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On Multiset Ordering

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Summary. Formalization of a part of [11]. Unfortunately, not all is possible to be formalized. Namely, in the paper there is a mistake in the proof of Lemma 3. It states that there exists $x \in M_1$ such that $M_1(x) > N_1(x)$ and $(\forall y \in N_1) x \not< y$. It should be $M_1(x) \geqslant N_1(x)$. Nevertheless we do not know whether $x \in N_1$ or not and cannot prove the contradiction. In the article we referred to [8], [9] and [10].

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1. Preliminaries

Now we state the propositions:

(1) Let us consider natural numbers m, n. Then $n = m -' (m -' n) + (n -' m)$.

(2) Let us consider natural numbers n, m. Then $m -' n \geqslant m - n$.

Let us consider natural numbers m, n, x, y. Now we state the propositions:

(3) If $n = m -' x + y$, then $m -' n \leqslant x$ and $n -' m \leqslant y$. The theorem is a consequence of (2).

(4) If $x \leqslant m$ and $n = m -' x + y$, then $x -' (m -' n) = y -' (n -' m)$. The theorem is a consequence of (3).

Now we state the propositions:

(5) Let us consider natural numbers k, x_1, x_2, y_1, y_2. Suppose $x_2 \leqslant k$ and $x_1 \leqslant k -' x_2 + y_2$. Then

(i) $x_2 + (x_1 -' y_2) \leqslant k$, and

(ii) $k -' x_2 + y_2 -' x_1 + y_1 = k -' (x_2 + (x_1 -' y_2)) + (y_2 -' x_1 + y_1)$.

PROOF: $x_2 + (x_1 -' y_2) \leqslant k$ by [12, (8)]. □

(6) Let us consider natural numbers x, y. If $x + y > 0$, then $x > 0$ or $y > 0$.

From now on a, b denote objects and I, J denote sets.

Let us consider I. Let J be a non empty set. Let us note that every function from I into J is total and there exists a relational structure which is asymmetric, transitive, and non empty.

Let us consider I. One can verify that there exists a binary relation on I which is asymmetric and transitive.

Let R be a transitive relational structure. Observe that the internal relation of R is transitive.

Let R be an asymmetric relational structure. Let us observe that the internal relation of R is asymmetric.

Let us consider I. Let p, q be I-valued finite sequences. Let us observe that $p \frown q$ is I-valued.

Now we state the proposition:

(7) Let us consider finite sequences p, q. Suppose $p \frown q$ is I-valued. Then

(i) p is I-valued, and

(ii) q is I-valued.

Let us consider I. Let f be an I-valued finite sequence and n be a natural number. Let us note that $f{\restriction}n$ is I-valued.

Now we state the propositions:

(8) Let us consider a finite sequence p. Suppose $a \in \operatorname{rng} p$. Then there exist finite sequences q, r such that $p = (q \frown \langle a \rangle) \frown r$.

(9) Let us consider finite sequences p, q. Then $p \subset q$ if and only if $\operatorname{len} p < \operatorname{len} q$ and for every natural number i such that $i \in \operatorname{dom} p$ holds $p(i) = q(i)$.

(10) Let us consider finite sequences p, q, r. Then $r \frown p \subset r \frown q$ if and only if $p \subset q$.

PROOF: If $r \frown p \subset r \frown q$, then $p \subset q$ by [4, (22)], (9), [15, (30)], [4, (28)]. □

Let R be an asymmetric, non empty relational structure and x, y be elements of R. Let us observe that the predicate $x \leqslant y$ is asymmetric.

Now we state the proposition:

(11) Let us consider an asymmetric, non empty relational structure R, and elements x, y of R. Then $x \leqslant y$ if and only if $x < y$.

2. Relational Extension

Let us consider I.

A multiset of I is an element of $I^{\otimes}$. Observe that every multiset of I is I-defined and natural-valued and every multiset of I is total.

Let m be a natural-valued function. Let us note that the functor support m is defined by the term

(Def. 1) $m^{-1}(\mathbb{N} \setminus \{0\})$.

Let us consider I. One can check that every multiset of I is finite-support.

Now we state the propositions:

(12) a is a multiset of I if and only if a is a bag of I.

(13) $1_{I^{\otimes}} = \mathrm{EmptyBag}\, I$.

Let R be a relational structure and x, y be elements of R. We say that $x \equiv y$ if and only if

(Def. 2) $x \nleqslant y$ and $y \nleqslant x$.

Observe that the predicate is symmetric.

We consider relational multiplicative magmas which extend multiplicative magmas and relational structures and are systems

$$\langle \text{a carrier}, \text{a multiplication}, \text{an internal relation} \rangle$$

where the carrier is a set, the multiplication is a binary operation on the carrier, the internal relation is a binary relation on the carrier.

We consider relational monoids which extend multiplicative loop structures and relational structures and are systems

$$\langle \text{a carrier}, \text{a multiplication}, \text{a one}, \text{an internal relation} \rangle$$

where the carrier is a set, the multiplication is a binary operation on the carrier, the one is an element of the carrier, the internal relation is a binary relation on the carrier.

Let M be a multiplicative loop structure.

A relational extension of M is a relational monoid and is defined by

(Def. 3) the multiplicative loop structure of it = the multiplicative loop structure of M.

Let M be a non empty multiplicative loop structure. Let us observe that every relational extension of M is non empty.

Let M be a multiplicative loop structure. One can check that there exists a relational extension of M which is strict.

Let us consider a multiplicative loop structure N and a relational extension M of N. Now we state the propositions:

(14) a is an element of M if and only if a is an element of N.

(15) $1_N = 1_M$.

Let us consider I. Let M be a relational extension of $I^{\otimes}$. Let us observe that every element of M is function-like and relation-like and every element of M is I-defined, natural-valued, and finite-support and every element of M is total.

Now we state the proposition:

(16) Let us consider a relational extension M of $I^{\otimes}$. Then the carrier of $M = \operatorname{Bags} I$. The theorem is a consequence of (12) and (14).

The scheme *RelEx* deals with a non empty multiplicative loop structure $\mathcal{M}$ and a binary predicate $\mathcal{R}$ and states that

(Sch. 1) There exists a strict relational extension N of $\mathcal{M}$ such that for every elements x, y of N, $x \leqslant y$ iff $\mathcal{R}[x, y]$.

Now we state the proposition:

(17) Let us consider a multiplicative loop structure N, and strict relational extensions M_1, M_2 of N. Suppose for every elements m, n of M_1 for every elements x, y of M_2 such that $m = x$ and $n = y$ holds $m \leqslant n$ iff $x \leqslant y$. Then $M_1 = M_2$.

Proof: The internal relation of M_1 = the internal relation of M_2 by [7, (87)]. □

3. Dershowitz-Manna Order

Let R be a non empty relational structure. The Dershowitz-Manna order R yielding a strict relational extension of (the carrier of $R)^{\otimes}$ is defined by

(Def. 4) for every elements m, n of *it*, $m \leqslant n$ iff there exist elements x, y of *it* such that $1_{it} \neq x \mid n$ and $m = n -' x + y$ and for every element b of R such that $y(b) > 0$ there exists an element a of R such that $x(a) > 0$ and $b \leqslant a$.

Now we state the proposition:

(18) Let us consider bags m, n of I. Then $n = m -' (m -' n) + (n -' m)$. The theorem is a consequence of (1).

Let us consider bags m, n, x, y of I. Now we state the propositions:

(19) If $n = m -' x + y$, then $m -' n \mid x$ and $n -' m \mid y$. The theorem is a consequence of (3).

(20) If $x \mid m$ and $n = m -' x + y$, then $x -' (m -' n) = y -' (n -' m)$. The theorem is a consequence of (4).

Now we state the propositions:

(21) Let us consider bags m, x, y of I. If $x \mid m$ and $x \neq y$, then $m \neq m -' x + y$.

(22) Let us consider a non empty set I, a binary relation R on I, and a reduction sequence r w.r.t. R. If $\operatorname{len} r > 1$, then $r(\operatorname{len} r) \in I$.

(23) Let us consider an asymmetric, transitive binary relation R on I. Then every reduction sequence w.r.t. R is one-to-one.

PROOF: For every natural numbers i, j such that $i > j$ and $i, j \in \operatorname{dom} r$ holds $r(i) \neq r(j)$ by [1, (13)], [13, (22)], [1, (11)], [15, (25)]. □

(24) Let us consider an asymmetric, transitive, non empty relational structure R, and a set X. Suppose X is finite and there exists an element x of R such that $x \in X$. Then there exists an element x of R such that x is maximal in X.

PROOF: Reconsider $X_1 = X$ as a finite set. Set $Y = \{r$, where r is an element of ${X_1}^*$: r is a reduction sequence w.r.t. the internal relation of $R\}$. Define $\mathcal{P}$[natural number] $\equiv$ there exists a reduction sequence r w.r.t. the internal relation of R such that $r \in Y$ and $\operatorname{len} r = \$_1$. For every natural number k such that $\mathcal{P}[k]$ holds $k \leqslant \overline{\overline{X_1}}$ by (23), [1, (43)]. $\mathcal{P}[1]$ by [2, (6)], [4, (74), (39)]. Consider k being a natural number such that $\mathcal{P}[k]$ and for every natural number n such that $\mathcal{P}[n]$ holds $n \leqslant k$ from [1, Sch. 6]. Consider r being a reduction sequence w.r.t. the internal relation of R such that $r \in Y$ and $\operatorname{len} r = k$. Consider q being an element of ${X_1}^*$ such that $r = q$ and q is a reduction sequence w.r.t. the internal relation of R. □

(25) Let us consider bags m, n of I. Then $m -' n \mid m$.

Let us consider I. Note that every element of Bags I is function-like and relation-like.

Now we state the proposition:

(26) Let us consider bags m, n of I. Then

(i) $m -' n \neq \operatorname{EmptyBag} I$, or

(ii) $m = n$, or

(iii) $n -' m \neq \operatorname{EmptyBag} I$.

Let R be an asymmetric, transitive, non empty relational structure. Let us observe that the Dershowitz-Manna order R is defined by

(Def. 5) for every elements m, n of *it*, $m \leqslant n$ iff $m \neq n$ and for every element a of R such that $m(a) > n(a)$ there exists an element b of R such that $a \leqslant b$ and $m(b) < n(b)$.

Now we state the proposition:

(27) Let us consider bags k, x_1, x_2, y_1, y_2 of I. Suppose $x_2 \mid k$ and $x_1 \mid k -' x_2 + y_2$. Then

(i) $x_2 + (x_1 -' y_2) \mid k$, and

(ii) $k -' x_2 + y_2 -' x_1 + y_1 = k -' (x_2 + (x_1 -' y_2)) + (y_2 -' x_1 + y_1)$.

The theorem is a consequence of (5).

Let R be an asymmetric, transitive, non empty relational structure. Let us observe that the Dershowitz-Manna order R is asymmetric and transitive.

Let us consider I. The functor DivOrder(I) yielding a binary relation on Bags I is defined by

(Def. 6) for every bags b_1, b_2 of I, $\langle b_1, b_2 \rangle \in it$ iff $b_1 \neq b_2$ and $b_1 \mid b_2$.

Now we state the proposition:

(28) Let us consider bags a, b, c of I. If $a \mid b \mid c$, then $a \mid c$.

Let us consider I. Note that DivOrder(I) is asymmetric and transitive.

Let us consider an asymmetric, transitive, non empty relational structure R. Now we state the propositions:

(29) DivOrder(the carrier of R) $\subseteq$ the internal relation of the Dershowitz-Manna order R. The theorem is a consequence of (12) and (14).

(30) Suppose the internal relation of R is empty. Then the internal relation of the Dershowitz-Manna order $R =$ DivOrder(the carrier of R). The theorem is a consequence of (29).

Now we state the proposition:

(31) Let us consider asymmetric, transitive, non empty relational structures R_1, R_2. Suppose the carrier of R_1 = the carrier of R_2 and the internal relation of $R_1 \subseteq$ the internal relation of R_2. Then the internal relation of the Dershowitz-Manna order $R_1 \subseteq$ the internal relation of the Dershowitz-Manna order R_2. The theorem is a consequence of (12) and (14).

4. Monoidal Order

Let us consider I. Let f be a (Bags I)-valued finite sequence. The functor $\sum f$ yielding a bag of I is defined by

(Def. 7) there exists a function F from $\mathbb{N}$ into Bags I such that $it = F(\operatorname{len} f)$ and $F(0) =$ EmptyBag I and for every natural number i and for every bag b of I such that $i < \operatorname{len} f$ and $b = f(i+1)$ holds $F(i+1) = F(i) + b$.

Now we state the proposition:

(32) $\sum \varepsilon_{\mathrm{Bags}\, I} =$ EmptyBag I.

Let us consider I. Let b be a bag of I. One can verify that $\langle b \rangle$ is (Bags I)-valued as a finite sequence.

Now we state the proposition:

(33) Let us consider a (Bags I)-valued finite sequence p, and a bag b of I. Then $\sum(p \frown \langle b \rangle) = \sum p + b$.

Proof: Set $f = p \frown \langle b \rangle$. Consider F being a function from $\mathbb{N}$ into Bags I such that $\sum f = F(\operatorname{len} f)$ and $F(0) = \operatorname{EmptyBag} I$ and for every natural number i and for every bag b of I such that $i < \operatorname{len} f$ and $b = f(i+1)$ holds $F(i+1) = F(i)+b$. Consider F_1 being a function from $\mathbb{N}$ into Bags I such that $\sum p = F_1(\operatorname{len} p)$ and $F_1(0) = \operatorname{EmptyBag} I$ and for every natural number i and for every bag b of I such that $i < \operatorname{len} p$ and $b = p(i+1)$ holds $F_1(i+1) = F_1(i)+b$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant \operatorname{len} p$, then $F(\$_1) = F_1(\$_1)$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [5, (16)], [1, (13), (11)], [15, (25)]. For every natural number i, $\mathcal{P}[i]$ from [1, Sch. 2]. □

From now on b denotes a bag of I.

Now we state the propositions:

(34) $\sum \langle b \rangle = b$. The theorem is a consequence of (33) and (32).

(35) Let us consider (Bags I)-valued finite sequences p, q. Then $\sum(p \frown q) = \sum p + \sum q$.

Proof: Set $f = p \frown q$. Consider F being a function from $\mathbb{N}$ into Bags I such that $\sum f = F(\operatorname{len} f)$ and $F(0) = \operatorname{EmptyBag} I$ and for every natural number i and for every bag b of I such that $i < \operatorname{len} f$ and $b = f(i+1)$ holds $F(i+1) = F(i)+b$. Consider F_1 being a function from $\mathbb{N}$ into Bags I such that $\sum p = F_1(\operatorname{len} p)$ and $F_1(0) = \operatorname{EmptyBag} I$ and for every natural number i and for every bag b of I such that $i < \operatorname{len} p$ and $b = p(i+1)$ holds $F_1(i+1) = F_1(i) + b$. Consider F_2 being a function from $\mathbb{N}$ into Bags I such that $\sum q = F_2(\operatorname{len} q)$ and $F_2(0) = \operatorname{EmptyBag} I$ and for every natural number i and for every bag b of I such that $i < \operatorname{len} q$ and $b = q(i+1)$ holds $F_2(i+1) = F_2(i) + b$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant \operatorname{len} p$, then $F(\$_1) = F_1(\$_1)$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [4, (22)], [1, (11), (13)], [15, (25)]. For every natural number i, $\mathcal{P}[i]$ from [1, Sch. 2]. Define $\mathcal{Q}[\text{natural number}] \equiv$ if $\$_1 \leqslant \operatorname{len} q$, then $F(\operatorname{len} p + \$_1) = \sum p + F_2(\$_1)$. For every natural number i such that $\mathcal{Q}[i]$ holds $\mathcal{Q}[i+1]$ by [4, (22)], [1, (13), (11)], [15, (25)]. For every natural number i, $\mathcal{Q}[i]$ from [1, Sch. 2]. □

Let us consider a (Bags I)-valued finite sequence p. Now we state the propositions:

(36) $\sum(\langle b \rangle \frown p) = b + \sum p$. The theorem is a consequence of (35) and (34).

(37) If $b \in \operatorname{rng} p$, then $b \mid \sum p$. The theorem is a consequence of (8), (7), (33), and (35).

Now we state the proposition:

(38) Let us consider a (Bags I)-valued finite sequence p, and an object i. Suppose $i \in \operatorname{support} \sum p$. Then there exists b such that

(i) $b \in \operatorname{rng} p$, and

(ii) $i \in \operatorname{support} b$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every (Bags I)-valued finite sequence p such that $\operatorname{len} p = \$_1$ for every object i such that $i \in \operatorname{support} \sum p$ there exists b such that $b \in \operatorname{rng} p$ and $i \in \operatorname{support} b$. $\mathcal{P}[0]$. For every natural number j such that $\mathcal{P}[j]$ holds $\mathcal{P}[j+1]$ by [3, (3)], (7), [4, (40)], [15, (25)]. For every natural number j, $\mathcal{P}[j]$ from [1, Sch. 2]. □

Let us consider I and b.

A partition of b is a (Bags I)-valued finite sequence and is defined by

(Def. 8) $b = \sum it$.

Observe that the functor $\langle b \rangle$ yields a partition of b. Let R be a relational structure, M be a relational extension of (the carrier of R)$^{\otimes}$, b be an element of M, and p be a partition of b. We say that p is co-ordered if and only if

(Def. 9) for every natural number i such that i, $i+1 \in \operatorname{dom} p$ for every elements b_1, b_2 of M such that $b_1 = p(i)$ and $b_2 = p(i+1)$ holds $b_2 \leqslant b_1$.

Let R be a non empty relational structure and b be a bag of the carrier of R. We say that p is ordered if and only if

(Def. 10) for every bag m of the carrier of R such that $m \in \operatorname{rng} p$ for every element x of R such that $m(x) > 0$ holds $m(x) = b(x)$ and for every bag m of the carrier of R such that $m \in \operatorname{rng} p$ for every elements x, y of R such that $m(x) > 0$ and $m(y) > 0$ and $x \neq y$ holds $x \equiv y$ and for every bag m of the carrier of R such that $m \in \operatorname{rng} p$ holds $m \neq$ EmptyBag(the carrier of R) and for every natural number i such that i, $i+1 \in \operatorname{dom} p$ for every element x of R such that $p_{i+1}(x) > 0$ there exists an element y of R such that $p_i(y) > 0$ and $x \leqslant y$.

In the sequel R denotes an asymmetric, transitive, non empty relational structure, a, b, c denote bags of the carrier of R, and x, y, z denote elements of R.

Now we state the propositions:

(39) $\langle a \rangle$ is ordered if and only if $a \neq$ EmptyBag(the carrier of R) and for every x and y such that $a(x) > 0$ and $a(y) > 0$ and $x \neq y$ holds $x \equiv y$.

(40) Let us consider a (Bags I)-valued finite sequence p, and bags a, b of I. Then $\langle a \rangle \frown p$ is a partition of b if and only if $a \mid b$ and p is a partition of $b -' a$. The theorem is a consequence of (36).

From now on p denotes a partition of $b -' a$ and q denotes a partition of b.

Now we state the proposition:

(41) If $q = \langle a \rangle \frown p$ and q is ordered, then p is ordered. The theorem is a consequence of (37) and (25).

Let us consider I. Let m be a bag of I and J be a set. The functor $m\restriction J$ yielding a bag of I is defined by

(Def. 11) for every object i such that $i \in I$ holds if $i \in J$, then $it(i) = m(i)$ and if $i \notin J$, then $it(i) = 0$.

From now on J denotes a set and m denotes a bag of I.

Now we state the propositions:

(42) $\mathrm{support}(m\restriction J) = J \cap \mathrm{support}\, m$.

(43) $m\restriction J + m\restriction(I \setminus J) = m$.

(44) $m\restriction J \mid m$.

(45) If $\mathrm{support}\, m \subseteq J$, then $m\restriction J = m$.

(46) $\mathrm{support}(m -' m\restriction J) = \mathrm{support}\, m \setminus J$.

(47) If q is ordered and $q = \langle a \rangle \frown p$ and $a(x) > 0$, then $a(x) = b(x)$.

(48) If q is ordered and $q = \langle a \rangle \frown p$ and $a(x) > 0$ and $a(y) > 0$ and $x \neq y$, then $x \equiv y$.

(49) If q is ordered and $q = \langle a \rangle \frown p$, then $a \neq \mathrm{EmptyBag}(\text{the carrier of } R)$.

(50) Let us consider a bag c of the carrier of R, and a (Bags(the carrier of R))-valued finite sequence r. Suppose q is ordered and $q = \langle a, c \rangle \frown r$ and $c(y) > 0$. Then there exists x such that

(i) $a(x) > 0$, and

(ii) $y \leqslant x$.

(51) If $x \in I$ and for every y such that $y \in I$ and $y \neq x$ holds $x \equiv y$, then x is maximal in I.

