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Torsion Part of $\mathbb{Z}$-module

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Summary. In this article, we formalize in Mizar [7] the definition of "torsion part" of $\mathbb{Z}$-module and its properties. We show $\mathbb{Z}$-module generated by the field of rational numbers as an example of torsion-free non free $\mathbb{Z}$-modules. We also formalize the rank-nullity theorem over finite-rank free $\mathbb{Z}$-modules (previously formalized in [1]). $\mathbb{Z}$-module is necessary for lattice problems, LLL (Lenstra, Lenstra and Lovász) base reduction algorithm [23] and cryptographic systems with lattices [24].

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The notation and terminology used in this paper have been introduced in the following articles: [27], [8], [2], [29], [6], [13], [9], [10], [17], [30], [22], [28], [25], [4], [5], [11], [20], [38], [39], [32], [37], [21], [33], [34], [35], [36], [12], [14], [15], [16], [26], and [19].

1. Torsion Part of $\mathbb{Z}$-module

From now on x, y, y_1, y_2 denote objects, V denotes a $\mathbb{Z}$-module, W, W_1, W_2 denote submodules of V, u, v denote vectors of V, and i, j, k, n denote elements of $\mathbb{N}$.

Now we state the proposition:

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(1) Let us consider an integer n. Suppose $n \neq 0$ and $n \neq -1$ and $n \neq -2$. Then $\frac{n}{n+1} \notin \mathbb{Z}$.

One can check that there exists an element of $\mathbb{Z}^{\mathrm{R}}$ which is prime and non zero and every element of $\mathbb{Z}^{\mathrm{R}}$ which is prime is also non zero.

Now we state the propositions:

(2) Let us consider a $\mathbb{Z}$-module V, and a subset A of V. Suppose A is linearly independent. Then there exists a subset B of V such that

   (i) $A \subseteq B$, and

   (ii) B is linearly independent, and

   (iii) for every vector v of V, there exists an element a of $\mathbb{Z}^{\mathrm{R}}$ such that $a \neq 0$ and $a \cdot v \in \mathrm{Lin}(B)$.

PROOF: Define $\mathcal{P}[\text{set}] \equiv$ there exists a subset B of V such that $B = \$_1$ and $A \subseteq B$ and B is linearly independent. Consider Q being a set such that For every set Z, $Z \in Q$ iff $Z \in 2^{\alpha}$ and $\mathcal{P}[Z]$, where α is the carrier of V. Consider X being a set such that $X \in Q$ and for every set Z such that $Z \in Q$ and $Z \neq X$ holds $X \not\subseteq Z$. Consider B being a subset of V such that $B = X$ and $A \subseteq B$ and B is linearly independent. Consider v being a vector of V such that for every element a of $\mathbb{Z}^{\mathrm{R}}$ such that $a \neq 0$ holds $a \cdot v \notin \mathrm{Lin}(B)$. $B \cup \{v\}$ is linearly independent by [10, (8)], [15, (39), (55)], [31, (61)]. $\square$

(3) Let us consider a $\mathbb{Z}$-module V, a finite subset I of V, and a submodule W of V. Suppose for every vector v of V such that $v \in I$ there exists an element a of $\mathbb{Z}^{\mathrm{R}}$ such that $a \neq 0_{\mathbb{Z}^{\mathrm{R}}}$ and $a \cdot v \in W$. Then there exists an element a of $\mathbb{Z}^{\mathrm{R}}$ such that

   (i) $a \neq 0_{\mathbb{Z}^{\mathrm{R}}}$, and

   (ii) for every vector v of V such that $v \in I$ holds $a \cdot v \in W$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite subset I of V such that $\overline{\overline{I}} = \$_1$ and for every vector v of V such that $v \in I$ there exists an element a of $\mathbb{Z}^{\mathrm{R}}$ such that $a \neq 0_{\mathbb{Z}^{\mathrm{R}}}$ and $a \cdot v \in W$ there exists an element a of $\mathbb{Z}^{\mathrm{R}}$ such that $a \neq 0_{\mathbb{Z}^{\mathrm{R}}}$ and for every vector v of V such that $v \in I$ holds $a \cdot v \in W$. $\mathcal{P}[0]$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [37, (41)], [3, (44)], [2, (30)], [14, (37)]. For every natural number n, $\mathcal{P}[n]$ from [4, Sch. 2]. $\square$

(4) Let us consider a finite rank, free $\mathbb{Z}$-module V. Then every linearly independent subset of V is finite.

