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Groups – Additive Notation

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Summary. We translate the articles covering group theory already available in the Mizar Mathematical Library from multiplicative into additive notation. We adapt the works of Wojciech A. Trybulec [41, 42, 43] and Artur Korniłowicz [25].

In particular, these authors have defined the notions of group, abelian group, power of an element of a group, order of a group and order of an element, subgroup, coset of a subgroup, index of a subgroup, conjugation, normal subgroup, topological group, dense subset and basis of a topological group. Lagrange's theorem and some other theorems concerning these notions [9, 24, 22] are presented.

Note that "The term $\mathbb{Z}$-module is simply another name for an additive abelian group" [27]. We take an approach different than that used by Futa et al. [21] to use in a future article the results obtained by Artur Korniłowicz [25]. Indeed, Hölzl et al. showed that it was possible to build "a generic theory of limits based on filters" in Isabelle/HOL [23, 10]. Our goal is to define the convergence of a sequence and the convergence of a series in an abelian topological group [11] using the notion of filters.

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The notation and terminology used in this paper have been introduced in the following articles: [12], [32], [31], [2], [18], [28], [33], [13], [19], [39], [14], [15], [1], [40], [26], [35], [36], [5], [6], [16], [30], [8], [46], [47], [44], [29], [37], [45], [25], [48], [20], [7], [38], and [17].

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1. Additive Notation for Groups – GROUP_1

From now on m, n denote natural numbers, i, j denote integers, S denotes a non empty additive magma, and r, r_1, r_2, s, s_1, s_2, t, t_1, t_2 denote elements of S.

The scheme *SeqEx2Dbis* deals with non empty sets $\mathcal{X}$, $\mathcal{Z}$ and a ternary predicate $\mathcal{P}$ and states that

(Sch. 1) There exists a function f from $\mathbb{N} \times \mathcal{X}$ into $\mathcal{Z}$ such that for every natural number x for every element y of $\mathcal{X}$, $\mathcal{P}[x, y, f(x, y)]$

provided

- for every natural number x and for every element y of $\mathcal{X}$, there exists an element z of $\mathcal{Z}$ such that $\mathcal{P}[x, y, z]$.

Let I_1 be an additive magma. We say that I_1 is add-unital if and only if

(Def. 1) there exists an element e of I_1 such that for every element h of I_1, $h + e = h$ and $e + h = h$.

We say that I_1 is additive group-like if and only if

(Def. 2) there exists an element e of I_1 such that for every element h of I_1, $h + e = h$ and $e + h = h$ and there exists an element g of I_1 such that $h + g = e$ and $g + h = e$.

Let us note that every additive magma which is additive group-like is also add-unital and there exists an additive magma which is strict, additive group-like, add-associative, and non empty.

An additive group is an additive group-like, add-associative, non empty additive magma. Now we state the propositions:

(1) Suppose for every r, s, and t, $(r + s) + t = r + (s + t)$ and there exists t such that for every s_1, $s_1 + t = s_1$ and $t + s_1 = s_1$ and there exists s_2 such that $s_1 + s_2 = t$ and $s_2 + s_1 = t$. Then S is an additive group.

(2) Suppose for every r, s, and t, $(r + s) + t = r + (s + t)$ and for every r and s, there exists t such that $r + t = s$ and there exists t such that $t + r = s$. Then S is add-associative and additive group-like.

(3) $\langle \mathbb{R}, +_{\mathbb{R}} \rangle$ is add-associative and additive group-like.

From now on G denotes an additive group-like, non empty additive magma and e, h denote elements of G.

Let G be an additive magma. Assume G is add-unital. The functor 0_G yielding an element of G is defined by

(Def. 3) for every element h of G, $h + it = h$ and $it + h = h$.

Now we state the proposition:

(4) If for every h, $h+e=h$ and $e+h=h$, then $e=0_G$.

From now on G denotes an additive group and f, g, h denote elements of G.

Let us consider G and h. The functor $-h$ yielding an element of G is defined by

(Def. 4) $h+it=0_G$ and $it+h=0_G$.

Let us note that the functor is involutive.

Now we state the propositions:

(5) If $h+g=0_G$ and $g+h=0_G$, then $g=-h$.

(6) If $h+g=h+f$ or $g+h=f+h$, then $g=f$.

(7) If $h+g=h$ or $g+h=h$, then $g=0_G$. The theorem is a consequence of (6).

(8) $-0_G=0_G$.

(9) If $-h=-g$, then $h=g$. The theorem is a consequence of (6).

(10) If $-h=0_G$, then $h=0_G$. The theorem is a consequence of (8).

(11) If $h+g=0_G$, then $h=-g$ and $g=-h$. The theorem is a consequence of (6).

(12) $h+f=g$ if and only if $f=-h+g$. The theorem is a consequence of (6).

(13) $f+h=g$ if and only if $f=g+-h$. The theorem is a consequence of (6).

(14) There exists f such that $g+f=h$. The theorem is a consequence of (12).

(15) There exists f such that $f+g=h$. The theorem is a consequence of (13).

(16) $-(h+g)=-g+-h$. The theorem is a consequence of (11).

(17) $g+h=h+g$ if and only if $-(g+h)=-g+-h$. The theorem is a consequence of (16) and (6).

(18) $g+h=h+g$ if and only if $-g+-h=-h+-g$. The theorem is a consequence of (16) and (17).

(19) $g+h=h+g$ if and only if $g+-h=-h+g$. The theorem is a consequence of (18), (11), and (16).

From now on u denotes a unary operation on G.

Let us consider G. The functor add inverse G yielding a unary operation on G is defined by

(Def. 5) $it(h)=-h$.

Let G be an add-associative, non empty additive magma. Let us note that the addition of G is associative.

Let us consider an add-unital, non empty additive magma G. Now we state the propositions:

(20) 0_G is a unity w.r.t. the addition of G.

(21) $\mathbf{1}_\alpha = 0_G$, where α is the addition of G. The theorem is a consequence of (20).

Let G be an add-unital, non empty additive magma. Let us note that the addition of G is unital.

Now we state the proposition:

(22) add inverse G is an inverse operation w.r.t. the addition of G. The theorem is a consequence of (21).

Let us consider G. One can verify that the addition of G has inverse operation.

Now we state the proposition:

(23) The inverse operation w.r.t. the addition of G = add inverse G. The theorem is a consequence of (22).

Let G be a non empty additive magma. The functor mult G yielding a function from $\mathbb{N} \times$ (the carrier of G) into the carrier of G is defined by

(Def. 6) for every element h of G, $it(0, h) = 0_G$ and for every natural number n, $it(n+1, h) = it(n, h) + h$.

Let us consider G, i, and h. The functor $i \cdot h$ yielding an element of G is defined by the term

(Def. 7) $\begin{cases} (\mathrm{mult}\, G)(|i|, h), & \textbf{if } 0 \leqslant i, \\ -(\mathrm{mult}\, G)(|i|, h), & \textbf{otherwise}. \end{cases}$

Let us consider n. One can check that the functor $n \cdot h$ is defined by the term

(Def. 8) $(\mathrm{mult}\, G)(n, h)$.

Now we state the propositions:

(24) $0 \cdot h = 0_G$.

(25) $1 \cdot h = h$.

(26) $2 \cdot h = h + h$. The theorem is a consequence of (25).

(27) $3 \cdot h = h + h + h$. The theorem is a consequence of (26).

(28) $2 \cdot h = 0_G$ if and only if $-h = h$. The theorem is a consequence of (26) and (11).

(29) If $i \leqslant 0$, then $i \cdot h = -|i| \cdot h$. The theorem is a consequence of (8).

(30) $i \cdot 0_G = 0_G$. The theorem is a consequence of (8).

(31) $(-1) \cdot h = -h$. The theorem is a consequence of (25).

(32) $(i + j) \cdot h = i \cdot h + j \cdot h$.

PROOF: Define $\mathcal{P}[\text{integer}] \equiv$ for every i, $(i + \$_1) \cdot h = i \cdot h + \$_1 \cdot h$. For every j such that $\mathcal{P}[j]$ holds $\mathcal{P}[j-1]$ and $\mathcal{P}[j+1]$. $\mathcal{P}[0]$. For every j, $\mathcal{P}[j]$ from [40, Sch. 4]. □

(33) (i) $(i+1) \cdot h = i \cdot h + h$, and

(ii) $(i+1) \cdot h = h + i \cdot h$.

The theorem is a consequence of (25) and (32).

(34) $(-i) \cdot h = -i \cdot h$.

Let us assume that $g + h = h + g$. Now we state the propositions:

(35) $i \cdot (g + h) = i \cdot g + i \cdot h$. The theorem is a consequence of (16).

(36) $i \cdot g + j \cdot h = j \cdot h + i \cdot g$. The theorem is a consequence of (19) and (16).

(37) $g + i \cdot h = i \cdot h + g$. The theorem is a consequence of (25) and (36).

Let us consider G and h. We say that h is of order 0 if and only if

(Def. 9) if $n \cdot h = 0_G$, then $n = 0$.

One can check that 0_G is non of order 0.

