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Compactness in Metric Spaces

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Summary. In this article, we mainly formalize in Mizar [2] the equivalence among a few compactness definitions of metric spaces, norm spaces, and the real line. In the first section, we formalized general topological properties of metric spaces. We discussed openness and closedness of subsets in metric spaces in terms of convergence of element sequences. In the second section, we firstly formalize the definition of sequentially compact, and then discuss the equivalence of compactness, countable compactness, sequential compactness, and totally boundedness with completeness in metric spaces.

In the third section, we discuss compactness in norm spaces. We formalize the equivalence of compactness and sequential compactness in norm space. In the fourth section, we formalize topological properties of the real line in terms of convergence of real number sequences. In the last section, we formalize the equivalence of compactness and sequential compactness in the real line. These formalizations are based on [20], [5], [17], [14], and [4].

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1. Topological Properties of Metric Spaces

Now we state the propositions:

(1) Let us consider a non empty set M, and a sequence x of M. Suppose $\operatorname{rng} x$ is finite. Then there exists an element z of M such that

(i) $x^{-1}(\{z\}) \subseteq \mathbb{N}$, and

(ii) $x^{-1}(\{z\})$ is infinite.

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Proof: Define $\mathcal{X}(\text{object}) = x^{-1}(\{\$_1\})$. Set $K = \{\mathcal{X}(w)$, where w is an element of $M : w \in \operatorname{rng} x\}$. K is finite from [18, Sch. 21]. For every set Y such that $Y \in K$ holds Y is finite. $\operatorname{dom} x \subseteq \bigcup K$ by [6, (3)]. □

(2) Let us consider a subset X of $\mathbb{N}$. Suppose X is infinite. Then there exists an increasing sequence N of $\mathbb{N}$ such that $\operatorname{rng} N \subseteq X$.

Proof: Reconsider $B = 2^X$ as a non empty set. Reconsider $N_0 = \min^* X$ as an element of $\mathbb{N}$. Reconsider $Y_0 = X$ as an element of B. Define $\mathcal{P}[\text{object}, \text{object}, \text{set}, \text{object}, \text{set}] \equiv \$_5 = \$_3 \setminus \{\$_2\}$ and $\$_4 = \min^* \$_5$. For every natural number n and for every element x of $\mathbb{N}$ and for every element y of B, there exists an element x_1 of $\mathbb{N}$ and there exists an element y_1 of B such that $\mathcal{P}[n, x, y, x_1, y_1]$. Consider N being a sequence of $\mathbb{N}$, Y being a sequence of B such that $N(0) = N_0$ and $Y(0) = Y_0$ and for every natural number n, $\mathcal{P}[n, N(n), Y(n), N(n+1), Y(n+1)]$ from [13, Sch. 3]. Define $\mathcal{Q}[\text{natural number}] \equiv N(\$_1) = \min^*(Y(\$_1))$ and $N(\$_1) \in Y(\$_1)$ and $Y(\$_1)$ is infinite and $Y(\$_1) \subseteq \mathbb{N}$. For every natural number i such that $\mathcal{Q}[i]$ holds $\mathcal{Q}[i+1]$ by [8, (31)]. For every natural number i, $\mathcal{Q}[i]$ from [1, Sch. 2]. $\operatorname{rng} N \subseteq X$ by [7, (11)]. For every natural number i, $N(i) < N(i+1)$. □

(3) Let us consider a non empty metric space M, and a subset V of M_{top}. Suppose V is open. Then there exists a family F of subsets of M such that

(i) $F = \{\text{Ball}(x, r)$, where x is an element of M, r is a real number : $r > 0$ and $\text{Ball}(x, r) \subseteq V\}$, and

(ii) $V = \bigcup F$.

Proof: Set $F = \{\text{Ball}(x, r)$, where x is an element of M, r is a real number: $r > 0$ and $\text{Ball}(x, r) \subseteq V\}$. For every object z such that $z \in F$ holds $z \in$ the open set family of M by [3, (29)]. Reconsider $Q = \bigcup F$ as a subset of M_{top}. For every object z, $z \in V$ iff $z \in Q$ by [9, (15)], [12, (1), (11)]. □

(4) Let us consider a non empty metric space M, a subset X of M_{top}, and an element p of M. Then $p \in \overline{X}$ if and only if for every real number r such that $0 < r$ holds X meets $\text{Ball}(p, r)$.

(5) Let us consider a non empty metric space M, a subset X of M_{top}, and an object x. Then $x \in \overline{X}$ if and only if there exists a sequence S of M such that for every natural number n, $S(n) \in X$ and S is convergent and $\lim S = x$.

(6) Let us consider a non empty metric space M, and a subset X of M_{top}. Then X is closed if and only if for every sequence S of M such that for every natural number n, $S(n) \in X$ and S is convergent holds $\lim S \in X$. The theorem is a consequence of (5).

(7) Let us consider non empty metric spaces X, Y, and a function f from $X_{\rm top}$ into $Y_{\rm top}$. Then f is continuous if and only if for every sequence S of X and for every sequence T of Y such that S is convergent and $T = f \cdot S$ holds T is convergent and $\lim T = f(\lim S)$.
PROOF: For every subset B of $Y_{\rm top}$ such that B is closed holds $f^{-1}(B)$ is closed by [7, (15)], (6). □

2. Compactness in Metric Spaces

Let M be a non empty metric space and X be a subset of M. We say that X is sequentially compact if and only if

(Def. 1) for every sequence S_1 of M such that $\operatorname{rng} S_1 \subseteq X$ there exists a sequence S_2 of M such that there exists an increasing sequence N of $\mathbb{N}$ such that $S_2 = S_1 \cdot N$ and S_2 is convergent and $\lim S_2 \in X$.

Let us observe that every subset of M which is empty is also sequentially compact.

We say that M is sequentially compact if and only if

(Def. 2) Ω_M is sequentially compact.

Now we state the proposition:

(8) Let us consider a non empty metric space M, a subset X of M, a subset Y of $M_{\rm top}$, an element x of M, and an element y of $M_{\rm top}$. Suppose $X = Y$ and $x = y$. Then y is an accumulation point of Y if and only if for every real number r such that $0 < r$ holds $\operatorname{Ball}(x, r)$ meets $X \setminus \{x\}$.

Let us consider a non empty metric space M. Now we state the propositions:

(9) If $M_{\rm top}$ is countably-compact, then M is sequentially compact.
PROOF: For every subset X of M such that X is infinite there exists an element x of M such that for every real number r such that $0 < r$ holds $\operatorname{Ball}(x, r)$ meets $X \setminus \{x\}$ by [16, (28)], [11, (16)], (8). For every sequence x of M such that $\operatorname{rng} x \subseteq \Omega_M$ there exists a sequence x_1 of M such that there exists an increasing sequence N of $\mathbb{N}$ such that $x_1 = x \cdot N$ and x_1 is convergent and $\lim x_1 \in \Omega_M$ by (1), (2), [7, (4), (38), (15)]. □

(10) If M is sequentially compact, then $M_{\rm top}$ is countably-compact.
PROOF: For every subset X of M such that X is infinite there exists an element x of M such that for every real number r such that $0 < r$ holds $\operatorname{Ball}(x, r)$ meets $X \setminus \{x\}$ by [15, (3)], [7, (2)], [19, (26)], [7, (112)]. For every subset A of $M_{\rm top}$ such that A is infinite holds $\operatorname{Der} A$ is not empty by (8), [11, (16)]. □

(11) $M_{\rm top}$ is compact if and only if M is sequentially compact. The theorem is a consequence of (9).

(12) M is totally bounded and complete if and only if M is sequentially compact. The theorem is a consequence of (11).

Let us consider a non empty metric space M and a non empty subset S of M. Now we state the propositions:

(13) S is sequentially compact if and only if $M \upharpoonright S$ is sequentially compact.
PROOF: For every sequence S_1 of M such that $\operatorname{rng} S_1 \subseteq S$ there exists a sequence S_2 of M such that there exists an increasing sequence N of $\mathbb{N}$ such that $S_2 = S_1 \cdot N$ and S_2 is convergent and $\lim S_2 \in S$ by [7, (6)]. $\square$

(14) S is sequentially compact if and only if $M \upharpoonright S$ is compact. The theorem is a consequence of (11) and (13).