Let V be a finite rank, free $\mathbb{Z}$-module. Let us observe that every subset of V which is linearly independent is also finite.

Let us consider a finite rank, free $\mathbb{Z}$-module V and a linearly independent subset A of V. Now we state the propositions:

(5) There exists a finite, linearly independent subset I of V and there exists an element a of $\mathbb{Z}^{\mathrm{R}}$ such that $a \neq 0_{\mathbb{Z}^{\mathrm{R}}}$ and $A \subseteq I$ and $a \circ V$ is a submodule of $\mathrm{Lin}(I)$.

(6) There exists a finite, linearly independent subset I of V such that

(i) $A \subseteq I$, and

(ii) $\mathrm{rank}\, V = \overline{\overline{I}}$.

The theorem is a consequence of (5).

Now we state the proposition:

(7) Let us consider a torsion-free $\mathbb{Z}$-module V, finite rank, free submodules W_1, W_2 of V, and a basis I_1 of W_1. Then there exists a finite, linearly independent subset I of V such that

(i) I is a subset of $W_1 + W_2$, and

(ii) $I_1 \subseteq I$, and

(iii) $\mathrm{rank}(W_1 + W_2) = \mathrm{rank}\, \mathrm{Lin}(I)$.

The theorem is a consequence of (6).

Let us consider a torsion-free $\mathbb{Z}$-module V and finite rank, free submodules W_1, W_2 of V. Now we state the propositions:

(8) Suppose W_2 is a submodule of W_1. Then there exists a finite rank, free submodule W_3 of V such that

(i) $\mathrm{rank}\, W_1 = \mathrm{rank}\, W_2 + \mathrm{rank}\, W_3$, and

(ii) $W_2 \cap W_3 = \mathbf{0}_V$, and

(iii) W_3 is a submodule of W_1.

Proof: Set $I_2 =$ the basis of W_2. Reconsider $J_2 = I_2$ as a subset of W_1. Consider J_1 being a finite, linearly independent subset of W_1 such that $J_2 \subseteq J_1$ and $\mathrm{rank}\, W_1 = \overline{\overline{J_1}}$. Set $J_3 = J_1 \setminus J_2$. Reconsider $I_3 = J_3$ as a subset of V. $W_2 \cap \mathrm{Lin}(I_3) = \mathbf{0}_V$ by [16, (20)], [14, (42)], [18, (23)], [19, (4)]. $\square$

(9) There exists a finite rank, free submodule W_3 of V such that

(i) $\mathrm{rank}(W_1 + W_2) = \mathrm{rank}\, W_1 + \mathrm{rank}\, W_3$, and

(ii) $W_1 \cap W_3 = \mathbf{0}_V$, and

(iii) W_3 is a submodule of $W_1 + W_2$.

PROOF: Set I_1 = the basis of W_1. Consider I being a finite, linearly independent subset of V such that I is a subset of $W_1 + W_2$ and $I_1 \subseteq I$ and $\operatorname{rank}(W_1 + W_2) = \operatorname{rank} \operatorname{Lin}(I)$. Set $I_2 = I \setminus I_1$. Reconsider $J_2 = I_2$ as a finite, linearly independent subset of V. $W_1 \cap \operatorname{Lin}(J_2) = \mathbf{0}_V$ by [16, (20)], [14, (42)], [18, (23)], [19, (4)]. □

Now we state the proposition:

(10) Let us consider a finite rank, free $\mathbb{Z}$-module V, and submodules W_1, W_2 of V. Then $\operatorname{rank}(W_1 \cap W_2) \geqslant \operatorname{rank} W_1 + \operatorname{rank} W_2 - \operatorname{rank} V$.

Let V be a $\mathbb{Z}$-module. The functor torsion-part(V) yielding a strict submodule of V is defined by

(Def. 1) the carrier of *it* $= \{v$, where v is a vector of $V : v$ is torsion$\}$.

Now we state the propositions:

(11) Let us consider a $\mathbb{Z}$-module V, and a vector v of V. Then v is torsion if and only if $v \in$ torsion-part(V).