Let us consider h. The functor $\operatorname{ord}(h)$ yielding an element of $\mathbb{N}$ is defined by

(Def. 10) (i) $it = 0$, **if** h is of order 0,

(ii) $it \cdot h = 0_G$ and $it \neq 0$ and for every m such that $m \cdot h = 0_G$ and $m \neq 0$ holds $it \leqslant m$, **otherwise**.

Now we state the propositions:

(38) $\operatorname{ord}(h) \cdot h = 0_G$.

(39) $\operatorname{ord}(0_G) = 1$.

(40) If $\operatorname{ord}(h) = 1$, then $h = 0_G$. The theorem is a consequence of (25).

Observe that there exists an additive group which is strict and Abelian.

Now we state the proposition:

(41) $\langle \mathbb{R}, +_{\mathbb{R}} \rangle$ is an Abelian additive group. The theorem is a consequence of (3).

In the sequel A denotes an Abelian additive group and a, b denote elements of A.

Now we state the propositions:

(42) $-(a + b) = -a + -b$.

(43) $i \cdot (a + b) = i \cdot a + i \cdot b$.

(44) $\langle \text{the carrier of } A, \text{the addition of } A, 0_A \rangle$ is Abelian, add-associative, right zeroed, and right complementable.

Let us consider an add-unital, non empty additive magma L and an element x of L. Now we state the propositions:

(45) $(\operatorname{mult} L)(1, x) = x$.

(46) $(\text{mult}\, L)(2, x) = x + x$. The theorem is a consequence of (45).

Now we state the proposition:

(47) Let us consider an add-associative, Abelian, add-unital, non empty additive magma L, elements x, y of L, and a natural number n. Then $(\text{mult}\, L)(n, x + y) = (\text{mult}\, L)(n, x) + (\text{mult}\, L)(n, y)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (\text{mult}\, L)(\$_1, x+y) = (\text{mult}\, L)(\$_1, x)+ (\text{mult}\, L)(\$_1, y)$. For every natural number n, $\mathcal{P}[n]$ from [5, Sch. 2]. □

Let G, H be additive magmas and I_1 be a function from G into H. We say that I_1 preserves zero if and only if

(Def. 11) $I_1(0_G) = 0_H$.

2. Subgroups and Lagrange Theorem – GROUP_2

In the sequel x denotes an object, y, y_1, y_2, Y, Z denote sets, k denotes a natural number, G denotes an additive group, a, g, h denote elements of G, and A denotes a subset of G.

Let us consider G and A. The functor $-A$ yielding a subset of G is defined by the term

(Def. 12) $\{-g : g \in A\}$.

One can check that the functor is involutive.

Now we state the propositions:

(48) $x \in -A$ if and only if there exists g such that $x = -g$ and $g \in A$.

(49) $-\{g\} = \{-g\}$.

(50) $-\{g, h\} = \{-g, -h\}$.

(51) $-\emptyset_\alpha = \emptyset$, where α is the carrier of G.

(52) $-\Omega_\alpha =$ the carrier of G, where α is the carrier of G.

(53) $A \neq \emptyset$ if and only if $-A \neq \emptyset$. The theorem is a consequence of (48).

Let us consider G. Let A be an empty subset of G. Observe that $-A$ is empty.

Let A be a non empty subset of G. One can check that $-A$ is non empty.

In the sequel G denotes a non empty additive magma, A, B, C denote subsets of G, and a, b, g, g_1, g_2, h, h_1, h_2 denote elements of G.

Let G be an Abelian, non empty additive magma and A, B be subsets of G. One can check that the functor $A + B$ is commutative.

(54) $x \in A + B$ if and only if there exists g and there exists h such that $x = g + h$ and $g \in A$ and $h \in B$.

(55) $A \neq \emptyset$ and $B \neq \emptyset$ if and only if $A+B \neq \emptyset$. The theorem is a consequence of (54).

(56) If G is add-associative, then $(A + B) + C = A + (B + C)$.

(57) Let us consider an additive group G, and subsets A, B of G. Then $-(A + B) = -B + -A$. The theorem is a consequence of (16).

(58) $A + (B \cup C) = A + B \cup (A + C)$.

(59) $(A \cup B) + C = A + C \cup (B + C)$.

(60) $A + B \cap C \subseteq (A + B) \cap (A + C)$.

(61) $A \cap B + C \subseteq (A + C) \cap (B + C)$.

(62) (i) $\emptyset_\alpha + A = \emptyset$, and

(ii) $A + \emptyset_\alpha = \emptyset$,

where α is the carrier of G. The theorem is a consequence of (54).

(63) Let us consider an additive group G, and a subset A of G. Suppose $A \neq \emptyset$. Then

(i) $\Omega_\alpha + A =$ the carrier of G, and

(ii) $A + \Omega_\alpha =$ the carrier of G,

where α is the carrier of G.

(64) $\{g\} + \{h\} = \{g + h\}$.

(65) $\{g\} + \{g_1, g_2\} = \{g + g_1, g + g_2\}$.

(66) $\{g_1, g_2\} + \{g\} = \{g_1 + g, g_2 + g\}$.

(67) $\{g, h\} + \{g_1, g_2\} = \{g + g_1, g + g_2, h + g_1, h + g_2\}$.

Let us consider an additive group G and a subset A of G. Now we state the propositions:

(68) Suppose for every elements g_1, g_2 of G such that $g_1, g_2 \in A$ holds $g_1 + g_2 \in A$ and for every element g of G such that $g \in A$ holds $-g \in A$. Then $A + A = A$.

(69) If for every element g of G such that $g \in A$ holds $-g \in A$, then $-A = A$.

(70) If for every a and b such that $a \in A$ and $b \in B$ holds $a + b = b + a$, then $A + B = B + A$.

(71) If G is an Abelian additive group, then $A + B = B + A$.

(72) Let us consider an Abelian additive group G, and subsets A, B of G. Then $-(A + B) = -A + -B$. The theorem is a consequence of (42).

Let us consider G, g, and A. The functors: $g + A$ and $A + g$ yielding subsets of G are defined by terms,

(Def. 13) $\{g\} + A$,

(Def. 14) $A + \{g\}$,

respectively. Now we state the propositions:

(73) $x \in g + A$ if and only if there exists h such that $x = g + h$ and $h \in A$.

(74) $x \in A + g$ if and only if there exists h such that $x = h + g$ and $h \in A$.

Let us assume that G is add-associative. Now we state the propositions:

(75) $(g + A) + B = g + (A + B)$.

(76) $(A + g) + B = A + (g + B)$.

(77) $(A + B) + g = A + (B + g)$.

(78) $(g + h) + A = g + (h + A)$. The theorem is a consequence of (64) and (56).

(79) $(g + A) + h = g + (A + h)$.

(80) $(A + g) + h = A + (g + h)$. The theorem is a consequence of (56) and (64).

(81) (i) $\emptyset_\alpha + a = \emptyset$, and

(ii) $a + \emptyset_\alpha = \emptyset$,

where α is the carrier of G.

From now on G denotes an additive group-like, non empty additive magma, h, g, g_1, g_2 denote elements of G, and A denotes a subset of G.

(82) Let us consider an additive group G, and an element a of G. Then

(i) $\Omega_\alpha + a =$ the carrier of G, and

(ii) $a + \Omega_\alpha =$ the carrier of G,

where α is the carrier of G.

(83) (i) $0_G + A = A$, and

(ii) $A + 0_G = A$.

The theorem is a consequence of (73) and (74).

(84) If G is an Abelian additive group, then $g + A = A + g$.

Let G be an additive group-like, non empty additive magma.

A subgroup of G is an additive group-like, non empty additive magma and is defined by

(Def. 15) the carrier of *it* $\subseteq$ the carrier of G and the addition of *it* $=$ (the addition of G) $\restriction$ (the carrier of *it*).

In the sequel H denotes a subgroup of G and h, h_1, h_2 denote elements of H.

Now we state the propositions:

(85) If G is finite, then H is finite.

(86) If $x \in H$, then $x \in G$.

(87) $h \in G$.

(88) h is an element of G.

(89) If $h_1 = g_1$ and $h_2 = g_2$, then $h_1 + h_2 = g_1 + g_2$.

Let G be an additive group. Let us observe that every subgroup of G is add-associative.

In the sequel G, G_1, G_2, G_3 denote additive groups, a, a_1, a_2, b, b_1, b_2, g, g_1, g_2 denote elements of G, A, B denote subsets of G, H, H_1, H_2, H_3 denote subgroups of G, and h, h_1, h_2 denote elements of H.

(90) $0_H = 0_G$. The theorem is a consequence of (87), (89), and (7).

(91) $0_{H_1} = 0_{H_2}$. The theorem is a consequence of (90).

(92) $0_G \in H$. The theorem is a consequence of (90).

(93) $0_{H_1} \in H_2$. The theorem is a consequence of (90) and (92).

(94) If $h = g$, then $-h = -g$. The theorem is a consequence of (87), (89), (90), and (11).

(95) add inverse H = add inverse $G \restriction$(the carrier of H). The theorem is a consequence of (87) and (94).

(96) If $g_1, g_2 \in H$, then $g_1 + g_2 \in H$. The theorem is a consequence of (89).

(97) If $g \in H$, then $-g \in H$. The theorem is a consequence of (94).

Let us consider G. Observe that there exists a subgroup of G which is strict.