(15) Let us consider a non empty metric space M, a subset S of M, and a subset T of M_{top}. If $T = S$, then T is compact iff S is sequentially compact. The theorem is a consequence of (11) and (13).

(16) Let us consider a non empty metric space M, a non empty subset S of M, and a non empty subset T of M_{top}. Suppose $T = S$. Then $M_{\text{top}} \upharpoonright T$ is countably-compact if and only if S is sequentially compact. The theorem is a consequence of (11) and (13).

(17) Let us consider a non empty metric space M, and a non empty subset S of M. Then $M \upharpoonright S$ is totally bounded and complete if and only if S is sequentially compact. The theorem is a consequence of (12) and (13).

3. Compactness in Norm Spaces

Now we state the propositions:

(18) Let us consider a real normed space N, a subset S of N, and a subset T of MetricSpaceNorm N. If $S = T$, then S is compact iff T is sequentially compact.

(19) Let us consider a real normed space N, a subset S of N, and a subset T of TopSpaceNorm N. If $S = T$, then S is compact iff T is compact. The theorem is a consequence of (15) and (18).

4. Topological Properties of the Real Line

Let us consider a sequence S_1 of the metric space of real numbers, a sequence s of real numbers, a real number g, and an element g_1 of the metric space of real numbers. Now we state the propositions:

(20) Suppose $S_1 = s$ and $g = g_1$. Then for every real number p such that $0 < p$ there exists a natural number n such that for every natural number

m such that $n \leqslant m$ holds $|s(m) - g| < p$ if and only if for every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $\rho(S_1(m), g_1) < p$.

Proof: For every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $|s(m) - g| < p$ by [9, (11)]. □

(21) If $S_1 = s$ and $g = g_1$, then s is convergent and $\lim s = g$ iff S_1 is convergent and $\lim S_1 = g_1$. The theorem is a consequence of (20).

(22) Let us consider a sequence S_1 of the metric space of real numbers, and a sequence s of real numbers. Suppose $S_1 = s$ and s is convergent. Then

(i) S_1 is convergent, and

(ii) $\lim S_1 = \lim s$.

The theorem is a consequence of (20).

5. Compactness in the Real Line

Now we state the propositions:

(23) Let us consider a subset N of $\mathbb{R}$, and a subset M of $\mathbb{R}^{\mathbf{1}}$. Suppose $N = M$. Then for every family F of subsets of $\mathbb{R}$ such that F is a cover of N and for every subset P of $\mathbb{R}$ such that $P \in F$ holds P is open there exists a family G of subsets of $\mathbb{R}$ such that $G \subseteq F$ and G is cover of N and finite if and only if for every family F_1 of subsets of $\mathbb{R}^{\mathbf{1}}$ such that F_1 is cover of M and open there exists a family G_1 of subsets of $\mathbb{R}^{\mathbf{1}}$ such that $G_1 \subseteq F_1$ and G_1 is cover of M and finite.

Proof: Reconsider $F_1 = F$ as a family of subsets of $\mathbb{R}^{\mathbf{1}}$. For every subset P_1 of $\mathbb{R}^{\mathbf{1}}$ such that $P_1 \in F_1$ holds P_1 is open by [10, (39)]. Consider G_1 being a family of subsets of $\mathbb{R}^{\mathbf{1}}$ such that $G_1 \subseteq F_1$ and G_1 is cover of M and finite. □

(24) Let us consider a subset N of $\mathbb{R}$. Then N is compact if and only if for every family F of subsets of $\mathbb{R}$ such that F is a cover of N and for every subset P of $\mathbb{R}$ such that $P \in F$ holds P is open there exists a family G of subsets of $\mathbb{R}$ such that $G \subseteq F$ and G is cover of N and finite. The theorem is a consequence of (23).

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Double Sequences and Iterated Limits in Regular Space

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Summary. First, we define in Mizar [5], the Cartesian product of two filters bases and the Cartesian product of two filters. After comparing the product of two Fréchet filters on $\mathbb{N}$ ($\mathcal{F}_1$) with the Fréchet filter on $\mathbb{N} \times \mathbb{N}$ ($\mathcal{F}_2$), we compare $lim_{\mathcal{F}_1}$ and $lim_{\mathcal{F}_2}$ for all double sequences in a non empty topological space.

Endou, Okazaki and Shidama formalized in [14] the "convergence in Pringsheim's sense" for double sequence of real numbers. We show some basic correspondences between the p-convergence and the filter convergence in a topological space. Then we formalize that the double sequence $(x_{m,n} = \frac{1}{m+1})_{(m,n)} \in \mathbb{N} \times \mathbb{N}$ converges in "Pringsheim's sense" but not in Frechet filter on $\mathbb{N} \times \mathbb{N}$ sense.

In the next section, we generalize some definitions: "is convergent in the first coordinate", "is convergent in the second coordinate", "the *lim* in the first coordinate of", "the *lim* in the second coordinate of" according to [14], in Hausdorff space.

Finally, we generalize two theorems: (3) and (4) from [14] in the case of double sequences and we formalize the "iterated limit" theorem ("Double limit" [7], p. 81, par. 8.5 *"Double limite"* [6] (TG I,57)), all in regular space. We were inspired by the exercises (2.11.4), (2.17.5) [17] and the corrections B.10 [18].

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1. Preliminaries

From now on x denotes an object, X, Y, Z denote sets, i, j, k, l, m, n denote natural numbers, r, s denote real numbers, n_1 denotes an element of the ordered $\mathbb{N}$, and A denotes a subset of $\mathbb{N} \times \mathbb{N}$.

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Now we state the propositions:

(1) Let us consider a finite subset W of X. If $X \setminus W \subseteq Z$, then $X \setminus Z$ is finite.

(2) If $Z \subseteq X$ and $X \setminus Z$ is finite, then there exists a finite subset W of X such that $X \setminus W = Z$.

(3) Let us consider sets X_1, X_2, a family S_1 of subsets of X_1, and a family S_2 of subsets of X_2. Then $\{s$, where s is a subset of $X_1 \times X_2$: there exist sets s_1, s_2 such that $s_1 \in S_1$ and $s_2 \in S_2$ and $s = s_1 \times s_2\}$ is a family of subsets of $X_1 \times X_2$.

(4) If $x \in X \times Y$, then x is pair.

(5) If $0 < r$, then there exists m such that m is not zero and $\frac{1}{m} < r$.

(6) Let us consider points x, y of the metric space of real numbers. Then there exist real numbers x_1, y_1 such that

   (i) $x = x_1$, and

   (ii) $y = y_1$, and

   (iii) $\rho(x, y) = \rho_{\mathbb{R}}(x, y)$, and

   (iv) $\rho(x, y) = \rho^1(\langle x\rangle, \langle y\rangle)$, and

   (v) $\rho(x, y) = |x_1 - y_1|$.

(7) Let us consider points x, y of $(\mathcal{E}^1)_{\mathrm{top}}$. Then there exist points x_2, y_2 of the metric space of real numbers and there exist real numbers x_1, y_1 such that $x_2 = x_1$ and $y_2 = y_1$ and $x = \langle x_1\rangle$ and $y = \langle y_1\rangle$ and $\rho(x_2, y_2) = \rho_{\mathbb{R}}(x_1, y_1)$ and $\rho(x_2, y_2) = \rho^1(\langle x_1\rangle, \langle y_1\rangle)$ and $\rho(x_2, y_2) = |x_1 - y_1|$.

(8) Let us consider points x, y of $\mathcal{E}^1$, and real numbers r, s. If $x = \langle r\rangle$ and $y = \langle s\rangle$, then $\rho(x, y) = |r - s|$. The theorem is a consequence of (7).

One can check that $\mathbb{N} \times \mathbb{N}$ is countable and $\mathbb{N} \times \mathbb{N}$ is denumerable.

Now we state the propositions:

(9) the set of all $\langle 0, n\rangle$ where n is a natural number is infinite.
PROOF: Define $\mathcal{F}(\text{object}) = \langle 0, \$_1\rangle$. Consider f being a function such that $\operatorname{dom} f = \mathbb{N}$ and for every object x such that $x \in \mathbb{N}$ holds $f(x) = \mathcal{F}(x)$ from [9, Sch. 3]. f is one-to-one. $\operatorname{rng} f =$ the set of all $\langle 0, n\rangle$ where n is a natural number by [9, (3)]. □

(10) If $i \leqslant k$ and $j \leqslant l$, then $\mathbb{Z}_i \times \mathbb{Z}_j \subseteq \mathbb{Z}_k \times \mathbb{Z}_l$.