(12) Let us consider a $\mathbb{Z}$-module V. Then V is torsion-free if and only if torsion-part$(V) = \mathbf{0}_V$. The theorem is a consequence of (11).

Let V be a $\mathbb{Z}$-module. Observe that $\mathbb{Z}$-ModuleQuot$(V,$ torsion-part$(V))$ is torsion-free.

Let W be a submodule of V. The functor $\mathbb{Z}$-QMorph(V, W) yielding a linear transformation from V to $\mathbb{Z}$-ModuleQuot(V, W) is defined by

(Def. 2) for every element v of V, $it(v) = v + W$.

One can check that $\mathbb{Z}$-QMorph(V, W) is onto.

Now we state the proposition:

(13) Let us consider $\mathbb{Z}$-modules V, W, a linear transformation T from V to W, a finite sequence s of elements of V, and a finite sequence t of elements of W. Suppose $\operatorname{len} s = \operatorname{len} t$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} s$ there exists a vector s_1 of V such that $s_1 = s(i)$ and $t(i) = T(s_1)$. Then $\sum t = T(\sum s)$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every finite sequence s of elements of V for every finite sequence t of elements of W such that $\operatorname{len} s = \$_1$ and $\operatorname{len} s = \operatorname{len} t$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} s$ there exists a vector s_1 of V such that $s_1 = s(i)$ and $t(i) = T(s_1)$ holds $\sum t = T(\sum s)$. $\mathcal{P}[0]$ by [32, (43)], [26, (19)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [6, (59)], [4, (11)], [6, (4)], [9, (3)]. For every natural number k, $\mathcal{P}[k]$ from [4, Sch. 2]. □

Let V be a finitely generated $\mathbb{Z}$-module and W be a submodule of V. Observe that $\mathbb{Z}$-ModuleQuot(V, W) is finitely generated and

$\mathbb{Z}$-ModuleQuot$(V,$ torsion-part$(V))$ is free.

2. $\mathbb{Z}$-module Generated by the Field of Rational Numbers

The functor $\mathbb{Z}$-module$\mathbb{Q}$ yielding a vector space structure over $\mathbb{Z}^{\mathrm{R}}$ is defined by the term

(Def. 3) $\langle$the carrier of $\mathbb{F}_{\mathbb{Q}}$, the addition of $\mathbb{F}_{\mathbb{Q}}$, the zero of $\mathbb{F}_{\mathbb{Q}}$, the left integer multiplication of $\mathbb{F}_{\mathbb{Q}}\rangle$.

One can verify that $\mathbb{Z}$-module$\mathbb{Q}$ is non empty and $\mathbb{Z}$-module$\mathbb{Q}$ is Abelian, add-associative, right zeroed, right complementable, scalar distributive, vector distributive, scalar associative, and scalar unital.

Now we state the propositions:

(14) Let us consider an element v of $\mathbb{F}_{\mathbb{Q}}$, and a rational number v_1. Suppose $v = v_1$. Let us consider a natural number n. Then (Nat-mult-left $\mathbb{F}_{\mathbb{Q}})(n, v) = n \cdot v_1$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ (Nat-mult-left $\mathbb{F}_{\mathbb{Q}})(\$_1, v) = \$_1 \cdot v_1$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n + 1]$. For every natural number n, $\mathcal{P}[n]$ from [4, Sch. 2]. □

(15) Let us consider an integer x, an element v of $\mathbb{F}_{\mathbb{Q}}$, and a rational number v_1. Suppose $v = v_1$. Then (the left integer multiplication of $\mathbb{F}_{\mathbb{Q}})(x, v) = x \cdot v_1$. The theorem is a consequence of (14).

Let us observe that $\mathbb{Z}$-module$\mathbb{Q}$ is torsion-free and $\mathbb{Z}$-module$\mathbb{Q}$ is non trivial.

Now we state the propositions:

(16) Let us consider an element s of $\mathbb{Z}$-module$\mathbb{Q}$. Then $\mathrm{Lin}(\{s\}) \neq \mathbb{Z}$-module$\mathbb{Q}$. The theorem is a consequence of (15) and (1).

(17) Let us consider elements s, t of $\mathbb{Z}$-module$\mathbb{Q}$. If $s \neq t$, then $\{s, t\}$ is not linearly independent. The theorem is a consequence of (15).

Let us observe that $\mathbb{Z}$-module$\mathbb{Q}$ is non free.