(98) Suppose $A \neq \emptyset$ and for every g_1 and g_2 such that $g_1, g_2 \in A$ holds $g_1 + g_2 \in A$ and for every g such that $g \in A$ holds $-g \in A$. Then there exists a strict subgroup H of G such that the carrier of $H = A$.

PROOF: Reconsider $D = A$ as a non empty set. Set $o =$ (the addition of G) $\restriction A$. $\operatorname{rng} o \subseteq A$ by [17, (87)], [14, (47)]. Set $H = \langle D, o \rangle$. H is additive group-like. $\square$

(99) If G is an Abelian additive group, then H is Abelian. The theorem is a consequence of (87) and (89).

Let G be an Abelian additive group. One can check that every subgroup of G is Abelian.

(100) G is a subgroup of G.

(101) Suppose G_1 is a subgroup of G_2 and G_2 is a subgroup of G_1. Then the additive magma of G_1 = the additive magma of G_2.

(102) If G_1 is a subgroup of G_2 and G_2 is a subgroup of G_3, then G_1 is a subgroup of G_3.

(103) If the carrier of $H_1 \subseteq$ the carrier of H_2, then H_1 is a subgroup of H_2.

(104) If for every g such that $g \in H_1$ holds $g \in H_2$, then H_1 is a subgroup of H_2. The theorem is a consequence of (87) and (103).

(105) Suppose the carrier of H_1 = the carrier of H_2. Then the additive magma of H_1 = the additive magma of H_2. The theorem is a consequence of (103) and (101).

(106) Suppose for every g, $g \in H_1$ iff $g \in H_2$. Then the additive magma of $H_1 =$ the additive magma of H_2. The theorem is a consequence of (104) and (101).

Let us consider G. Let H_1, H_2 be strict subgroups of G. One can check that $H_1 = H_2$ if and only if the condition (Def. 16) is satisfied.

(Def. 16) for every g, $g \in H_1$ iff $g \in H_2$.

Now we state the propositions:

(107) Let us consider an additive group G, and a subgroup H of G. Suppose the carrier of $G \subseteq$ the carrier of H. Then the additive magma of $H =$ the additive magma of G. The theorem is a consequence of (100) and (105).

(108) Suppose for every element g of G, $g \in H$. Then the additive magma of $H =$ the additive magma of G. The theorem is a consequence of (100) and (106).

Let us consider G. The functor $\mathbf{0}_G$ yielding a strict subgroup of G is defined by

(Def. 17) the carrier of *it* $= \{0_G\}$.

The functor Ω_G yielding a strict subgroup of G is defined by the term

(Def. 18) the additive magma of G.

Note that the functor is projective.

Now we state the propositions:

(109) $\mathbf{0}_H = \mathbf{0}_G$. The theorem is a consequence of (90) and (102).

(110) $\mathbf{0}_{H_1} = \mathbf{0}_{H_2}$. The theorem is a consequence of (109).

(111) $\mathbf{0}_G$ is a subgroup of H. The theorem is a consequence of (109).

(112) Let us consider a strict additive group G. Then every subgroup of G is a subgroup of Ω_G.

(113) Every strict additive group is a subgroup of Ω_G.

(114) $\mathbf{0}_G$ is finite.

Let us consider G. Note that $\mathbf{0}_G$ is finite and there exists a subgroup of G which is strict and finite and there exists an additive group which is strict and finite.

Let G be a finite additive group. One can verify that every subgroup of G is finite.

Now we state the propositions:

(115) $\overline{\overline{\mathbf{0}_G}} = 1$.

(116) Let us consider a strict, finite subgroup H of G. If $\overline{\overline{H}} = 1$, then $H = \mathbf{0}_G$. The theorem is a consequence of (92).

(117) $\overline{\overline{H}} \subseteq \overline{\overline{G}}$.

Let us consider a finite additive group G and a subgroup H of G. Now we state the propositions:

(118) $\overline{\overline{H}} \leqslant \overline{\overline{G}}$.

(119) Suppose $\overline{\overline{G}} = \overline{\overline{H}}$. Then the additive magma of H = the additive magma of G.

PROOF: The carrier of H = the carrier of G by [3, (48)]. □

Let us consider G and H. The functor $\overline{H}$ yielding a subset of G is defined by the term

(Def. 19) the carrier of H.

Now we state the propositions:

(120) If $g_1, g_2 \in \overline{H}$, then $g_1 + g_2 \in \overline{H}$. The theorem is a consequence of (96).

(121) If $g \in \overline{H}$, then $-g \in \overline{H}$. The theorem is a consequence of (97).

(122) $\overline{H} + \overline{H} = \overline{H}$. The theorem is a consequence of (121), (120), and (68).

(123) $-\overline{H} = \overline{H}$. The theorem is a consequence of (121) and (69).

(124) (i) if $\overline{H_1} + \overline{H_2} = \overline{H_2} + \overline{H_1}$, then there exists a strict subgroup H of G such that the carrier of $H = \overline{H_1} + \overline{H_2}$, and

(ii) if there exists H such that the carrier of $H = \overline{H_1} + \overline{H_2}$, then $\overline{H_1} + \overline{H_2} = \overline{H_2} + \overline{H_1}$.

The theorem is a consequence of (121), (16), (120), (55), and (98).

(125) Suppose G is an Abelian additive group. Then there exists a strict subgroup H of G such that the carrier of $H = \overline{H_1} + \overline{H_2}$. The theorem is a consequence of (71) and (124).

Let us consider G, H_1, and H_2. The functor $H_1 \cap H_2$ yielding a strict subgroup of G is defined by

(Def. 20) the carrier of *it* $= \overline{H_1} \cap \overline{H_2}$.

Now we state the propositions:

(126) (i) for every subgroup H of G such that $H = H_1 \cap H_2$ holds the carrier of H = (the carrier of H_1) $\cap$ (the carrier of H_2), and

(ii) for every strict subgroup H of G such that the carrier of H = (the carrier of H_1) $\cap$ (the carrier of H_2) holds $H = H_1 \cap H_2$.

(127) $\overline{H_1 \cap H_2} = \overline{H_1} \cap \overline{H_2}$.

(128) $x \in H_1 \cap H_2$ if and only if $x \in H_1$ and $x \in H_2$.

(129) Let us consider a strict subgroup H of G. Then $H \cap H = H$. The theorem is a consequence of (105).

Let us consider G, H_1, and H_2. Note that the functor $H_1 \cap H_2$ is commutative.

(130) $(H_1 \cap H_2) \cap H_3 = H_1 \cap (H_2 \cap H_3)$. The theorem is a consequence of (105).

(131) (i) $\mathbf{0}_G \cap H = \mathbf{0}_G$, and

(ii) $H \cap \mathbf{0}_G = \mathbf{0}_G$.

The theorem is a consequence of (111).

(132) Let us consider a strict additive group G, and a strict subgroup H of G. Then

(i) $H \cap \Omega_G = H$, and

(ii) $\Omega_G \cap H = H$.

(133) Let us consider a strict additive group G. Then $\Omega_G \cap \Omega_G = G$.

(134) $H_1 \cap H_2$ is subgroup of H_1 and subgroup of H_2.

(135) Let us consider a subgroup H_1 of G. Then H_1 is a subgroup of H_2 if and only if the additive magma of $H_1 \cap H_2 =$ the additive magma of H_1.

(136) If H_1 is a subgroup of H_2, then $H_1 \cap H_3$ is a subgroup of H_2. The theorem is a consequence of (102).

(137) If H_1 is subgroup of H_2 and subgroup of H_3, then H_1 is a subgroup of $H_2 \cap H_3$. The theorem is a consequence of (86), (128), and (104).

(138) If H_1 is a subgroup of H_2, then $H_1 \cap H_3$ is a subgroup of $H_2 \cap H_3$. The theorem is a consequence of (126) and (103).

(139) If H_1 is finite or H_2 is finite, then $H_1 \cap H_2$ is finite.

Let us consider G, H, and A. The functors: $A + H$ and $H + A$ yielding subsets of G are defined by terms,

(Def. 21) $A + \overline{H}$,

(Def. 22) $\overline{H} + A$,

respectively. Now we state the propositions:

(140) $x \in A + H$ if and only if there exists g_1 and there exists g_2 such that $x = g_1 + g_2$ and $g_1 \in A$ and $g_2 \in H$.

(141) $x \in H + A$ if and only if there exists g_1 and there exists g_2 such that $x = g_1 + g_2$ and $g_1 \in H$ and $g_2 \in A$.

(142) $(A + B) + H = A + (B + H)$.

(143) $(A + H) + B = A + (H + B)$.

(144) $(H + A) + B = H + (A + B)$.

(145) $(A + H_1) + H_2 = A + (H_1 + \overline{H_2})$.

(146) $(H_1 + A) + H_2 = H_1 + (A + H_2)$.

(147) $(H_1 + \overline{H_2}) + A = H_1 + (H_2 + A)$.

(148) If G is an Abelian additive group, then $A + H = H + A$.

Let us consider G, H, and a. The functors: $a+H$ and $H+a$ yielding subsets of G are defined by terms,

(Def. 23) $a+\overline{H}$,

(Def. 24) $\overline{H}+a$,

respectively. Now we state the propositions:

(149) $x \in a+H$ if and only if there exists g such that $x = a+g$ and $g \in H$. The theorem is a consequence of (73).