(11) $(\mathbb{N} \setminus \mathbb{Z}_m) \times (\mathbb{N} \setminus \mathbb{Z}_n) \subseteq \mathbb{N} \times \mathbb{N} \setminus \mathbb{Z}_m \times \mathbb{Z}_n$.

(12) If $n = n_1$ and $n \leqslant m$, then $m \in {\uparrow}n_1$.

(13) If $n = n_1$ and $m \in {\uparrow}n_1$, then $n \leqslant m$.

(14) If $n = n_1$, then ${\uparrow}n_1 = \mathbb{N} \setminus \mathbb{Z}_n$.

Proof: $\uparrow n_1 \subseteq \mathbb{N} \setminus \mathbb{Z}_n$ by [12, (50)], (13), [1, (44)]. $\mathbb{N} \setminus \mathbb{Z}_n \subseteq \uparrow n_1$ by [1, (44)], [12, (50)]. □

(15) $\pi_1(A) = \{x$, where x is an element of $\mathbb{N}$: there exists an element y of $\mathbb{N}$ such that $\langle x, y \rangle \in A\}$.

(16) $\pi_2(A) = \{y$, where y is an element of $\mathbb{N}$: there exists an element x of $\mathbb{N}$ such that $\langle x, y \rangle \in A\}$.

(17) Let us consider a finite subset A of $\mathbb{N} \times \mathbb{N}$. Then there exists m and there exists n such that $A \subseteq \mathbb{Z}_m \times \mathbb{Z}_n$. The theorem is a consequence of (15) and (16).

(18) Let us consider a non empty set X. Then every filter of X is a proper filter of $2^X_\subseteq$.

(19) Let us consider a non empty set X, and a filter $\mathcal{F}$ of X. Then there exists a filter base $\mathcal{B}$ of X such that

(i) $\mathcal{B} = \mathcal{F}$, and

(ii) $[\mathcal{B}) = \mathcal{F}$.

(20) Let us consider a non empty topological space T, and a filter $\mathcal{F}$ of the carrier of T. If $x \in \mathrm{LimFilter}(\mathcal{F})$, then x is a cluster point of $\mathcal{F}$,T.

(21) Let us consider an element B of the base of Frechet filter. Then there exists n such that $B = \mathbb{N} \setminus \mathbb{Z}_n$. The theorem is a consequence of (14).

(22) Let us consider a subset B of $\mathbb{N}$. Suppose $B = \mathbb{N} \setminus \mathbb{Z}_n$. Then B is an element of the base of Frechet filter. The theorem is a consequence of (14).

2. Cartesian Product of Two Filters

From now on X, Y, X_1, X_2 denote non empty sets, $\mathcal{A}_1$, $\mathcal{B}_1$ denote filter bases of X_1, $\mathcal{A}_2$, $\mathcal{B}_2$ denote filter bases of X_2, $\mathcal{F}_1$ denotes a filter of X_1, $\mathcal{F}_2$ denotes a filter of X_2, $\mathcal{B}_3$ denotes a generalized basis of $\mathcal{F}_1$.

Let X_1, X_2 be non empty sets, $\mathcal{B}_1$ be a filter base of X_1, and $\mathcal{B}_2$ be a filter base of X_2. The functor $\mathcal{B}_1 \times \mathcal{B}_2$ yielding a filter base of $X_1 \times X_2$ is defined by the term

(Def. 1) the set of all $B_1 \times B_2$ where B_1 is an element of $\mathcal{B}_1$, B_2 is an element of $\mathcal{B}_2$.

Now we state the propositions:

(23) Suppose $\mathcal{F}_1 = [\mathcal{B}_1)$ and $\mathcal{F}_1 = [\mathcal{A}_1)$ and $\mathcal{F}_2 = [\mathcal{B}_2)$ and $\mathcal{F}_2 = [\mathcal{A}_2)$. Then $[\mathcal{B}_1 \times \mathcal{B}_2) = [\mathcal{A}_1 \times \mathcal{A}_2)$.

(24) If $\mathcal{B}_3 = \mathcal{B}_1$, then $[\mathcal{B}_1] = \mathcal{F}_1$.

(25) There exists $\mathcal{B}_1$ such that $[\mathcal{B}_1) = \mathcal{F}_1$. The theorem is a consequence of (24).

Let X_1, X_2 be non empty sets, $\mathcal{F}_1$ be a filter of X_1, and $\mathcal{F}_2$ be a filter of X_2. The functor $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle$ yielding a filter of $X_1 \times X_2$ is defined by

(Def. 2) there exists a filter base $\mathcal{B}_1$ of X_1 and there exists a filter base $\mathcal{B}_2$ of X_2 such that $[\mathcal{B}_1) = \mathcal{F}_1$ and $[\mathcal{B}_2) = \mathcal{F}_2$ and $it = [\mathcal{B}_1 \times \mathcal{B}_2)$.

Let $\mathcal{B}_1$ be a generalized basis of $\mathcal{F}_1$ and $\mathcal{B}_2$ be a generalized basis of $\mathcal{F}_2$. The functor $\mathcal{B}_1 \times \mathcal{B}_2$ yielding a generalized basis of $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle$ is defined by

(Def. 3) there exists a filter base $\mathcal{B}_3$ of X_1 and there exists a filter base $\mathcal{B}_4$ of X_2 such that $\mathcal{B}_1 = \mathcal{B}_3$ and $\mathcal{B}_2 = \mathcal{B}_4$ and $it = \mathcal{B}_3 \times \mathcal{B}_4$.

Let n be a natural number. The functor $\uparrow^2(n)$ yielding a subset of $\mathbb{N} \times \mathbb{N}$ is defined by

(Def. 4) for every element x of $\mathbb{N} \times \mathbb{N}$, $x \in it$ iff there exist natural numbers n_1, n_2 such that $n_1 = (x)_{\mathbf{1}}$ and $n_2 = (x)_{\mathbf{2}}$ and $n \leqslant n_1$ and $n \leqslant n_2$.

Now we state the proposition:

(26) $\langle n, n \rangle \in \uparrow^2(n)$.

Let us consider n. One can check that $\uparrow^2(n)$ is non empty.

Now we state the propositions:

(27) If $\langle i, j \rangle \in \uparrow^2(n)$, then $\langle i + k, j \rangle$, $\langle i, j + l \rangle \in \uparrow^2(n)$.

(28) $\uparrow^2(n)$ is an infinite subset of $\mathbb{N} \times \mathbb{N}$. The theorem is a consequence of (17).

(29) If $n_1 = n$, then $\uparrow^2(n) = \uparrow n_1 \times \uparrow n_1$. The theorem is a consequence of (12) and (13).

(30) If $m = n - 1$, then $\uparrow^2(n) \subseteq \mathbb{N} \times \mathbb{N} \setminus \operatorname{Seg} m \times \operatorname{Seg} m$.

PROOF: Reconsider $y = x$ as an element of $\mathbb{N} \times \mathbb{N}$. Consider n_1, n_2 being natural numbers such that $n_1 = (y)_{\mathbf{1}}$ and $n_2 = (y)_{\mathbf{2}}$ and $n \leqslant n_1$ and $n \leqslant n_2$. $x \notin \operatorname{Seg} m \times \operatorname{Seg} m$ by [3, (1)]. □

(31) $\uparrow^2(n) \subseteq \mathbb{N} \times \mathbb{N} \setminus \mathbb{Z}_n \times \mathbb{Z}_n$.

PROOF: Reconsider $y = x$ as an element of $\mathbb{N} \times \mathbb{N}$. Consider n_1, n_2 being natural numbers such that $n_1 = (y)_{\mathbf{1}}$ and $n_2 = (y)_{\mathbf{2}}$ and $n \leqslant n_1$ and $n \leqslant n_2$. $x \notin \mathbb{Z}_n \times \mathbb{Z}_n$ by [16, (10)]. □

(32) $\uparrow^2(n) = (\mathbb{N} \setminus \mathbb{Z}_n) \times (\mathbb{N} \setminus \mathbb{Z}_n)$. The theorem is a consequence of (14) and (29).

