princeton University Press, 2014.
[17] Robert Milewski. Trigonometric form of complex numbers. *Formalized Mathematics*, 9 (**3**):455–460, 2001.
[18] Cletus O. Oakley and Justine C. Baker. The Morley trisector theorem. *American Mathematical Monthly*, pages 737–745, 1978.
[19] Marco Riccardi. Heron's formula and Ptolemy's theorem. *Formalized Mathematics*, 16 (**2**):97–101, 2008. doi:10.2478/v10037-008-0014-2.

[20] Brian Stonebridge. A simple geometric proof of Morley's trisector theorem. *Applied Probability Trust*, 2009.

[21] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[22] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[23] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1 (**1**):73–83, 1990.

[24] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Formalized Mathematics*, 7(**2**):255–263, 1998.

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