(150) $x \in H+a$ if and only if there exists g such that $x = g+a$ and $g \in H$. The theorem is a consequence of (74).

(151) $(a+b)+H = a+(b+H)$.

(152) $(a+H)+b = a+(H+b)$.

(153) $(H+a)+b = H+(a+b)$.

(154) (i) $a \in a+H$, and

(ii) $a \in H+a$.

The theorem is a consequence of (92), (149), and (150).

(155) (i) $0_G+H = \overline{H}$, and

(ii) $H+0_G = \overline{H}$.

(156) (i) $\mathbf{0}_G+a = \{a\}$, and

(ii) $a+\mathbf{0}_G = \{a\}$.

The theorem is a consequence of (64).

(157) (i) $a+\Omega_G =$ the carrier of G, and

(ii) $\Omega_G+a =$ the carrier of G.

The theorem is a consequence of (63).

(158) If G is an Abelian additive group, then $a+H = H+a$.

(159) $a \in H$ if and only if $a+H = \overline{H}$. The theorem is a consequence of (149), (96), (97), and (92).

(160) $a+H = b+H$ if and only if $-b+a \in H$. The theorem is a consequence of (78), (83), and (159).

(161) $a+H = b+H$ if and only if $a+H$ meets $b+H$. The theorem is a consequence of (154), (149), (97), (13), (12), (96), and (160).

(162) $(a+b)+H \subseteq a+H+(b+H)$. The theorem is a consequence of (149) and (92).

(163) (i) $\overline{H} \subseteq a+H+(-a+H)$, and

(ii) $\overline{H} \subseteq -a+H+(a+H)$.

The theorem is a consequence of (83) and (162).

(164) $2 \cdot a + H \subseteq a + H + (a + H)$. The theorem is a consequence of (26) and (162).

(165) $a \in H$ if and only if $H + a = \overline{H}$. The theorem is a consequence of (150), (96), (97), and (92).

(166) $H + a = H + b$ if and only if $b + -a \in H$. The theorem is a consequence of (83), (80), and (165).

(167) $H + a = H + b$ if and only if $H + a$ meets $H + b$. The theorem is a consequence of (154), (150), (97), (12), (13), (96), and (166).

(168) $(H + a) + b \subseteq H + a + (H + b)$. The theorem is a consequence of (92), (150), and (80).

(169) (i) $\overline{H} \subseteq H + a + (H + -a)$, and

(ii) $\overline{H} \subseteq H + -a + (H + a)$.

The theorem is a consequence of (80), (83), and (168).

(170) $H + 2 \cdot a \subseteq H + a + (H + a)$. The theorem is a consequence of (80), (26), and (168).

(171) $a + H_1 \cap H_2 = (a + H_1) \cap (a + H_2)$. The theorem is a consequence of (149), (128), and (6).

(172) $H_1 \cap H_2 + a = (H_1 + a) \cap (H_2 + a)$. The theorem is a consequence of (150), (128), and (6).

(173) There exists a strict subgroup H_1 of G such that the carrier of $H_1 = a + H_2 + -a$. The theorem is a consequence of (154), (74), (149), (97), (150), (16), (73), (56), (96), and (98).

(174) $a + H \approx b + H$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists g_1 such that $\$_1 = g_1$ and $\$_2 = b + -a + g_1$. For every object x such that $x \in a + H$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = a + H$ and for every object x such that $x \in a + H$ holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. $\operatorname{rng} f = b + H$. f is one-to-one. □

(175) $a + H \approx H + b$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists g_1 such that $\$_1 = g_1$ and $\$_2 = -a + g_1 + b$. For every object x such that $x \in a + H$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = a + H$ and for every object x such that $x \in a + H$ holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. $\operatorname{rng} f = H + b$. f is one-to-one. □

(176) $H + a \approx H + b$. The theorem is a consequence of (175).

(177) (i) $\overline{H} \approx a + H$, and

(ii) $\overline{H} \approx H + a$.

The theorem is a consequence of (83), (174), and (176).

(178) (i) $\overline{\overline{H}} = \overline{\overline{a + H}}$, and

(ii) $\overline{\overline{H}} = \overline{\overline{H + a}}$.

(179) Let us consider a finite subgroup H of G. Then there exist finite sets B, C such that

(i) $B = a + H$, and

(ii) $C = H + a$, and

(iii) $\overline{\overline{H}} = \overline{\overline{B}}$, and

(iv) $\overline{\overline{H}} = \overline{\overline{C}}$.

The theorem is a consequence of (177).

Let us consider G and H. The functors: the left cosets of H and the right cosets of H yielding families of subsets of G are defined by conditions,

(Def. 25) $A \in$ the left cosets of H iff there exists a such that $A = a + H$,

(Def. 26) $A \in$ the right cosets of H iff there exists a such that $A = H + a$,

respectively. Now we state the propositions:

(180) If G is finite, then the right cosets of H is finite and the left cosets of H is finite.

(181) (i) $\overline{H} \in$ the left cosets of H, and

(ii) $\overline{H} \in$ the right cosets of H.

The theorem is a consequence of (83).

(182) The left cosets of $H \approx$ the right cosets of H.

Proof: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists g such that $\$_1 = g + H$ and $\$_2 = H + -g$. For every object x such that $x \in$ the left cosets of H there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f =$ the left cosets of H and for every object x such that $x \in$ the left cosets of H holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. $\operatorname{rng} f =$ the right cosets of H. f is one-to-one. □

(183) (i) $\bigcup$(the left cosets of H) = the carrier of G, and

(ii) $\bigcup$(the right cosets of H) = the carrier of G.

The theorem is a consequence of (87), (149), and (150).

(184) The left cosets of $\mathbf{0}_G$ = the set of all $\{a\}$. The theorem is a consequence of (156).

(185) The right cosets of $\mathbf{0}_G$ = the set of all $\{a\}$. The theorem is a consequence of (156).

Let us consider a strict subgroup H of G. Now we state the propositions:

(186) If the left cosets of H = the set of all $\{a\}$, then $H = \mathbf{0}_G$. The theorem is a consequence of (87), (149), (92), and (6).

(187) If the right cosets of $H =$ the set of all $\{a\}$, then $H = \mathbf{0}_G$. The theorem is a consequence of (87), (150), (92), and (6).

(188) (i) the left cosets of $\Omega_G = \{\text{the carrier of } G\}$, and

(ii) the right cosets of $\Omega_G = \{\text{the carrier of } G\}$.

The theorem is a consequence of (157).

Let us consider a strict additive group G and a strict subgroup H of G. Now we state the propositions:

(189) If the left cosets of $H = \{\text{the carrier of } G\}$, then $H = G$. The theorem is a consequence of (149), (6), and (108).

(190) If the right cosets of $H = \{\text{the carrier of } G\}$, then $H = G$. The theorem is a consequence of (150), (6), and (108).

Let us consider G and H. The functor $|\bullet : H|$ yielding a cardinal number is defined by the term

(Def. 27) $\overline{\overline{\alpha}}$, where α is the left cosets of H.

Now we state the proposition:

(191) (i) $|\bullet : H| = \overline{\overline{\alpha}}$, and

(ii) $|\bullet : H| = \overline{\overline{\beta}}$,

where α is the left cosets of H and β is the right cosets of H.

Let us consider G and H. Assume the left cosets of H is finite. The functor $|\bullet : H|_{\mathbb{N}}$ yielding an element of $\mathbb{N}$ is defined by

(Def. 28) there exists a finite set B such that $B =$ the left cosets of H and $it = \overline{\overline{B}}$.

Now we state the proposition:

(192) Suppose the left cosets of H is finite. Then

(i) there exists a finite set B such that $B =$ the left cosets of H and $|\bullet : H|_{\mathbb{N}} = \overline{\overline{B}}$, and

(ii) there exists a finite set C such that $C =$ the right cosets of H and $|\bullet : H|_{\mathbb{N}} = \overline{\overline{C}}$.

The theorem is a consequence of (182).

Let us consider a finite additive group G and a subgroup H of G. Now we state the propositions:

(193) Lagrange theorem for additive groups: $\overline{\overline{G}} = \overline{\overline{H}} \cdot |\bullet : H|_{\mathbb{N}}$. The theorem is a consequence of (179), (174), (161), and (183).

(194) $\overline{\overline{H}} \mid \overline{\overline{G}}$. The theorem is a consequence of (193).

(195) Let us consider a finite additive group G, subgroups I, H of G, and a subgroup J of H. Suppose $I = J$. Then $|\bullet : I|_{\mathbb{N}} = |\bullet : J|_{\mathbb{N}} \cdot |\bullet : H|_{\mathbb{N}}$. The theorem is a consequence of (193).

(196) $|\bullet : \Omega_G|_{\mathbb{N}} = 1$. The theorem is a consequence of (188).

(197) Let us consider a strict additive group G, and a strict subgroup H of G. Suppose the left cosets of H is finite and $|\bullet : H|_{\mathbb{N}} = 1$. Then $H = G$. The theorem is a consequence of (183) and (189).

(198) $|\bullet : \mathbf{0}_G| = \overline{\overline{G}}$.