(33) There exists n such that $\uparrow^2(n) \subseteq (\mathbb{N} \setminus \mathbb{Z}_i) \times (\mathbb{N} \setminus \mathbb{Z}_j)$. The theorem is a consequence of (4).

(34) If $n = \max(i, j)$, then $\uparrow^2(n) \subseteq (\uparrow^2(i)) \cap (\uparrow^2(j))$.

Let n be a natural number. The functor $\downarrow^2(n)$ yielding a subset of $\mathbb{N} \times \mathbb{N}$ is defined by

(Def. 5) for every element x of $\mathbb{N} \times \mathbb{N}$, $x \in it$ iff there exist natural numbers n_1, n_2 such that $n_1 = (x)_{\mathbf{1}}$ and $n_2 = (x)_{\mathbf{2}}$ and $n_1 < n$ and $n_2 < n$.

Now we state the propositions:

(35) $\downarrow^2(n) = \mathbb{Z}_n \times \mathbb{Z}_n$.
PROOF: $\downarrow^2(n) \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$ by [1, (44)]. Consider y_2, y_1 being objects such that $y_2 \in \mathbb{Z}_n$ and $y_1 \in \mathbb{Z}_n$ and $x = \langle y_2, y_1 \rangle$. □

(36) Let us consider a finite subset A of $\mathbb{N} \times \mathbb{N}$. Then there exists n such that $A \subseteq \downarrow^2(n)$.
PROOF: Consider m, n such that $A \subseteq \mathbb{Z}_m \times \mathbb{Z}_n$. Reconsider $m_1 = \max(m, n)$ as a natural number. $A \subseteq \downarrow^2(m_1)$ by [1, (39)], [11, (96)], (35). □

(37) $\downarrow^2(n)$ is a finite subset of $\mathbb{N} \times \mathbb{N}$. The theorem is a consequence of (35).

3. COMPARISON BETWEEN CARTESIAN PRODUCT OF FRECHET FILTER ON $\mathbb{N}$ AND THE FRECHET FILTER OF $\mathbb{N} \times \mathbb{N}$

Let us consider an element x of (the base of Frechet filter) $\times$ (the base of Frechet filter). Now we state the propositions:

(38) There exists i and there exists j such that $x = (\mathbb{N} \setminus \mathbb{Z}_i) \times (\mathbb{N} \setminus \mathbb{Z}_j)$. The theorem is a consequence of (21).

(39) There exists n such that $\uparrow^2(n) \subseteq x$. The theorem is a consequence of (38) and (33).

(40) (The base of Frechet filter) $\times$ (the base of Frechet filter) is a filter base of $\mathbb{N} \times \mathbb{N}$.

(41) There exists a generalized basis $\mathcal{B}$ of FrechetFilter($\mathbb{N}$) such that

(i) $\mathcal{B} =$ the base of Frechet filter, and

(ii) $\mathcal{B} \times \mathcal{B}$ is a generalized basis of $\langle$FrechetFilter($\mathbb{N}$), FrechetFilter($\mathbb{N}$)$\rangle$.

The functor $\uparrow^2_{\mathbb{N}}$ yielding a filter base of $\mathbb{N} \times \mathbb{N}$ is defined by the term

(Def. 6) the set of all $\uparrow^2(n)$ where n is a natural number.

Now we state the propositions:

(42) $\uparrow^2_{\mathbb{N}}$ and (the base of Frechet filter) $\times$ (the base of Frechet filter) are equivalent generators. The theorem is a consequence of (22), (32), and (39).

(43) $[$(the base of Frechet filter) $\times$ (the base of Frechet filter)$) = \langle$FrechetFilter($\mathbb{N}$), FrechetFilter($\mathbb{N}$)$\rangle$. The theorem is a consequence of (41).

(44) $[\uparrow^2_{\mathbb{N}}) = \langle$FrechetFilter($\mathbb{N}$), FrechetFilter($\mathbb{N}$)$\rangle$.

(45) $\langle$FrechetFilter($\mathbb{N}$), FrechetFilter($\mathbb{N}$)$\rangle$ is finer than FrechetFilter($\mathbb{N} \times \mathbb{N}$). The theorem is a consequence of (17), (11), (22), and (43).

(46) (i) $\mathbb{N}\times\mathbb{N}\setminus$the set of all $\langle 0, n\rangle$ where n is a natural number $\in \langle$Frechet Filter($\mathbb{N}$), FrechetFilter($\mathbb{N}$)$\rangle$, and

(ii) $\mathbb{N} \times \mathbb{N} \setminus$ the set of all $\langle 0, n\rangle$ where n is a natural number $\notin$ Frechet Filter($\mathbb{N} \times \mathbb{N}$).

PROOF: Set $X = \mathbb{N}\times\mathbb{N}\setminus$the set of all $\langle 0, n\rangle$ where n is a natural number. $\uparrow^2(1) \subseteq X$ by (32), [1, (44)]. $X \notin$ FrechetFilter($\mathbb{N} \times \mathbb{N}$) by [12, (51)], [15, (5)], (9). $\square$

(47) FrechetFilter($\mathbb{N} \times \mathbb{N}$) $\neq \langle$FrechetFilter($\mathbb{N}$), FrechetFilter($\mathbb{N}$)$\rangle$.

4. Topological Space and Double Sequence

In the sequel T denotes a non empty topological space, s denotes a function from $\mathbb{N} \times \mathbb{N}$ into the carrier of T, M denotes a subset of the carrier of T, and $\mathcal{F}_1$, $\mathcal{F}_2$ denote filters of the carrier of T. Now we state the propositions:

(48) If $\mathcal{F}_2$ is finer than $\mathcal{F}_1$, then LimFilter($\mathcal{F}_1$) $\subseteq$ LimFilter($\mathcal{F}_2$).

(49) Let us consider a function f from X into Y, and filters $\mathcal{F}_1$, $\mathcal{F}_2$ of X. Suppose $\mathcal{F}_2$ is finer than $\mathcal{F}_1$. Then the image of filter $\mathcal{F}_2$ under f is finer than the image of filter $\mathcal{F}_1$ under f.

(50) $s^{-1}(M) \in$ FrechetFilter($\mathbb{N} \times \mathbb{N}$) if and only if there exists a finite subset A of $\mathbb{N} \times \mathbb{N}$ such that $s^{-1}(M) = \mathbb{N} \times \mathbb{N} \setminus A$.

(51) $s^{-1}(M) \in \langle$FrechetFilter($\mathbb{N}$), FrechetFilter($\mathbb{N}$)$\rangle$ if and only if there exists n such that $\uparrow^2(n) \subseteq s^{-1}(M)$. The theorem is a consequence of (43), (39), and (42).

(52) The image of filter FrechetFilter($\mathbb{N} \times \mathbb{N}$) under $s = \{M$, where M is a subset of the carrier of T : there exists a finite subset A of $\mathbb{N} \times \mathbb{N}$ such that $s^{-1}(M) = \mathbb{N} \times \mathbb{N} \setminus A\}$. The theorem is a consequence of (50).

(53) The image of filter $\langle$FrechetFilter($\mathbb{N}$), FrechetFilter($\mathbb{N}$)$\rangle$ under $s = \{M$, where M is a subset of the carrier of T : there exists a natural number n such that $\uparrow^2(n) \subseteq s^{-1}(M)\}$. The theorem is a consequence of (51).

Let us consider a point x of T. Now we state the propositions:

(54) $x \in \lim_{\text{FrechetFilter}(\mathbb{N}\times\mathbb{N})} s$ if and only if for every neighbourhood A of x, there exists a finite subset B of $\mathbb{N} \times \mathbb{N}$ such that $s^{-1}(A) = \mathbb{N} \times \mathbb{N} \setminus B$. The theorem is a consequence of (52).

(55) $x \in \lim_{\text{FrechetFilter}(\mathbb{N}\times\mathbb{N})} s$ if and only if for every neighbourhood A of x, $\mathbb{N} \times \mathbb{N} \setminus s^{-1}(A)$ is finite. The theorem is a consequence of (54), (1), and (2).

(56) $x \in \lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} s$ if and only if for every neighbourhood A of x, there exists a natural number n such that $\uparrow^2(n) \subseteq s^{-1}(A)$. The theorem is a consequence of (53).

