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Stone Lattices

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Summary. The article continues the formalization of the lattice theory (as structures with two binary operations, not in terms of ordering relations). In the paper, the notion of a pseudocomplement in a lattice is formally introduced in Mizar, and based on this we define the notion of the skeleton and the set of dense elements in a pseudocomplemented lattice, giving the meet-decomposition of arbitrary element of a lattice as the infimum of two elements: one belonging to the skeleton, and the other which is dense.

The core of the paper is of course the idea of Stone identity

$$a^\star \sqcup a^{\star\star} = \top,$$

which is fundamental for us: Stone lattices are those lattices L, which are distributive, bounded, and satisfy Stone identity for all elements $a \in L$. Stone algebras were introduced by Grätzer and Schmidt in [18]. Of course, the pseudocomplement is unique (if exists), so in a pseudcomplemented lattice we defined $a^\star$ as the Mizar functor (unary operation mapping every element to its pseudocomplement). In Section 2 we prove formally a collection of ordinary properties of pseudocomplemented lattices.

All Boolean lattices are Stone, and a natural example of the lattice which is Stone, but not Boolean, is the lattice of all natural divisors of p^2 for arbitrary prime number p (Section 6). At the end we formalize the notion of the Stone lattice $B^{[2]}$ (of pairs of elements a, b of B such that $a \leqslant b$) constructed as a sublattice of B^2, where B is arbitrary Boolean algebra (and we describe skeleton and the set of dense elements in such lattices). In a natural way, we deal with Cartesian product of pseudocomplemented lattices.

Our formalization was inspired by [17], and is an important step in formalizing Jouni Järvinen *Lattice theory for rough sets* [19], so it follows rather the latter paper. We deal essentially with Section 4.3, pages 423–426. The description of handling complemented structures in Mizar [6] can be found in [12]. The

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current article together with [15] establishes the formal background for algebraic structures which are important for [10], [16] by means of mechanisms of merging theories as described in [11].

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The notation and terminology used in this paper have been introduced in the following articles: [1], [25], [2], [3], [9], [26], [23], [4], [13], [28], [20], [14], [8], [5], [27], and [7].

1. Preliminaries

Now we state the proposition:

(1) Let us consider a distributive lattice L. Then every sublattice of L is distributive.

Let L be a distributive lattice. One can verify that every sublattice of L is distributive.

Let L_1, L_2 be bounded lattices. One can check that $L_1 \times L_2$ is bounded.

From now on L denotes a lattice and I, P denote non empty closed subset of L.

Now we state the propositions:

(2) If L is lower-bounded and $\bot_L \in I$, then $\mathbb{L}_I^L$ is lower-bounded and $\bot_{\mathbb{L}_I^L} = \bot_L$.

PROOF: Set $c = \bot_L$. Reconsider $c' = c$ as an element of $\mathbb{L}_I^L$. There exists an element c' of $\mathbb{L}_I^L$ such that for every element a' of $\mathbb{L}_I^L$, $c' \sqcap a' = c'$ and $a' \sqcap c' = c'$ by [3, (68), (73)]. For every element a' of $\mathbb{L}_I^L$, $c' \sqcap a' = c'$ and $a' \sqcap c' = c'$ by [3, (68), (73)]. □

(3) If L is upper-bounded and $\top_L \in I$, then $\mathbb{L}_I^L$ is upper-bounded and $\top_{\mathbb{L}_I^L} = \top_L$.

PROOF: Set $c = \top_L$. Reconsider $c' = c$ as an element of $\mathbb{L}_I^L$. There exists an element c' of $\mathbb{L}_I^L$ such that for every element a' of $\mathbb{L}_I^L$, $c' \sqcup a' = c'$ and $a' \sqcup c' = c'$ by [3, (68), (73)]. For every element a' of $\mathbb{L}_I^L$, $c' \sqcup a' = c'$ and $a' \sqcup c' = c'$ by [3, (68), (73)]. □

2. Pseudocomplements in Lattices

Let L be a non empty lattice structure and a, b be elements of L. We say that a is a pseudocomplement of b if and only if

(Def. 1) $a \sqcap b = \bot_L$ and for every element x of L such that $b \sqcap x = \bot_L$ holds $x \sqsubseteq a$.

We say that L is pseudocomplemented if and only if

(Def. 2) for every element x of L, there exists an element y of L such that y is a pseudocomplement of x.

Now we state the proposition:

(4) Every Boolean lattice is pseudocomplemented.

Let us note that every lattice which is Boolean is also pseudocomplemented and there exists a lattice which is Boolean, pseudocomplemented, and bounded.

Now we state the proposition:

(5) Let us consider a pseudocomplemented, lower-bounded lattice L, and elements a, b, x of L. If a is a pseudocomplement of x and b is a pseudocomplement of x, then $a = b$.