PROOF: Define $\mathcal{F}(\text{object}) = \{\$_1\}$. Consider f being a function such that $\operatorname{dom} f =$ the carrier of G and for every object x such that $x \in$ the carrier of G holds $f(x) = \mathcal{F}(x)$ from [14, Sch. 3]. $\operatorname{rng} f =$ the left cosets of $\mathbf{0}_G$. f is one-to-one by [17, (3)]. □

(199) Let us consider a finite additive group G. Then $|\bullet : \mathbf{0}_G|_{\mathbb{N}} = \overline{\overline{G}}$. The theorem is a consequence of (193) and (115).

(200) Let us consider a finite additive group G, and a strict subgroup H of G. Suppose $|\bullet : H|_{\mathbb{N}} = \overline{\overline{G}}$. Then $H = \mathbf{0}_G$. The theorem is a consequence of (193) and (116).

(201) Let us consider a strict subgroup H of G. Suppose the left cosets of H is finite and $|\bullet : H| = \overline{\overline{G}}$. Then

(i) G is finite, and

(ii) $H = \mathbf{0}_G$.

The theorem is a consequence of (200).

3. CLASSES OF CONJUGATION AND NORMAL SUBGROUPS – GROUP_3

From now on x, y, y_1, y_2 denote sets, G denotes an additive group, a, b, c, d, g, h denote elements of G, A, B, C, D denote subsets of G, H, H_1, H_2, H_3 denote subgroups of G, n denotes a natural number, and i denotes an integer.

Now we state the propositions:

(202) (i) $a + b + -b = a$, and

(ii) $a + -b + b = a$, and

(iii) $-b + b + a = a$, and

(iv) $b + -b + a = a$, and

(v) $a + (b + -b) = a$, and

(vi) $a + (-b + b) = a$, and

(vii) $-b + (b + a) = a$, and

(viii) $b + (-b + a) = a$.

(203) G is an Abelian additive group if and only if the addition of G is commutative.

(204) $\mathbf{0}_G$ is Abelian.

(205) If $A \subseteq B$ and $C \subseteq D$, then $A + C \subseteq B + D$.

(206) If $A \subseteq B$, then $a + A \subseteq a + B$ and $A + a \subseteq B + a$.

(207) If H_1 is a subgroup of H_2, then $a + H_1 \subseteq a + H_2$ and $H_1 + a \subseteq H_2 + a$. The theorem is a consequence of (205).

(208) $a + H = \{a\} + H$.

(209) $H + a = H + \{a\}$.

(210) $(A + a) + H = A + (a + H)$. The theorem is a consequence of (142).

(211) $(a + H) + A = a + (H + A)$. The theorem is a consequence of (143).

(212) $(A + H) + a = A + (H + a)$. The theorem is a consequence of (143).

(213) $(H + a) + A = H + (a + A)$. The theorem is a consequence of (144).

(214) $(H_1 + a) + H_2 = H_1 + (a + H_2)$.

Let us consider G. The functor $\mathrm{SubGr}\, G$ yielding a set is defined by

(Def. 29) for every object x, $x \in it$ iff x is a strict subgroup of G.

Note that $\mathrm{SubGr}\, G$ is non empty.

Now we state the propositions:

(215) Let us consider a strict additive group G. Then $G \in \mathrm{SubGr}\, G$. The theorem is a consequence of (100).

(216) If G is finite, then $\mathrm{SubGr}\, G$ is finite.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a strict subgroup H of G such that $\$_1 = H$ and $\$_2 =$ the carrier of H. For every object x such that $x \in \mathrm{SubGr}\, G$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\mathrm{dom}\, f = \mathrm{SubGr}\, G$ and for every object x such that $x \in \mathrm{SubGr}\, G$ holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. $\mathrm{rng}\, f \subseteq 2^{\alpha}$, where α is the carrier of G. f is one-to-one. □

Let us consider G, a, and b. The functor $a \cdot b$ yielding an element of G is defined by the term

(Def. 30) $-b + a + b$.

Now we state the propositions:

(217) If $a \cdot g = b \cdot g$, then $a = b$. The theorem is a consequence of (6).

(218) $0_G \cdot a = 0_G$.

(219) If $a \cdot b = 0_G$, then $a = 0_G$. The theorem is a consequence of (11) and (7).

(220) $a \cdot 0_G = a$. The theorem is a consequence of (8).

(221) $a \cdot a = a$.

(222) (i) $a \cdot (-a) = a$, and

(ii) $(-a) \cdot a = -a$.

(223) $a \cdot b = a$ if and only if $a + b = b + a$. The theorem is a consequence of (12).

(224) $(a + b) \cdot g = a \cdot g + b \cdot g$.

(225) $a \cdot g \cdot h = a \cdot (g + h)$. The theorem is a consequence of (16).

(226) (i) $a \cdot b \cdot (-b) = a$, and

(ii) $a \cdot (-b) \cdot b = a$.

The theorem is a consequence of (225) and (220).

(227) $(-a) \cdot b = -a \cdot b$. The theorem is a consequence of (16).

(228) $(n \cdot a) \cdot b = n \cdot (a \cdot b)$.

(229) $(i \cdot a) \cdot b = i \cdot (a \cdot b)$. The theorem is a consequence of (29) and (227).

(230) If G is an Abelian additive group, then $a \cdot b = a$. The theorem is a consequence of (202).

(231) If for every a and b, $a \cdot b = a$, then G is Abelian. The theorem is a consequence of (223).

Let us consider G, A, and B. The functor $A \cdot B$ yielding a subset of G is defined by the term

(Def. 31) $\{g \cdot h : g \in A \text{ and } h \in B\}$.

Now we state the propositions:

(232) $x \in A \cdot B$ if and only if there exists g and there exists h such that $x = g \cdot h$ and $g \in A$ and $h \in B$.

(233) $A \cdot B \neq \emptyset$ if and only if $A \neq \emptyset$ and $B \neq \emptyset$. The theorem is a consequence of (232).

(234) $A \cdot B \subseteq -B + A + B$.

(235) $(A + B) \cdot C \subseteq A \cdot C + B \cdot C$. The theorem is a consequence of (224).

(236) $A \cdot B \cdot C = A \cdot (B + C)$. The theorem is a consequence of (225).

(237) $(-A) \cdot B = -A \cdot B$. The theorem is a consequence of (227).

(238) $\{a\} \cdot \{b\} = \{a \cdot b\}$. The theorem is a consequence of (49), (64), (233), and (234).

(239) $\{a\} \cdot \{b, c\} = \{a \cdot b, a \cdot c\}$.

(240) $\{a, b\} \cdot \{c\} = \{a \cdot c, b \cdot c\}$.

(241) $\{a, b\} \cdot \{c, d\} = \{a \cdot c, a \cdot d, b \cdot c, b \cdot d\}$.

Let us consider G, A, and g. The functors: $A \cdot g$ and $g \cdot A$ yielding subsets of G are defined by terms, (Def. 32) $A \cdot \{g\}$,

(Def. 33) $\{g\} \cdot A$,

respectively. Now we state the propositions:

(242) $x \in A \cdot g$ if and only if there exists h such that $x = h \cdot g$ and $h \in A$.

(243) $x \in g \cdot A$ if and only if there exists h such that $x = g \cdot h$ and $h \in A$.

(244) $g \cdot A \subseteq -A + g + A$. The theorem is a consequence of (243) and (74).

(245) $A \cdot B \cdot g = A \cdot (B + g)$.

(246) $A \cdot g \cdot B = A \cdot (g + B)$.

(247) $g \cdot A \cdot B = g \cdot (A + B)$.

(248) $A \cdot a \cdot b = A \cdot (a + b)$. The theorem is a consequence of (236) and (64).

(249) $a \cdot A \cdot b = a \cdot (A + b)$.

(250) $a \cdot b \cdot A = a \cdot (b + A)$. The theorem is a consequence of (238) and (236).

(251) $A \cdot g = -g + A + g$. The theorem is a consequence of (234), (49), (74), (73), and (242).

(252) $(A + B) \cdot a \subseteq A \cdot a + B \cdot a$.

(253) $A \cdot 0_G = A$. The theorem is a consequence of (251), (83), and (8).

(254) If $A \neq \emptyset$, then $0_G \cdot A = \{0_G\}$. The theorem is a consequence of (243) and (218).

(255) (i) $A \cdot a \cdot (-a) = A$, and

(ii) $A \cdot (-a) \cdot a = A$.

The theorem is a consequence of (248) and (253).

(256) G is an Abelian additive group if and only if for every A and B such that $B \neq \emptyset$ holds $A \cdot B = A$. The theorem is a consequence of (230), (238), and (231).

(257) G is an Abelian additive group if and only if for every A and g, $A \cdot g = A$. The theorem is a consequence of (256), (238), and (231).

(258) G is an Abelian additive group if and only if for every A and g such that $A \neq \emptyset$ holds $g \cdot A = \{g\}$. The theorem is a consequence of (256), (238), and (231).

Let us consider G, H, and a. The functor $H \cdot a$ yielding a strict subgroup of G is defined by

(Def. 34) the carrier of $it = \overline{H} \cdot a$.

Now we state the propositions:

(259) $x \in H \cdot a$ if and only if there exists g such that $x = g \cdot a$ and $g \in H$. The theorem is a consequence of (242).