Let us consider a point x of T and a generalized basis $\mathcal{B}$ of BooleanFilter ToFilter(the neighborhood system of x). Now we state the propositions:

(57) $x \in \lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} s$ if and only if for every element B of $\mathcal{B}$, there exists a natural number n such that $\uparrow^2(n) \subseteq s^{-1}(B)$. The theorem is a consequence of (56).

(58) $x \in \lim_{\mathrm{FrechetFilter}(\mathbb{N} \times \mathbb{N})} s$ if and only if for every element B of $\mathcal{B}$, there exists a finite subset A of $\mathbb{N} \times \mathbb{N}$ such that $s^{-1}(B) = \mathbb{N} \times \mathbb{N} \setminus A$. The theorem is a consequence of (54), (1), and (55).

(59) $x \in \lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} s$ if and only if for every element B of $\mathcal{B}$, there exists a natural number n such that $s^\circ(\uparrow^2(n)) \subseteq B$. The theorem is a consequence of (57).

(60) $x \in \lim_{\mathrm{FrechetFilter}(\mathbb{N} \times \mathbb{N})} s$ if and only if for every element B of $\mathcal{B}$, there exists a finite subset A of $\mathbb{N} \times \mathbb{N}$ such that $s^\circ(\mathbb{N} \times \mathbb{N} \setminus A) \subseteq B$.

Proof: For every neighbourhood A of x, $\mathbb{N} \times \mathbb{N} \setminus s^{-1}(A)$ is finite by [4, (2)], [19, (143)], [9, (76)]. □

(61) $x \in \lim_{\mathrm{FrechetFilter}(\mathbb{N} \times \mathbb{N})} s$ if and only if for every element B of $\mathcal{B}$, there exists n and there exists m such that $s^\circ(\mathbb{N} \times \mathbb{N} \setminus \mathbb{Z}_n \times \mathbb{Z}_m) \subseteq B$. The theorem is a consequence of (60) and (17).

(62) $x \in s^\circ(\uparrow^2(n))$ if and only if there exists i and there exists j such that $n \leqslant i$ and $n \leqslant j$ and $x = s(i, j)$.

(63) $x \in s^\circ(\mathbb{N} \times \mathbb{N} \setminus \mathbb{Z}_i \times \mathbb{Z}_j)$ if and only if there exist natural numbers n, m such that ($i \leqslant n$ or $j \leqslant m$) and $x = s(n, m)$.

Proof: Consider n, m being natural numbers such that $i \leqslant n$ or $j \leqslant m$ and $x = s(n, m)$. $\langle n, m \rangle \notin \mathbb{Z}_i \times \mathbb{Z}_j$ by [1, (44)]. □

Let us consider a point x of T and a generalized basis $\mathcal{B}$ of BooleanFilter ToFilter(the neighborhood system of x). Now we state the propositions:

(64) $x \in \lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} s$ if and only if for every element B of $\mathcal{B}$, there exists a natural number n such that for every natural numbers n_1, n_2 such that $n \leqslant n_1$ and $n \leqslant n_2$ holds $s(n_1, n_2) \in B$. The theorem is a consequence of (62) and (59).

(65) $x \in \lim_{\mathrm{FrechetFilter}(\mathbb{N} \times \mathbb{N})} s$ if and only if for every element B of $\mathcal{B}$, there exists i and there exists j such that for every m and n such that $i \leqslant m$ or $j \leqslant n$ holds $s(m, n) \in B$. The theorem is a consequence of (61).

(66) $\lim_{\mathrm{FrechetFilter}(\mathbb{N} \times \mathbb{N})} s \subseteq \lim_{[\uparrow^2_{\mathbb{N}})} s$. The theorem is a consequence of (42), (43), (45), (48), and (49).

5. Metric Space and Double Sequence

Now we state the propositions:

(67) Let us consider a non empty metric space M, a point p of M, a point x of M_{top}, and a function s from $\mathbb{N} \times \mathbb{N}$ into M_{top}. Suppose $x = p$. Then $x \in \lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} s$ if and only if for every non zero natural number m, there exists a natural number n such that for every natural numbers n_1, n_2 such that $n \leqslant n_1$ and $n \leqslant n_2$ holds $s(n_1, n_2) \in \{q$, where q is a point of $M : \rho(p, q) < \frac{1}{m}\}$.

Proof: $x \in \lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} s$ iff for every non zero natural number m, there exists a natural number n such that for every natural numbers n_1, n_2 such that $n \leqslant n_1$ and $n \leqslant n_2$ holds $s(n_1, n_2) \in \{q$, where q is a point of $M : \rho(p, q) < \frac{1}{m}\}$ by [13, (6)], (64). □

(68) Let us consider a non empty metric space M, a point p of M, a point x of M_{top}, a function s from $\mathbb{N} \times \mathbb{N}$ into M_{top}, and a function s_2 from $\mathbb{N} \times \mathbb{N}$ into M. Suppose $x = p$ and $s = s_2$. Then $x \in \lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} s$ if and only if for every non zero natural number m, there exists a natural number n such that for every natural numbers n_1, n_2 such that $n \leqslant n_1$ and $n \leqslant n_2$ holds $s_2(n_1, n_2) \in \{q$, where q is a point of $M : \rho(p, q) < \frac{1}{m}\}$.

6. One-dimensional Euclidean Metric Space and Double Sequence

In the sequel R denotes a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$.

Now we state the proposition:

(69) Let us consider a point x of $(\mathcal{E}^1)_{\mathrm{top}}$, a point y of $\mathcal{E}^1$, a generalized basis $\mathcal{B}$ of BooleanFilterToFilter(the neighborhood system of x), and an element b of $\mathcal{B}$. Suppose $x = y$ and $\mathcal{B} = \mathrm{Balls}\, x$. Then there exists a natural number n such that $b = \{q$, where q is an element of $\mathcal{E}^1 : \rho(y, q) < \frac{1}{n}\}$.

Let s be a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. The functor $\# s$ yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}^{\mathbf{1}}$ is defined by the term

(Def. 7) s.

Now we state the propositions:

(70) Let us consider a function s from $\mathbb{N} \times \mathbb{N}$ into $(\mathcal{E}^1)_{\mathrm{top}}$, and a point y of $\mathcal{E}^1$. Then $s^\circ(\uparrow^2(n)) \subseteq \{q$, where q is an element of $\mathcal{E}^1 : \rho(y, q) < \frac{1}{m}\}$ if and only if for every object x such that $x \in s^\circ(\uparrow^2(n))$ there exist real numbers r_1, r_2 such that $x = \langle r_1 \rangle$ and $y = \langle r_2 \rangle$ and $|r_2 - r_1| < \frac{1}{m}$. The theorem is a consequence of (8).

(71) $r \in \lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} \# R$ if and only if for every non zero natural number m, there exists a natural number n such that for every

natural numbers n_1, n_2 such that $n \leqslant n_1$ and $n \leqslant n_2$ holds $|R(n_1, n_2) - r| < \frac{1}{m}$.

Proof: Reconsider $p = r$ as a point of the metric space of real numbers. for every non zero natural number m, there exists a natural number n such that for every natural numbers n_1, n_2 such that $n \leqslant n_1$ and $n \leqslant n_2$ holds $R(n_1, n_2) \in \{q$, where q is a point of the metric space of real numbers : $\rho(p, q) < \frac{1}{m}\}$ iff for every non zero natural number m, there exists a natural number n such that for every natural numbers n_1, n_2 such that $n \leqslant n_1$ and $n \leqslant n_2$ holds $|R(n_1, n_2) - r| < \frac{1}{m}$ by (6), [8, (60)]. □

7. Basic Relations Convergence in Pringsheim's Sense and Filter Convergence

Now we state the propositions:

(72) Suppose $\lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} \# R \neq \emptyset$. Then there exists a real number x such that $\lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} \# R = \{x\}$.

(73) If R is P-convergent, then $\text{P-lim}\, R \in \lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} \# R$. The theorem is a consequence of (71).

(74) R is P-convergent if and only if $\lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} \# R \neq \emptyset$. The theorem is a consequence of (71) and (5).

(75) Suppose R is P-convergent. Then $\{\text{P-lim}\, R\} = \lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} \# R$. The theorem is a consequence of (73) and (72).

(76) Suppose $\lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} \# R$ is not empty. Then

(i) R is P-convergent, and

(ii) $\{\text{P-lim}\, R\} = \lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} \# R$.