Now we state the proposition:

(18) Let us consider a finite subset A of $\mathbb{Z}$-module$\mathbb{Q}$. Then there exists an integer n such that

(i) $n \neq 0$, and

(ii) for every element s of $\mathbb{Z}$-module$\mathbb{Q}$ such that $s \in \mathrm{Lin}(A)$ there exists an integer m such that $s = \frac{m}{n}$.

Proof: Set $S = \mathbb{Z}$-module$\mathbb{Q}$. Define $\mathcal{P}$[natural number] $\equiv$ for every finite subset A of S such that $\overline{\overline{A}} = \$_1$ there exists an integer n such that $n \neq 0$ and for every element s of S such that $s \in \mathrm{Lin}(A)$ there exists an integer m such that $s = \frac{m}{n}$. $\mathcal{P}[0]$ by [15, (67)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [37, (41)], [3, (44)], [2, (30)], [20, (1)]. For every natural number k, $\mathcal{P}[k]$ from [4, Sch. 2]. □

One can verify that $\mathbb{Z}$-module$\mathbb{Q}$ is non finitely generated.

Now we state the proposition:

(19) Let us consider a finite subset A of $\mathbb{Z}$-module$\mathbb{Q}$. Then $\operatorname{rank}\operatorname{Lin}(A) \leqslant 1$.
PROOF: Set $S = \mathbb{Z}$-module$\mathbb{Q}$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite subset A of S such that $\overline{\overline{A}} = \$_1$ holds $\operatorname{rank}\operatorname{Lin}(A) \leqslant 1$. $\mathcal{P}[0]$ by [15, (67)], [14, (51)], [26, (1)]. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [12, (31)], [3, (44)], [2, (30)], [15, (72)]. For every natural number n, $\mathcal{P}[n]$ from [4, Sch. 2]. □

3. The Rank-Nullity Theorem

In the sequel V, W denote finite rank, free $\mathbb{Z}$-modules and T denotes a linear transformation from V to W.

Let W be a finite rank, free $\mathbb{Z}$-module, V be a $\mathbb{Z}$-module, and T be a linear transformation from V to W. Observe that $\operatorname{im} T$ is finite rank and free.

The functor $\operatorname{rank} T$ yielding a natural number is defined by the term

(Def. 4) $\operatorname{rank}\operatorname{im} T$.

Let V be a finite rank, free $\mathbb{Z}$-module and W be a $\mathbb{Z}$-module. The functor $\operatorname{nullity} T$ yielding a natural number is defined by the term

(Def. 5) $\operatorname{rank}\ker T$.

Now we state the propositions:

(20) Let us consider a finite rank, free $\mathbb{Z}$-module V, a subset A of V, a linearly independent subset B of V, and a linear transformation T from V to W. Suppose $\operatorname{rank} V = \overline{\overline{B}}$ and A is a basis of $\ker T$ and $A \subseteq B$. Then $T \restriction (B \setminus A)$ is one-to-one.

(21) Let us consider a finite rank, free $\mathbb{Z}$-module V, a subset A of V, a linearly independent subset B of V, a linear transformation T from V to W, and a linear combination l of $B \setminus A$. Suppose $\operatorname{rank} V = \overline{\overline{B}}$ and A is a basis of $\ker T$ and $A \subseteq B$. Then $T(\sum l) = \sum(T \mathbin{@*} l)$. The theorem is a consequence of (20).

(22) Let us consider $\mathbb{Z}$-modules V, W, a linear transformation T from V to W, and a subset A of V. Suppose $A \subseteq$ the carrier of $\ker T$. Then $\operatorname{Lin}(T^\circ A) = \mathbf{0}_W$.

(23) Let us consider $\mathbb{Z}$-modules V, W, a linear transformation T from V to W, and subsets A, B, X of V. Suppose $A \subseteq$ the carrier of $\ker T$ and $X = B \cup A$. Then $\operatorname{Lin}(T^\circ X) = \operatorname{Lin}(T^\circ B)$. The theorem is a consequence of (22).

Let us consider finite rank, free $\mathbb{Z}$-modules V, W and a linear transformation T from V to W. Now we state the propositions:

(24) $\operatorname{rank} V = \operatorname{rank} T + \operatorname{nullity} T$.