(52) If q is ordered and $q = \langle a \rangle \frown p$ and $c \in \mathrm{rng}\, p$ and $c(x) > 0$, then there exists y such that $a(y) > 0$ and $x \leqslant y$.

Proof: Consider i being an object such that $i \in \mathrm{dom}\, p$ and $c = p(i)$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \in \mathrm{dom}\, p$, then for every x such that $p_{\$_1}(x) > 0$ there exists y such that $a(y) > 0$ and $x \leqslant y$. $\mathcal{P}[1]$ by [4, (28)], [15, (25)], [4, (40)]. For every natural number i such that $i \geqslant 1$ and $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [1, (13)], [15, (25)], [4, (28)], [16, (3)]. For every natural number i such that $i \geqslant 1$ holds $\mathcal{P}[i]$ from [1, Sch. 8]. □

Let us assume that q is ordered and $q = \langle a \rangle \frown p$. Now we state the propositions:

(53) x is maximal in $\mathrm{support}\, b$ if and only if $a(x) > 0$.

Proof: $a \mid \sum q = b$. There exists no y such that $y \in \mathrm{support}\, b$ and $x < y$ by (48), (38), [4, (31), (39)]. □

(54) $a = b \restriction \{x : x$ is maximal in support $b\}$. The theorem is a consequence of (53) and (47).

Now we state the propositions:

(55) Let us consider a (Bags I)-valued finite sequence p. Suppose $\sum p =$ EmptyBag I and for every bag a of I such that $a \in \operatorname{rng} p$ holds $a \neq$ EmptyBag I. Then $p = \emptyset$. The theorem is a consequence of (37).

(56) Let us consider bags a, b of I. If $a \neq$ EmptyBag I, then $a + b \neq$ EmptyBag I.

(57) Let us consider partitions p, q of b. If p is ordered and q is ordered, then $p = q$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every b and q such that $\operatorname{len} q = \$_1$ and q is ordered for every partition p of b such that p is ordered holds $q = p$. $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [5, (130)], (40), (49), (36). For every natural number i, $\mathcal{P}[i]$ from [1, Sch. 2]. $\square$

Let us consider I. Let a, b be bags of I. One can verify that the functor $\langle a, b\rangle$ yields an element of Bags $I \times$ Bags I. Now we state the proposition:

(58) Suppose $a \neq$ EmptyBag(the carrier of R). Then $\{x : x$ is maximal in support $a\} \neq \emptyset$. The theorem is a consequence of (24).

Let us consider R and b. The ordered partition of b yielding a (Bags(the carrier of R))-valued finite sequence is defined by

(Def. 12) there exist functions F, G from $\mathbb{N}$ into Bags(the carrier of R) such that $F(0) = b$ and $G(0) =$ EmptyBag(the carrier of R) and for every natural number i, $G(i+1) = F(i) \restriction \{x : x$ is maximal in support$(F(i))\}$ and $F(i+1) = F(i) -' G(i+1)$ and there exists a natural number i such that $F(i) =$ EmptyBag(the carrier of R) and *it* $= G \restriction \operatorname{Seg} i$ and for every natural number j such that $j < i$ holds $F(j) \neq$ EmptyBag(the carrier of R).

One can verify that the ordered partition of b yields a partition of b. Let us note that the ordered partition of b is ordered as a partition of b.

Now we state the proposition:

(59) $b =$ EmptyBag(the carrier of R) if and only if the ordered partition of $b = \emptyset$. The theorem is a consequence of (32).

Let us consider R. The functor $\prec_{\mathcal{M}} R$ yielding a strict relational extension of (the carrier of $R)^{\otimes}$ is defined by

(Def. 13) for every elements m, n of *it*, $m \leqslant n$ iff $m \neq n$ and for every x such that $m(x) > 0$ holds $m(x) < n(x)$ or there exists y such that $n(y) > 0$ and $x \leqslant y$.

Let us note that $\prec_{\mathcal{M}} R$ is asymmetric and transitive.

Let us consider I. Let R be a relation between I and I.

The functor LexOrder(I, R) yielding a binary relation on I^* is defined by

(Def. 14) for every I-valued finite sequences p, q, $\langle p, q\rangle \in it$ iff $p \subset q$ or there exists a natural number k such that $k \in \operatorname{dom} p$ and $k \in \operatorname{dom} q$ and $\langle p(k), q(k)\rangle \in R$ and for every natural number n such that $1 \leqslant n < k$ holds $p(n) = q(n)$.

Let R be a transitive binary relation on I. One can verify that LexOrder(I, R) is transitive.

Let R be an asymmetric binary relation on I. Note that LexOrder(I, R) is asymmetric.

Now we state the proposition:

(60) Let us consider an asymmetric binary relation R on I, and I-valued finite sequences p, q, r. Then $\langle p, q\rangle \in$ LexOrder(I, R) if and only if $\langle r \frown p, r \frown q\rangle \in$ LexOrder(I, R). The theorem is a consequence of (10).

Let us consider R. The functor $\prec\prec_{\mathcal{M}} R$ yielding a strict relational extension of (the carrier of $R)^{\otimes}$ is defined by

(Def. 15) for every elements m, n of it, $m \leqslant n$ iff $\langle$the ordered partition of m, the ordered partition of $n\rangle \in$ LexOrder((the carrier of $\prec_{\mathcal{M}} R$), (the internal relation of $\prec_{\mathcal{M}} R$)).

Observe that $\prec\prec_{\mathcal{M}} R$ is asymmetric and transitive.

Now we state the propositions:

(61) Let us consider elements a, b of the Dershowitz-Manna order R. Suppose $a \leqslant b$. Then $b \neq$ EmptyBag(the carrier of R). The theorem is a consequence of (29).

(62) Let us consider elements a, b, c, d of the Dershowitz-Manna order R, and a bag e of the carrier of R. Suppose $a \leqslant b$ and $e \mid a$ and $e \mid b$. If $c = a -' e$ and $d = b -' e$, then $c \leqslant d$.

(63) Let us consider a (Bags I)-valued finite sequence p, and an object x. Suppose $x \in I$ and $(\sum p)(x) > 0$. Then there exists a natural number i such that

(i) $i \in \operatorname{dom} p$, and

(ii) $p_i(x) > 0$.

Proof: Define $\mathcal{P}[\text{object}] \equiv$ for every (Bags I)-valued finite sequence p such that $p = \$_1$ and $(\sum p)(x) > 0$ there exists a natural number i such that $i \in \operatorname{dom} p$ and $p_i(x) > 0$. $\mathcal{P}[\emptyset]$ by (32), [14, (7)]. For every finite sequence p and for every object a such that $\mathcal{P}[p]$ holds $\mathcal{P}[p \frown \langle a\rangle]$ by (7), [4, (40)], [15, (25)], [6, (102)]. For every finite sequence p, $\mathcal{P}[p]$ from [4, Sch. 3]. □

(64) If q is ordered and $q_1(x) = 0$ and $b(x) > 0$, then there exists y such that $q_1(y) > 0$ and $x \leqslant y$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \in \operatorname{dom} q$, then for every x such that $q_{\$_1}(x) > 0$ there exists y such that $q_1(y) > 0$ and $x \leqslant y$. $\mathcal{P}[2]$ by [15, (25)]. For every natural number i such that $2 \leqslant i$ and $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [1, (11)], [15, (25)], [16, (3)]. For every natural number i such that $i \geqslant 2$ holds $\mathcal{P}[i]$ from [1, Sch. 8]. Consider i being a natural number such that $i \in \operatorname{dom} q$ and $q_i(x) > 0$. $\square$

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Cousin's Lemma

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Summary. We formalize, in two different ways, that "the n-dimensional Euclidean metric space is a complete metric space" (version 1. with the results obtained in [13], [26], [25] and version 2., the results obtained in [13], [14], (*registrations*) [24]).

With the Cantor's theorem - in complete metric space (proof by Karol Pąk in [22]), we formalize "The Nested Intervals Theorem in 1-dimensional Euclidean metric space".

Pierre Cousin's proof in 1892 [18] the lemma, published in 1895 [9] states that:

> "Soit, sur le plan YOX, une aire connexe S limitée par un contour fermé simple ou complexe; on suppose qu'à chaque point de S ou de son périmètre correspond un cercle, de rayon non nul, ayant ce point pour centre : il est alors toujours possible de subdiviser S en régions, en nombre fini et assez petites pour que chacune d'elles soit complétement intérieure au cercle correspondant à un point convenablement choisi dans S ou sur son périmètre."

(In the plane YOX let S be a connected area bounded by a closed contour, simple or complex; one supposes that at each point of S or its perimeter there is a circle, of non-zero radius, having this point as its centre; it is then always possible to subdivide S into regions, finite in number and sufficiently small for each one of them to be entirely inside a circle corresponding to a suitably chosen point in S or on its perimeter) [23].

Cousin's Lemma, used in Henstock and Kurzweil integral [29] (generalized Riemann integral), state that: "for any gauge δ, there exists at least one δ-fine tagged partition". In the last section, we formalize this theorem. We use the suggestions given to the Cousin's Theorem p.11 in [5] and with notations: [4], [29], [19], [28] and [12].

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1. Preliminaries

Now we state the proposition:

(1) Let us consider non empty, increasing finite sequences p, q of elements of $\mathbb{R}$. Suppose $p(\operatorname{len} p) < q(1)$. Then $p \frown q$ is a non empty, increasing finite sequence of elements of $\mathbb{R}$.

Let us consider real numbers a, b. Now we state the propositions:

(2) If $1 < a$ and $0 < b < 1$, then $\log_a b < 0$.

(3) If $1 < a$ and $1 < b$, then $0 < \log_a b$.

Let us consider a finite sequence p and a natural number i.

Let us assume that $i \in \operatorname{dom} p$. Now we state the propositions:

(4) (i) $i = 1$, or

(ii) $1 < i$.

(5) (i) $i = \operatorname{len} p$, or

(ii) $i < \operatorname{len} p$.

Now we state the propositions:

(6) Let us consider an object x. Then $\prod\langle\{x\}\rangle = \{\langle x\rangle\}$.

(7) Let us consider an element x of $\mathcal{R}^1$. Then there exists a real number r_3 such that $x = \langle r_3\rangle$.

(8) Let us consider a real number a. Then $\langle a\rangle$ is a point of $\mathcal{E}^1$.

(9) Let us consider real numbers a, b. If $a \leqslant b$, then $a \leqslant \frac{a+b}{2} \leqslant b$.

(10) Let us consider real numbers a, b, c. If $a \leqslant b < c$, then $a < \frac{b+c}{2}$.

Let us consider real numbers a, b. Now we state the propositions:

(11) If $a < b$, then $\frac{a+b}{2} < b$.

(12) If $a \leqslant b$, then $[a, b]$ is a non empty, compact subset of $\mathbb{R}$.

(13) Let us consider a finite sequence f. Suppose $2 \leqslant \operatorname{len} f$. Then $f_{\lfloor 1}(\operatorname{len} f_{\lfloor 1}) = f(\operatorname{len} f)$.

2. $\mathcal{E}^n$ is Complete - Proof Version 1

From now on n denotes a natural number, s_1 denotes a sequence of $\mathcal{E}^n$, and s_2 denotes a sequence of $\langle \mathcal{E}^n, \|\cdot\| \rangle$.

Now we state the propositions:

(14) Let us consider elements x, y of $\mathcal{E}^n$, and points g, h of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $x = g$ and $y = h$, then $\rho(x, y) = \|g - h\|$.

(15) (i) s_1 is a sequence of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and

(ii) s_2 is a sequence of $\mathcal{E}^n$.

PROOF: s_1 is a sequence of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ by [10, (67), (22)]. s_2 is a sequence of $\mathcal{E}^n$ by [10, (22), (67)]. □

Let us assume that $s_1 = s_2$. Now we state the propositions:

(16) s_1 is Cauchy if and only if s_2 is Cauchy sequence by norm. The theorem is a consequence of (14).

(17) s_1 is convergent if and only if s_2 is convergent. The theorem is a consequence of (14).

(18) Let us consider a sequence S_1 of $\mathcal{E}^n$. If S_1 is Cauchy, then S_1 is convergent. The theorem is a consequence of (15), (16), and (17).

(19) $\mathcal{E}^n$ is complete.

3. $\mathcal{E}^n$ is Complete - Proof Version 2

Now we state the propositions:

(20) The distance by norm of $\langle \mathcal{E}^n, \|\cdot\| \rangle = \rho^n$. The theorem is a consequence of (14).

(21) MetricSpaceNorm$\langle \mathcal{E}^n, \|\cdot\| \rangle = \mathcal{E}^n$. The theorem is a consequence of (20).

(22) $\mathcal{E}^n$ is complete. The theorem is a consequence of (21).

Let n be a natural number. Let us note that $\mathcal{E}^n$ is complete.

4. The Nested Intervals Theorem (1-dimensional Euclidean Space)

Let a, b be sequences of real numbers. The functor IntervalSeq(a, b) yielding a sequence of subsets of $\mathcal{R}^1$ is defined by

(Def. 1) for every natural number i, $it(i) = \prod\langle [a(i), b(i)] \rangle$.

Now we state the propositions:

(23) Let us consider sequences a, b of real numbers, and a natural number i. Then $(\text{IntervalSeq}(a, b))(i) = \prod\langle [a(i), b(i)] \rangle$.

(24) Let us consider sequences a, b of real numbers. Then IntervalSeq(a, b) is a sequence of subsets of $\mathcal{E}^1$.

(25) $\prod\langle\mathbb{R}\rangle = \mathcal{R}^1$.

(26) Let us consider real numbers a, b, and points x_1, x_2 of $\mathcal{E}^1$. Suppose $x_1 = \langle a\rangle$ and $x_2 = \langle b\rangle$. Then $\rho(x_1, x_2) = |a - b|$.

(27) Let us consider real numbers a, b, and a subset S of $\mathcal{E}^1$. Suppose $a \leqslant b$ and $S = \prod\langle[a, b]\rangle$. Let us consider points x, y of $\mathcal{E}^1$. If x, $y \in S$, then $\rho(x, y) \leqslant b - a$.

PROOF: Set $s = \prod\langle[a, b]\rangle$. For every points x, y of $\mathcal{E}^1$ such that x, $y \in s$ holds $\rho(x, y) \leqslant b - a$ by (6), [10, (67), (22)], (7). □

(28) Let us consider real numbers a, b, and a subset S of $\mathcal{E}^1$. If $a \leqslant b$ and $S = \prod\langle[a, b]\rangle$, then S is bounded.

PROOF: Set $s = \prod\langle[a, b]\rangle$. There exists a real number r such that $0 < r$ and for every points x, y of $\mathcal{E}^1$ such that x, $y \in s$ holds $\rho(x, y) \leqslant r$ by (6), [10, (67), (22)], (7). □

Let us consider sequences a, b of real numbers.

Let us assume that for every natural number i, $a(i) \leqslant b(i)$ and $a(i) \leqslant a(i+1)$ and $b(i+1) \leqslant b(i)$. Now we state the propositions:

(29) IntervalSeq(a, b) is a non-empty, pointwise bounded, closed sequence of subsets of $\mathcal{E}^1$.

PROOF: Reconsider s = IntervalSeq(a, b) as a sequence of subsets of $\mathcal{E}^1$. s is non-empty by (23), [1, (26)], [3, (2)]. s is pointwise bounded by (23), (6), [10, (67), (22)]. s is closed by (23), [10, (67), (22)], (25). □

(30) IntervalSeq(a, b) is non ascending. The theorem is a consequence of (23).

(31) Let us consider real numbers a, b, x. If $a \leqslant x \leqslant b$, then $\langle x\rangle \in \prod\langle[a, b]\rangle$.

PROOF: Reconsider $P = \langle x\rangle$ as a point of $\mathcal{E}^1$. There exists a function g such that $g = P$ and $\operatorname{dom} g = \operatorname{dom}\langle[a, b]\rangle$ and for every object y such that $y \in \operatorname{dom}\langle[a, b]\rangle$ holds $g(y) \in \langle[a, b]\rangle(y)$ by [3, (2)]. □

(32) Let us consider real numbers a, b, and a subset S of $\mathcal{E}^1$. If $a \leqslant b$ and $S = \prod\langle[a, b]\rangle$, then $\varnothing S = b - a$. The theorem is a consequence of (28), (31), (27), (8), and (26).

(33) Let us consider sequences a, b of real numbers. Suppose for every natural number i, $a(i) \leqslant b(i)$ and a is non-decreasing and b is non-increasing. Then

(i) a is convergent, and

(ii) b is convergent.

(34) Let us consider sequences a, b of real numbers. Suppose $a(0) \leqslant b(0)$ and for every natural number i, $a(i + 1) = a(i)$ and $b(i + 1) = \frac{a(i)+b(i)}{2}$ or

$a(i+1) = \frac{a(i)+b(i)}{2}$ and $b(i+1) = b(i)$. Let us consider a natural number i. Then $a(i) \leqslant b(i)$.

PROOF: Define $\mathcal{P}[\text{object}] \equiv$ there exists a natural number i such that $\$_1 = i$ and $a(i) \leqslant b(i)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$ from [2, Sch. 2]. □

Let us consider sequences a, b of real numbers, a sequence S of subsets of $\mathcal{E}^1$, and a natural number i. Now we state the propositions:

(35) Suppose $a(0) \leqslant b(0)$ and $S = \text{IntervalSeq}(a, b)$ and for every natural number i, $a(i+1) = a(i)$ and $b(i+1) = \frac{a(i)+b(i)}{2}$ or $a(i+1) = \frac{a(i)+b(i)}{2}$ and $b(i+1) = b(i)$. Then

(i) $a(i) \leqslant b(i)$, and

(ii) $a(i) \leqslant a(i+1)$, and

(iii) $b(i+1) \leqslant b(i)$, and

(iv) $(\varnothing S)(i) = b(i) - a(i)$.

The theorem is a consequence of (34), (9), (24), (23), and (32).

(36) Suppose $a(0) = b(0)$ and $S = \text{IntervalSeq}(a, b)$ and for every natural number i, $a(i+1) = a(i)$ and $b(i+1) = \frac{a(i)+b(i)}{2}$ or $a(i+1) = \frac{a(i)+b(i)}{2}$ and $b(i+1) = b(i)$. Then

(i) $a(i) = a(0)$, and

(ii) $b(i) = b(0)$, and

(iii) $(\varnothing S)(i) = 0$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv a(\$_1) = a(0)$ and $b(\$_1) = b(0)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$ from [2, Sch. 2]. □

(37) Let us consider sequences a, b of real numbers. Suppose for every natural number i, $a(i+1) = a(i)$ and $b(i+1) = \frac{a(i)+b(i)}{2}$ or $a(i+1) = \frac{a(i)+b(i)}{2}$ and $b(i+1) = b(i)$. Let us consider a natural number i, and a real number r. If $r = 2^i$ and $r \neq 0$, then $b(i) - a(i) \leqslant \frac{b(0)-a(0)}{r}$.

PROOF: Define $\mathcal{P}[\text{object}] \equiv$ there exists a natural number i and there exists a real number r such that $\$_1 = i$ and $r = 2^i$ and $r \neq 0$ and $b(i) - a(i) \leqslant \frac{b(0)-a(0)}{r}$. $\mathcal{P}[0]$ by [17, (4)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [17, (87), (6)]. For every natural number k, $\mathcal{P}[k]$ from [2, Sch. 2]. Consider i_1 being a natural number, r_1 being a real number such that $i = i_1$ and $r_1 = 2^{i_1}$ and $r_1 \neq 0$ and $b(i_1) - a(i_1) \leqslant \frac{b(0)-a(0)}{r_1}$. □

(38) Let us consider sequences a, b of real numbers, and a sequence S of subsets of $\mathcal{E}^1$. Suppose $a(0) \leqslant b(0)$ and $S = \text{IntervalSeq}(a, b)$ and for every

natural number i, $a(i+1) = a(i)$ and $b(i+1) = \frac{a(i)+b(i)}{2}$ or $a(i+1) = \frac{a(i)+b(i)}{2}$ and $b(i+1) = b(i)$. Then

(i) $\varnothing S$ is convergent, and

(ii) $\lim \varnothing S = 0$.

The theorem is a consequence of (36), (35), (34), (33), (3), and (37).

(39) Let us consider sequences a, b of real numbers. Suppose $a(0) \leqslant b(0)$ and for every natural number i, $a(i+1) = a(i)$ and $b(i+1) = \frac{a(i)+b(i)}{2}$ or $a(i+1) = \frac{a(i)+b(i)}{2}$ and $b(i+1) = b(i)$. Then $\bigcap \mathrm{IntervalSeq}(a, b)$ is not empty. The theorem is a consequence of (24), (35), (29), (30), and (38).

(40) Let us consider a real number r, and sequences a, b of real numbers. Suppose $0 < r$ and $a(0) \leqslant b(0)$ and for every natural number i, $a(i+1) = a(i)$ and $b(i+1) = \frac{a(i)+b(i)}{2}$ or $a(i+1) = \frac{a(i)+b(i)}{2}$ and $b(i+1) = b(i)$. Then there exists a real number c such that

(i) for every natural number j, $a(j) \leqslant c \leqslant b(j)$, and

(ii) there exists a natural number k such that $c-r < a(k)$ and $b(k) < c+r$.