8. Example: Double Sequence Converges in Pringsheim's Sense but not in Frechet Filter of $\mathbb{N} \times \mathbb{N}$ Sense

The functor DblSeq-ex1 yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ is defined by

(Def. 8) for every natural numbers m, n, $it(m, n) = \frac{1}{m+1}$.

Now we state the propositions:

(77) Let us consider a non zero natural number m. Then there exists a natural number n such that for every natural numbers n_1, n_2 such that $n \leqslant n_1$ and $n \leqslant n_2$ holds $|(\text{DblSeq-ex1})(n_1, n_2) - 0| < \frac{1}{m}$.

(78) $0 \in \lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} \# \text{DblSeq-ex1}$.

(79) $\lim_{\text{FrechetFilter}(\mathbb{N}\times\mathbb{N})} \#\,\text{DblSeq-ex1} = \emptyset$. The theorem is a consequence of (66), (42), (43), (72), (78), and (65).

(80) $\lim_{\langle\text{FrechetFilter}(\mathbb{N}),\text{FrechetFilter}(\mathbb{N})\rangle} \#\,\text{DblSeq-ex1} \neq \lim_{\text{FrechetFilter}(\mathbb{N}\times\mathbb{N})} \#\,\text{DblSeq-ex1}$.

9. Correspondence with some Definitions from [14]

Let X_1, X_2 be non empty sets, $\mathcal{F}_1$ be a filter of X_1, Y be a Hausdorff, non empty topological space, and f be a function from $X_1 \times X_2$ into Y. Assume for every element x of X_2, $\lim_{\mathcal{F}_1} \text{curry}'(f, x) \neq \emptyset$. The functor $\lim_1(f, \mathcal{F}_1)$ yielding a function from X_2 into Y is defined by

(Def. 9) for every element x of X_2, $\{it(x)\} = \lim_{\mathcal{F}_1} \text{curry}'(f, x)$.

Let $\mathcal{F}_2$ be a filter of X_2. Assume for every element x of X_1, $\lim_{\mathcal{F}_2} \text{curry}(f, x) \neq \emptyset$. The functor $\lim_2(f, \mathcal{F}_2)$ yielding a function from X_1 into Y is defined by

(Def. 10) for every element x of X_1, $\{it(x)\} = \lim_{\mathcal{F}_2} \text{curry}(f, x)$.

Now we state the propositions:

(81) Every function from X into $\mathbb{R}$ is a function from X into $\mathbb{R}^{\mathbf{1}}$.

(82) Every sequence of $\mathbb{R}$ is a function from $\mathbb{N}$ into $\mathbb{R}^{\mathbf{1}}$.

From now on f denotes a function from $\Omega_{\text{the ordered }\mathbb{N}}$ into $\mathbb{R}^{\mathbf{1}}$ and s_1 denotes a function from $\mathbb{N}$ into $\mathbb{R}$.

Now we state the propositions:

(83) Suppose $f = s_1$ and $\text{LimF}(f) \neq \emptyset$. Then

(i) s_1 is convergent, and

(ii) there exists a real number z such that $z \in \text{LimF}(f)$ and for every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $|s_1(m) - z| < p$.

Proof: Consider x being an object such that $x \in \text{LimF}(f)$. Reconsider $y = x$ as a point of (the metric space of real numbers)$_{\text{top}}$. Reconsider $z = y$ as a real number. Consider y_1 being a point of the metric space of real numbers such that $y_1 = y$ and $\text{Balls}\, y = \{\text{Ball}(y_1, \frac{1}{n})$, where n is a natural number : $n \neq 0\}$. For every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $|s_1(m) - z| < p$ by (5), [12, (84), (50)], [2, (18)]. □

(84) If $f = s_1$ and $\text{LimF}(f) \neq \emptyset$, then $\text{LimF}(f) = \{\lim s_1\}$.

Proof: Consider x being an object such that $x \in \text{LimF}(f)$. Consider u being an object such that $\text{LimF}(f) = \{u\}$. $\text{LimF}(f) = \{\lim s_1\}$ by (83), [11, (3)]. □

(85) Let us consider a function f from Ω_α into T, and a sequence s of T. If $f = s$, then $\mathrm{LimF}(f) = \mathrm{LimF}(s)$, where α is the ordered $\mathbb{N}$.

(86) Let us consider a function f from Ω_α into T, and a function g from $\mathbb{N}$ into T. If $f = g$, then $\mathrm{LimF}(f) = \mathrm{LimF}(g)$, where α is the ordered $\mathbb{N}$.

(87) Let us consider a function f from $\mathbb{N}$ into $\mathbb{R}^{\mathbf{1}}$. Suppose $f = s_1$ and $\mathrm{LimF}(f) \neq \emptyset$. Then $\mathrm{LimF}(f) = \{\lim s_1\}$. The theorem is a consequence of (84).

(88) for every element x of $\mathbb{N}$, $\lim_{\mathrm{FrechetFilter}(\mathbb{N})} \mathrm{curry}'(\# R, x) \neq \emptyset$ if and only if R is convergent in the first coordinate. The theorem is a consequence of (5).

(89) for every element x of $\mathbb{N}$, $\lim_{\mathrm{FrechetFilter}(\mathbb{N})} \mathrm{curry}(\# R, x) \neq \emptyset$ if and only if R is convergent in the second coordinate. The theorem is a consequence of (5).

Let us consider an element t of $\mathbb{N}$, a function f from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}^{\mathbf{1}}$, and a function s_1 from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Now we state the propositions:

(90) Suppose $f = s_1$ and for every element x of $\mathbb{N}$, $\lim_{\mathrm{FrechetFilter}(\mathbb{N})} \mathrm{curry}(f, x) \neq \emptyset$. Then $\lim_{\mathrm{FrechetFilter}(\mathbb{N})} \mathrm{curry}(f, t) = \{\lim \mathrm{curry}(s_1, t)\}$. The theorem is a consequence of (87).

(91) Suppose $f = s_1$ and for every element x of $\mathbb{N}$, $\lim_{\mathrm{FrechetFilter}(\mathbb{N})} \mathrm{curry}'(f, x) \neq \emptyset$. Then $\lim_{\mathrm{FrechetFilter}(\mathbb{N})} \mathrm{curry}'(f, t) = \{\lim \mathrm{curry}'(s_1, t)\}$. The theorem is a consequence of (87).

(92) Let us consider a Hausdorff, non empty topological space Y, and a function f from $\mathbb{N}\times\mathbb{N}$ into Y. Suppose for every element x of $\mathbb{N}$, $\lim_{\mathrm{FrechetFilter}(\mathbb{N})} \mathrm{curry}'(f, x) \neq \emptyset$ and $f = R$ and $Y = \mathbb{R}^{\mathbf{1}}$. Then $\lim_1(f, \mathrm{FrechetFilter}(\mathbb{N})) =$ the lim in the first coordinate of R. The theorem is a consequence of (91).

(93) Let us consider a non empty, Hausdorff topological space Y, and a function f from $\mathbb{N}\times\mathbb{N}$ into Y. Suppose for every element x of $\mathbb{N}$, $\lim_{\mathrm{FrechetFilter}(\mathbb{N})} \mathrm{curry}(f, x) \neq \emptyset$ and $f = R$ and $Y = \mathbb{R}^{\mathbf{1}}$. Then $\lim_2(f, \mathrm{FrechetFilter}(\mathbb{N})) =$ the lim in the second coordinate of R. The theorem is a consequence of (90).

10. Regular Space, Double Limit and Iterated Limit

From now on Y denotes a non empty topological space, x denotes a point of Y, and f denotes a function from $X_1 \times X_2$ into Y.

Now we state the proposition:

(94) Suppose $x \in \lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f$ and $[\mathcal{B}_1) = \mathcal{F}_1$ and $[\mathcal{B}_2) = \mathcal{F}_2$. Let us consider a subset V of Y. Suppose V is open and $x \in V$. Then there exists an ele-

ment B_1 of $\mathcal{B}_1$ and there exists an element B_2 of $\mathcal{B}_2$ such that $f^\circ(B_1 \times B_2) \subseteq V$.