Let L be a non empty lattice structure and x be an element of L. Assume L is a pseudocomplemented, lower-bounded lattice. The functor x^* yielding an element of L is defined by

(Def. 3) *it* is a pseudocomplement of x.

Now we state the proposition:

(6) Let us consider a pseudocomplemented, lower-bounded lattice L, and an element x of L. Then $x^* \sqcap x = \bot_L$.

From now on L denotes a lower-bounded, pseudocomplemented lattice.

Now we state the propositions:

(7) Let us consider an element a of L. Then $a \sqsubseteq (a^*)^*$.

(8) Let us consider elements a, b of L. If $a \sqsubseteq b$, then $b^* \sqsubseteq a^*$. The theorem is a consequence of (6).

(9) Let us consider an element a of L. Then $a^* = ((a^*)^*)^*$. The theorem is a consequence of (8) and (7).

Let us consider a pseudocomplemented, bounded lattice L. Now we state the propositions:

(10) $(\bot_L)^* = \top_L$.

(11) $(\top_L)^* = \bot_L$.

(12) Let us consider a Boolean lattice L, and an element x of L. Then $x^{\mathrm{c}} = x^*$.
Proof: $x^* \sqsubseteq x^{\mathrm{c}}$ by (6), [28, (25)]. $x^{\mathrm{c}} \sqsubseteq x^*$ by [28, (20)]. □

(13) Let us consider a pseudocomplemented, bounded lattice L, and elements x, y of L. Suppose y is a pseudocomplement of x. Then $y \in$ the set of pseudo-complements of x.

(14) Let us consider a pseudocomplemented, bounded lattice L, and an element x of L. Then $x^* \in$ the set of pseudo-complements of x. The theorem is a consequence of (13).

3. Skeleton of a Pseudocomplemented Lattice

Let L be a lower-bounded, pseudocomplemented lattice. The functor Skeleton L yielding a subset of L is defined by the term

(Def. 4) the set of all a^* where a is an element of L.

Now we state the propositions:

(15) Let us consider a lower-bounded, pseudocomplemented lattice L. Then $\mathrm{Skeleton}\, L = \{a$, where a is an element of $L : (a^*)^* = a\}$. The theorem is a consequence of (9).

(16) Let us consider a lower-bounded, pseudocomplemented lattice L, and an element x of L. Then $x \in \mathrm{Skeleton}\, L$ if and only if $(x^*)^* = x$. The theorem is a consequence of (9).

Let L be a bounded, pseudocomplemented lattice. Let us note that Skeleton L is non empty.

Now we state the proposition:

(17) Let us consider a pseudocomplemented, distributive, lower-bounded lattice L, and elements a, b of L. If $a, b \in \mathrm{Skeleton}\, L$, then $a \sqcap b \in \mathrm{Skeleton}\, L$. The theorem is a consequence of (16), (8), and (7).

4. Stone Identity

Let L be a non empty lattice structure. We say that L satisfies the Stone identity if and only if

(Def. 5) for every element x of L, $x^* \sqcup (x^*)^* = \top_L$.

Now we state the proposition:

(18) Every Boolean lattice satisfies the Stone identity.
Proof: $x^* \sqcup (x^*)^* = \top_L$ by (12), [28, (21)]. $\square$

Let us note that every lattice which is Boolean satisfies also the Stone identity and there exists a lattice which is pseudocomplemented and Boolean and satisfies the Stone identity.

Now we state the proposition:

(19) Let us consider a pseudocomplemented, distributive, bounded lattice L. Then L satisfies the Stone identity if and only if for every elements a, b of L, $(a \sqcap b)^* = a^* \sqcup b^*$. The theorem is a consequence of (6) and (10).

Let L be a lattice. We say that L is Stone if and only if

(Def. 6) L is pseudocomplemented, distributive, and bounded and satisfies the Stone identity.

Let us note that every lattice which is Stone is also pseudocomplemented, distributive, and bounded and satisfies also the Stone identity and every lattice which is pseudocomplemented, distributive, and bounded and satisfies the Stone identity is also Stone.

Now we state the proposition:

(20) Let us consider a pseudocomplemented, distributive, bounded lattice L. Then L satisfies the Stone identity if and only if for every elements a, b of L such that a, $b \in \mathrm{Skeleton}\, L$ holds $a \sqcup b \in \mathrm{Skeleton}\, L$. The theorem is a consequence of (19), (16), (8), (9), (6), and (10).

In the sequel L denotes a Stone lattice.

Now we state the proposition:

(21) $\top_L$, $\bot_L \in \mathrm{Skeleton}\, L$. The theorem is a consequence of (11) and (10).

Let L be a Stone lattice and a be an element of L. We say that a is skeletal if and only if

(Def. 7) $a \in \mathrm{Skeleton}\, L$.