(260) The carrier of $H \cdot a = -a + H + a$. The theorem is a consequence of (251).

(261) $H \cdot a \cdot b = H \cdot (a + b)$. The theorem is a consequence of (248) and (105).

Let us consider a strict subgroup H of G. Now we state the propositions:

(262) $H \cdot 0_G = H$. The theorem is a consequence of (253) and (105).

(263) (i) $H \cdot a \cdot (-a) = H$, and

(ii) $H \cdot (-a) \cdot a = H$.

The theorem is a consequence of (261) and (262).

Now we state the propositions:

(264) $(H_1 \cap H_2) \cdot a = H_1 \cdot a \cap (H_2 \cdot a)$. The theorem is a consequence of (259), (128), and (217).

(265) $\overline{\overline{H}} = \overline{\overline{H \cdot a}}$.

PROOF: Define $\mathcal{F}(\text{element of } G) = \$_1 \cdot a$. Consider f being a function from the carrier of G into the carrier of G such that for every g, $f(g) = \mathcal{F}(g)$ from [15, Sch. 4]. Set $g = f \restriction (\text{the carrier of } H)$. $\operatorname{rng} g =$ the carrier of $H \cdot a$ by [46, (62)], (88), (242), [14, (47)]. g is one-to-one by [46, (62)], (88), [14, (47)], (217). □

(266) H is finite if and only if $H \cdot a$ is finite. The theorem is a consequence of (265).

Let us consider G and a. Let H be a finite subgroup of G. Observe that $H \cdot a$ is finite.

Now we state the propositions:

(267) Let us consider a finite subgroup H of G. Then $\overline{\overline{H}} = \overline{\overline{H \cdot a}}$.

(268) $\mathbf{0}_G \cdot a = \mathbf{0}_G$. The theorem is a consequence of (238) and (218).

(269) Let us consider a strict subgroup H of G. If $H \cdot a = \mathbf{0}_G$, then $H = \mathbf{0}_G$. The theorem is a consequence of (266), (115), (265), and (116).

(270) Let us consider an additive group G, and an element a of G. Then $\Omega_G \cdot a = \Omega_G$. The theorem is a consequence of (225), (220), and (259).

(271) Let us consider a strict subgroup H of G. If $H \cdot a = G$, then $H = G$. The theorem is a consequence of (259), (217), and (108).

(272) $|\bullet : H| = |\bullet : H \cdot a|$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists b such that $\$_1 = b + H$ and $\$_2 = b \cdot a + H \cdot a$. For every object x such that $x \in$ the left cosets of H there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f =$ the left cosets of H and for every object x such that $x \in$ the left cosets of H holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. For every x, y_1, and y_2 such that $x \in$ the left cosets of H and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. $\operatorname{rng} f =$ the left cosets of $H \cdot a$. f is one-to-one. □

(273) If the left cosets of H is finite, then $|\bullet : H|_{\mathbb{N}} = |\bullet : H \cdot a|_{\mathbb{N}}$. The theorem is a consequence of (272).

(274) If G is an Abelian additive group, then for every strict subgroup H of G and for every a, $H \cdot a = H$. The theorem is a consequence of (260), (158), (153), (155), and (105).

Let us consider G, a, and b. We say that a and b are conjugated if and only if

(Def. 35) there exists g such that $a = b \cdot g$.

Now we state the proposition:

(275) a and b are conjugated if and only if there exists g such that $b = a \cdot g$. The theorem is a consequence of (226).

Let us consider G, a, and b. Observe that a and b are conjugated is reflexive and symmetric.

Now we state the propositions:

(276) If a and b are conjugated and b and c are conjugated, then a and c are conjugated. The theorem is a consequence of (225).

(277) If a and 0_G are conjugated or 0_G and a are conjugated, then $a = 0_G$. The theorem is a consequence of (275) and (219).

(278) $a \cdot \overline{\Omega_G} = \{b : a$ and b are conjugated$\}$. The theorem is a consequence of (243).

Let us consider G and a. The functor $a^\bullet$ yielding a subset of G is defined by the term

(Def. 36) $a \cdot \overline{\Omega_G}$.

Now we state the propositions:

(279) $x \in a^\bullet$ if and only if there exists b such that $b = x$ and a and b are conjugated. The theorem is a consequence of (278).

(280) $a \in b^\bullet$ if and only if a and b are conjugated. The theorem is a consequence of (279).

(281) $a \cdot g \in a^\bullet$.

(282) $a \in a^\bullet$.

(283) If $a \in b^\bullet$, then $b \in a^\bullet$. The theorem is a consequence of (280).

(284) $a^\bullet = b^\bullet$ if and only if $a^\bullet$ meets $b^\bullet$. The theorem is a consequence of (280), (279), and (276).

(285) $a^\bullet = \{0_G\}$ if and only if $a = 0_G$. The theorem is a consequence of (280), (279), and (277).

(286) $a^\bullet + A = A + a^\bullet$. The theorem is a consequence of (280), (202), (226), (224), (221), (225), (279), and (275).

Let us consider G, A, and B. We say that A and B are conjugated if and only if

(Def. 37) there exists g such that $A = B \cdot g$.

Now we state the propositions:

(287) A and B are conjugated if and only if there exists g such that $B = A \cdot g$. The theorem is a consequence of (255).

(288) A and A are conjugated. The theorem is a consequence of (253).

(289) If A and B are conjugated, then B and A are conjugated. The theorem is a consequence of (255).

Let us consider G, A, and B. Let us observe that A and B are conjugated is reflexive and symmetric.

Now we state the propositions:

(290) If A and B are conjugated and B and C are conjugated, then A and C are conjugated. The theorem is a consequence of (248).

(291) $\{a\}$ and $\{b\}$ are conjugated if and only if a and b are conjugated.
PROOF: If $\{a\}$ and $\{b\}$ are conjugated, then a and b are conjugated by (287), (238), (275), [17, (3)]. Consider g such that $a \cdot g = b$. $\{b\} = \{a\} \cdot g$. $\square$

(292) If A and $\overline{H_1}$ are conjugated, then there exists a strict subgroup H_2 of G such that the carrier of $H_2 = A$.

Let us consider G and A. The functor $A^\bullet$ yielding a family of subsets of G is defined by the term

(Def. 38) $\{B : A$ and B are conjugated$\}$.

Now we state the propositions:

(293) $x \in A^\bullet$ if and only if there exists B such that $x = B$ and A and B are conjugated.

(294) $A \in B^\bullet$ if and only if A and B are conjugated.

(295) $A \cdot g \in A^\bullet$. The theorem is a consequence of (287).

(296) $A \in A^\bullet$.

(297) If $A \in B^\bullet$, then $B \in A^\bullet$. The theorem is a consequence of (294).

(298) $A^\bullet = B^\bullet$ if and only if $A^\bullet$ meets $B^\bullet$. The theorem is a consequence of (294) and (290).

(299) $\{a\}^\bullet = \{\{b\} : b \in a^\bullet\}$. The theorem is a consequence of (287), (275), (280), (238), and (291).

(300) If G is finite, then $A^\bullet$ is finite.

Let us consider G, H_1, and H_2. We say that H_1 and H_2 are conjugated if and only if

(Def. 39) there exists g such that the additive magma of $H_1 = H_2 \cdot g$.

Now we state the propositions:

(301) Let us consider strict subgroups H_1, H_2 of G. Then H_1 and H_2 are conjugated if and only if there exists g such that $H_2 = H_1 \cdot g$. The theorem is a consequence of (263).

(302) Let us consider a strict subgroup H_1 of G. Then H_1 and H_1 are conjugated. The theorem is a consequence of (262).

(303) Let us consider strict subgroups H_1, H_2 of G. If H_1 and H_2 are conjugated, then H_2 and H_1 are conjugated. The theorem is a consequence of (263).

Let us consider G. Let H_1, H_2 be strict subgroups of G. Observe that H_1 and H_2 are conjugated is reflexive and symmetric.

Now we state the proposition:

(304) Let us consider strict subgroups H_1, H_2 of G. Suppose H_1 and H_2 are conjugated and H_2 and H_3 are conjugated. Then H_1 and H_3 are conjugated. The theorem is a consequence of (261).

In the sequel L denotes a subset of $\mathrm{SubGr}\, G$.

Let us consider G and H. The functor $H^\bullet$ yielding a subset of $\mathrm{SubGr}\, G$ is defined by

(Def. 40) for every object x, $x \in it$ iff there exists a strict subgroup H_1 of G such that $x = H_1$ and H and H_1 are conjugated.

Now we state the propositions:

(305) If $x \in H^\bullet$, then x is a strict subgroup of G.

(306) Let us consider strict subgroups H_1, H_2 of G. Then $H_1 \in {H_2}^\bullet$ if and only if H_1 and H_2 are conjugated.

Let us consider a strict subgroup H of G. Now we state the propositions:

(307) $H \cdot g \in H^\bullet$. The theorem is a consequence of (301).

(308) $H \in H^\bullet$.

Let us consider strict subgroups H_1, H_2 of G. Now we state the propositions:

(309) If $H_1 \in {H_2}^\bullet$, then $H_2 \in {H_1}^\bullet$. The theorem is a consequence of (306).