The theorem is a consequence of (39), (23), (24), (35), (29), and (38).

5. Tagged Partition

Now we state the propositions:

(41) Let us consider a non empty, closed interval subset I of $\mathbb{R}$. Then there exist real numbers a, b such that

(i) $a \leqslant b$, and

(ii) $I = [a, b]$.

(42) Let us consider non empty, closed interval subsets I_1, I_2 of $\mathbb{R}$. Suppose $\sup I_1 = \inf I_2$. Then there exist real numbers a, b, c such that

(i) $a \leqslant c \leqslant b$, and

(ii) $I_1 = [a, c]$, and

(iii) $I_2 = [c, b]$.

The theorem is a consequence of (41).

Let A be a non empty, closed interval subset of $\mathbb{R}$ and D be a partition of A. The set of tagged partitions of D yielding a subset of $\mathbb{R}^*$ is defined by

(Def. 2) for every object x, $x \in it$ iff there exists a non empty, non-decreasing finite sequence s of elements of $\mathbb{R}$ such that $x = s$ and $\operatorname{dom} s = \operatorname{dom} D$ and for every natural number i such that $i \in \operatorname{dom} s$ holds $s(i) \in \operatorname{divset}(D, i)$.

Now we state the propositions:

(43) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, and a partition D of A. Then $D \in$ the set of tagged partitions of D.
Proof: For every natural number i such that $i \in \operatorname{dom} D$ holds $D(i) \in \operatorname{divset}(D, i)$ by [15, (19)], (4). □

(44) Let us consider real numbers a, b, and a non empty, closed interval subset I_4 of $\mathbb{R}$. If $I_4 = [a, b]$, then $\langle b \rangle$ is a partition of I_4.
Proof: $\langle b \rangle$ is a partition of I_4 by [3, (39)], [15, (19)]. □

Let I be a non empty, closed interval subset of $\mathbb{R}$ and φ be a positive yielding function from I into $\mathbb{R}$.

A tagged partition of I and φ is defined by

(Def. 3) there exists a partition D of I and there exists an element T of the set of tagged partitions of D such that $it = \langle D, T \rangle$.

Let T_1 be a tagged partition of I and φ. We say that T_1 is δ-fine if and only if

(Def. 4) there exists a partition D of I and there exists an element T of the set of tagged partitions of D such that $T_1 = \langle D, T \rangle$ and for every natural number i such that $i \in \operatorname{dom} D$ holds $\operatorname{vol}(\operatorname{divset}(D, i)) \leqslant \varphi(T(i))$.

6. Partition Composition

Let us consider a real number r. Now we state the propositions:

(45) (i) $\sup\{r\} = r$, and

(ii) $\inf\{r\} = r$.

(46) $\operatorname{vol}(\{r\}) = 0$. The theorem is a consequence of (45).

(47) Let us consider non empty, closed interval subsets I_1, I_2 of $\mathbb{R}$, and a positive yielding function φ from I_1 into $\mathbb{R}$. Suppose $I_2 \subseteq I_1$. Then $\varphi \restriction I_2$ is a positive yielding function from I_2 into $\mathbb{R}$.

(48) Let us consider a non empty, closed interval subset I of $\mathbb{R}$, and a real number c. Suppose $c \in I$. Then

(i) $[\inf I, c]$ is a non empty, closed interval subset of $\mathbb{R}$, and

(ii) $[c, \sup I]$ is a non empty, closed interval subset of $\mathbb{R}$, and

(iii) $\sup[\inf I, c] = \inf[c, \sup I]$.

The theorem is a consequence of (41).

Let I_5, I_6 be non empty, closed interval subsets of $\mathbb{R}$, D_4 be a partition of I_5, and D_6 be a partition of I_6. Assume $\sup I_5 \leqslant \inf I_6$. The functor $D_4 \cdot D_6$

yielding a non empty, increasing finite sequence of elements of $\mathbb{R}$ is defined by the term

(Def. 5) $\begin{cases} D_4 \frown D_6, & \textbf{if } D_6(1) \neq \sup I_5, \\ D_4 \frown D_{6\restriction 1}, & \textbf{otherwise}. \end{cases}$

Now we state the propositions:

(49) Let us consider non empty, closed interval subsets I_5, I_6 of $\mathbb{R}$, a partition D_4 of I_5, and a partition D_6 of I_6. Suppose $\sup I_5 = \inf I_6$ and $\operatorname{len} D_6 = 1$ and $D_6(1) = \inf I_6$. Then $D_4 \cdot D_6 = D_4$.

(50) Let us consider non empty, closed interval subsets I_1, I_2, I of $\mathbb{R}$. Suppose $\sup I_1 \leqslant \inf I_2$ and $\inf I \leqslant \inf I_1$ and $\sup I_2 \leqslant \sup I$. Then $I_1 \cup I_2 \subseteq I$.

(51) Let us consider non empty, closed interval subsets I_1, I_2, I of $\mathbb{R}$, a partition D_1 of I_1, and a partition D_2 of I_2. Suppose $\sup I_1 \leqslant \inf I_2$ and $I = [\inf I_1, \sup I_2]$. Then $D_1 \cdot D_2$ is a partition of I. The theorem is a consequence of (50).

(52) Let us consider a non empty, closed interval subset I of $\mathbb{R}$, and a partition D of I. Then the set of tagged partitions of D is not empty.

(53) Let us consider a non empty, increasing finite sequence s of elements of $\mathbb{R}$, and a real number r. Suppose $s(\operatorname{len} s) < r$. Then $s \frown \langle r \rangle$ is a non empty, increasing finite sequence of elements of $\mathbb{R}$. The theorem is a consequence of (1).

(54) Let us consider non empty, increasing finite sequences s_1, s_2 of elements of $\mathbb{R}$, and a real number r. Suppose $s_1(\operatorname{len} s_1) < r < s_2(1)$. Then $(s_1 \frown \langle r \rangle) \frown s_2$ is a non empty, increasing finite sequence of elements of $\mathbb{R}$. The theorem is a consequence of (53) and (1).

(55) Let us consider non empty, closed interval subsets I_1, I_2, I of $\mathbb{R}$. Suppose $\sup I_1 = \inf I_2$ and $I = I_1 \cup I_2$. Then

  (i) $\inf I = \inf I_1$, and

  (ii) $\sup I = \sup I_2$.

(56) Let us consider a non empty, closed interval subset I of $\mathbb{R}$, and a partition D of I. Then

  (i) $\operatorname{divset}(D, 1) = [\inf I, D(1)]$, and

  (ii) for every natural number j such that $j \in \operatorname{dom} D$ and $j \neq 1$ holds $\operatorname{divset}(D, j) = [D(j-1), D(j)]$.

PROOF: For every natural number j such that $j \in \operatorname{dom} D$ and $j \neq 1$ holds $\operatorname{divset}(D, j) = [D(j-1), D(j)]$ by [12, (4)]. □

(57) Let us consider a real number r, and finite sequences p, q of elements of $\mathbb{R}$. Then $\operatorname{len}((p \frown \langle r \rangle) \frown q) = \operatorname{len} p + \operatorname{len} q + 1$.

(58) Let us consider a non empty, closed interval subset I of $\mathbb{R}$, and a partition D of I. Then every element of the set of tagged partitions of D is not empty. The theorem is a consequence of (43).

(59) Let us consider a non empty, closed interval subset I of $\mathbb{R}$, a partition D of I, and an element T of the set of tagged partitions of D. Then $\operatorname{rng} T \subseteq \mathbb{R}$. The theorem is a consequence of (43).

Let I be a non empty, closed interval subset of $\mathbb{R}$, φ be a positive yielding function from I into $\mathbb{R}$, and T_1 be a tagged partition of I and φ. The functor T_1-partition yielding a partition of I is defined by

(Def. 6) there exists a partition D of I and there exists an element T of the set of tagged partitions of D such that $it = D$ and $T_1 = \langle D, T \rangle$.

7. Examples of Partitions

In the sequel r, s denote real numbers.

Now we state the proposition:

(60) Let us consider a function φ from $[r, s]$ into $]0, +\infty[$. Suppose $r \leqslant s$. Then the set of all $]x - \varphi(x), x + \varphi(x)[\cap [r, s]$ where x is an element of $[r, s]$ is a family of subsets of $[r, s]_{\mathrm{T}}$.

Let us consider a function φ from $[r, s]$ into $]0, +\infty[$ and a family S of subsets of $[r, s]_{\mathrm{T}}$.

Let us assume that $r \leqslant s$ and $S =$ the set of all $]x - \varphi(x), x + \varphi(x)[\cap [r, s]$ where x is an element of $[r, s]$. Now we state the propositions:

(61) S is a cover of $[r, s]_{\mathrm{T}}$.
Proof: $[r, s] \subseteq \bigcup S$ by [8, (3)]. □

(62) S is open.
Proof: For every subset P of $[r, s]_{\mathrm{T}}$ such that $P \in S$ holds P is open by [11, (17)], [20, (35)], [11, (15), (9), (10)]. □

(63) Suppose $S =$ the set of all $]x - \varphi(x), x + \varphi(x)[\cap [r, s]$ where x is an element of $[r, s]$. Then S is connected.
Proof: For every subset X of $[r, s]_{\mathrm{T}}$ such that $X \in S$ holds X is connected by [16, (43)]. □

(64) Let us consider a function φ from $[r, s]$ into $]0, +\infty[$, and a family S of subsets of $[r, s]_{\mathrm{T}}$. Suppose $r \leqslant s$ and $S =$ the set of all $]x - \varphi(x), x + \varphi(x)[\cap [r, s]$ where x is an element of $[r, s]$. Let us consider an interval cover I of S. Then

(i) I is a finite sequence of elements of $2^{\mathbb{R}}$, and

(ii) $\operatorname{rng} I \subseteq S$, and

(iii) $\bigcup \operatorname{rng} I = [r, s]$, and

(iv) for every natural number n such that $1 \leqslant n$ holds if $n \leqslant \operatorname{len} I$, then I_n is not empty and if $n + 1 \leqslant \operatorname{len} I$, then $\inf I_n \leqslant \inf I_{n+1}$ and $\sup I_n \leqslant \sup I_{n+1}$ and $\inf I_{n+1} < \sup I_n$ and if $n + 2 \leqslant \operatorname{len} I$, then $\sup I_n \leqslant \inf I_{n+2}$, and

(v) if $[r, s] \in S$, then $I = \langle [r, s] \rangle$, and

(vi) if $[r, s] \notin S$, then there exists a real number p such that $r < p \leqslant s$ and $I(1) = [r, p[$ and there exists a real number p such that $r \leqslant p < s$ and $I(\operatorname{len} I) =]p, s]$ and for every natural number n such that $1 < n < \operatorname{len} I$ there exist real numbers p, q such that $r \leqslant p < q \leqslant s$ and $I(n) =]p, q[$.

The theorem is a consequence of (61), (62), and (63).

(65) Let us consider real numbers r, s, t, x. Then

(i) if $r \leqslant x - t$ and $x + t \leqslant s$, then $]x - t, x + t[\cap [r, s] =]x - t, x + t[$, and

(ii) if $r \leqslant x - t$ and $s < x + t$, then $]x - t, x + t[\cap [r, s] =]x - t, s]$, and

(iii) if $x - t < r$ and $x + t \leqslant s$, then $]x - t, x + t[\cap [r, s] = [r, x + t[$, and

(iv) if $x - t < r$ and $s < x + t$, then $]x - t, x + t[\cap [r, s] = [r, s]$.

(66) Let us consider real numbers r, s, t, x, and a subset X_1 of $\mathbb{R}$. Suppose $0 < t$ and $r \leqslant x \leqslant s$ and $X_1 =]x - t, x + t[\cap [r, s]$. Then

(i) if $r \leqslant x - t$ and $x + t \leqslant s$, then $\inf X_1 = x - t$ and $\sup X_1 = x + t$, and

(ii) if $r \leqslant x - t$ and $s < x + t$, then $\inf X_1 = x - t$ and $\sup X_1 = s$, and

(iii) if $x - t < r$ and $x + t \leqslant s$, then $\inf X_1 = r$ and $\sup X_1 = x + t$, and

(iv) if $x - t < r$ and $s < x + t$, then $\inf X_1 = r$ and $\sup X_1 = s$.

The theorem is a consequence of (65).

Let us consider real numbers a, b, c, non empty, compact subsets I_5, I_6 of $\mathbb{R}$, a partition D_4 of I_5, a partition D_6 of I_6, and natural numbers i, j.

Let us assume that $a \leqslant c \leqslant b$ and $I_5 = [a, c]$ and $I_6 = [c, b]$. Now we state the propositions:

(67) Suppose $i \in \operatorname{dom} D_4$ and $j \in \operatorname{dom} D_6$. Then

(i) if $i < \operatorname{len} D_4$, then $D_4(i) < D_6(j)$, and

(ii) if $i = \operatorname{len} D_4$ and $c < D_6(1)$, then $D_4(i) < D_6(j)$, and

(iii) if $D_6(1) = c$, then $D_4(\operatorname{len} D_4) = D_6(1)$.

Proof: If $i < \operatorname{len} D_4$, then $D_4(i) < D_6(j)$ by [3, (3)]. If $i = \operatorname{len} D_4$ and $c < D_6(1)$, then $D_4(i) < D_6(j)$ by [7, (6)], [3, (91)]. □

(68) If $i \in \operatorname{dom} D_4$ and $j \in \operatorname{dom} D_6$, then if $c < D_6(1)$, then $D_4(i) < D_6(j)$. The theorem is a consequence of (67).

(69) Let us consider real numbers a, b, c, and non empty, compact subsets I_4, I_5, I_6 of $\mathbb{R}$. Suppose $a \leqslant c \leqslant b$ and $I_4 = [a, b]$ and $I_5 = [a, c]$ and $I_6 = [c, b]$. Let us consider a partition D_4 of I_5, and a partition D_6 of I_6. Suppose $c < D_6(1)$. Then $D_4 \frown D_6$ is a partition of I_4.

Proof: Set $D_5 = D_4 \frown D_6$. For every extended reals e_1, e_2 such that e_1, $e_2 \in \operatorname{dom} D_5$ and $e_1 < e_2$ holds $D_5(e_1) < D_5(e_2)$ by [3, (25)], (68), [2, (11)], [3, (1)]. $\operatorname{rng} D_5 \subseteq I_4$ by [3, (31)]. $D_5(\operatorname{len} D_5) = \sup I_4$ by [3, (3), (22)], [15, (19)]. □

(70) Let us consider real numbers a, b, and a non empty, closed interval subset I_4 of $\mathbb{R}$. Suppose $a \leqslant b$ and $I_4 = [a, b]$. Let us consider a partition D_3 of I_4. If $\operatorname{len} D_3 = 1$, then $D_3 = \langle b \rangle$.

(71) Let us consider real numbers a, b, a non empty, compact subset I_4 of $\mathbb{R}$, and a partition D_3 of I_4. Suppose $2 \leqslant \operatorname{len} D_3$. Then $D_{3\restriction 1}$ is a partition of I_4.

Proof: Set $D = D_{3\restriction 1}$. D is a non empty, increasing finite sequence of elements of $\mathbb{R}$ by [3, (60)]. $\operatorname{rng} D \subseteq I_4$ by [7, (33)]. $D(\operatorname{len} D) = \sup I_4$ by [3, (3)]. □

(72) Let us consider real numbers a, b. Suppose $a < b$. Then $\langle a, b \rangle$ is a non empty, increasing finite sequence of elements of $\mathbb{R}$.

Proof: Set $s = \langle a, b \rangle$. s is increasing by [3, (44), (2)]. □

(73) Let us consider real numbers a, b, and a non empty, closed interval subset I_4 of $\mathbb{R}$. Suppose $a < b$ and $I_4 = [a, b]$. Then $\langle a, b \rangle$ is a partition of I_4.

Proof: $\langle a, b \rangle$ is a partition of I_4 by (72), [6, (127)], [3, (44)], [15, (19)]. □

8. Cousin's Lemma

Now we state the proposition:

(74) Let us consider real numbers a, b, and a positive yielding function φ from $[a, b]$ into $\mathbb{R}$. Suppose $a \leqslant b$. Then there exists a non empty, increasing finite sequence x of elements of $\mathbb{R}$ and there exists a non empty finite sequence t of elements of $\mathbb{R}$ such that $x(1) = a$ and $x(\operatorname{len} x) = b$ and $t(1) = a$ and $\operatorname{dom} x = \operatorname{dom} t$ and for every natural number i such that $i - 1$, $i \in \operatorname{dom} t$ holds $t(i) - \varphi(t(i)) \leqslant x(i-1) \leqslant t(i)$ and for every natural number i such that $i \in \operatorname{dom} t$ holds $t(i) \leqslant x(i) \leqslant t(i) + \varphi(t(i))$.

Proof: Define $\mathcal{P}[\text{object}] \equiv$ there exists a non empty, increasing finite sequence x of elements of $\mathbb{R}$ and there exists a non empty finite sequence t of elements of $\mathbb{R}$ such that $x(1) = a$ and $x(\operatorname{len} x) = \$_1$ and $t(1) = a$ and $\operatorname{dom} x = \operatorname{dom} t$ and for every natural number i such that $i - 1$, $i \in \operatorname{dom} t$ holds $t(i) - \varphi(t(i)) \leqslant x(i - 1) \leqslant t(i)$ and for every natural number i such that $i \in \operatorname{dom} t$ holds $t(i) \leqslant x(i) \leqslant t(i) + \varphi(t(i))$. Consider C being a set such that for every object x, $x \in C$ iff $x \in [a, b]$ and $\mathcal{P}[x]$. For every object x such that $x \in C$ holds x is real. Reconsider $c = \sup C$ as a real number. $c \in [a, b]$. Consider d being an element of $\overline{\mathbb{R}}$ such that $d \in C$ and $c - \varphi(c) < d$. Consider D_0 being a non empty, increasing finite sequence of elements of $\mathbb{R}$, T_0 being a non empty finite sequence of elements of $\mathbb{R}$ such that $D_0(1) = a$ and $D_0(\operatorname{len} D_0) = d$ and $T_0(1) = a$ and $\operatorname{dom} D_0 = \operatorname{dom} T_0$ and for every natural number i such that $i - 1$, $i \in \operatorname{dom} T_0$ holds $T_0(i) - \varphi(T_0(i)) \leqslant D_0(i-1) \leqslant T_0(i)$ and for every natural number i such that $i \in \operatorname{dom} T_0$ holds $T_0(i) \leqslant D_0(i) \leqslant T_0(i) + \varphi(T_0(i))$. $c \in C$ and $\mathcal{P}[c]$ by (1), [27, (32)], [3, (22), (39), (1)]. $c = b$ by (1), [27, (32)], [3, (22), (39), (1)]. □

(75) Cousin's Lemma:

Let us consider a non empty, closed interval subset I of $\mathbb{R}$, and a positive yielding function φ from I into $\mathbb{R}$. Then there exists a tagged partition T_1 of I and φ such that T_1 is δ-fine.

Proof: Consider a, b being real numbers such that $a \leqslant b$ and $I = [a, b]$. Reconsider $r = \frac{1}{2}$ as a positive real number. Reconsider $\phi = r \cdot \varphi$ as a positive yielding function from I into $\mathbb{R}$. Consider x being a non empty, increasing finite sequence of elements of $\mathbb{R}$, t being a non empty finite sequence of elements of $\mathbb{R}$ such that $x(1) = a$ and $x(\operatorname{len} x) = b$ and $t(1) = a$ and $\operatorname{dom} x = \operatorname{dom} t$ and for every natural number i such that $i-1, i \in \operatorname{dom} t$ holds $t(i) - \phi(t(i)) \leqslant x(i - 1) \leqslant t(i)$ and for every natural number i such that $i \in \operatorname{dom} t$ holds $t(i) \leqslant x(i) \leqslant t(i) + \phi(t(i))$. Reconsider $D = x$ as a partition of I. Reconsider $T = t$ as an element of the set of tagged partitions of D. Reconsider $T_1 = \langle D, T\rangle$ as a tagged partition of I and φ. T_1 is δ-fine by [15, (19)], (4), [8, (3)], [21, (20)]. □

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Chebyshev Distance

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Summary. In [21], Marco Riccardi formalized that $\mathbb{R}$N-basis n is a basis (in the algebraic sense defined in [26]) of $\mathcal{E}_T^n$ and in [20] he has formalized that $\mathcal{E}_T^n$ is second-countable, we build (in the topological sense defined in [23]) a denumerable base of $\mathcal{E}_T^n$.

Then we introduce the n-dimensional intervals (interval in n-dimensional Euclidean space, *pavé (borné) de* $\mathbb{R}^n$ [16], *semi-intervalle (borné) de* $\mathbb{R}^n$ [22]).

We conclude with the definition of Chebyshev distance [11].