Proof: Set $A =$ the finite basis of $\ker T$. Reconsider $A' = A$ as a subset of V. Consider B' being a finite, linearly independent subset of V, a being an element of $\mathbb{Z}^{\mathrm{R}}$ such that $a \neq 0_{\mathbb{Z}^{\mathrm{R}}}$ and $A' \subseteq B'$ and $a \circ V$ is a submodule of $\mathrm{Lin}(B')$. Reconsider $X = B' \setminus A'$ as a finite subset of B'. Reconsider $C = T^{\circ}X$ as a finite subset of W. $T{\restriction}X$ is one-to-one. C is linearly independent by [26, (60)], (21), [26, (20)], [16, (20)]. Reconsider $a_1 = a \circ \operatorname{im} T$ as a submodule of W. $\mathrm{Lin}(T^{\circ}B') = \mathrm{Lin}(T^{\circ}X)$. For every vector v of W such that $v \in a_1$ holds $v \in \mathrm{Lin}(C)$ by [14, (25)], [26, (23)], [14, (29), (24)]. $\square$

(25) If T is one-to-one, then $\operatorname{rank} V = \operatorname{rank} T$. The theorem is a consequence of (24).

Let V, W be $\mathbb{Z}$-modules and T be a linear transformation from V to W. The functor $\mathbb{Z}$-decom(T) yielding a linear transformation from $\mathbb{Z}$-ModuleQuot$(V, \ker T)$ to $\operatorname{im} T$ is defined by

(Def. 6) *it* is bijective and for every element v of V, $it((\mathbb{Z}\text{-QMorph}(V, \ker T))(v)) = T(v)$.

Now we state the propositions:

(26) Let us consider $\mathbb{Z}$-modules V, W, and a linear transformation T from V to W. Then $T = \mathbb{Z}\text{-decom}(T) \cdot \mathbb{Z}\text{-QMorph}(V, \ker T)$.

Proof: Set $g = \mathbb{Z}\text{-decom}(T) \cdot \mathbb{Z}\text{-QMorph}(V, \ker T)$. For every element z of V, $T(z) = g(z)$ by [10, (15)]. $\square$

(27) Let us consider $\mathbb{Z}$-modules V, U, W, a linear transformation f from V to U, and a linear transformation g from U to W. Then $g \cdot f$ is a linear transformation from V to W.

Proof: Set $\mathfrak{f} = g \cdot f$. For every elements x, y of V, $\mathfrak{f}(x + y) = \mathfrak{f}(x) + \mathfrak{f}(y)$ by [10, (15)]. For every element a of $\mathbb{Z}^{\mathrm{R}}$ and for every element x of V, $\mathfrak{f}(a \cdot x) = a \cdot \mathfrak{f}(x)$ by [10, (15)]. $\square$

Let V, U, W be $\mathbb{Z}$-modules, f be a linear transformation from V to U, and g be a linear transformation from U to W. One can check that the functor $g \cdot f$ yields a linear transformation from V to W. Now we state the propositions:

(28) Let us consider $\mathbb{Z}$-modules V, W, and a linear transformation f from V to W. Then the carrier of $\ker f = f^{-1}(\{0_W\})$.

Proof: For every object x, $x \in$ the carrier of $\ker f$ iff $x \in f^{-1}(\{0_W\})$ by [10, (38)]. $\square$

(29) Let us consider $\mathbb{Z}$-modules V, U, W, a linear transformation f from V to U, and a linear transformation g from U to W. Then the carrier of

$\ker g \cdot f = f^{-1}$(the carrier of $\ker g$). The theorem is a consequence of (28).

(30) Let us consider $\mathbb{Z}$-modules V, W, and a linear transformation f from V to W. If f is onto, then $\operatorname{im} f = \Omega_W$.

(31) Let us consider a $\mathbb{Z}$-module V, and a submodule W of V. Then $\ker \mathbb{Z}\text{-QMorph}(V, W) = \Omega_W$.

PROOF: Set $f = \mathbb{Z}\text{-QMorph}(V, W)$. Reconsider $W_1 = \Omega_W$ as a strict submodule of V. For every object x, $x \in f^{-1}(\{0_{\mathbb{Z}\text{-ModuleQuot}(V,W)}\})$ iff $x \in$ the carrier of W by [10, (38)], [14, (63)]. $\ker f = W_1$. □

(32) Let us consider a $\mathbb{Z}$-module V, a submodule W of V, a strict submodule W_1 of V, and a vector v of V. If $W_1 = \Omega_W$, then $v + W = v + W_1$.