One can verify that $\top_L$ is skeletal and $\bot_L$ is skeletal and $\mathrm{Skeleton}\, L$ is join-closed and meet-closed.

Let us observe that the functor $\mathrm{Skeleton}\, L$ yields a closed subset of L. The functor $\mathrm{SkelLatt}\, L$ yielding a sublattice of L is defined by the term

(Def. 8) $\mathbb{L}^{L}_{\mathrm{Skeleton}\, L}$.

Observe that $\mathrm{SkelLatt}\, L$ is distributive.

Now we state the proposition:

(22) (i) $\bot_L = \bot_{\mathrm{SkelLatt}\, L}$, and

(ii) $\top_L = \top_{\mathrm{SkelLatt}\, L}$.

The theorem is a consequence of (21), (2), and (3).

Let L be a Stone lattice. Observe that $\mathrm{SkelLatt}\, L$ is Boolean.

5. Dense Elements in Lattices

Let L be a lower-bounded lattice. The functor DenseElements L yielding a subset of L is defined by the term

(Def. 9) $\{a$, where a is an element of $L : a^* = \bot_L\}$.

Now we state the proposition:

(23) $\top_L \in$ DenseElements L. The theorem is a consequence of (11).

Let L be a Stone lattice. Note that DenseElements L is non empty.

Let a be an element of L. We say that a is dense if and only if

(Def. 10) $a \in$ DenseElements L.

Note that $\top_L$ is dense.

Now we state the proposition:

(24) Let us consider a Stone lattice L, and an element x of L. If $x \in$ DenseElements L, then $x^* = \bot_L$.

Let L be a Stone lattice. Note that DenseElements L is join-closed and meet-closed.

Let us note that the functor DenseElements L yields a closed subset of L. The functor DenseLatt L yielding a sublattice of L is defined by the term

(Def. 11) $\mathbb{L}^{L}_{\mathrm{DenseElements}\, L}$.

Note that DenseLatt L is distributive.

Now we state the proposition:

(25) Let us consider a Stone lattice L, and an element a of L. Then there exist elements b, c of L such that

(i) $a = b \sqcap c$, and

(ii) $b \in$ Skeleton L, and

(iii) $c \in$ DenseElements L.

The theorem is a consequence of (7), (6), and (8).

6. An Example: Lattice of Natural Divisors

Let us consider a prime number p. Now we state the propositions:

(26) The set of positive divisors of $p = \{1, p\}$.
Proof: $\{p^k$, where k is an element of $\mathbb{N} : k \leqslant 1\} = \{1, p\}$ by [22, (4)]. □

(27) The set of positive divisors of $p \cdot p = \{1, p, p \cdot p\}$.
Proof: $\{p^k$, where k is an element of $\mathbb{N} : k \leqslant 2\} = \{1, p, p \cdot p\}$ by [22, (81), (4)]. □

Let n be a non zero natural number. Let us observe that the lattice of positive divisors of n is finite and there exists a Boolean lattice which is complete.

Let p be a prime number. One can check that the lattice of positive divisors of p is Boolean and the lattice of positive divisors of $p \cdot p$ is pseudocomplemented.

Now we state the proposition:

(28) Let us consider a lattice L, a prime number p, and an element x of L. Suppose $L =$ the lattice of positive divisors of $p \cdot p$ and $x = p$. Then $x^* = \bot_L$.

Proof: Reconsider $y_1 = \bot_L$ as an element of L. For every element y of L such that $x \sqcap y = \bot_L$ holds $y \sqsubseteq y_1$ by (27), [14, (64)]. □

Let p be a prime number. Observe that the lattice of positive divisors of $p \cdot p$ satisfies the Stone identity and the lattice of positive divisors of $p \cdot p$ is non Boolean and Stone and there exists a lattice which is Stone and non Boolean.

7. Products of Pseudocomplemented Lattices

From now on L_1, L_2 denote lattices, p_1, q_1 denote elements of L_1, and p_2, q_2 denote elements of L_2.

Let us assume that L_1 is a bounded lattice and L_2 is a bounded lattice. Now we state the propositions:

(29) p_1 is a pseudocomplement of q_1 and p_2 is a pseudocomplement of q_2 if and only if $\langle p_1, p_2 \rangle$ is a pseudocomplement of $\langle q_1, q_2 \rangle$.

Proof: If p_1 is a pseudocomplement of q_1 and p_2 is a pseudocomplement of q_2, then $\langle p_1, p_2 \rangle$ is a pseudocomplement of $\langle q_1, q_2 \rangle$ by [2, (35), (42), (36)]. For every element x_3 of L_1 such that $q_1 \sqcap x_3 = \bot_{L_1}$ holds $x_3 \sqsubseteq p_1$ by [2, (42), (35), (36)]. For every element x_4 of L_2 such that $q_2 \sqcap x_4 = \bot_{L_2}$ holds $x_4 \sqsubseteq p_2$ by [2, (42), (35), (36)]. □

(30) L_1 is pseudocomplemented and L_2 is pseudocomplemented if and only if $L_1 \times L_2$ is pseudocomplemented. The theorem is a consequence of (29).