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1. Preliminaries

From now on n denotes a natural number, r, s denote real numbers, x, y denote elements of $\mathcal{R}^n$, p, q denote points of $\mathcal{E}_{\mathrm{T}}^n$, and e denotes a point of $\mathcal{E}^n$.

Now we state the propositions:

(1) $|x-y| = |y-x|$.

(2) Let us consider a natural number i. If $i \in \operatorname{Seg} n$, then $|x|(i) \in \mathbb{R}$.

(3) Let us consider elements x, y of $\mathbb{R}$, and extended reals x_1, y_1. If $x \leqslant x_1$ and $y \leqslant y_1$, then $x+y \leqslant x_1+y_1$.

(4) Let us consider real numbers a, c, and an extended real number b. Suppose $a < b$ and $[a,b[\subseteq [a,c[$. Then b is a real number.

(5) Let us consider a non empty set D, and a non empty subset D_1 of D. Then ${D_1}^n \subseteq D^n$.

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(6) Let us consider a non empty set X, and a function f. Suppose $f = \operatorname{Seg} n \longmapsto X$. Then f is a non-empty, n-element finite sequence.

Let n be a natural number. The functor $\mathbb{R}(n)$ yielding a non-empty, n-element finite sequence is defined by the term

(Def. 1) $\operatorname{Seg} n \longmapsto \mathbb{R}$.

Now we state the propositions:

(7) $\mathbb{R}(n) = \operatorname{Seg} n \longmapsto$ the carrier of $\mathbb{R}^{\mathbf{1}}$.

(8) $\prod(\operatorname{Seg} n \longmapsto \mathbb{R}) = \mathcal{R}^n$.

(9) $\prod \mathbb{R}(n) = \mathcal{R}^n$.

(10) Let us consider a set X. Then $\prod(\operatorname{Seg} n \longmapsto X) = X^n$.

(11) Let us consider a non empty set D, and an n-tuple x of D. Then $x \in D^{\operatorname{Seg} n}$.

(12) Let us consider a subset O_1 of $\mathcal{E}^n_{\mathrm{T}}$, and a subset O_2 of $(\mathcal{E}^n)_{\mathrm{top}}$. If $O_1 = O_2$, then O_1 is open iff O_2 is open.

(13) Suppose $e = p$. Then the set of all OpenHypercube$(e, \frac{1}{m})$ where m is a non zero element of $\mathbb{N}$ = the set of all OpenHypercube$(p, \frac{1}{m})$ where m is a non zero element of $\mathbb{N}$.

(14) If $q \in$ OpenHypercube(p, r), then $p \in$ OpenHypercube(q, r).

(15) If $q \in$ OpenHypercube$(p, \frac{r}{2})$, then OpenHypercube$(q, \frac{r}{2}) \subseteq$ OpenHypercube(p, r).

Let x be an element of $\mathbb{R} \times \mathbb{R}$. The functors: $(x)_{\mathbf{1}}$ and $(x)_{\mathbf{2}}$ yield elements of $\mathbb{R}$. Let n be a natural number and x be an element of $\mathcal{R}^n \times \mathcal{R}^n$. The functors: $(x)_{\mathbf{1}}$ and $(x)_{\mathbf{2}}$ yield elements of $\mathcal{R}^n$. Now we state the proposition:

(16) Let us consider an n-element finite sequence f of elements of $\mathbb{R} \times \mathbb{R}$. Then there exists an element x of $\mathcal{R}^n \times \mathcal{R}^n$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $(x)_{\mathbf{1}}(i) = (f_i)_{\mathbf{1}}$ and $(x)_{\mathbf{2}}(i) = (f_i)_{\mathbf{2}}$.

2. The Set of n-Tuples of Rational Numbers

Let us consider n. The functor $\mathcal{Q}^n$ yielding a set of finite sequences of $\mathbb{Q}$ is defined by the term

(Def. 2) $\mathbb{Q}^n$.

Now we state the proposition:

(17) $\mathcal{Q}^0 = \{0\}$.

One can check that $\mathcal{Q}^0$ is trivial.

Let us consider n. One can check that $\mathcal{Q}^n$ is non empty and every element of $\mathcal{Q}^n$ is n-element and $\mathcal{Q}^n$ is countable.

Let n be a positive natural number. Let us note that $\mathcal{Q}^n$ is infinite and $\mathcal{Q}^n$ is denumerable.

Now we state the proposition:

(18) $\mathcal{Q}^n$ is a dense subset of $\mathcal{E}^n_{\mathrm{T}}$.

Proof: $\mathcal{Q}^n$ is a subset of $\mathcal{R}^n$. Reconsider $R = \mathcal{Q}^n$ as a subset of $\mathcal{E}^n_{\mathrm{T}}$. For every subset Q of $\mathcal{E}^n_{\mathrm{T}}$ such that $Q \neq \emptyset$ and Q is open holds R meets Q by [10, (67)], (12), [15, (23)], [13, (39)]. □

Let us consider n. One can check that $\mathcal{Q}^n$ is countable and dense as a subset of $\mathcal{E}^n_{\mathrm{T}}$.

3. A Countable Base of an n-Dimensional Euclidean Space

(Version 1: [20]):

Let n be a natural number. Let us observe that there exists a basis of $\mathcal{E}^n_{\mathrm{T}}$ which is countable.

Let us consider n and e. Note that OpenHypercubes e is countable.

The functor OpenHypercubes-$\mathbb{Q}(n)$ yielding a non empty set is defined by the term

(Def. 3) $\{\text{OpenHypercubes}\, q$, where q is a point of $\mathcal{E}^n : q \in \mathcal{Q}^n\}$.

Let q be an element of $\mathcal{Q}^n$. The functor ${}^{@}q$ yielding a point of $\mathcal{E}^n$ is defined by the term

(Def. 4) q.

Let q be an object. Assume $q \in \mathcal{Q}^n$. The functor object2$\mathbb{Q}(q, n)$ yielding an element of $\mathcal{Q}^n$ is defined by the term

(Def. 5) q.

Let us note that OpenHypercubes-$\mathbb{Q}(n)$ is countable and $\bigcup$ OpenHypercubes-$\mathbb{Q}(n)$ is countable.

Now we state the propositions:

(19) $\bigcup$ OpenHypercubes-$\mathbb{Q}(n)$ is an open family of subsets of $\mathcal{E}^n_{\mathrm{T}}$. The theorem is a consequence of (12).

(20) Let us consider a non empty, open subset O of $\mathcal{E}^n_{\mathrm{T}}$. Then there exists an element p of $\mathcal{Q}^n$ such that $p \in O$. The theorem is a consequence of (18).

(21) Let us consider a family $\mathcal{B}$ of subsets of $\mathcal{E}^n_{\mathrm{T}}$.
Suppose $\mathcal{B} = \bigcup$ OpenHypercubes-$\mathbb{Q}(n)$. Then $\mathcal{B}$ is quasi basis.
Proof: F is quasi basis by (12), [15, (23)], [10, (67)], (20). □

Let us consider n. Observe that $\bigcup$ OpenHypercubes-$\mathbb{Q}(n)$ is non empty.

The functor OpenHypercubesℚUnion(n) yielding a countable, open family of subsets of $\mathcal{E}^n_{\mathrm{T}}$ is defined by the term

(Def. 6) $\bigcup$ OpenHypercubes-ℚ(n).

Now we state the proposition:

(22) OpenHypercubesℚUnion(n) $= \{$OpenHypercube$(q, \frac{1}{m})$, where q is a point of $\mathcal{E}^n, m$ is a positive natural number : $q \in \mathcal{Q}^n\}$.

(Version 2):

Let n be a natural number. Observe that there exists a basis of $\mathcal{E}^n_{\mathrm{T}}$ which is countable.

Now we state the propositions:

(23) OpenHypercubesℚUnion(n) is a countable basis of $\mathcal{E}^n_{\mathrm{T}}$.

(24) Let us consider an open subset O of $\mathcal{E}^n_{\mathrm{T}}$. Then there exists a subset Y of OpenHypercubesℚUnion(n) such that

(i) Y is countable, and

(ii) $O = \bigcup Y$.

The theorem is a consequence of (21).

Let us consider an open, non empty subset O of $\mathcal{E}^n_{\mathrm{T}}$. Now we state the propositions:

(25) There exists a subset Y of OpenHypercubesℚUnion(n) such that

(i) Y is not empty, and

(ii) $O = \bigcup Y$, and

(iii) there exists a function g from $\mathbb{N}$ into Y such that for every object x, $x \in O$ iff there exists an object y such that $y \in \mathbb{N}$ and $x \in g(y)$.

The theorem is a consequence of (24).

(26) There exists a sequence s of OpenHypercubesℚUnion(n) such that for every object x, $x \in O$ iff there exists an object y such that $y \in \mathbb{N}$ and $x \in s(y)$. The theorem is a consequence of (25).

(27) There exists a sequence s of OpenHypercubesℚUnion(n) such that $O = \bigcup s$. The theorem is a consequence of (26).

4. The Set of All Left Open Real Bounded Intervals

The set of all left open real bounded intervals yielding a family of subsets of $\mathbb{R}$ is defined by the term

(Def. 7) the set of all $]a, b]$ where a, b are real numbers.

Let us note that the set of all left open real bounded intervals is non empty.

Now we state the propositions:

(28) The set of all left open real bounded intervals $\subseteq \{I$, where I is a subset of $\mathbb{R} : I$ is left open interval$\}$.

(29) The set of all left open real bounded intervals is $\cap$-closed and $\setminus_{fp}$-closed and has the empty element and countable cover.

(30) The set of all left open real bounded intervals is a semiring of $\mathbb{R}$.

5. The Set of All Right Open Real Bounded Intervals

The set of all right open real bounded intervals yielding a family of subsets of $\mathbb{R}$ is defined by the term

(Def. 8) the set of all $[a, b[$ where a, b are real numbers.

Observe that the set of all right open real bounded intervals is non empty.

Now we state the propositions:

(31) The set of all right open real bounded intervals $\subseteq \{I$, where I is a subset of $\mathbb{R} : I$ is right open interval$\}$.

(32) The set of all right open real bounded intervals has the empty element.

(33) (i) the set of all right open real bounded intervals is $\cap$-closed, and

(ii) the set of all right open real bounded intervals is $\setminus_{fp}$-closed and has the empty element.

The theorem is a consequence of (31), (32), and (4).

(34) The set of all right open real bounded intervals has countable cover.

Proof: Define $\mathcal{F}[\text{object}, \text{object}] \equiv \$_1$ is an element of $\mathbb{N}$ and $\$_2 \in$ the set of all right open real bounded intervals and there exists a real number x such that $x = \$_1$ and $\$_2 = [-x, x[$. For every object x such that $x \in \mathbb{N}$ there exists an object y such that $y \in$ the set of all right open real bounded intervals and $\mathcal{F}[x, y]$. Consider f being a function such that $\mathrm{dom}\, f = \mathbb{N}$ and $\mathrm{rng}\, f \subseteq$ the set of all right open real bounded intervals and for every object x such that $x \in \mathbb{N}$ holds $\mathcal{F}[x, f(x)]$ from [7, Sch. 6]. $\mathrm{rng}\, f$ is countable by [27, (4)], [14, (58)]. $\mathrm{rng}\, f$ is a cover of $\mathbb{R}$ by [2, (2)], [12, (8)], [3, (13)], [17, (45)]. □

(35) The set of all right open real bounded intervals is a semiring of $\mathbb{R}$.

6. Finite Product of Left Open Intervals

In the sequel n denotes a non zero natural number.

Let n be a non zero natural number. The functor LeftOpenIntervals(n) yielding a classical semiring family of $\mathbb{R}(n)$ is defined by the term

(Def. 9) $\operatorname{Seg} n \longmapsto$ (the set of all left open real bounded intervals).

Now we state the propositions:

(36) LeftOpenIntervals(n) $= \operatorname{Seg} n \longmapsto$ the set of all $]a, b]$ where a, b are real numbers.

(37) MeasurableRectangle LeftOpenIntervals(n) is a semiring of $\mathcal{R}^n$. The theorem is a consequence of (8).

Let us consider an object x.

Let us assume that $x \in$ MeasurableRectangle LeftOpenIntervals(n). Now we state the propositions:

(38) There exists a subset y of $\mathcal{R}^n$ such that

(i) $x = y$, and

(ii) for every natural number i such that $i \in \operatorname{Seg} n$ there exist real numbers a, b such that for every element t of $\mathcal{R}^n$ such that $t \in y$ holds $t(i) \in]a, b]$.

The theorem is a consequence of (37).

(39) There exists a subset y of $\mathcal{R}^n$ and there exists an n-element finite sequence f of elements of $\mathbb{R} \times \mathbb{R}$ such that $x = y$ and for every element t of $\mathcal{R}^n$, $t \in y$ iff for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in](f_i)_{\mathbf{1}}, (f_i)_{\mathbf{2}}]$.

Proof: MeasurableRectangle LeftOpenIntervals(n) is a family of subsets of $\mathcal{R}^n$. Reconsider $y = x$ as a subset of $\mathcal{R}^n$. Consider g being a function such that $x = \prod g$ and $g \in \prod$ LeftOpenIntervals(n). Define $\mathcal{P}$[natural number, set] $\equiv$ there exists an element x of $\mathbb{R} \times \mathbb{R}$ such that $\$_2 = x$ and $g(\$_1) =](x)_{\mathbf{1}}, (x)_{\mathbf{2}}]$. For every natural number i such that $i \in \operatorname{Seg} n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$. There exists a finite sequence f_1 of elements of $\mathbb{R} \times \mathbb{R}$ such that $\operatorname{len} f_1 = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, {f_1}_i]$ from [25, Sch. 1]. Consider f_1 being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that $\operatorname{len} f_1 = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ there exists an element x of $\mathbb{R} \times \mathbb{R}$ such that ${f_1}_i = x$ and $g(i) =](x)_{\mathbf{1}}, (x)_{\mathbf{2}}]$. For every natural number i such that $i \in \operatorname{Seg} n$ holds $g(i) =]({f_1}_i)_{\mathbf{1}}, ({f_1}_i)_{\mathbf{2}}]$. For every element t of $\mathcal{R}^n$ such that $t \in y$ for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in]({f_1}_i)_{\mathbf{1}}, ({f_1}_i)_{\mathbf{2}}]$. For every element t of $\mathcal{R}^n$ such that for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $t(i) \in](f1_i)_{\mathbf{1}}, (f1_i)_{\mathbf{2}}]$ holds $t \in y$ by [6, (93)]. □

(40) There exists a subset y of $\mathcal{R}^n$ and there exist elements a, b of $\mathcal{R}^n$ such that $x = y$ and for every object s, $s \in y$ iff there exists an element t of $\mathcal{R}^n$ such that $s = t$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $t(i) \in]a(i), b(i)]$. The theorem is a consequence of (39) and (16).

Now we state the proposition:

(41) Let us consider a set x. Suppose $x \in$ MeasurableRectangle LeftOpenIntervals(n). Then there exist elements a, b of $\mathcal{R}^n$ such that for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $t(i) \in]a(i), b(i)]$. The theorem is a consequence of (39) and (16).

7. Finite Product of Right Open Intervals

Let n be a non zero natural number. The functor RightOpenIntervals(n) yielding a classical semiring family of $\mathbb{R}(n)$ is defined by the term

(Def. 10) $\mathrm{Seg}\, n \longmapsto$ (the set of all right open real bounded intervals).

From now on n denotes a non zero natural number.

Now we state the propositions:

(42) RightOpenIntervals(n) $=$ $\mathrm{Seg}\, n \longmapsto$ the set of all $[a, b[$ where a, b are real numbers.

(43) MeasurableRectangle RightOpenIntervals(n) is a semiring of $\mathcal{R}^n$. The theorem is a consequence of (8).

Let us consider an object x.

Let us assume that $x \in$ MeasurableRectangle RightOpenIntervals(n). Now we state the propositions:

(44) There exists a subset y of $\mathcal{R}^n$ such that

(i) $x = y$, and

(ii) for every natural number i such that $i \in \mathrm{Seg}\, n$ there exist real numbers a, b such that for every element t of $\mathcal{R}^n$ such that $t \in y$ holds $t(i) \in [a, b[$.

The theorem is a consequence of (43).

(45) There exists a subset y of $\mathcal{R}^n$ and there exists an n-element finite sequence f of elements of $\mathbb{R} \times \mathbb{R}$ such that $x = y$ and for every element t of $\mathcal{R}^n$, $t \in y$ iff for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $t(i) \in [(f_i)_{\mathbf{1}}, (f_i)_{\mathbf{2}}[$.

Proof: MeasurableRectangle RightOpenIntervals(n) is a family of subsets of $\mathcal{R}^n$. Reconsider $y = x$ as a subset of $\mathcal{R}^n$. Consider g being a function such that $x = \prod g$ and $g \in \prod \mathrm{RightOpenIntervals}(n)$. Define $\mathcal{P}$[natural number, set] $\equiv$ there exists an element x of $\mathbb{R} \times \mathbb{R}$ such that $\$_2 = x$ and $g(\$_1) = [(x)_{\mathbf{1}}, (x)_{\mathbf{2}}[$. For every natural number i such that $i \in \mathrm{Seg}\, n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$. There exists a finite sequence f_1 of elements of $\mathbb{R} \times \mathbb{R}$ such that $\mathrm{len}\, f_1 = n$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $\mathcal{P}[i, {f_1}_i]$ from [25, Sch. 1]. Consider f_1 being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that $\mathrm{len}\, f_1 = n$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ there exists an element x of $\mathbb{R} \times \mathbb{R}$ such that ${f_1}_i = x$ and $g(i) = [(x)_{\mathbf{1}}, (x)_{\mathbf{2}}[$. For every natural number i such that $i \in \mathrm{Seg}\, n$ holds $g(i) = [({f_1}_i)_{\mathbf{1}}, ({f_1}_i)_{\mathbf{2}}[$. For every element t of $\mathcal{R}^n$ such that $t \in y$ for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $t(i) \in [({f_1}_i)_{\mathbf{1}}, ({f_1}_i)_{\mathbf{2}}[$. For every element t of $\mathcal{R}^n$ such that for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $t(i) \in [({f_1}_i)_{\mathbf{1}}, ({f_1}_i)_{\mathbf{2}}[$ holds $t \in y$ by [6, (93)]. □

(46) There exists a subset y of $\mathcal{R}^n$ and there exist elements a, b of $\mathcal{R}^n$ such that $x = y$ and for every object s, $s \in y$ iff there exists an element t of $\mathcal{R}^n$ such that $s = t$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $t(i) \in [a(i), b(i)[$. The theorem is a consequence of (45) and (16).

Now we state the proposition:

(47) Let us consider a set x. Suppose $x \in$ MeasurableRectangle RightOpenIntervals(n). Then there exist elements a, b of $\mathcal{R}^n$ such that for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $t(i) \in [a(i), b(i)[$. The theorem is a consequence of (45) and (16).

8. n-Dimensional Product of Subset Family

In the sequel n denotes a natural number, X denotes a set, and S denotes a family of subsets of X.

Let us consider n and X. The functor $\mathrm{Product}(n, X)$ yielding a set is defined by

(Def. 11) for every object f, $f \in it$ iff there exists a function g such that $f = \prod g$ and $g \in \prod(\mathrm{Seg}\, n \longmapsto X)$.

Now we state the propositions:

(48) $\mathrm{Product}(n, X) \subseteq 2^{(\bigcup \bigcup (\mathrm{Seg}\, n \longmapsto X))^{\mathrm{dom}(\mathrm{Seg}\, n \longmapsto X)}}$.

(49) $\mathrm{Product}(n, S)$ is a family of subsets of $\prod(\mathrm{Seg}\, n \longmapsto X)$.

Proof: Reconsider $S_1 = \mathrm{Product}(n, S)$ as a subset of $2^{(\bigcup \bigcup (\mathrm{Seg}\, n \longmapsto S))^{\mathrm{dom}(\mathrm{Seg}\, n \longmapsto S)}}$. $S_1 \subseteq 2^{\prod(\mathrm{Seg}\, n \longmapsto X)}$ by [1, (9)], [24, (13), (7)], [9, (77), (81)]. □

(50) Let us consider a non empty family S of subsets of X. Then $\mathrm{Product}(n, S) =$ the set of all $\prod f$ where f is an n-tuple of S.
PROOF: $\mathrm{Product}(n, S) \subseteq$ the set of all $\prod f$ where f is an n-tuple of S by (10), [6, (131)]. the set of all $\prod f$ where f is an n-tuple of $S \subseteq \mathrm{Product}(n, S)$ by [6, (131)], (10). □

(51) Let us consider a non zero natural number n. Then $\mathrm{Product}(n, X) \subseteq 2^{(\bigcup X)^{\mathrm{Seg}\, n}}$.

Let us consider a non zero natural number n, a non empty set X, and a non empty family S of subsets of X.