PROOF: For every object x, $x \in v + W$ iff $x \in v + W_1$ by [14, (72)]. □

(33) Let us consider a $\mathbb{Z}$-module V, a submodule W of V, a strict submodule W_1 of V, and an object A. If $W_1 = \Omega_W$, then A is a coset of W iff A is a coset of W_1. The theorem is a consequence of (32).

Let us consider a $\mathbb{Z}$-module V, a submodule W of V, and a strict submodule W_1 of V.

Let us assume that $W_1 = \Omega_W$. Now we state the propositions:

(34) $\text{CosetSet}(V, W) = \text{CosetSet}(V, W_1)$. The theorem is a consequence of (33).

(35) $\text{addCoset}(V, W) = \text{addCoset}(V, W_1)$. The theorem is a consequence of (34) and (32).

(36) $\text{lmultCoset}(V, W) = \text{lmultCoset}(V, W_1)$. The theorem is a consequence of (34) and (32).

(37) $\mathbb{Z}\text{-ModuleQuot}(V, W) = \mathbb{Z}\text{-ModuleQuot}(V, W_1)$. The theorem is a consequence of (34), (35), and (36).

Now we state the propositions:

(38) Let us consider $\mathbb{Z}$-modules V, U, a submodule V_1 of V, a submodule U_1 of U, and a linear transformation f from V to U. Suppose f is onto and the carrier of $V_1 = f^{-1}$(the carrier of U_1). Then there exists a linear transformation F from $\mathbb{Z}\text{-ModuleQuot}(V, V_1)$ to $\mathbb{Z}\text{-ModuleQuot}(U, U_1)$ such that F is bijective. The theorem is a consequence of (37), (29), (31), and (30).

(39) Let us consider a $\mathbb{Z}$-module V, submodules W_1, W_2 of V, a submodule U_1 of $W_1 + W_2$, and a strict submodule U_2 of W_1. Suppose $U_1 = W_2$ and $U_2 = W_1 \cap W_2$. Then there exists a linear transformation F from $\mathbb{Z}\text{-ModuleQuot}(W_1 + W_2, U_1)$ to $\mathbb{Z}\text{-ModuleQuot}(W_1, U_2)$ such that F is bijective.

PROOF: Set $Z_1 = \mathbb{Z}\text{-ModuleQuot}(W_1 + W_2, U_1)$. Set $Z_2 = \mathbb{Z}\text{-ModuleQuot}$

(W_1, U_2). Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an element v of $W_1 + W_2$ such that $\$_1 = v$ and $\$_2 = v + U_1$. For every element z of W_1, there exists an element y of Z_1 such that $\mathcal{P}[z, y]$ by [14, (25), (93)]. Consider f being a function from the carrier of W_1 into the carrier of Z_1 such that for every element z of W_1, $\mathcal{P}[z, f(z)]$ from [10, Sch. 3]. f is a linear transformation from W_1 to Z_1 by [14, (25), (28), (29)]. $\ker f = U_2$ by [26, (20)], [14, (63), (94), (46)]. $\operatorname{im} f = \mathbb{Z}\text{-ModuleQuot}(W_1 + W_2, U_1)$ by [14, (92), (93), (28)]. Reconsider $F = \mathbb{Z}\text{-decom}(f)$ as a linear transformation from Z_2 to Z_1. Consider F_1 being a linear transformation from Z_1 to Z_2 such that $F_1 = F^{-1}$ and F_1 is bijective. $\square$