(310) ${H_1}^\bullet = {H_2}^\bullet$ if and only if ${H_1}^\bullet$ meets ${H_2}^\bullet$. The theorem is a consequence of (308), (305), (306), and (304).

Now we state the propositions:

(311) If G is finite, then $H^\bullet$ is finite.

(312) Let us consider a strict subgroup H_1 of G. Then H_1 and H_2 are conjugated if and only if $\overline{H_1}$ and $\overline{H_2}$ are conjugated.

Let us consider G. Let I_1 be a subgroup of G. We say that I_1 is normal if and only if

(Def. 41) for every a, $I_1 \cdot a =$ the additive magma of I_1.

Let us note that there exists a subgroup of G which is strict and normal.

From now on N_2 denotes a normal subgroup of G.

Now we state the propositions:

(313) (i) $\mathbf{0}_G$ is normal, and

(ii) Ω_G is normal.

(314) Let us consider strict, normal subgroups N_1, N_2 of G. Then $N_1 \cap N_2$ is normal. The theorem is a consequence of (264).

(315) Let us consider a strict subgroup H of G. If G is an Abelian additive group, then H is normal.

(316) H is a normal subgroup of G if and only if for every a, $a + H = H + a$. The theorem is a consequence of (260), (79), (151), (83), (153), (155), and (105).

Let us consider a subgroup H of G. Now we state the propositions:

(317) H is a normal subgroup of G if and only if for every a, $a + H \subseteq H + a$. The theorem is a consequence of (316), (205), (151), (155), (152), (80), and (83).

(318) H is a normal subgroup of G if and only if for every a, $H + a \subseteq a + H$. The theorem is a consequence of (316), (205), (151), (155), (152), (80), and (83).

(319) H is a normal subgroup of G if and only if for every A, $A + H = H + A$. The theorem is a consequence of (140), (149), (316), (150), and (141).

Let us consider a strict subgroup H of G. Now we state the propositions:

(320) H is a normal subgroup of G if and only if for every a, H is a subgroup of $H \cdot a$. The theorem is a consequence of (100), (260), (80), (83), (207), and (318).

(321) H is a normal subgroup of G if and only if for every a, $H \cdot a$ is a subgroup of H. The theorem is a consequence of (100), (260), (80), (83), (207), and (317).

(322) H is a normal subgroup of G if and only if $H^\bullet = \{H\}$.

PROOF: If H is a normal subgroup of G, then $H^\bullet = \{H\}$ by (301), (308), [17, (31)]. H is normal. $\square$

(323) H is a normal subgroup of G if and only if for every a such that $a \in H$ holds $a^\bullet \subseteq \overline{H}$. The theorem is a consequence of (279), (275), (259), and (226).

Let us consider strict, normal subgroups N_1, N_2 of G. Now we state the propositions:

(324) $\overline{N_1} + \overline{N_2} = \overline{N_2} + \overline{N_1}$.

(325) There exists a strict, normal subgroup N of G such that the carrier of $N = \overline{N_1} + \overline{N_2}$. The theorem is a consequence of (124), (75), (316), (76), and (77).

Now we state the propositions:

(326) Let us consider a normal subgroup N of G. Then the left cosets of $N =$ the right cosets of N. The theorem is a consequence of (316).

(327) Let us consider a subgroup H of G. Suppose the left cosets of H is finite and $|\bullet : H|_{\mathbb{N}} = 2$. Then H is a normal subgroup of G.

PROOF: There exists a finite set B such that $B =$ the left cosets of H and $|\bullet : H|_{\mathbb{N}} = \overline{\overline{B}}$. Consider x, y being objects such that $x \neq y$ and the left cosets of $H = \{x, y\}$. $\overline{H} \in$ the left cosets of H. Consider z_3 being an object such that $\{x, y\} = \{\overline{H}, z_3\}$. $\overline{H}$ misses z_3 by (155), (161), [34, (29)], [17, (4)]. $\bigcup$(the left cosets of H) = the carrier of G and $\bigcup$(the left cosets of H) $= \overline{H} \cup z_3$. $\bigcup$(the right cosets of H) = the carrier of G and $z_3 =$ (the carrier of G) $\setminus \overline{H}$. There exists a finite set C such that $C =$ the right cosets of H and $|\bullet : H|_{\mathbb{N}} = \overline{\overline{C}}$. Consider z_1, z_2 being objects such that $z_1 \neq z_2$ and the right cosets of $H = \{z_1, z_2\}$. $\overline{H} \in$ the right cosets of H. Consider z_4 being an object such that $\{z_1, z_2\} = \{\overline{H}, z_4\}$. $\overline{H}$ misses z_4 by (155), (167), [34, (29)], [17, (4)]. □

Let us consider G and A. The functor $\mathrm{N}(A)$ yielding a strict subgroup of G is defined by

(Def. 42) the carrier of *it* $= \{h : A \cdot h = A\}$.

Now we state the propositions:

(328) $x \in \mathrm{N}(A)$ if and only if there exists h such that $x = h$ and $A \cdot h = A$.

(329) $\overline{\overline{A^\bullet}} = |\bullet : \mathrm{N}(A)|$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a such that $\$_1 = A \cdot a$ and $\$_2 = \mathrm{N}(A) + a$. For every object x such that $x \in A^\bullet$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = A^\bullet$ and for every object x such that $x \in A^\bullet$ holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. For every x, y_1, and y_2 such that $x \in A^\bullet$ and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. $\operatorname{rng} f =$ the right cosets of $\mathrm{N}(A)$. f is one-to-one. □

(330) Suppose $A^\bullet$ is finite or the left cosets of $\mathrm{N}(A)$ is finite. Then there exists a finite set C such that

(i) $C = A^\bullet$, and

(ii) $\overline{\overline{C}} = |\bullet : \mathrm{N}(A)|_{\mathbb{N}}$.

The theorem is a consequence of (329).

(331) $\overline{\overline{a^\bullet}} = |\bullet : \mathrm{N}(\{a\})|$.

Proof: Define $\mathcal{F}(\text{object}) = \{\$_1\}$. Consider f being a function such that $\operatorname{dom} f = a^\bullet$ and for every object x such that $x \in a^\bullet$ holds $f(x) = \mathcal{F}(x)$ from [14, Sch. 3]. $\operatorname{rng} f = \{a\}^\bullet$. f is one-to-one by [17, (3)]. □

(332) Suppose $a^\bullet$ is finite or the left cosets of $\mathrm{N}(\{a\})$ is finite. Then there exists a finite set C such that

(i) $C = a^\bullet$, and

(ii) $\overline{\overline{C}} = |\bullet : \mathrm{N}(\{a\})|_{\mathbb{N}}$.

The theorem is a consequence of (331).

Let us consider G and H. The functor $\mathrm{N}(H)$ yielding a strict subgroup of G is defined by the term

(Def. 43) $\mathrm{N}(\overline{H})$.

Let us consider a strict subgroup H of G. Now we state the propositions:

(333) $x \in \mathrm{N}(H)$ if and only if there exists h such that $x = h$ and $H \cdot h = H$. The theorem is a consequence of (328).

(334) $\overline{\overline{H^\bullet}} = |\bullet : \mathrm{N}(H)|$.

Proof: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a strict subgroup H_1 of G such that $\$_1 = H_1$ and $\$_2 = \overline{H_1}$. For every object x such that $x \in H^\bullet$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = H^\bullet$ and for every object x such that $x \in H^\bullet$ holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. $\operatorname{rng} f = \overline{H}^\bullet$. f is one-to-one. □

(335) Suppose $H^\bullet$ is finite or the left cosets of $\mathrm{N}(H)$ is finite. Then there exists a finite set C such that

(i) $C = H^\bullet$, and

(ii) $\overline{\overline{C}} = |\bullet : \mathrm{N}(H)|_{\mathbb{N}}$.

The theorem is a consequence of (334).

Now we state the proposition:

(336) Let us consider a strict additive group G, and a strict subgroup H of G. Then H is a normal subgroup of G if and only if $\mathrm{N}(H) = G$. The theorem is a consequence of (333) and (108).

Let us consider a strict additive group G. Now we state the propositions:

(337) $\mathrm{N}(\mathbf{0}_G) = G$. The theorem is a consequence of (313) and (336).

(338) $\mathrm{N}(\Omega_G) = G$. The theorem is a consequence of (313) and (336).

4. Topological Groups – TOPGRP_1

In the sequel S, R denote 1-sorted structures, X denotes a subset of R, T denotes a topological structure, x denotes a set, H denotes a non empty additive magma, P, Q, P_1, Q_1 denote subsets of H, and h denotes an element of H.

Now we state the proposition:

(339) If $P \subseteq P_1$ and $Q \subseteq Q_1$, then $P + Q \subseteq P_1 + Q_1$.

Let us assume that $P \subseteq Q$. Now we state the propositions:

(340) $P + h \subseteq Q + h$. The theorem is a consequence of (74).

(341) $h + P \subseteq h + Q$. The theorem is a consequence of (73).

From now on a denotes an element of G.

Now we state the propositions:

(342) $a \in -A$ if and only if $-a \in A$.

(343) $A \subseteq B$ if and only if $-A \subseteq -B$.

(344) $(\text{add inverse}\, G)^{\circ} A = -A$.