Let us assume that $S \neq \{\emptyset\}$. Now we state the propositions:

(52) $\mathrm{Product}(n, S) \subseteq 2^{X^{\mathrm{Seg}\, n}}$. The theorem is a consequence of (51) and (5).

(53) $\bigcup \mathrm{Product}(n, S) \subseteq X^{\mathrm{Seg}\, n}$. The theorem is a consequence of (52).

Let n be a natural number and X be a non empty set. Let us note that $\mathrm{Product}(n, X)$ is non empty.

Now we state the proposition:

(54) Let us consider a non empty set X, a non empty family S of subsets of X, and a subset x of $X^{\mathrm{Seg}\, n}$. Then x is an element of $\mathrm{Product}(n, S)$ if and only if there exists an n-tuple s of S such that for every element t of $X^{\mathrm{Seg}\, n}$, for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $t(i) \in s(i)$ iff $t \in x$.

9. The Set of All Closed Real Bounded Intervals

The set of all closed real bounded intervals yielding a family of subsets of $\mathbb{R}$ is defined by the term

(Def. 12) the set of all $[a, b]$ where a, b are real numbers.

Now we state the proposition:

(55) The set of all closed real bounded intervals $= \{I$, where I is a subset of $\mathbb{R} : I$ is closed interval$\}$.

Let us note that the set of all closed real bounded intervals is non empty.

Now we state the propositions:

(56) The set of all closed real bounded intervals is $\cap$-closed and has the empty element and countable cover.
PROOF: The set of all closed real bounded intervals is $\cap$-closed. There exists a countable subset X of the set of all closed real bounded intervals such that X is a cover of $\mathbb{R}$ by [27, (4)], [14, (58)], [2, (2)], [12, (8)]. □

(57) Let us consider a natural number n. Then $\mathrm{Seg}\, n \longmapsto$ (the set of all closed real bounded intervals) is an n-element finite sequence.

10. The Set of All Open Real Bounded Intervals

The set of all open real bounded intervals yielding a family of subsets of $\mathbb{R}$ is defined by the term

(Def. 13) the set of all $]a, b[$ where a, b are real numbers.

Now we state the proposition:

(58) The set of all open real bounded intervals $\subseteq \{I$, where I is a subset of $\mathbb{R} : I$ is open interval$\}$.

Let us observe that the set of all open real bounded intervals is non empty. Now we state the propositions:

(59) The set of all open real bounded intervals is $\cap$-closed and has the empty element and countable cover.

PROOF: The set of all open real bounded intervals is $\cap$-closed. There exists a countable subset X of the set of all open real bounded intervals such that X is a cover of $\mathbb{R}$ by [27, (4)], [14, (58)], [2, (2)], [12, (8)]. $\square$

(60) Let us consider a natural number n. Then $\operatorname{Seg} n \longmapsto$ (the set of all open real bounded intervals) is an n-element finite sequence.

11. n-Dimensional Subset Family of $\mathbb{R}$

From now on n denotes a natural number and S denotes a family of subsets of $\mathbb{R}$.

Now we state the proposition:

(61) $\operatorname{Product}(n, S)$ is a family of subsets of $\mathcal{R}^n$. The theorem is a consequence of (49) and (8).

Let us consider n and S. One can check that the functor $\operatorname{Product}(n, S)$ yields a family of subsets of $\mathcal{R}^n$. Now we state the propositions:

(62) Let us consider a non empty family S of subsets of $\mathbb{R}$, and a subset x of $\mathcal{R}^n$. Then x is an element of $\operatorname{Product}(n, S)$ if and only if there exists an n-tuple s of S such that for every element t of $\mathcal{R}^n$, for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in s(i)$ iff $t \in x$.

PROOF: If x is an element of $\operatorname{Product}(n, S)$, then there exists an n-tuple s of S such that for every element t of $\mathcal{R}^n$, for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in s(i)$ iff $t \in x$ by [6, (93)]. If there exists an n-tuple s of S such that for every element t of $\mathcal{R}^n$, for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in s(i)$ iff $t \in x$, then x is an element of $\operatorname{Product}(n, S)$ by [6, (93)]. $\square$

(63) Let us consider a non zero natural number n, and an n-tuple s of the set of all closed real bounded intervals. Then there exist elements a, b of $\mathcal{R}^n$ such that for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $s(i) = [a(i), b(i)]$.

PROOF: $s \in$ (the set of all closed real bounded intervals)$^{\mathrm{Seg}\, n}$. Consider f being a function such that $s = f$ and $\mathrm{dom}\, f = \mathrm{Seg}\, n$ and $\mathrm{rng}\, f \subseteq$ the set of all closed real bounded intervals. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an element f of $\mathbb{R} \times \mathbb{R}$ such that $f = \$_2$ and $s(\$_1) = [(f)_{\mathbf{1}}, (f)_{\mathbf{2}}]$. For every natural number i such that $i \in \mathrm{Seg}\, n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$ by [7, (3)]. Consider f being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that $\mathrm{len}\, f = n$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $\mathcal{P}[i, f_i]$ from [25, Sch. 1]. Consider z being an element of $\mathcal{R}^n \times \mathcal{R}^n$ such that for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $(z)_{\mathbf{1}}(i) = (f_i)_{\mathbf{1}}$ and $(z)_{\mathbf{2}}(i) = (f_i)_{\mathbf{2}}$. Reconsider $a = (z)_{\mathbf{1}}$, $b = (z)_{\mathbf{2}}$ as an element of $\mathcal{R}^n$. For every natural number i such that $i \in \mathrm{Seg}\, n$ holds $s(i) = [a(i), b(i)]$. $\square$

(64) Let us consider a non zero natural number n, and an element x of $\mathrm{Product}(n, \text{the set of all closed real bounded intervals})$. Then there exist elements a, b of $\mathcal{R}^n$ such that for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $t(i) \in [a(i), b(i)]$. The theorem is a consequence of (62) and (63).

Let us consider a non zero natural number n, a subset x of $\mathcal{R}^n$, and elements a, b of $\mathcal{R}^n$. Now we state the propositions:

(65) Suppose for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $t(i) \in [a(i), b(i)]$. Then x is an element of $\mathrm{Product}(n, \text{the set of all closed real bounded intervals})$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a natural number n such that $\$_1 = n$ and $\$_2 = [a(n), b(n)]$. For every natural number i such that $i \in \mathrm{Seg}\, n$ there exists an element d of the set of all closed real bounded intervals such that $\mathcal{P}[i, d]$. There exists a finite sequence g of elements of the set of all closed real bounded intervals such that $\mathrm{len}\, g = n$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $\mathcal{P}[i, g_i]$ from [25, Sch. 1]. Consider g being a finite sequence of elements of the set of all closed real bounded intervals such that $\mathrm{len}\, g = n$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $\mathcal{P}[i, g_i]$. For every natural number i such that $i \in \mathrm{Seg}\, n$ holds $g(i) = [a(i), b(i)]$. There exists a function g such that $x = \prod g$ and $g \in \prod(\mathrm{Seg}\, n \longmapsto (\text{the set of all closed real bounded intervals}))$ by [4, (89)], [24, (13), (7)], [1, (9)]. $\square$

(66) Suppose for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number

i such that $i \in \operatorname{Seg} n$ holds $t(i) \in]a(i), b(i)]$. Then x is an element of Product(n, the set of all left open real bounded intervals).

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a natural number n such that $\$_1 = n$ and $\$_2 =]a(n), b(n)]$. For every natural number i such that $i \in \operatorname{Seg} n$ there exists an element d of the set of all left open real bounded intervals such that $\mathcal{P}[i, d]$. There exists a finite sequence g of elements of the set of all left open real bounded intervals such that $\operatorname{len} g = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, g_i]$ from [25, Sch. 1]. Consider g being a finite sequence of elements of the set of all left open real bounded intervals such that $\operatorname{len} g = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, g_i]$. For every natural number i such that $i \in \operatorname{Seg} n$ holds $g(i) =]a(i), b(i)]$. There exists a function g such that $x = \prod g$ and $g \in \prod(\operatorname{Seg} n \longmapsto$ (the set of all left open real bounded intervals)) by [4, (89)], [24, (13), (7)], [1, (9)]. □

(67) Suppose for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in [a(i), b(i)[$. Then x is an element of Product(n, the set of all right open real bounded intervals).

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a natural number n such that $\$_1 = n$ and $\$_2 = [a(n), b(n)[$. For every natural number i such that $i \in \operatorname{Seg} n$ there exists an element d of the set of all right open real bounded intervals such that $\mathcal{P}[i, d]$. There exists a finite sequence g of elements of the set of all right open real bounded intervals such that $\operatorname{len} g = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, g_i]$ from [25, Sch. 1]. Consider g being a finite sequence of elements of the set of all right open real bounded intervals such that $\operatorname{len} g = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, g_i]$. For every natural number i such that $i \in \operatorname{Seg} n$ holds $g(i) = [a(i), b(i)[$. There exists a function g such that $x = \prod g$ and $g \in \prod(\operatorname{Seg} n \longmapsto$ (the set of all right open real bounded intervals)) by [4, (89)], [24, (13), (7)], [1, (9)]. □

Now we state the propositions:

(68) Let us consider a non zero natural number n, and an n-tuple s of the set of all left open real bounded intervals. Then there exist elements a, b of $\mathcal{R}^n$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $s(i) =]a(i), b(i)]$.

PROOF: $s \in$ (the set of all left open real bounded intervals)$^{\operatorname{Seg} n}$. Consider f being a function such that $s = f$ and $\operatorname{dom} f = \operatorname{Seg} n$ and $\operatorname{rng} f \subseteq$ the set of all left open real bounded intervals. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an element f of $\mathbb{R} \times \mathbb{R}$ such that $f = \$_2$ and $s(\$_1) =](f)_{\mathbf{1}}, (f)_{\mathbf{2}}]$. For every natural number i such that $i \in \operatorname{Seg} n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$ by [7, (3)]. Consider f being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that len $f = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, f_i]$ from [25, Sch. 1]. Consider z being an element of $\mathcal{R}^n \times \mathcal{R}^n$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $(z)_{\mathbf{1}}(i) = (f_i)_{\mathbf{1}}$ and $(z)_{\mathbf{2}}(i) = (f_i)_{\mathbf{2}}$. Reconsider $a = (z)_{\mathbf{1}}$, $b = (z)_{\mathbf{2}}$ as an element of $\mathcal{R}^n$. For every natural number i such that $i \in \operatorname{Seg} n$ holds $s(i) =]a(i), b(i)]$. $\square$

(69) Let us consider a non zero natural number n, and an element x of Product(n, the set of all left open real bounded intervals). Then there exist elements a, b of $\mathcal{R}^n$ such that for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in]a(i), b(i)]$. The theorem is a consequence of (62) and (68).

(70) Let us consider a non zero natural number n, and an n-tuple s of the set of all right open real bounded intervals. Then there exist elements a, b of $\mathcal{R}^n$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $s(i) = [a(i), b(i)[$.

PROOF: $s \in$ (the set of all right open real bounded intervals)$^{\operatorname{Seg} n}$. Consider f being a function such that $s = f$ and dom $f = \operatorname{Seg} n$ and rng $f \subseteq$ the set of all right open real bounded intervals. Define $\mathcal{P}$[object, object] $\equiv$ there exists an element f of $\mathbb{R} \times \mathbb{R}$ such that $f = \$_2$ and $s(\$_1) = [(f)_{\mathbf{1}}, (f)_{\mathbf{2}}[$. For every natural number i such that $i \in \operatorname{Seg} n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$ by [7, (3)]. Consider f being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that len $f = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, f_i]$ from [25, Sch. 1]. Consider z being an element of $\mathcal{R}^n \times \mathcal{R}^n$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $(z)_{\mathbf{1}}(i) = (f_i)_{\mathbf{1}}$ and $(z)_{\mathbf{2}}(i) = (f_i)_{\mathbf{2}}$. Reconsider $a = (z)_{\mathbf{1}}$, $b = (z)_{\mathbf{2}}$ as an element of $\mathcal{R}^n$. For every natural number i such that $i \in \operatorname{Seg} n$ holds $s(i) = [a(i), b(i)[$. $\square$

(71) Let us consider a non zero natural number n, and an element x of Product(n, the set of all right open real bounded intervals). Then there exist elements a, b of $\mathcal{R}^n$ such that for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in [a(i), b(i)[$. The theorem is a consequence of (62) and (70).

12. Closed/Open/Left-Open/Right-Open – Hyper Interval

From now on n denotes a natural number and a, b, c, d denote elements of $\mathcal{R}^n$.

Let us consider n, a, and b. The functor ClosedHyperInterval(a, b) yielding a subset of $\mathcal{R}^n$ is defined by

(Def. 14) for every object x, $x \in it$ iff there exists an element y of $\mathcal{R}^n$ such that $x = y$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $y(i) \in [a(i), b(i)]$.

The functor OpenHyperInterval(a, b) yielding a subset of $\mathcal{R}^n$ is defined by

(Def. 15) for every object x, $x \in it$ iff there exists an element y of $\mathcal{R}^n$ such that $x = y$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $y(i) \in]a(i), b(i)[$.

The functor LeftOpenHyperInterval(a, b) yielding a subset of $\mathcal{R}^n$ is defined by

(Def. 16) for every object x, $x \in it$ iff there exists an element y of $\mathcal{R}^n$ such that $x = y$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $y(i) \in]a(i), b(i)]$.

The functor RightOpenHyperInterval(a, b) yielding a subset of $\mathcal{R}^n$ is defined by

(Def. 17) for every object x, $x \in it$ iff there exists an element y of $\mathcal{R}^n$ such that $x = y$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $y(i) \in [a(i), b(i)[$.

Now we state the proposition:

(72) ClosedHyperInterval$(a, a) = \{a\}$.
PROOF: ClosedHyperInterval$(a, a) \subseteq \{a\}$ by [6, (124)].
$\{a\} \subseteq$ ClosedHyperInterval(a, a). $\square$

Let us consider n and a. Let us observe that ClosedHyperInterval(a, a) is trivial.

Now we state the proposition:

(73) (i) OpenHyperInterval$(a, b) \subseteq$ LeftOpenHyperInterval(a, b), and

(ii) OpenHyperInterval$(a, b) \subseteq$ RightOpenHyperInterval(a, b), and

(iii) LeftOpenHyperInterval$(a, b) \subseteq$ ClosedHyperInterval(a, b), and

(iv) RightOpenHyperInterval$(a, b) \subseteq$ ClosedHyperInterval(a, b).

Let us consider n, a, and b. We say that $a \leqslant b$ if and only if

(Def. 18) for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $a(i) \leqslant b(i)$.

One can verify that the predicate is reflexive.

Now we state the propositions:

(74) If $a \leqslant b \leqslant c$, then $a \leqslant c$.

(75) If $a \leqslant c$ and $d \leqslant b$,
then ClosedHyperInterval$(c, d) \subseteq$ ClosedHyperInterval(a, b).

(76) If $a \leqslant b$, then ClosedHyperInterval(a, b) is not empty. The theorem is a consequence of (75) and (72).

Let us consider n, a, and b. We say that $a < b$ if and only if

(Def. 19) for every natural number i such that $i \in \operatorname{Seg} n$ holds $a(i) < b(i)$.

Now we state the propositions:

(77) If $a < b < c$, then $a < c$.

(78) If $b < a$ and n is not zero, then ClosedHyperInterval(a, b) is empty.

(79) $n \mapsto r$ is an element of $\mathcal{R}^n$.

PROOF: Set $f = n \mapsto r$. $f \in \mathbb{R}^{\operatorname{Seg} n}$ by [6, (112), (93)]. $\square$

Let us consider n and r. Note that the functor $n \mapsto r$ yields an element of $\mathcal{R}^n$. One can check that the functor $\langle r \rangle$ yields an element of $\mathcal{R}^1$. Now we state the propositions:

(80) Let us consider a non zero natural number n, and a point e of $\mathcal{E}^n$. Then there exists an element a of $\mathcal{R}^n$ such that

(i) $a = e$, and

(ii) OpenHypercube(e, r) = OpenHyperInterval$(a - n \mapsto r, a + n \mapsto r)$.

PROOF: Reconsider $a = e$ as an element of $\mathcal{R}^n$. Reconsider $p = e$ as a point of $\mathcal{E}^n_{\mathrm{T}}$. Consider e_0 being a point of $\mathcal{E}^n$ such that $p = e_0$ and OpenHypercube(e_0, r) = OpenHypercube(p, r). OpenHypercube$(e, r) \subseteq$ OpenHyperInterval$(a - n \mapsto r, a + n \mapsto r)$ by [8, (27)], [6, (57)], [8, (11)], [18, (4)]. OpenHyperInterval$(a - n \mapsto r, a + n \mapsto r) \subseteq$ OpenHypercube(e, r) by [10, (22)], [8, (27)], [6, (57)], [8, (11)]. $\square$

(81) Let us consider a point p of $\mathcal{E}^n_{\mathrm{T}}$. Then there exists an element a of $\mathcal{R}^n$ such that

(i) $a = p$, and

(ii) ClosedHypercube(p, b) = ClosedHyperInterval$(a - b, a + b)$.

PROOF: Reconsider $a = p$ as an element of $\mathcal{R}^n$. ClosedHypercube$(p, b) \subseteq$ ClosedHyperInterval$(a-b, a+b)$ by [10, (22)], [8, (11), (27)]. ClosedHyperInterval$(a - b, a + b) \subseteq$ ClosedHypercube(p, b) by [10, (22)], [8, (11), (27)]. $\square$

13. Correspondance between Interval and 1-Dimensional Hyper Interval

Let us consider a real number x. Now we state the propositions:

(82) $x \in [r, s]$ if and only if $1 \mapsto x \in$ ClosedHyperInterval$(\langle r \rangle, \langle s \rangle)$.

PROOF: Set $a_1 = \langle r \rangle$. Set $b_1 = \langle s \rangle$. For every real number x such that $x \in [r, s]$ holds $1 \mapsto x \in$ ClosedHyperInterval(a_1, b_1) by [4, (2)], [24, (7)]. For every real number x such that $1 \mapsto x \in$ ClosedHyperInterval(a_1, b_1) holds $x \in [r, s]$ by [24, (7)]. $\square$

(83) $x \in]r, s[$ if and only if $1 \mapsto x \in \mathrm{OpenHyperInterval}(\langle r \rangle, \langle s \rangle)$.
PROOF: Set $a_1 = \langle r \rangle$. Set $b_1 = \langle s \rangle$. For every real number x such that $x \in]r, s[$ holds $1 \mapsto x \in \mathrm{OpenHyperInterval}(a_1, b_1)$ by [4, (2)], [24, (7)]. For every real number x such that $1 \mapsto x \in \mathrm{OpenHyperInterval}(a_1, b_1)$ holds $x \in]r, s[$ by [24, (7)]. □

(84) $x \in]r, s]$ if and only if $1 \mapsto x \in \mathrm{LeftOpenHyperInterval}(\langle r \rangle, \langle s \rangle)$.
PROOF: Set $a_1 = \langle r \rangle$. Set $b_1 = \langle s \rangle$. For every real number x such that $x \in]r, s]$ holds $1 \mapsto x \in \mathrm{LeftOpenHyperInterval}(a_1, b_1)$ by [4, (2)], [24, (7)]. For every real number x such that $1 \mapsto x \in \mathrm{LeftOpenHyperInterval}(a_1, b_1)$ holds $x \in]r, s]$ by [24, (7)]. □

(85) $x \in [r, s[$ if and only if $1 \mapsto x \in \mathrm{RightOpenHyperInterval}(\langle r \rangle, \langle s \rangle)$.
PROOF: Set $a_1 = \langle r \rangle$. Set $b_1 = \langle s \rangle$. For every real number x such that $x \in [r, s[$ holds $1 \mapsto x \in \mathrm{RightOpenHyperInterval}(a_1, b_1)$ by [4, (2)], [24, (7)]. For every real number x such that $1 \mapsto x \in \mathrm{RightOpenHyperInterval}(a_1, b_1)$ holds $x \in [r, s[$ by [24, (7)]. □

14. Correspondance between Measurable Rectangle and Product

From now on n denotes a non zero natural number.

Now we state the propositions:

(86) Let us consider an n-tuple s of the set of all open real bounded intervals. Then there exist elements a, b of $\mathcal{R}^n$ such that for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $s(i) =]a(i), b(i)[$.
PROOF: $s \in (\text{the set of all open real bounded intervals})^{\mathrm{Seg}\, n}$. Consider f being a function such that $s = f$ and $\mathrm{dom}\, f = \mathrm{Seg}\, n$ and $\mathrm{rng}\, f \subseteq$ the set of all open real bounded intervals. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an element f of $\mathbb{R} \times \mathbb{R}$ such that $f = \$_2$ and $s(\$_1) =](f)_{\mathbf{1}}, (f)_{\mathbf{2}}[$. For every natural number i such that $i \in \mathrm{Seg}\, n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$ by [7, (3)]. Consider f being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that $\mathrm{len}\, f = n$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $\mathcal{P}[i, f_i]$ from [25, Sch. 1]. Consider z being an element of $\mathcal{R}^n \times \mathcal{R}^n$ such that for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $(z)_{\mathbf{1}}(i) = (f_i)_{\mathbf{1}}$ and $(z)_{\mathbf{2}}(i) = (f_i)_{\mathbf{2}}$. Reconsider $a = (z)_{\mathbf{1}}$, $b = (z)_{\mathbf{2}}$ as an element of $\mathcal{R}^n$. For every natural number i such that $i \in \mathrm{Seg}\, n$ holds $s(i) =]a(i), b(i)[$. □

(87) Let us consider an element x of $\mathrm{Product}(n, \text{the set of all open real bounded intervals})$. Then there exist elements a, b of $\mathcal{R}^n$ such that for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number i such that

$i \in \operatorname{Seg} n$ holds $t(i) \in]a(i), b(i)[$. The theorem is a consequence of (62) and (86).