Let us consider a neighbourhood U of x. Now we state the propositions:

(95) Suppose $x \in \lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f$ and $[\mathcal{B}_1) = \mathcal{F}_1$ and $[\mathcal{B}_2) = \mathcal{F}_2$. Then suppose U is closed. Then there exists an element B_1 of $\mathcal{B}_1$ and there exists an element B_2 of $\mathcal{B}_2$ such that $f^\circ(B_1 \times B_2) \subseteq \operatorname{Int} U$.

(96) Suppose $x \in \lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f$ and $[\mathcal{B}_1) = \mathcal{F}_1$ and $[\mathcal{B}_2) = \mathcal{F}_2$. Then suppose U is closed. Then there exists an element B_1 of $\mathcal{B}_1$ and there exists an element B_2 of $\mathcal{B}_2$ such that for every element y of B_1, $f^\circ(\{y\} \times B_2) \subseteq \operatorname{Int} U$. The theorem is a consequence of (95).

(97) Suppose $x \in \lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f$ and $[\mathcal{B}_1) = \mathcal{F}_1$ and $[\mathcal{B}_2) = \mathcal{F}_2$. Then suppose U is closed. Then there exists an element B_1 of $\mathcal{B}_1$ and there exists an element B_2 of $\mathcal{B}_2$ such that for every element z of X_1 for every element y of Y such that $z \in B_1$ and $y \in \lim_{\mathcal{F}_2} \operatorname{curry}(f, z)$ holds $y \in \overline{\operatorname{Int} U}$.

PROOF: Consider B_1 being an element of $\mathcal{B}_1$, B_2 being an element of $\mathcal{B}_2$ such that $f^\circ(B_1 \times B_2) \subseteq \operatorname{Int} U$. For every element y of B_1, $f^\circ(\{y\} \times B_2) \subseteq \operatorname{Int} U$ by [11, (95)], [19, (125)]. For every element z of B_1 and for every element y of Y such that $y \in \lim_{\mathcal{F}_2} \operatorname{curry}(f, z)$ holds the image of filter $\mathcal{F}_2$ under $\operatorname{curry}(f, z)$ is a proper filter of $2^{\Omega_Y}_{\subseteq}$ and $\operatorname{Int} U \in$ the image of filter $\mathcal{F}_2$ under $\operatorname{curry}(f, z)$ and y is a cluster point of the image of filter $\mathcal{F}_2$ under $\operatorname{curry}(f, z)$,$Y$ by (18), [19, (132)], [10, (95)], (20). For every element z of B_1 and for every element y of Y such that $y \in \lim_{\mathcal{F}_2} \operatorname{curry}(f, z)$ holds $y \in \overline{\operatorname{Int} U}$ by [4, (25)]. □

(98) Suppose $x \in \lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f$ and $[\mathcal{B}_1) = \mathcal{F}_1$ and $[\mathcal{B}_2) = \mathcal{F}_2$. Then suppose U is closed. Then there exists an element B_1 of $\mathcal{B}_1$ and there exists an element B_2 of $\mathcal{B}_2$ such that for every element z of X_2 for every element y of Y such that $z \in B_2$ and $y \in \lim_{\mathcal{F}_1} \operatorname{curry}'(f, z)$ holds $y \in \overline{\operatorname{Int} U}$.

PROOF: Consider B_1 being an element of $\mathcal{B}_1$, B_2 being an element of $\mathcal{B}_2$ such that $f^\circ(B_1 \times B_2) \subseteq \operatorname{Int} U$. For every element y of B_2, $f^\circ(B_1 \times \{y\}) \subseteq \operatorname{Int} U$ by [11, (95)], [19, (125)]. For every element z of B_2 and for every element y of Y such that $y \in \lim_{\mathcal{F}_1} \operatorname{curry}'(f, z)$ holds the image of filter $\mathcal{F}_1$ under $\operatorname{curry}'(f, z)$ is a proper filter of $2^{\Omega_Y}_{\subseteq}$ and $\operatorname{Int} U \in$ the image of filter $\mathcal{F}_1$ under $\operatorname{curry}'(f, z)$ and y is a cluster point of the image of filter $\mathcal{F}_1$ under $\operatorname{curry}'(f, z)$,$Y$ by (18), [19, (132)], [10, (95)], (20). For every element z of B_2 and for every element y of Y such that $y \in \lim_{\mathcal{F}_1} \operatorname{curry}'(f, z)$ holds $y \in \overline{\operatorname{Int} U}$ by [4, (25)]. □

Let us consider a Hausdorff, regular, non empty topological space Y and a function f from $X_1 \times X_2$ into Y. Now we state the propositions:

(99) Suppose for every element x of X_2, $\lim_{\mathcal{F}_1} \operatorname{curry}'(f, x) \neq \emptyset$. Then $\lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle}$

$f \subseteq \lim_{\mathcal{F}_2} \lim_1(f, \mathcal{F}_1)$. The theorem is a consequence of (19) and (98).

(100) Suppose for every element x of X_1, $\lim_{\mathcal{F}_2} \mathrm{curry}(f, x) \neq \emptyset$. Then $\lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f \subseteq \lim_{\mathcal{F}_1} \lim_2(f, \mathcal{F}_2)$. The theorem is a consequence of (19) and (97).

Let us consider non empty sets X_1, X_2, a filter $\mathcal{F}_1$ of X_1, a filter $\mathcal{F}_2$ of X_2, a Hausdorff, regular, non empty topological space Y, and a function f from $X_1 \times X_2$ into Y. Now we state the propositions:

(101) Suppose $\lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f \neq \emptyset$ and for every element x of X_1, $\lim_{\mathcal{F}_2} \mathrm{curry}(f, x) \neq \emptyset$. Then $\lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f = \lim_{\mathcal{F}_1} \lim_2(f, \mathcal{F}_2)$. The theorem is a consequence of (100).

(102) Suppose $\lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f \neq \emptyset$ and for every element x of X_2, $\lim_{\mathcal{F}_1} \mathrm{curry}'(f, x) \neq \emptyset$. Then $\lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f = \lim_{\mathcal{F}_2} \lim_1(f, \mathcal{F}_1)$. The theorem is a consequence of (99).

(103) Suppose $\lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f \neq \emptyset$ and for every element x of X_1, $\lim_{\mathcal{F}_2} \mathrm{curry}(f, x) \neq \emptyset$ and for every element x of X_2, $\lim_{\mathcal{F}_1} \mathrm{curry}'(f, x) \neq \emptyset$. Then $\lim_{\mathcal{F}_1} \lim_2 (f, \mathcal{F}_2) = \lim_{\mathcal{F}_2} \lim_1(f, \mathcal{F}_1)$. The theorem is a consequence of (102) and (101).

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Prime Factorization of Sums and Differences of Two Like Powers

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Summary. Representation of a non zero integer as a signed product of primes is unique similarly to its representations in various types of positional notations [4], [3]. The study focuses on counting the prime factors of integers in the form of sums or differences of two equal powers (thus being represented by 1 and a series of zeroes in respective digital bases).

Although the introduced theorems are not particularly important, they provide a couple of shortcuts useful for integer factorization, which could serve in further development of Mizar projects [2]. This could be regarded as one of the important benefits of proof formalization [9].

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From now on a, b, c, d, x, j, k, l, m, n, o denote natural numbers, p, q, t, z, u, v denote integers, and a_1, b_1, c_1, d_1 denote complexes.

Now we state the propositions:

(1) $a_1{}^{n+k} + b_1{}^{n+k} = a_1{}^n \cdot (a_1{}^k + b_1{}^k) + b_1{}^k \cdot (b_1{}^n - a_1{}^n)$.

(2) $a_1{}^{n+k} - b_1{}^{n+k} = a_1{}^n \cdot (a_1{}^k - b_1{}^k) + b_1{}^k \cdot (a_1{}^n - b_1{}^n)$.

(3) $a_1{}^{m+2} + b_1{}^{m+2} = (a_1 + b_1) \cdot (a_1{}^{m+1} + b_1{}^{m+1}) - a_1 \cdot b_1 \cdot (a_1{}^m + b_1{}^m)$.

Let a be a natural number. Let us note that a is trivial if and only if the condition (Def. 1) is satisfied.

(Def. 1) $a \leqslant 1$.

Let a be a complex. Let us note that the functor $a^{\mathbf{2}}$ yields a set and is defined by the term

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(Def. 2) a^2.