(345) $(\text{add inverse}\, G)^{-1}(A) = -A$.

(346) add inverse G is one-to-one. The theorem is a consequence of (9).

(347) $\text{rng add inverse}\, G =$ the carrier of G.

Let G be an additive group. One can verify that add inverse G is one-to-one and onto.

Now we state the propositions:

(348) $(\text{add inverse}\, G)^{-1} = \text{add inverse}\, G$.

(349) (The addition of $H)^{\circ}(P \times Q) = P + Q$.

Let G be a non empty additive magma and a be an element of G. The functors: a^{+} and ${}^{+}a$ yielding functions from G into G are defined by conditions,

(Def. 44) for every element x of G, $a^{+}(x) = a + x$,

(Def. 45) for every element x of G, ${}^{+}a(x) = x + a$,

respectively. Let G be an additive group. One can verify that a^{+} is one-to-one and onto and ${}^{+}a$ is one-to-one and onto.

Now we state the propositions:

(350) $(h^{+})^{\circ} P = h + P$. The theorem is a consequence of (73).

(351) $({}^{+}h)^{\circ} P = P + h$. The theorem is a consequence of (74).

(352) $(a^{+})^{-1} = (-a)^{+}$.

(353) $({}^{+}a)^{-1} = {}^{+}(-a)$.

We consider topological additive group structures which extend additive magmas and topological structures and are systems

$$\langle \text{a carrier}, \text{an addition}, \text{a topology} \rangle$$

where the carrier is a set, the addition is a binary operation on the carrier, the topology is a family of subsets of the carrier.

Let A be a non empty set, R be a binary operation on A, and T be a family of subsets of A. Let us observe that $\langle A, R, T\rangle$ is non empty.

Let x be a set, R be a binary operation on $\{x\}$, and T be a family of subsets of $\{x\}$. Observe that $\langle\{x\}, R, T\rangle$ is trivial and every 1-element additive magma is additive group-like, add-associative, and Abelian and there exists a topological additive group structure which is strict and non empty and there exists a topological additive group structure which is strict, topological space-like, and 1-element.

Let G be an additive group-like, add-associative, non empty topological additive group structure. We say that G is inverse-continuous if and only if

(Def. 46) add inverse G is continuous.

Let G be a topological space-like topological additive group structure. We say that G is continuous if and only if

(Def. 47) for every function f from $G \times G$ into G such that $f =$ the addition of G holds f is continuous.

One can check that there exists a topological space-like, additive group-like, add-associative, 1-element topological additive group structure which is strict, Abelian, inverse-continuous, and continuous.

A semi additive topological group is a topological space-like, additive group-like, add-associative, non empty topological additive group structure.

A topological additive group is an inverse-continuous, continuous semi additive topological group. Now we state the propositions:

(354) Let us consider a continuous, non empty, topological space-like topological additive group structure T, elements a, b of T, and a neighbourhood W of $a + b$. Then there exists an open neighbourhood A of a and there exists an open neighbourhood B of b such that $A + B \subseteq W$.

(355) Let us consider a topological space-like, non empty topological additive group structure T. Suppose for every elements a, b of T for every neighbourhood W of $a + b$, there exists a neighbourhood A of a and there exists a neighbourhood B of b such that $A + B \subseteq W$. Then T is continuous.

PROOF: For every point W of $T \times T$ and for every neighbourhood G of $f(W)$, there exists a neighbourhood H of W such that $f^{\circ}H \subseteq G$ by [32, (10)], (349). □

(356) Let us consider an inverse-continuous semi additive topological group T, an element a of T, and a neighbourhood W of $-a$. Then there exists an open neighbourhood A of a such that $-A \subseteq W$.

(357) Let us consider a semi additive topological group T. Suppose for every

element a of T for every neighbourhood W of $-a$, there exists a neighbourhood A of a such that $-A \subseteq W$. Then T is inverse-continuous. The theorem is a consequence of (344).

(358) Let us consider a topological additive group T, elements a, b of T, and a neighbourhood W of $a+-b$. Then there exists an open neighbourhood A of a and there exists an open neighbourhood B of b such that $A+-B \subseteq W$. The theorem is a consequence of (354) and (356).

(359) Let us consider a semi additive topological group T. Suppose for every elements a, b of T for every neighbourhood W of $a+-b$, there exists a neighbourhood A of a and there exists a neighbourhood B of b such that $A+-B \subseteq W$. Then T is a topological additive group.

PROOF: For every element a of T and for every neighbourhood W of $-a$, there exists a neighbourhood A of a such that $-A \subseteq W$ by [28, (4)]. For every elements a, b of T and for every neighbourhood W of $a+b$, there exists a neighbourhood A of a and there exists a neighbourhood B of b such that $A+B \subseteq W$. □

Let G be a continuous, non empty, topological space-like topological additive group structure and a be an element of G. One can check that a^+ is continuous and ${}^+a$ is continuous.

Let us consider a continuous semi additive topological group G and an element a of G. Now we state the propositions:

(360) a^+ is a homeomorphism of G. The theorem is a consequence of (352).

(361) ${}^+a$ is a homeomorphism of G. The theorem is a consequence of (353).

Let G be a continuous semi additive topological group and a be an element of G. The functors: a^+ and ${}^+a$ yield homeomorphisms of G. Now we state the proposition:

(362) Let us consider an inverse-continuous semi additive topological group G. Then add inverse G is a homeomorphism of G. The theorem is a consequence of (348).

Let G be an inverse-continuous semi additive topological group. Let us note that the functor add inverse G yields a homeomorphism of G. Let us note that every semi additive topological group which is continuous is also homogeneous.

Let us consider a continuous semi additive topological group G, a closed subset F of G, and an element a of G. Now we state the propositions:

(363) $F+a$ is closed. The theorem is a consequence of (351).

(364) $a+F$ is closed. The theorem is a consequence of (350).

Let G be a continuous semi additive topological group, F be a closed subset of G, and a be an element of G. Let us note that $F+a$ is closed and $a+F$ is closed.

Now we state the proposition:

(365) Let us consider an inverse-continuous semi additive topological group G, and a closed subset F of G. Then $-F$ is closed. The theorem is a consequence of (344).

Let G be an inverse-continuous semi additive topological group and F be a closed subset of G. One can verify that $-F$ is closed.

Let us consider a continuous semi additive topological group G, an open subset O of G, and an element a of G. Now we state the propositions:

(366) $O + a$ is open. The theorem is a consequence of (351).

(367) $a + O$ is open. The theorem is a consequence of (350).

Let G be a continuous semi additive topological group, A be an open subset of G, and a be an element of G. One can check that $A + a$ is open and $a + A$ is open.

Now we state the proposition:

(368) Let us consider an inverse-continuous semi additive topological group G, and an open subset O of G. Then $-O$ is open. The theorem is a consequence of (344).

Let G be an inverse-continuous semi additive topological group and A be an open subset of G. Observe that $-A$ is open.

Let us consider a continuous semi additive topological group G and subsets A, O of G.

Let us assume that O is open. Now we state the propositions:

(369) $O + A$ is open.

Proof: $\mathrm{Int}(O + A) = O + A$ by [48, (16)], (74), [48, (22)]. □

(370) $A + O$ is open.

Proof: $\mathrm{Int}(A + O) = A + O$ by [48, (16)], (73), [48, (22)]. □

Let G be a continuous semi additive topological group, A be an open subset of G, and B be a subset of G. Note that $A + B$ is open and $B + A$ is open.

Now we state the propositions:

(371) Let us consider an inverse-continuous semi additive topological group G, a point a of G, and a neighbourhood A of a. Then $-A$ is a neighbourhood of $-a$. The theorem is a consequence of (343).

(372) Let us consider a topological additive group G, a point a of G, and a neighbourhood A of $a + -a$. Then there exists an open neighbourhood B of a such that $B + -B \subseteq A$. The theorem is a consequence of (358) and (342).

(373) Let us consider an inverse-continuous semi additive topological group G, and a dense subset A of G. Then $-A$ is dense. The theorem is a consequence of (345).

Let G be an inverse-continuous semi additive topological group and A be a dense subset of G. Observe that $-A$ is dense.

Let us consider a continuous semi additive topological group G, a dense subset A of G, and a point a of G. Now we state the propositions:

(374) $a + A$ is dense. The theorem is a consequence of (350).

(375) $A + a$ is dense. The theorem is a consequence of (351).

Let G be a continuous semi additive topological group, A be a dense subset of G, and a be a point of G. Let us observe that $A + a$ is dense and $a + A$ is dense.

Now we state the proposition:

(376) Let us consider a topological additive group G, a basis B of 0_G, and a dense subset M of G. Then $\{V + x$, where V is a subset of G, x is a point of $G : V \in B$ and $x \in M\}$ is a basis of G.

Proof: Set $Z = \{V + x$, where V is a subset of G, x is a point of $G : V \in B$ and $x \in M\}$. $Z \subseteq$ the topology of G by [38, (12)]. For every subset W of G such that W is open for every point a of G such that $a \in W$ there exists a subset V of G such that $V \in Z$ and $a \in V$ and $V \subseteq W$ by (8), [28, (3)], (74), (372). $Z \subseteq 2^{\alpha}$, where α is the carrier of G. □

One can check that every topological additive group is regular.

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