(88) Let us consider an n-tuple s of the set of all left open real bounded intervals. Then there exist elements a, b of $\mathcal{R}^n$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $s(i) =]a(i), b(i)]$.

PROOF: $s \in$ (the set of all left open real bounded intervals)$^{\operatorname{Seg} n}$. Consider f being a function such that $s = f$ and $\operatorname{dom} f = \operatorname{Seg} n$ and $\operatorname{rng} f \subseteq$ the set of all left open real bounded intervals. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an element f of $\mathbb{R} \times \mathbb{R}$ such that $f = \$_2$ and $s(\$_1) =](f)_{\mathbf{1}}, (f)_{\mathbf{2}}]$. For every natural number i such that $i \in \operatorname{Seg} n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$ by [7, (3)]. Consider f being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that $\operatorname{len} f = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, f_i]$ from [25, Sch. 1]. Consider z being an element of $\mathcal{R}^n \times \mathcal{R}^n$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $(z)_{\mathbf{1}}(i) = (f_i)_{\mathbf{1}}$ and $(z)_{\mathbf{2}}(i) = (f_i)_{\mathbf{2}}$. Reconsider $a = (z)_{\mathbf{1}}$, $b = (z)_{\mathbf{2}}$ as an element of $\mathcal{R}^n$. For every natural number i such that $i \in \operatorname{Seg} n$ holds $s(i) =]a(i), b(i)]$. $\square$

(89) Let us consider an element x of Product(n, the set of all left open real bounded intervals). Then there exist elements a, b of $\mathcal{R}^n$ such that for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in]a(i), b(i)]$. The theorem is a consequence of (62) and (88).

(90) Let us consider an n-tuple s of the set of all right open real bounded intervals. Then there exist elements a, b of $\mathcal{R}^n$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $s(i) = [a(i), b(i)[$.

PROOF: $s \in$ (the set of all right open real bounded intervals)$^{\operatorname{Seg} n}$. Consider f being a function such that $s = f$ and $\operatorname{dom} f = \operatorname{Seg} n$ and $\operatorname{rng} f \subseteq$ the set of all right open real bounded intervals. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an element f of $\mathbb{R} \times \mathbb{R}$ such that $f = \$_2$ and $s(\$_1) = [(f)_{\mathbf{1}}, (f)_{\mathbf{2}}[$. For every natural number i such that $i \in \operatorname{Seg} n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$ by [7, (3)]. Consider f being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that $\operatorname{len} f = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, f_i]$ from [25, Sch. 1]. Consider z being an element of $\mathcal{R}^n \times \mathcal{R}^n$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $(z)_{\mathbf{1}}(i) = (f_i)_{\mathbf{1}}$ and $(z)_{\mathbf{2}}(i) = (f_i)_{\mathbf{2}}$. Reconsider $a = (z)_{\mathbf{1}}$, $b = (z)_{\mathbf{2}}$ as an element of $\mathcal{R}^n$. For every natural number i such that $i \in \operatorname{Seg} n$ holds $s(i) = [a(i), b(i)[$. $\square$

(91) Let us consider an element x of Product(n, the set of all right open real bounded intervals). Then there exist elements a, b of $\mathcal{R}^n$ such that

for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in [a(i), b(i)[$. The theorem is a consequence of (62) and (90).

(92) MeasurableRectangle LeftOpenIntervals(n) = Product$(n,$ the set of all left open real bounded intervals). The theorem is a consequence of (40) and (66).

(93) MeasurableRectangle RightOpenIntervals(n) = Product$(n,$ the set of all right open real bounded intervals). The theorem is a consequence of (46) and (67).

15. Chebyshev Distance

In the sequel n denotes a non zero natural number and x, y, z denote elements of $\mathcal{R}^n$.

Let us consider n. The functor $D^n_{\mathrm{Chebyshev}}$ yielding a function from $\mathcal{R}^n \times \mathcal{R}^n$ into $\mathbb{R}$ is defined by

(Def. 20) for every elements x, y of $\mathcal{R}^n$, $it(x, y) = \sup \operatorname{rng}|x - y|$.

Now we state the propositions:

(94) (i) the set of all $|x(i) - y(i)|$ where i is an element of $\operatorname{Seg} n$ is real-membered, and

(ii) the set of all $|x(i) - y(i)|$ where i is an element of $\operatorname{Seg} n = \operatorname{rng}|x - y|$.

Proof: Set S = the set of all $|x(i) - y(i)|$ where i is an element of $\operatorname{Seg} n$. $S \subseteq \operatorname{rng}|x - y|$ by [8, (27)], [6, (124)]. For every object t such that $t \in \operatorname{rng}|x - y|$ holds $t \in S$ by [6, (124)], [8, (27)]. □

(95) There exists an extended real-membered set S such that

(i) S = the set of all $|x(i) - y(i)|$ where i is an element of $\operatorname{Seg} n$, and

(ii) $(D^n_{\mathrm{Chebyshev}})(x, y) = \sup S$.

The theorem is a consequence of (94).

(96) $(D^n_{\mathrm{Chebyshev}})(x, y) = |x - y|(\text{max-diff-index}(x, y))$.

Proof: $(D^n_{\mathrm{Chebyshev}})(x, y) \leqslant |x - y|(\text{max-diff-index}(x, y))$ by [15, (5)]. □

(97) $(D^n_{\mathrm{Chebyshev}})(x, y) = 0$ if and only if $x = y$.

Proof: Consider S being an extended real-membered set such that S = the set of all $|x(i) - y(i)|$ where i is an element of $\operatorname{Seg} n$ and $(D^n_{\mathrm{Chebyshev}})(x, y) = \sup S$. $S = \{0\}$ by [19, (2)], [3, (53)], [4, (1)]. □

(98) $(D^n_{\mathrm{Chebyshev}})(x, y) = (D^n_{\mathrm{Chebyshev}})(y, x)$. The theorem is a consequence of (1).

(99) $(D^n_{\text{Chebyshev}})(x,y) \leqslant (D^n_{\text{Chebyshev}})(x,z) + (D^n_{\text{Chebyshev}})(z,y)$.

Proof: Reconsider $s_1 = \sup \operatorname{rng}|x-y|$, $s_2 = \sup \operatorname{rng}|x-z|$, $s_3 = \sup \operatorname{rng}|z-y|$ as a real number. $s_1 \leqslant s_2 + s_3$ by [8, (27)], [5, (56)], [6, (124)], (2). □

(100) $D^n_{\text{Chebyshev}}$ is a metric of $\mathcal{R}^n$. The theorem is a consequence of (97), (98), and (99).

(101) $\rho^2([0,0],[1,1]) = \sqrt{2}$.

(102) $(D^2_{\text{Chebyshev}})([0,0],[1,1]) = 1$.

Proof: Consider S being an extended real-membered set such that $S =$ the set of all $|[0,0](i) - [1,1](i)|$ where i is an element of $\operatorname{Seg} 2$ and $(D^2_{\text{Chebyshev}})([0,0],[1,1]) = \sup S$. $S = \{|0-1|\}$ by [4, (2), (44)]. □

Let us consider elements x, y of $\mathcal{R}^1$. Now we state the propositions:

(103) $(D^1_{\text{Chebyshev}})(x,y) = |x(1) - y(1)|$.

Proof: Consider S being an extended real-membered set such that $S =$ the set of all $|x(i) - y(i)|$ where i is an element of $\operatorname{Seg} 1$ and $(D^1_{\text{Chebyshev}})(x,y) = \sup S$. $S = \{|x(1) - y(1)|\}$ by [4, (2)]. □

(104) $\rho^1(x,y) = |x(1) - y(1)|$.

Now we state the propositions:

(105) $\rho^1 = D^1_{\text{Chebyshev}}$. The theorem is a consequence of (104) and (103).

(106) $\rho^2 \neq D^2_{\text{Chebyshev}}$. The theorem is a consequence of (101) and (102).

Let n be a non zero natural number. The functor $\mathsf{L}_\infty(n)$ yielding a strict metric space is defined by the term

(Def. 21) $\langle \mathcal{R}^n, D^n_{\text{Chebyshev}} \rangle$.

Let us observe that $\mathsf{L}_\infty(n)$ is non empty.

The functor $\mathcal{E}^n_\infty(n)$ yielding a strict real linear topological structure is defined by

(Def. 22) the topological structure of $it = (\mathsf{L}_\infty(n))_{\text{top}}$ and the RLS structure of $it = \mathbb{R}_{\mathbb{R}}^{\operatorname{Seg} n}$.

Now we state the proposition:

(107) The RLS structure of $\mathcal{E}^n_{\mathrm{T}} =$ the RLS structure of $\mathcal{E}^n_\infty(n)$.

Let n be a non zero natural number. Let us note that $\mathcal{E}^n_\infty(n)$ is non empty.

Now we state the propositions:

(108) Let us consider an element x of $\mathcal{R}^0$. Then

(i) $\operatorname{Intervals}(x,r)$ is empty, and

(ii) $\prod \operatorname{Intervals}(x,r) = \{\emptyset\}$.

(109) If $r \leqslant 0$, then $\prod \operatorname{Intervals}(x,r)$ is empty.

In the sequel p denotes an element of $\mathsf{L}_\infty(n)$.

Let n be a non zero natural number and p be an element of $\mathsf{L}_\infty(n)$. The functor ${}^@p$ yielding an element of $\mathcal{R}^n$ is defined by the term

(Def. 23) p.

Now we state the propositions:

(110) $\mathrm{Ball}(p,r) = \prod \mathrm{Intervals}({}^@p, r)$. The theorem is a consequence of (109), (95), and (96).

(111) Let us consider a point e of $\mathcal{E}^n$. If $e = p$, then $\mathrm{Ball}(p,r) = \mathrm{OpenHypercube}(e,r)$. The theorem is a consequence of (110).

Let n be a non zero natural number, p be an element of $\mathsf{L}_\infty(n)$, and r be a negative real number. Let us note that $\overline{\mathrm{Ball}}(p,r)$ is empty.

Now we state the propositions:

(112) Let us consider an object t. Then $t \in \overline{\mathrm{Ball}}(p,r)$ if and only if there exists a function f such that $t = f$ and $\mathrm{dom}\, f = \mathrm{Seg}\, n$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $f(i) \in [({}^@p)(i) - r, ({}^@p)(i) + r]$. The theorem is a consequence of (95).

(113) Let us consider a point p of $\mathcal{E}^n_{\mathrm{T}}$, and an element q of $\mathsf{L}_\infty(n)$. Suppose $q = p$. Then $\overline{\mathrm{Ball}}(q,r) = \mathrm{ClosedHypercube}(p, n \mapsto r)$.
PROOF: For every object x such that $x \in \overline{\mathrm{Ball}}(q,r)$ holds $x \in \mathrm{ClosedHypercube}(p, n \mapsto r)$ by (112), [6, (57), (93)], [10, (22)]. For every object x such that $x \in \mathrm{ClosedHypercube}(p, n \mapsto r)$ holds $x \in \overline{\mathrm{Ball}}(q,r)$ by [10, (22)], [6, (131), (124), (57)]. □

(114) $\mathrm{Ball}(p,r) = \mathrm{OpenHyperInterval}({}^@p - n \mapsto r, {}^@p + n \mapsto r)$. The theorem is a consequence of (80) and (110).

(115) $\overline{\mathrm{Ball}}(p,r) = \mathrm{ClosedHyperInterval}({}^@p - n \mapsto r, {}^@p + n \mapsto r)$. The theorem is a consequence of (81) and (113).

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Binary Relations-based Rough Sets – an Automated Approach

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Summary. Rough sets, developed by Zdzisław Pawlak [12], are an important tool to describe the state of incomplete or partially unknown information. In this article, which is essentially the continuation of [8], we try to give the characterization of approximation operators in terms of ordinary properties of underlying relations (some of them, as serial and mediate relations, were not available in the Mizar Mathematical Library [11]). Here we drop the classical equivalence- and tolerance-based models of rough sets trying to formalize some parts of [18].

The main aim of this Mizar article is to provide a formal counterpart for the rest of the paper of William Zhu [18]. In order to do this, we recall also Theorem 3 from Y.Y. Yao's paper [17]. The first part of our formalization (covering first seven pages) is contained in [8]. Now we start from page 5003, sec. 3.4. [18]. We formalized almost all numbered items (definitions, propositions, theorems, and corollaries), with the exception of Proposition 7, where we stated our theorem only in terms of singletons. We provided more thorough discussion of the property *positive alliance* and its connection with seriality and reflexivity (and also transitivity). Examples were not covered as a rule as we tried to construct a more general mechanism of finding appropriate models for approximation spaces in Mizar providing more automatization than it is now [10].

Of course, we can see some more general applications of some registrations of clusters, essentially not dealing with the notion of an approximation: the notions of an alliance binary relation were not defined in the Mizar Mathematical Library before, and we should think about other properties which are also absent but needed in the context of rough approximations [9], [5]. Via theory merging, using mechanisms described in [6] and [7], such elementary constructions can be extended to other frameworks.

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1. Preliminaries

From now on X, a, b, c, x, y, z, t denote sets and R denotes a binary relation.

Let X be a non empty set. Let us note that 2^X is closed under directed unions.

The scheme *FinSubIndA1* deals with a non empty, finite set $\mathcal{X}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 1) For every subset B of $\mathcal{X}$, $\mathcal{P}[B]$

provided

- $\mathcal{P}[\emptyset_{\mathcal{X}}]$ and
- for every subset B' of $\mathcal{X}$ and for every element b of $\mathcal{X}$ such that $\mathcal{P}[B']$ and $b \notin B'$ holds $\mathcal{P}[B' \cup \{b\}]$.

The scheme *FinSubIndA2* deals with a non empty, finite set $\mathcal{X}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 2) For every non empty subset B of $\mathcal{X}$, $\mathcal{P}[B]$

provided

- for every element x of $\mathcal{X}$, $\mathcal{P}[\{x\}]$ and
- for every non empty subsets B_1, B_2 of $\mathcal{X}$ such that $\mathcal{P}[B_1]$ and $\mathcal{P}[B_2]$ holds $\mathcal{P}[B_1 \cup B_2]$.

Let us consider a function f and sets a, y. Now we state the propositions:

(1) Suppose $\operatorname{dom} f$ is subset-closed and closed under directed unions and f is preserving directed unions. Then if $a \in \operatorname{dom} f$ and $y \in f(a)$, then there exists a set b such that b is finite and $b \subseteq a$ and $y \in f(b)$.

Proof: Reconsider $C = \operatorname{dom} f$ as a closed under directed unions, subset-closed set. Reconsider $A = \{b$, where b is a subset of $a : b$ is finite$\}$ as a set. A is $\cup$-directed by [3, (76)], [4, (7)]. $\bigcup A = a$ by [3, (31)]. $A \subseteq C$. Consider B being a set such that $y \in B$ and $B \in f^{\circ} A$. Consider b being an object such that $b \in \operatorname{dom} f$ and $b \in A$ and $B = f(b)$. $\square$

(2) Suppose dom f is subset-closed and f is preserving arbitrary unions and dom f is closed under directed unions. Then if $a \in \operatorname{dom} f$ and $y \in f(a)$, then there exists a set x such that $x \in a$ and $y \in f(\{x\})$.
Proof: Consider b being a set such that b is finite and $b \subseteq a$ and $y \in f(b)$. Reconsider $A =$ the set of all $\{x\}$ where x is an element of b as a set. $A \subseteq \operatorname{dom} f$. $b \subseteq \bigcup A$ by [3, (74), (31)]. Consider Y being a set such that $y \in Y$ and $Y \in f^{\circ}A$. Consider X being an object such that $X \in \operatorname{dom} f$ and $X \in A$ and $Y = f(X)$. Consider x being an element of b such that $X = \{x\}$. □

2. On the Union and the Intersection of Two Relational Structures

Let R_1, R_2 be relational structures. The functor Union(R_1, R_2) yielding a strict relational structure is defined by

(Def. 1) the carrier of *it* = (the carrier of R_1)∪(the carrier of R_2) and the internal relation of *it* = (the internal relation of R_1) ∪ (the internal relation of R_2).

One can check that the functor is commutative. The functor Meet(R_1, R_2) yielding a strict relational structure is defined by

(Def. 2) the carrier of *it* = (the carrier of R_1)∩(the carrier of R_2) and the internal relation of *it* = (the internal relation of R_1) ∩ (the internal relation of R_2).

Note that the functor is commutative.

Let R_1 be a relational structure and R_2 be a non empty relational structure. Let us observe that Union(R_1, R_2) is non empty.

3. Ordinary Properties of Maps

Let A be a set. Let us note that there exists a function from 2^A into 2^A which preserves ∩ and ∪.

Let f be a function from 2^A into 2^A preserves ∩. Observe that Flip f preserves ∪.

Let f be a function from 2^A into 2^A preserves ∪. Note that Flip f preserves ∩.

Now we state the proposition:

(3) Let us consider a non empty set A, and functions f, g from 2^A into 2^A. Suppose $f \dot{\subseteq} g$. Then Flip $g \dot{\subseteq}$ Flip f.
Proof: Set $f_1 =$ Flip f. Set $\mathfrak{g} =$ Flip g. For every set x such that $x \in \operatorname{dom} \mathfrak{g}$ holds $\mathfrak{g}(x) \subseteq f_1(x)$ by [15, (12)]. □

One can verify that there exists a relational structure which is non empty, mediate, and transitive.

Let R be a total, mediate relational structure. One can verify that the internal relation of R is mediate.

Let us consider relational structures L_1, L_2. Now we state the propositions:

(4) Suppose the relational structure of L_1 = the relational structure of L_2 and L_1 is mediate. Then L_2 is mediate.

(5) Suppose the relational structure of L_1 = the relational structure of L_2 and L_1 is serial. Then L_2 is serial.

Now we state the propositions:

(6) Let us consider a non empty set A, and functions L, U from 2^A into 2^A. Suppose $U = \mathrm{Flip}\, L$ and for every subset X of A, $L(X) \subseteq L(L(X))$. Let us consider a subset X of A. Then $U(U(X)) \subseteq U(X)$.

(7) Let us consider a non empty relational structure R, and elements a, b of R. Suppose $\langle a, b \rangle \in$ the internal relation of R. Then $a \in \mathrm{UAp}(\{b\})$.

Let us consider a non empty relational structure R and subsets A, B of R. Now we state the propositions:

(8) $(\mathrm{UAp}(R))(A \cup B) = (\mathrm{UAp}(R))(A) \cup (\mathrm{UAp}(R))(B)$.

(9) $(\mathrm{LAp}(R))(A \cap B) = (\mathrm{LAp}(R))(A) \cap (\mathrm{LAp}(R))(B)$.

(10) Let us consider a non empty relational structure R. Then $(\mathrm{UAp}(R))(\emptyset) = \emptyset$.

Let us consider non empty relational structures R_1, R_2, a subset X of R_1, and a subset Y of R_2.

Let us assume that the relational structure of R_1 = the relational structure of R_2 and $X = Y$. Now we state the propositions:

(11) $\mathrm{UAp}(X) = \mathrm{UAp}(Y)$.

(12) $\mathrm{LAp}(X) = \mathrm{LAp}(Y)$.

4. On the Relational Structure Generated by Rough Approximation

Let R be a non empty relational structure and H be a function from $2^{(\text{the carrier of } R)}$ into $2^{(\text{the carrier of } R)}$. The functor GeneratedRelation(R, H) yielding a binary relation on the carrier of R is defined by

(Def. 3) for every elements x, y of R, $\langle x, y \rangle \in$ *it* iff $x \in H(\{y\})$.

The functor GeneratedRelStr H yielding a relational structure is defined by the term

(Def. 4) $\langle$the carrier of R, GeneratedRelation$(R, H)\rangle$.

Let us note that GeneratedRelStr H is non empty.

Now we state the proposition:

(13) Let us consider a finite, non empty relational structure R, and a function H from 2^{α} into 2^{α}. Suppose $H(\emptyset) = \emptyset$ and H preserves $\cup$. Then UAp(GeneratedRelStr H) $= H$, where α is the carrier of R.