(40) Let us consider a $\mathbb{Z}$-module V, a submodule W_1 of V, a submodule W_2 of W_1, a submodule U_1 of V, and a submodule U_2 of $\mathbb{Z}\text{-ModuleQuot}(V, U_1)$. Suppose $U_1 = W_2$ and $U_2 = \mathbb{Z}\text{-ModuleQuot}(W_1, W_2)$. Then there exists a linear transformation F from $\mathbb{Z}\text{-ModuleQuot}(\mathbb{Z}\text{-ModuleQuot}(V, U_1), U_2)$ to $\mathbb{Z}\text{-ModuleQuot}(V, W_1)$ such that F is bijective.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an element v of V such that $\$_1 = v + U_1$ and $\$_2 = v + W_1$. For every element z of $\mathbb{Z}\text{-ModuleQuot}(V, U_1)$, there exists an element y of $\mathbb{Z}\text{-ModuleQuot}(V, W_1)$ such that $\mathcal{P}[z, y]$ by [10, (113)]. Consider f being a function from $\mathbb{Z}\text{-ModuleQuot}(V, U_1)$ into $\mathbb{Z}\text{-ModuleQuot}(V, W_1)$ such that for every element z of $\mathbb{Z}\text{-ModuleQuot}(V, U_1)$, $\mathcal{P}[z, f(z)]$ from [10, Sch. 3]. f is a linear transformation from $\mathbb{Z}\text{-ModuleQuot}(V, U_1)$ to $\mathbb{Z}\text{-ModuleQuot}(V, W_1)$ by [14, (58), (24), (68)]. $\ker f = U_2$ by [26, (20)], [14, (63), (24), (28)]. $\operatorname{im} f = \mathbb{Z}\text{-ModuleQuot}(V, W_1)$ by [14, (58), (24), (68)], [10, (38), (41)]. $\square$

Let V be a $\mathbb{Z}$-module and a be a non zero element of $\mathbb{Z}^{\mathrm{R}}$. Let us observe that $\mathbb{Z}\text{-ModuleQuot}(V, a \circ V)$ is torsion.

Now we state the propositions:

(41) Let us consider a trivial $\mathbb{Z}$-module V. Then $\Omega_V = \mathbf{0}_V$.

(42) Let us consider a $\mathbb{Z}$-module V, and a vector v of V. If $v \neq 0_V$, then $\operatorname{Lin}(\{v\})$ is not trivial. The theorem is a consequence of (41).

(43) There exists a $\mathbb{Z}$-module V and there exists an element p of $\mathbb{Z}^{\mathrm{R}}$ such that $p \neq 0_{\mathbb{Z}^{\mathrm{R}}}$ and $\mathbb{Z}\text{-ModuleQuot}(V, p \circ V)$ is not trivial.

PROOF: Reconsider $V = \langle$the carrier of $\mathbb{Z}^{\mathrm{R}}$, the addition of $\mathbb{Z}^{\mathrm{R}}$, the zero of $\mathbb{Z}^{\mathrm{R}}$, the left integer multiplication of $(\mathbb{Z}^{\mathrm{R}})\rangle$ as a $\mathbb{Z}$-module. Reconsider $p = 2$ as an element of $\mathbb{Z}^{\mathrm{R}}$. $\mathbb{Z}\text{-ModuleQuot}(V, p \circ V)$ is not trivial by [14, (63)], [19, (14)]. $\square$

Note that there exists a torsion $\mathbb{Z}$-module which is non trivial and there exists a $\mathbb{Z}$-module which is non torsion-free.

Let V be a non torsion-free $\mathbb{Z}$-module. Let us note that there exists a vector of V which is non zero and torsion and there exists a finitely generated $\mathbb{Z}$-module which is non trivial.

Now we state the proposition:

(44) Let us consider a $\mathbb{Z}$-module V. Then V is torsion-free if and only if Ω_V is torsion-free.

Observe that every non torsion-free $\mathbb{Z}$-module is non trivial and there exists a finitely generated, torsion-free $\mathbb{Z}$-module which is non trivial.

Let V be a non trivial, finitely generated, torsion-free $\mathbb{Z}$-module and p be a prime element of $\mathbb{Z}^{\mathrm{R}}$. Let us note that $\mathbb{Z}$-ModuleQuot$(V, p \circ V)$ is non trivial and there exists a torsion $\mathbb{Z}$-module which is finitely generated and there exists a finitely generated, torsion $\mathbb{Z}$-module which is non trivial.

Let V be a non trivial, finitely generated, torsion-free $\mathbb{Z}$-module and p be a prime element of $\mathbb{Z}^{\mathrm{R}}$. Note that $\mathbb{Z}$-ModuleQuot$(V, p \circ V)$ is finitely generated and torsion.

Let V be a non torsion $\mathbb{Z}$-module.

One can verify that $\mathbb{Z}$-ModuleQuot$(V, \text{torsion-part}(V))$ is non trivial.

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