Let L_1, L_2 be pseudocomplemented bounded lattices. Let us observe that $L_1 \times L_2$ is pseudocomplemented.

Now we state the proposition:

(31) Suppose L_1 is a pseudocomplemented bounded lattice and L_2 is a pseudocomplemented bounded lattice. Then $\langle p_1, p_2 \rangle^* = \langle p_1{}^*, p_2{}^* \rangle$. The theorem is a consequence of (29).

In the sequel L_1, L_2 denote non empty lattices.

Now we state the propositions:

(32) If L_1 is a pseudocomplemented bounded lattice and L_2 is a pseudocomplemented bounded lattice, then $L_1 \times L_2$ satisfies the Stone identity.

PROOF: Set $L = L_1 \times L_2$. For every element x of L, $x^* \sqcup (x^*)^* = \top_L$ by (31), [2, (43), (35)]. □

(33) If L_1 is Stone and L_2 is Stone, then $L_1 \times L_2$ is Stone.

Let L_1, L_2 be Stone lattices. Let us observe that $L_1 \times L_2$ is Stone.

8. Special Construction: $B^{[2]}$

From now on B denotes a Boolean lattice.

Let B be a Boolean lattice. The functor carrier($B^{[2]}$) yielding a subset of $B \times B$ is defined by the term

(Def. 12) $\{\langle a, b \rangle$, where a, b are elements of $B : a \sqsubseteq b\}$.

Let us note that carrier($B^{[2]}$) is non empty and carrier($B^{[2]}$) is join-closed and meet-closed.

Observe that the functor carrier($B^{[2]}$) yields a non empty closed subset of $B \times B$. The functor $B^{[2]}$ yielding a lattice is defined by the term

(Def. 13) $\mathbb{L}^{B \times B}_{\mathrm{carrier}(B^{[2]})}$.

Now we state the propositions:

(34) The carrier of $B^{[2]} = \mathrm{carrier}(B^{[2]})$.

(35) $\langle \bot_B, \bot_B \rangle \in$ the carrier of $B^{[2]}$. The theorem is a consequence of (34).

(36) $\langle \top_B, \top_B \rangle \in$ the carrier of $B^{[2]}$. The theorem is a consequence of (34).

Let B be a Boolean lattice. One can verify that $B^{[2]}$ is lower-bounded and $B^{[2]}$ is upper-bounded.

Now we state the propositions:

(37) $\bot_{B^{[2]}} = \langle \bot_B, \bot_B \rangle$. The theorem is a consequence of (2).

(38) $\top_{B^{[2]}} = \langle \top_B, \top_B \rangle$. The theorem is a consequence of (3).

Let B be a Boolean lattice. One can check that $B^{[2]}$ is pseudocomplemented.

Now we state the proposition:

(39) Let us consider a lattice L, elements x_1, x_2 of B, and an element x of L. Suppose $L = B^{[2]}$ and $x = \langle x_1, x_2 \rangle$. Then $x^* = \langle {x_2}^{\mathrm{c}}, {x_2}^{\mathrm{c}} \rangle$.

PROOF: $x \in \mathrm{carrier}(B^{[2]})$. Consider x_3, x_4 being elements of B such that $x = \langle x_3, x_4 \rangle$ and $x_3 \sqsubseteq x_4$. Reconsider $y = \langle {x_2}^{\mathrm{c}}, {x_2}^{\mathrm{c}} \rangle$ as an element of L. For every element w of L such that $x \sqcap w = \bot_L$ holds $w \sqsubseteq y$ by (34), [24, (11)], (37), [2, (35)]. y is a pseudocomplement of x. □

Let B be a Boolean lattice. One can verify that $B^{[2]}$ satisfies the Stone identity and $B^{[2]}$ is Stone.

Now we state the propositions:

(40) Skeleton $B^{[2]}$ = the set of all $\langle a, a\rangle$ where a is an element of B.

PROOF: Skeleton $B^{[2]}$ = the set of all $\langle a, a\rangle$ where a is an element of B by (34), (39), [3, (72)]. □

(41) DenseElements $B^{[2]}$ = the set of all $\langle a, \top_B\rangle$ where a is an element of B.

PROOF: Set $L = B^{[2]}$. DenseElements $L \subseteq$ the set of all $\langle a, \top_B\rangle$ where a is an element of B by (34), (37), (39), [21, (30)]. Consider a being an element of B such that $x = \langle a, \top_B\rangle$. Reconsider $y = x$ as an element of L. $y^* = \langle(\top_B)^c, (\top_B)^c\rangle$. □

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