PROOF: For every subset A of dom H such that $\bigcup A \in$ dom H holds $H(\bigcup A) = \bigcup(H^{\circ}A)$ by [3, (2)], [14, (14)], [3, (25)], [1, (59)]. Set $H_1 =$ UAp(GeneratedRelStr H). For every subset X of R, $H_1(X) = H(X)$ by [8, (7)], [13, (9)], [3, (31)], (2). □

5. Construction Revisited: Yao's [17] Theorem 3

Now we state the proposition:

(14) Let us consider a finite, non empty set A, and functions L, H from 2^A into 2^A. Suppose $L =$ Flip H. Then $H(\emptyset) = \emptyset$ and for every subsets X, Y of A, $H(X \cup Y) = H(X) \cup H(Y)$ if and only if there exists a non empty, finite relational structure R such that the carrier of $R = A$ and LAp$(R) = L$ and UAp$(R) = H$ and for every elements x, y of R, $\langle x, y\rangle \in$ the internal relation of R iff $x \in H(\{y\})$.

PROOF: If $H(\emptyset) = \emptyset$ and for every subsets X, Y of A, $H(X \cup Y) = H(X) \cup H(Y)$, then there exists a non empty, finite relational structure R such that the carrier of $R = A$ and LAp$(R) = L$ and UAp$(R) = H$ and for every elements x, y of R, $\langle x, y\rangle \in$ the internal relation of R iff $x \in H(\{y\})$ by [3, (31)], [2, (5)], [3, (50), (48), (116)]. □

6. Transitive Binary Relations

Let us consider a non empty, transitive relational structure R and a subset X of R. Now we state the propositions:

(15) LAp$(X) \subseteq$ LAp(LAp(X)).

PROOF: Consider y being an element of R such that $y = x$ and $[y]_{\alpha} \subseteq X$, where α is the internal relation of R. $[y]_{\alpha} \subseteq$ LAp(X), where α is the internal relation of R by [16, (169)]. □

(16) UAp(UAp(X)) $\subseteq$ UAp(X).

(17) Let us consider a finite, non empty set A, and a function L from 2^A into 2^A. Suppose $L(A) = A$ and for every subset X of A, $L(X) \subseteq L(L(X))$ and for every subsets X, Y of A, $L(X \cap Y) = L(X) \cap L(Y)$. Then there exists a non empty, finite, transitive relational structure R such that

(i) the carrier of $R = A$, and

(ii) $L = \mathrm{LAp}(R)$.

Proof: Set $H = \mathrm{Flip}\, L$. Consider R being a non empty, finite relational structure such that the carrier of $R = A$ and $\mathrm{LAp}(R) = L$ and $\mathrm{UAp}(R) = H$ and for every elements x, y of R, $\langle x, y\rangle \in$ the internal relation of R iff $x \in H(\{y\})$. For every objects x, y, z such that x, y, $z \in$ the carrier of R and $\langle x, y\rangle$, $\langle y, z\rangle \in$ the internal relation of R holds $\langle x, z\rangle \in$ the internal relation of R by [3, (31)], [2, (5)], (6). □

(18) Let us consider a non empty, finite set A, and a function U from 2^A into 2^A. Suppose $U(\emptyset) = \emptyset$ and for every subset X of A, $U(U(X)) \subseteq U(X)$ and for every subsets X, Y of A, $U(X \cup Y) = U(X) \cup U(Y)$. Then there exists a non empty, finite, transitive relational structure R such that

(i) the carrier of $R = A$, and

(ii) $U = \mathrm{UAp}(R)$.

The theorem is a consequence of (17).

7. Mediate and Transitive Binary Relations

Let us consider a non empty, mediate, transitive relational structure R and a subset X of R. Now we state the propositions:

(19) $\mathrm{LAp}(X) = \mathrm{LAp}(\mathrm{LAp}(X))$. The theorem is a consequence of (15).

(20) $\mathrm{UAp}(X) = \mathrm{UAp}(\mathrm{UAp}(X))$. The theorem is a consequence of (16).

(21) Let us consider a non empty, finite set A, and a function L from 2^A into 2^A. Suppose $L(A) = A$ and for every subset X of A, $L(X) = L(L(X))$ and for every subsets X, Y of A, $L(X \cap Y) = L(X) \cap L(Y)$. Then there exists a non empty, mediate, finite, transitive relational structure R such that

(i) the carrier of $R = A$, and

(ii) $L = \mathrm{LAp}(R)$.

The theorem is a consequence of (17), (13), and (4).

(22) Let us consider a non empty, finite set A, and a function U from 2^A into 2^A. Suppose $U(\emptyset) = \emptyset$ and for every subset X of A, $U(U(X)) = U(X)$ and for every subsets X, Y of A, $U(X \cup Y) = U(X) \cup U(Y)$. Then there exists a non empty, mediate, finite, transitive relational structure R such that

(i) the carrier of $R = A$, and

(ii) $U = \mathrm{UAp}(R)$.

PROOF: Consider R being a non empty, finite, transitive relational structure such that the carrier of $R = A$ and $U = \mathrm{UAp}(R)$. For every objects x, y such that $x, y \in$ the carrier of R holds if $\langle x, y\rangle \in$ the internal relation of R, then there exists an object z such that $z \in$ the carrier of R and $\langle x, z\rangle$, $\langle z, y\rangle \in$ the internal relation of R by [3, (31)], [16, (169)], [8, (5)]. □

8. Alliance Binary Relations

Let X be a set and R be a binary relation on X. We say that R is a positive alliance in X if and only if

(Def. 5) for every objects x, y such that $x, y \in X$ and $\langle x, y\rangle \notin R$ there exists an object z such that $z \in X$ and $\langle x, z\rangle \in R$ and $\langle z, y\rangle \notin R$.

We say that R is a negative alliance in X if and only if

(Def. 6) for every objects x, y such that $x, y \in X$ holds if there exists an object z such that $z \in X$ and $\langle x, z\rangle \in R$ and $\langle z, y\rangle \notin R$, then $\langle x, y\rangle \notin R$.

We say that R is an alliance in X if and only if

(Def. 7) R is a negative alliance in X and R is a positive alliance in X.

Let R be a non empty relational structure. We say that R is positive alliance if and only if

(Def. 8) the internal relation of R is a positive alliance in the carrier of R.

We say that R is negative alliance if and only if

(Def. 9) the internal relation of R is a negative alliance in the carrier of R.

We say that R is alliance if and only if

(Def. 10) the internal relation of R is an alliance in the carrier of R.

Let us observe that every non empty relational structure which is reflexive is also positive alliance and every non empty relational structure which is discrete is also negative alliance.

There exists a non empty relational structure which is positive alliance and negative alliance and every non empty relational structure which is alliance is also positive alliance and negative alliance and every non empty relational structure which is positive alliance and negative alliance is also alliance.

Every non empty relational structure which is positive alliance is also serial and every non empty relational structure which is transitive and serial is also positive alliance.

Let us consider non empty relational structures L_1, L_2. Now we state the propositions:

(23) Suppose the relational structure of $L_1 =$ the relational structure of L_2 and L_1 is negative alliance. Then L_2 is negative alliance.

(24) Suppose the relational structure of L_1 = the relational structure of L_2 and L_1 is positive alliance. Then L_2 is positive alliance.

(25) Suppose the relational structure of L_1 = the relational structure of L_2 and L_1 is alliance. Then L_2 is alliance.

9. Preparation for Translation of Proposition 10 (7H')

Let R be a non empty relational structure. We say that R is satisfying (7H') if and only if

(Def. 11) for every subset X of R, $(\mathrm{UAp}(X))^{\mathrm{c}} \subseteq \mathrm{UAp}((\mathrm{UAp}(X))^{\mathrm{c}})$.

We say that R is satisfying (7L') if and only if

(Def. 12) for every subset X of R, $\mathrm{LAp}((\mathrm{LAp}(X))^{\mathrm{c}}) \subseteq (\mathrm{LAp}(X))^{\mathrm{c}}$.

Let us consider a finite, non empty relational structure R. Now we state the propositions:

(26) If R is satisfying (7L'), then R is satisfying (7H').
Proof: For every subset X of R, $(\mathrm{UAp}(X))^{\mathrm{c}} \subseteq \mathrm{UAp}((\mathrm{UAp}(X))^{\mathrm{c}})$ by [8, (8)], [15, (12)], [8, (9)]. □

(27) If R is satisfying (7H'), then R is serial.
Proof: Set $U = \mathrm{UAp}(R)$. For every subsets X, Y of R, $U(X \cup Y) = U(X) \cup U(Y)$ by [8, (13)]. Consider S being a non empty, finite, serial relational structure such that the carrier of S = the carrier of R and $U = \mathrm{UAp}(S)$. □

(28) If R is satisfying (7L'), then R is serial.

Let us observe that every finite, non empty relational structure which is satisfying (7H') is also serial.

Now we state the proposition:

(29) Let us consider a non empty relational structure R. Suppose for every subset X of R, $\mathrm{UAp}((\mathrm{UAp}(X))^{\mathrm{c}}) \subseteq (\mathrm{UAp}(X))^{\mathrm{c}}$. Let us consider a subset X of R. Then $(\mathrm{LAp}(X))^{\mathrm{c}} \subseteq \mathrm{LAp}((\mathrm{LAp}(X))^{\mathrm{c}})$.

Let us consider a non empty set A, functions L, U from 2^A into 2^A, and a subset X of A. Now we state the propositions:

(30) Suppose $U = \mathrm{Flip}\, L$ and for every subset X of A, $L(X)^{\mathrm{c}} \subseteq L(L(X)^{\mathrm{c}})$. Then $U(U(X)^{\mathrm{c}}) \subseteq U(X)^{\mathrm{c}}$.

(31) Suppose $U = \mathrm{Flip}\, L$ and for every subset X of A, $U(U(X)^{\mathrm{c}}) \subseteq U(X)^{\mathrm{c}}$. Then $L(X)^{\mathrm{c}} \subseteq L(L(X)^{\mathrm{c}})$.

(32) Suppose $U = \mathrm{Flip}\, L$ and for every subset X of A, $L(L(X)^{\mathrm{c}}) \subseteq L(X)^{\mathrm{c}}$. Then $U(X)^{\mathrm{c}} \subseteq U(U(X)^{\mathrm{c}})$.

10. Translation Continued

Now we state the propositions:

(33) Let us consider a finite, positive alliance, non empty relational structure R, and an element x of R. Then $(\mathrm{UAp}(R))(\{x\})^{\mathrm{c}} \subseteq (\mathrm{UAp}(R))((\mathrm{UAp}(R))(\{x\})^{\mathrm{c}})$. The theorem is a consequence of (10), (8), and (13).

(34) Let us consider a non empty, finite set A, and a function U from 2^A into 2^A. Suppose $U(\emptyset) = \emptyset$ and for every subset X of A, $U(X)^{\mathrm{c}} \subseteq U(U(X)^{\mathrm{c}})$ and for every subsets X, Y of A, $U(X \cup Y) = U(X) \cup U(Y)$. Then there exists a positive alliance, finite, non empty relational structure R such that

(i) the carrier of $R = A$, and

(ii) $U = \mathrm{UAp}(R)$.

PROOF: Consider R being a non empty, finite relational structure such that the carrier of $R = A$ and $\mathrm{LAp}(R) = \mathrm{Flip}\,U$ and $\mathrm{UAp}(R) = U$ and for every elements x, y of R, $\langle x, y\rangle \in$ the internal relation of R iff $x \in U(\{y\})$. Set $X =$ the carrier of R. Set $I =$ the internal relation of R. For every objects x, y such that $x, y \in X$ and $\langle x, y\rangle \notin I$ there exists an object z such that $z \in X$ and $\langle x, z\rangle \in I$ and $\langle z, y\rangle \notin I$ by [8, (7)], [16, (169)]. □

(35) Let us consider a non empty, finite set A, and a function L from 2^A into 2^A. Suppose $L(A) = A$ and for every subset X of A, $L(L(X)^{\mathrm{c}}) \subseteq L(X)^{\mathrm{c}}$ and for every subsets X, Y of A, $L(X \cap Y) = L(X) \cap L(Y)$. Then there exists a positive alliance, finite, non empty relational structure R such that

(i) the carrier of $R = A$, and

(ii) $L = \mathrm{LAp}(R)$.

The theorem is a consequence of (32) and (34).

(36) Let us consider a finite, negative alliance, non empty relational structure R, and an element x of R. Then $(\mathrm{UAp}(R))((\mathrm{UAp}(R))(\{x\})^{\mathrm{c}}) \subseteq (\mathrm{UAp}(R))(\{x\})^{\mathrm{c}}$. The theorem is a consequence of (10), (8), and (13).

Let us consider a finite, negative alliance, non empty relational structure R and a subset X of R. Now we state the propositions:

(37) $\mathrm{UAp}((\mathrm{UAp}(X))^{\mathrm{c}}) \subseteq (\mathrm{UAp}(X))^{\mathrm{c}}$.

PROOF: Define $\mathcal{P}[\text{subset of } R] \equiv \mathrm{UAp}((\mathrm{UAp}(\$_1))^{\mathrm{c}}) \subseteq (\mathrm{UAp}(\$_1))^{\mathrm{c}}$. For every subset B of R and for every element b of R such that $\mathcal{P}[B]$ and $b \notin B$ holds $\mathcal{P}[B \cup \{b\}]$ by [8, (13)], (36). For every subset B of R, $\mathcal{P}[B]$ from *FinSubIndA1*. □

(38) $(\mathrm{LAp}(X))^{\mathrm{c}} \subseteq \mathrm{LAp}((\mathrm{LAp}(X))^{\mathrm{c}})$. The theorem is a consequence of (37) and (29).

(39) Let us consider a non empty, finite set A, and a function U from 2^A into 2^A. Suppose $U(\emptyset) = \emptyset$ and for every subset X of A, $U(U(X)^{\mathrm{c}}) \subseteq U(X)^{\mathrm{c}}$ and for every subsets X, Y of A, $U(X \cup Y) = U(X) \cup U(Y)$. Then there exists a negative alliance, finite, non empty relational structure R such that

  (i) the carrier of $R = A$, and

  (ii) $U = \mathrm{UAp}(R)$.

PROOF: Consider R being a non empty, finite relational structure such that the carrier of $R = A$ and $\mathrm{LAp}(R) = \mathrm{Flip}\, U$ and $\mathrm{UAp}(R) = U$ and for every elements x, y of R, $\langle x, y\rangle \in$ the internal relation of R iff $x \in U(\{y\})$. Set $X =$ the carrier of R. Set $I =$ the internal relation of R. For every objects x, y such that $x, y \in X$ holds if there exists an object z such that $z \in X$ and $\langle x, z\rangle \in I$ and $\langle z, y\rangle \notin I$, then $\langle x, y\rangle \notin I$ by [16, (169)]. □

(40) Let us consider a non empty, finite set A, and a function L from 2^A into 2^A. Suppose $L(A) = A$ and for every subset X of A, $L(X)^{\mathrm{c}} \subseteq L(L(X)^{\mathrm{c}})$ and for every subsets X, Y of A, $L(X \cap Y) = L(X) \cap L(Y)$. Then there exists a negative alliance, finite, non empty relational structure R such that

  (i) the carrier of $R = A$, and

  (ii) $L = \mathrm{LAp}(R)$.

The theorem is a consequence of (30) and (39).

(41) Let us consider a non empty, finite set A, and a function U from 2^A into 2^A. Suppose $U(\emptyset) = \emptyset$ and for every subset X of A, $U(U(X)^{\mathrm{c}}) = U(X)^{\mathrm{c}}$ and for every subsets X, Y of A, $U(X \cup Y) = U(X) \cup U(Y)$. Then there exists an alliance, finite, non empty relational structure R such that

  (i) the carrier of $R = A$, and

  (ii) $U = \mathrm{UAp}(R)$.

The theorem is a consequence of (39), (34), (24), and (23).

(42) Let us consider a non empty, finite set A, and a function L from 2^A into 2^A. Suppose $L(A) = A$ and for every subset X of A, $L(X)^{\mathrm{c}} = L(L(X)^{\mathrm{c}})$ and for every subsets X, Y of A, $L(X \cap Y) = L(X) \cap L(Y)$. Then there exists an alliance, finite, non empty relational structure R such that

  (i) the carrier of $R = A$, and

  (ii) $L = \mathrm{LAp}(R)$.

Proof: Set $U = \mathrm{Flip}\, L$. For every subset X of A, $U(U(X)^{\mathrm{c}}) = U(X)^{\mathrm{c}}$ by (30), [8, (23)], (31). Consider R being an alliance, finite, non empty relational structure such that the carrier of $R = A$ and $U = \mathrm{UAp}(R)$. □

11. On the Uniqueness of Binary Relations to Generate Rough Sets

Let us consider non empty relational structures R_1, R_2, R, a subset X of R, a subset X_1 of R_1, and a subset X_2 of R_2.

Let us assume that $R = \mathrm{Union}(R_1, R_2)$ and $X = X_1$ and $X = X_2$ and the carrier of $R_1 =$ the carrier of R_2. Now we state the propositions:

(43) $\mathrm{UAp}(X) = \mathrm{UAp}(X_1) \cup \mathrm{UAp}(X_2)$.
Proof: $\mathrm{UAp}(X) \subseteq \mathrm{UAp}(X_1) \cup \mathrm{UAp}(X_2)$ by [16, (169)]. $\mathrm{UAp}(X_1) \cup \mathrm{UAp}(X_2) \subseteq \mathrm{UAp}(X)$ by [16, (169)]. □

(44) $\mathrm{LAp}(X) = \mathrm{LAp}(X_1) \cap \mathrm{LAp}(X_2)$.
Proof: $\mathrm{LAp}(X) \subseteq \mathrm{LAp}(X_1) \cap \mathrm{LAp}(X_2)$ by [16, (169)]. $\mathrm{LAp}(X_1) \cap \mathrm{LAp}(X_2) \subseteq \mathrm{LAp}(X)$ by [16, (169)]. □

Let us consider non empty relational structures R_1, R_2.

Let us assume that the carrier of $R_1 =$ the carrier of R_2 and the internal relation of $R_1 \subseteq$ the internal relation of R_2. Now we state the propositions:

(45) $\mathrm{UAp}(R_1) \dot{\subseteq} \mathrm{UAp}(R_2)$.
Proof: For every set x such that $x \in \mathrm{dom}\, \mathrm{UAp}(R_1)$ holds $(\mathrm{UAp}(R_1))(x) \subseteq (\mathrm{UAp}(R_2))(x)$ by [16, (124)]. □

(46) $\mathrm{LAp}(R_2) \dot{\subseteq} \mathrm{LAp}(R_1)$.
Proof: For every set x such that $x \in \mathrm{dom}\, \mathrm{LAp}(R_2)$ holds $(\mathrm{LAp}(R_2))(x) \subseteq (\mathrm{LAp}(R_1))(x)$ by [16, (124)]. □

Let us consider non empty relational structures R_1, R_2, R, a subset X of R, a subset X_1 of R_1, and a subset X_2 of R_2.

Let us assume that $R = \mathrm{Meet}(R_1, R_2)$ and $X = X_1$ and $X = X_2$ and the carrier of $R_1 =$ the carrier of R_2. Now we state the propositions:

(47) $\mathrm{UAp}(X) \subseteq \mathrm{UAp}(X_1) \cap \mathrm{UAp}(X_2)$. The theorem is a consequence of (45).

(48) $\mathrm{LAp}(X_1) \cup \mathrm{LAp}(X_2) \subseteq \mathrm{LAp}(X)$. The theorem is a consequence of (46).

Let us consider non empty relational structures R_1, R_2. Now we state the propositions:

(49) Suppose the carrier of $R_1 =$ the carrier of R_2 and $\mathrm{UAp}(R_1) \dot{\subseteq} \mathrm{UAp}(R_2)$. Then the internal relation of $R_1 \subseteq$ the internal relation of R_2.

(50) Suppose the carrier of $R_1 =$ the carrier of R_2 and $\mathrm{UAp}(R_1) = \mathrm{UAp}(R_2)$. Then the internal relation of $R_1 =$ the internal relation of R_2.

(51) Suppose the carrier of R_1 = the carrier of R_2. Then $\mathrm{UAp}(R_1) = \mathrm{UAp}(R_2)$ if and only if the internal relation of R_1 = the internal relation of R_2.

(52) Suppose the carrier of R_1 = the carrier of R_2 and $\mathrm{LAp}(R_1) \dot{\subseteq} \mathrm{LAp}(R_2)$. Then the internal relation of $R_2 \subseteq$ the internal relation of R_1.

(53) Suppose the carrier of R_1 = the carrier of R_2 and $\mathrm{LAp}(R_1) = \mathrm{LAp}(R_2)$. Then the internal relation of R_2 = the internal relation of R_1.

(54) Suppose the carrier of R_1 = the carrier of R_2. Then $\mathrm{LAp}(R_1) = \mathrm{LAp}(R_2)$ if and only if the internal relation of R_1 = the internal relation of R_2. The theorem is a consequence of (53) and (12).

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Tarski Geometry Axioms – Part II

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Summary. In our earlier article [12], the first part of axioms of geometry proposed by Alfred Tarski [14] was formally introduced by means of Mizar proof assistant [9]. We defined a structure `TarskiPlane` with the following predicates:

- of betweenness `between` (a ternary relation),
- of congruence of segments `equiv` (quarternary relation),

which satisfy the following properties:

- congruence symmetry (A1),
- congruence equivalence relation (A2),
- congruence identity (A3),
- segment construction (A4),
- SAS (A5),
- betweenness identity (A6),
- Pasch (A7).

Also a simple model, which satisfies these axioms, was previously constructed, and described in [6]. In this paper, we deal with four remaining axioms, namely:

- the lower dimension axiom (A8),
- the upper dimension axiom (A9),
- the Euclid axiom (A10),
- the continuity axiom (A11).

They were introduced in the form of Mizar attributes. Additionally, the relation of congruence of triangles `cong` is introduced via congruence of sides (SSS).

In order to show that the structure which satisfies all eleven Tarski's axioms really exists, we provided a proof of the registration of a cluster that the Euclidean

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plane, or rather a natural [5] extension of ordinary metric structure `Euclid 2` satisfies all these attributes.

Although the tradition of the mechanization of Tarski's geometry in Mizar is not as long as in Coq [11], first approaches to this topic were done in Mizar in 1990 [16] (even if this article started formal Hilbert axiomatization of geometry, and parallel development was rather unlikely at that time [8]). Connection with another proof assistant should be mentioned – we had some doubts about the proof of the Euclid's axiom and inspection of the proof taken from Archive of Formal Proofs of Isabelle [10] clarified things a bit. Our development allows for the future faithful mechanization of [13] and opens the possibility of automatically generated Prover9 proofs which was useful in the case of lattice theory [7].

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1. Preliminaries

Now we state the propositions:

(1) Let us consider real numbers r, s, t, u. Suppose $s \neq 0$ and $t \neq 0$ and $r^{\mathbf{2}} = s^{\mathbf{2}} + t^{\mathbf{2}} - 2 \cdot s \cdot t \cdot u$. Then $u = \frac{r^{\mathbf{2}} - s^{\mathbf{2}} - t^{\mathbf{2}}}{-2 \cdot s \cdot t}$.

(2) Let us consider a natural number n, and elements u, v of $\mathcal{E}^n_{\mathrm{T}}$. Then $u + 0 \cdot v = u$.

(3) Let us consider a natural number n, real numbers r, s, and elements u, v, w of $\mathcal{E}^n_{\mathrm{T}}$. If $r \cdot u - r \cdot v = s \cdot w - s \cdot u$, then $(r + s) \cdot u = r \cdot v + s \cdot w$. The theorem is a consequence of (2).

(4) Let us consider real numbers r, s. If $0 < r$ and $0 < s$, then $0 \leqslant \frac{r}{r+s} \leqslant 1$.

(5) Let us consider a real number a. Then $\cos(3 \cdot \pi - a) = -\cos a$.

Let us consider a natural number n and elements a, b, c of $\mathcal{E}^n_{\mathrm{T}}$. Now we state the propositions:

(6) If $a - c = b - c$, then $a = b$.

(7) $c - a - (b - a) = c - b$.

(8) Let us consider real numbers a, b, c, d. Then $\rho([a, b], [c, d]) = \sqrt{(a - c)^{\mathbf{2}} + (b - d)^{\mathbf{2}}}$.

(9) $\rho([0, 0], [1, 0]) = 1$. The theorem is a consequence of (8).

(10) $\rho([0, 0], [0, 1]) = 1$. The theorem is a consequence of (8).

(11) $\rho([1, 0], [0, 1]) = \sqrt{2}$. The theorem is a consequence of (8).

Let n be a natural number. The functor TarskiEuclidSpace n yielding a metric Tarski structure is defined by the term

(Def. 1) the naturally generated Tarski extension of $\mathcal{E}^n$.

The functor TarskiEuclid2Space yielding a metric Tarski structure is defined by the term

(Def. 2) TarskiEuclidSpace 2.

2. Basic Properties of the Euclidean Plane

Let n be a natural number. Let us observe that TarskiEuclidSpace n is non empty and TarskiEuclid2Space is reflexive, symmetric, and discernible.

Let n be a natural number. One can check that TarskiEuclidSpace n is reflexive, symmetric, and discernible.

Let P be a point of TarskiEuclidSpace n. The functor $\hat{P}$ yielding an element of $\mathcal{E}^n_{\mathrm{T}}$ is defined by the term

(Def. 3) P.

Let P be a point of TarskiEuclid2Space. The functor $\hat{P}$ yielding an element of $\mathcal{E}^2_{\mathrm{T}}$ is defined by the term

(Def. 4) P.

The functor $\tilde{P}$ yielding a point of $\mathcal{E}^2$ is defined by the term

(Def. 5) P.

The functor $\breve{P}$ yielding an element of $\mathcal{R}^2$ is defined by the term

(Def. 6) P.

Now we state the propositions:

(12) Let us consider a natural number n, points p, q of TarskiEuclidSpace n, and elements p_1, q_1 of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $p = p_1$ and $q = q_1$. Then

(i) $\rho(p,q) = \rho^n(p_1,q_1)$, and

(ii) $\rho(p,q) = |p_1 - q_1|$.

(13) Let us consider points a, b, c of TarskiEuclid2Space. Then $(\rho(c,a))^{\mathbf{2}} = (\rho(a,b))^{\mathbf{2}} + (\rho(b,c))^{\mathbf{2}} - 2 \cdot \rho(a,b) \cdot \rho(b,c) \cdot \cos\measuredangle(\hat{a},\hat{b},\hat{c})$. The theorem is a consequence of (12).

(14) Let us consider points a, b, c, e, f, g of TarskiEuclid2Space. Suppose $\hat{a}$, $\hat{b}$, $\hat{c}$ form a triangle and $\measuredangle(\hat{a},\hat{b},\hat{c}) < \pi$ and $\measuredangle(\hat{e},\hat{c},\hat{a}) = \frac{\measuredangle(\hat{b},\hat{c},\hat{a})}{3}$ and $\measuredangle(\hat{c},\hat{a},\hat{e}) = \frac{\measuredangle(\hat{c},\hat{a},\hat{b})}{3}$ and $\measuredangle(\hat{a},\hat{b},\hat{f}) = \frac{\measuredangle(\hat{a},\hat{b},\hat{c})}{3}$ and $\measuredangle(\hat{f},\hat{a},\hat{b}) = \frac{\measuredangle(\hat{c},\hat{a},\hat{b})}{3}$ and $\measuredangle(\hat{b},\hat{c},\hat{g}) = \frac{\measuredangle(\hat{b},\hat{c},\hat{a})}{3}$ and $\measuredangle(\hat{g},\hat{b},\hat{c}) = \frac{\measuredangle(\hat{a},\hat{b},\hat{c})}{3}$. Then

(i) $\rho(f,e) = \rho(g,f)$, and

(ii) $\rho(f,e) = \rho(e,g)$, and

(iii) $\rho(g, f) = \rho(e, g)$.

The theorem is a consequence of (12).

(15) Let us consider a natural number n, elements p, q of TarskiEuclidSpace n, and elements p_1, q_1 of $\mathcal{E}^n$. If $p = p_1$ and $q = q_1$, then $\rho(p, q) = \rho(p_1, q_1)$.

(16) Let us consider points p, q of TarskiEuclid2Space. Then $\rho(p, q) = \sqrt{((\hat{p})_{\mathbf{1}} - (\hat{q})_{\mathbf{1}})^{\mathbf{2}} + ((\hat{p})_{\mathbf{2}} - (\hat{q})_{\mathbf{2}})^{\mathbf{2}}}$.

(17) Let us consider points A, B of TarskiEuclid2Space. Then

(i) $\rho(A, B) = |\hat{A} - \hat{B}|$, and

(ii) $\rho(A, B) = |\breve{A} - \breve{B}|$.

(18) Let us consider points a, b, c, d of TarskiEuclid2Space. Then $|\hat{a} - \hat{b}| = |\hat{c} - \hat{d}|$ if and only if $\overline{ab} \cong \overline{cd}$. The theorem is a consequence of (17).

(19) Let us consider points p, q, r of TarskiEuclid2Space. Then p is between q and r if and only if $\hat{p} \in \mathcal{L}(\hat{q}, \hat{r})$. The theorem is a consequence of (15).

From now on n denotes a natural number.

Now we state the propositions:

(20) Let us consider points p, q, r of TarskiEuclid2Space. Then q lies between p and r if and only if $\hat{q} \in \mathcal{L}(\hat{p}, \hat{r})$. The theorem is a consequence of (19).

(21) Let us consider points a, b of TarskiEuclid2Space. Then

(i) a lies between a and b, and

(ii) b lies between a and b.

The theorem is a consequence of (20).

(22) Let us consider points a, b, c of TarskiEuclid2Space. If b lies between a and c, then b lies between c and a. The theorem is a consequence of (20).

(23) Let us consider points a, b of TarskiEuclid2Space. If b lies between a and a, then $a = b$. The theorem is a consequence of (20).

(24) Let us consider points a, b of TarskiEuclid2Space. Then $a = b$ if and only if $\rho(a, b) = 0$. The theorem is a consequence of (12).

(25) Let us consider points a, b, c, d of TarskiEuclid2Space. If $\rho(a, b) + \rho(c, d) = 0$, then $a = b$ and $c = d$. The theorem is a consequence of (24).

(26) Let us consider points a, b, c, a_1, b_1, c_1 of TarskiEuclid2Space. Then $\triangle abc \cong \triangle a_1b_1c_1$ if and only if $\rho(a, b) = \rho(a_1, b_1)$ and $\rho(a, c) = \rho(a_1, c_1)$ and $\rho(b, c) = \rho(b_1, c_1)$.

(27) Let us consider points a, b, c of TarskiEuclid2Space. Then b lies between a and c if and only if $\rho(a, c) = \rho(a, b) + \rho(b, c)$.

(28) Let us consider points a, b, c, d of TarskiEuclid2Space. Then $(\rho(a,b))^{\mathbf{2}} = (\rho(c,d))^{\mathbf{2}}$ if and only if $\overline{ab} \cong \overline{cd}$.

(29) Let us consider a point a of TarskiEuclid2Space. Then a lies between a and a.

3. Ordered Affine Space Generated by $\mathcal{E}^2_{\rm T}$

Now we state the proposition:

(30) OASpace $\mathcal{E}^2_{\rm T}$ is an ordered affine space.
Proof: There exist vectors u, v of $\mathcal{E}^2_{\rm T}$ such that for every real numbers a, b such that $a \cdot u + b \cdot v = 0_{\mathcal{E}^2_{\rm T}}$ holds $a = 0$ and $b = 0$ by [4, (58), (56), (52)]. □

Let us consider elements a, b, c of OASpace $\mathcal{E}^2_{\rm T}$. Now we state the propositions:

(31) b is a midpoint of a, c if and only if $a = b$ or $b = c$ or there exist points u, v of $\mathcal{E}^2_{\rm T}$ such that $u = a$ and $v = c$ and $b \in \mathcal{L}(u, v)$. The theorem is a consequence of (3), (4), (30), and (2).

(32) b is a midpoint of a, c if and only if there exist points u, v of $\mathcal{E}^2_{\rm T}$ such that $u = a$ and $v = c$ and $b \in \mathcal{L}(u, v)$. The theorem is a consequence of (31).

(33) Let us consider elements a, b, c of OASpace $\mathcal{E}^2_{\rm T}$, and points a_1, b_1, c_1 of TarskiEuclid2Space. Suppose $a = a_1$ and $b = b_1$ and $c = c_1$. Then b is a midpoint of a, c if and only if b_1 lies between a_1 and c_1. The theorem is a consequence of (32) and (20).

4. Eucldiean Plane Satisfies First 7 Tarski's Axioms

Let us consider elements A, B, C, D of $\mathcal{E}^2_{\rm T}$. Now we state the propositions:

(34) If $B \in \mathcal{L}(A, C)$ and $C \in \mathcal{L}(A, D)$, then $B \in \mathcal{L}(A, D)$.

(35) If $B \neq C$ and $B \in \mathcal{L}(A, C)$ and $C \in \mathcal{L}(B, D)$, then $C \in \mathcal{L}(A, D)$. The theorem is a consequence of (30) and (32).

(36) Let us consider points p, q, r, s of TarskiEuclid2Space. If q lies between p and r and r lies between p and s, then q lies between p and s. The theorem is a consequence of (20) and (34).

(37) Let us consider points A, B, C, D of $\mathcal{E}^2_{\rm T}$. If $B \in \mathcal{L}(A, C)$ and $D \in \mathcal{L}(A, B)$, then $B \in \mathcal{L}(D, C)$. The theorem is a consequence of (34).

Let us consider points p, q, r, s of TarskiEuclid2Space. Now we state the proposition:

(38) If q lies between p and r and s lies between p and q, then q lies between s and r. The theorem is a consequence of (20) and (37).

Let us assume that $q \neq r$ and q lies between p and r and r lies between q and s. Now we state the propositions:

(39) q lies between p and s. The theorem is a consequence of (20) and (35).

(40) r lies between p and s. The theorem is a consequence of (20) and (35).

Note that TarskiEuclid2Space satisfies the axiom of congruence symmetry, the axiom of congruence equivalence relation, the axiom of congruence identity, the axiom of segment construction, the axiom of SAS, the axiom of betweenness identity, and the axiom of Pasch and TarskiEuclid2Space satisfies seven Tarski's geometry axioms.

5. Preparation for The Rest of Tarski's Axioms

Now we state the propositions:

(41) Let us consider points P, Q, R of $\mathcal{E}^2_{\mathrm{T}}$, and an element L of Lines($\mathcal{R}^2$). If P, Q, $R \in L$, then $P \in \mathcal{L}(Q, R)$ or $Q \in \mathcal{L}(R, P)$ or $R \in \mathcal{L}(P, Q)$.

(42) Let us consider elements a, b, c of TarskiEuclid2Space. Suppose $\hat{b} \in \mathcal{L}(\hat{a}, \hat{c})$. Then there exists a real number r such that

 (i) $0 \leqslant r \leqslant 1$, and

 (ii) $\hat{b} - \hat{a} = r \cdot (\hat{c} - \hat{a})$.

(43) Let us consider a natural number n, and elements a, b, c of TarskiEuclid-Space n. Suppose $\hat{b} \in \mathcal{L}(\hat{a}, \hat{c})$. Then there exists a real number r such that

 (i) $0 \leqslant r \leqslant 1$, and

 (ii) $\hat{b} - \hat{a} = r \cdot (\hat{c} - \hat{a})$.

(44) Let us consider elements a, b, c of TarskiEuclid2Space. Suppose there exists a real number r such that $0 \leqslant r \leqslant 1$ and $\hat{b} - \hat{a} = r \cdot (\hat{c} - \hat{a})$. Then $\hat{b} \in \mathcal{L}(\hat{a}, \hat{c})$.

6. Four Remaining Axioms of Tarski

Let S be a Tarski plane. We say that S satisfies (A8) if and only if

(Def. 7) there exist points a, b, c of S such that b does not lie between a and c and c does not lie between b and a and a does not lie between c and b.

We say that S satisfies (A9) if and only if

(Def. 8) for every points a, b, c, p, q of S such that $p \neq q$ and $\overline{ap} \cong \overline{aq}$ and $\overline{bp} \cong \overline{bq}$ and $\overline{cp} \cong \overline{cq}$ holds b lies between a and c or c lies between b and a or a lies between c and b.

We say that S satisfies (A10) if and only if

(Def. 9) for every points a, b, c, d, t of S such that d lies between a and t and d lies between b and c and $a \neq d$ there exist points x, y of S such that b lies between a and x and c lies between a and y and t lies between x and y.

We say that S satisfies (A11) if and only if

(Def. 10) for every subsets X, Y of S such that there exists a point a of S such that for every points x, y of S such that $x \in X$ and $y \in Y$ holds x lies between a and y there exists a point b of S such that for every points x, y of S such that $x \in X$ and $y \in Y$ holds b lies between x and y.

We introduce the notation S satisfies Lower Dimension Axiom as a synonym of S satisfies (A8) and S satisfies Upper Dimension Axiom as a synonym of S satisfies (A9) and S satisfies Euclid Axiom as a synonym of S satisfies (A10) and S satisfies Continuity Axiom as a synonym of S satisfies (A11).

Now we state the proposition:

(45) Lower dimension axiom:

There exist points a, b, c of TarskiEuclid2Space such that

(i) b does not lie between a and c, and

(ii) c does not lie between b and a, and

(iii) a does not lie between c and b.

Proof: Reconsider $a = [0, 0]$, $b = [1, 0]$, $c = [0, 1]$ as a point of TarskiEuclid-2Space. b does not lie between a and c by (20), [3, (12)], [15, (19)], (9). c does not lie between b and a by (20), [3, (12)], [15, (19)], (9). $\hat{a} \in \mathcal{L}(\hat{c}, \hat{b})$. $\square$

(46) Upper dimension axiom:

Let us consider points a, b, c, p, q of TarskiEuclid2Space. Suppose $p \neq q$ and $\overline{ap} \cong \overline{aq}$ and $\overline{bp} \cong \overline{bq}$ and $\overline{cp} \cong \overline{cq}$. Then

(i) b lies between a and c, or

(ii) c lies between b and a, or

(iii) a lies between c and b.

The theorem is a consequence of (18), (41), and (20).

(47) Axiom of Euclid:

Let us consider elements a, b, c, d, t of TarskiEuclid2Space. Suppose d lies between a and t and d lies between b and c and $a \neq d$. Then there exist elements x, y of TarskiEuclid2Space such that

(i) b lies between a and x, and

(ii) c lies between a and y, and

(iii) t lies between x and y.

PROOF: $\hat{d} \in \mathcal{L}(\hat{a}, \hat{t})$. Set $v = \hat{a}$. Set $w = \hat{t}$. Consider r being a real number such that $0 \leqslant r \leqslant 1$ and $\hat{d} = (1 - r) \cdot v + r \cdot w$. Set $r_1 = \frac{1}{r}$. $r \neq 0$ by [17, (10), (21)]. Set $x_1 = r_1 \cdot (\hat{b} - \hat{a}) + \hat{a}$. Reconsider $x_2 = x_1$ as an element of TarskiEuclid2Space. $\hat{b} \in \mathcal{L}(\hat{a}, \hat{x_2})$. b lies between a and x_2. Set $y_1 = r_1 \cdot (\hat{c} - \hat{a}) + \hat{a}$. Reconsider $y_2 = y_1$ as an element of TarskiEuclid2Space. $\hat{c} \in \mathcal{L}(\hat{a}, \hat{y_2})$. c lies between a and y_2. $\hat{d} \in \mathcal{L}(\hat{b}, \hat{c})$. Consider k being a real number such that $0 \leqslant k \leqslant 1$ and $\hat{d} - \hat{b} = k \cdot (\hat{c} - \hat{b})$. $\hat{t} \in \mathcal{L}(\hat{x_2}, \hat{y_2})$. t lies between x_2 and y_2. □

7. Axiom A11 – Axiom Schema of Continuity

Now we state the proposition:

(48) Axiom schema of continuity:
Let us consider subsets X, Y of TarskiEuclid2Space. Suppose there exists an element a of TarskiEuclid2Space such that for every elements x, y of TarskiEuclid2Space such that $x \in X$ and $y \in Y$ holds x lies between a and y. Then there exists an element b of TarskiEuclid2Space such that for every elements x, y of TarskiEuclid2Space such that $x \in X$ and $y \in Y$ holds b lies between x and y. The theorem is a consequence of (20), (42), (2), and (44).

Let us observe that TarskiEuclid2Space satisfies Lower Dimension Axiom, Upper Dimension Axiom, Euclid Axiom, and Continuity Axiom.

8. Corrolaries

In the sequel X, Y denote subsets of TarskiEuclid2Space.

Let us consider an element a of TarskiEuclid2Space. Now we state the propositions:

(49) Suppose for every elements x, y of TarskiEuclid2Space such that $x \in X$ and $y \in Y$ holds x lies between a and y and $a \in Y$. Then

(i) $X = \{a\}$, or

(ii) X is empty.

(50) Suppose for every elements x, y of TarskiEuclid2Space such that $x \in X$ and $y \in Y$ holds x lies between a and y and X is not empty and Y is not empty and if X is trivial, then $X \neq \{a\}$. Then there exists an element b of TarskiEuclid2Space such that

(i) $X \subseteq \mathrm{Line}(\hat{a}, \hat{b})$, and

(ii) $Y \subseteq \mathrm{Line}(\hat{a}, \hat{b})$.

Proof: Consider x_0 being an object such that $x_0 \in X$. Consider c being an object such that $c \in Y$. $X \subseteq \mathcal{L}(\hat{a}, \hat{c})$. $Y \subseteq \mathrm{Line}(\hat{a}, \hat{c})$ by [2, (131)], (20), [1, (73), (72), (75)]. □

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