Let a, b be integers. The functors: $\gcd(a,b)$ and $\operatorname{lcm}(a,b)$ yielding natural numbers are defined by terms

(Def. 3) $\gcd(|a|,|b|)$,

(Def. 4) $\operatorname{lcm}(|a|,|b|)$,

respectively. Let a, b be positive real numbers. Note that $\max(a,b)$ is positive and $\min(a,b)$ is positive.

Let a be a non zero integer and b be an integer. One can check that $\gcd(a,b)$ is non zero.

Let a be a non zero complex and n be a natural number. Let us observe that a^n is non zero.

Let a be a non trivial natural number and n be a non zero natural number. Note that a^n is non trivial.

Let a be an integer. One can check that $|a|$ is natural.

Let a be an even integer. Note that $|a|$ is even.

Let a be a natural number. Let us note that $\operatorname{lcm}(a,a)$ reduces to a and $\gcd(a,a)$ reduces to a.

Let a be a non zero integer and b be an integer. Note that $\gcd(a,b)$ is positive.

Let a, b be integers. One can check that $\gcd(a,\gcd(a,b))$ reduces to $\gcd(a,b)$ and $\operatorname{lcm}(a,\operatorname{lcm}(a,b))$ reduces to $\operatorname{lcm}(a,b)$.

Let a be an integer. Observe that $\gcd(a,1)$ reduces to 1 and $\gcd(a+1,a)$ reduces to 1.

Now we state the proposition:

(4) Let us consider integers t, z. Then $\gcd(t^n,z^n)=(\gcd(t,z))^n$.

Let a be an integer and n be a natural number.

One can verify that $\gcd((a+1)^n,a^n)$ reduces to 1.

Let us consider a_1 and b_1. One can verify that ${a_1}^0-{b_1}^0$ reduces to 0.

Let a be a non negative real number and n be a natural number. One can verify that a^n is non negative and there exists an odd natural number which is non trivial and there exists an even natural number which is non trivial.

Let a be a positive real number and n be a natural number. One can verify that a^n is positive.

Let a be an integer. One can verify that $a\cdot a$ is square and $\frac{a}{a}$ is square and there exists an element of $\mathbb{N}$ which is non square and every element of $\mathbb{N}$ which is prime is also non square and there exists a prime natural number which is even and there exists a prime natural number which is odd and every integer which is prime is also non square.

Let a be a square element of $\mathbb{N}$. Observe that $\sqrt{a}$ is natural.

Let a be an integer. Let us note that a^2 is square and $a \cdot a$ is square and there exists an integer which is non square and every natural number which is zero is also trivial and there exists a natural number which is square and there exists an element of $\mathbb{N}$ which is non zero and there exists a square element of $\mathbb{N}$ which is non trivial and every natural number which is trivial is also square and every integer which is non square is also non zero.

Now we state the propositions:

(5) Let us consider integers a, b, c, d. If $a \mid b$ and $c \mid d$, then $a \cdot c \mid b \cdot d$.

(6) Let us consider integers a, b. Then $a \mid b$ if and only if $\operatorname{lcm}(a, b) = |b|$.
PROOF: If $a \mid b$, then $\operatorname{lcm}(a, b) = |b|$ by [8, (16)], [7, (44)]. □

Let a be an integer. Observe that $\operatorname{lcm}(a, 0)$ reduces to 0.

Let a be a natural number. Note that $\operatorname{lcm}(a, 1)$ reduces to a.

Let us consider a and b. Let us observe that $\operatorname{lcm}(a \cdot b, a)$ reduces to $a \cdot b$ and $\operatorname{lcm}(\gcd(a, b), b)$ reduces to b and $\gcd(a, \operatorname{lcm}(a, b))$ reduces to a.

Let us consider integers a, b. Now we state the propositions:

(7) $|a \cdot b| = (\gcd(a, b)) \cdot \operatorname{lcm}(a, b)$.

(8) $\operatorname{lcm}(a^n, b^n) = \operatorname{lcm}(a, b)^n$. The theorem is a consequence of (4) and (7).

Let a be a square element of $\mathbb{N}$ and b be a square element of $\mathbb{N}$. One can check that $\gcd(a, b)$ is square and $\operatorname{lcm}(a, b)$ is square.

Let a, b be square integers. One can verify that $\gcd(a, b)$ is square and $\operatorname{lcm}(a, b)$ is square.

Now we state the proposition:

(9) Let us consider an integer t. Then t is odd if and only if $\gcd(t, 2) = 1$.
PROOF: If t is odd, then $\gcd(t, 2) = 1$ by [13, (1)], [14, (5)]. □

Let t be an integer. One can check that t is odd if and only if the condition (Def. 5) is satisfied.

(Def. 5) $\gcd(t, 2) = 1$.

Let a be an odd integer. Let us observe that $|a|$ is odd and $-a$ is odd.

Let a, b be even integers. Note that $\gcd(a, b)$ is even.

Let a be an integer and b be an odd integer. Note that $\gcd(a, b)$ is odd.

Let a be a natural number. One can check that $|-a|$ reduces to a.

Let t, z be even integers. One can check that $t + z$ is even and $t - z$ is even and $t \cdot z$ is even.

Let t, z be odd integers. Note that $t + z$ is even and $t - z$ is even and $t \cdot z$ is odd.

Let t be an odd integer and z be an even integer. Let us observe that $t + z$ is odd and $t - z$ is odd and $t \cdot z$ is even.

Now we state the proposition:

(10) Let us consider a non zero, square integer a, and an integer b. If $a \cdot b$ is square, then b is square.

Let a be a square element of $\mathbb{N}$ and n be a natural number. Let us observe that a^n is square.

Let a be a square integer. Note that a^n is square.

Let a be a non zero, square integer and b be a non square integer. Let us note that $a \cdot b$ is non square.

Let a be an element of $\mathbb{N}$ and b be an even natural number. Note that a^b is square.

Let a be a non square element of $\mathbb{N}$ and b be an odd natural number. Note that a^b is non square.

Let a be a non zero, square integer. Note that $a + 1$ is non square.

Let a be a non zero, square element of $\mathbb{N}$. Let us observe that $a + 1$ is non square.

Let a be a non zero, square object and b be a non square element of $\mathbb{N}$. Let us observe that $a \cdot b$ is non square.

Let a be a non zero, square integer and n, m be natural numbers. Let us observe that $a^n + a^m$ is non square.

Let a be a non zero, square element of $\mathbb{N}$. Let us note that $a^n + a^m$ is non square.

Let a be a non zero, square integer and p be a prime natural number. Note that $p \cdot a$ is non square.

Let a be a non trivial element of $\mathbb{N}$. One can verify that $a - 1$ is non zero.

Let q be a square integer. Let us observe that $|q|$ is square.

Let x be a non zero integer. Let us observe that $|x|$ is non zero.

Let a be a non trivial, square element of $\mathbb{N}$. Let us observe that $a - 1$ is non square.

Let a be a non trivial element of $\mathbb{N}$. Let us note that $a \cdot (a-1)$ is non square.

Let a, b be integers and n, m be natural numbers. One can verify that $(a^n + b^n) \cdot (a^m - b^m) + (a^m + b^m) \cdot (a^n - b^n)$ is even and $(a^n + b^n) \cdot (a^m + b^m) + (a^m - b^m) \cdot (a^n - b^n)$ is even.

Let a be an even integer. Let us note that $\frac{a}{2}$ is integer.

Let a, b be non zero natural numbers. Note that $a + b$ is non trivial.

Let b be a non zero natural number and a, c be non trivial natural numbers. Let us observe that $c\text{-count}(c^{a\text{-count}(b)})$ reduces to $a\text{-count}(b)$.

Let a, b be non zero integers. Let us note that $\frac{a}{\gcd(a,b)}$ is integer and $\frac{\operatorname{lcm}(a,b)}{b}$ is integer and $\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}$ is integer.

Let a be an even integer. One can verify that $\gcd(a, 2)$ reduces to 2.

Let us observe that there exists an even natural number which is non zero.

Let a be an even integer and n be a non zero natural number. Let us observe that $a \cdot n$ is even and a^n is even.

Let a be an integer and n be a zero natural number. One can check that $a \cdot n$ is even and a^n is odd.

Let a be an element of $\mathbb{N}$. Note that $|a|$ reduces to a.

One can check that every integer which is non negative is also natural.

Let a be a non negative real number and n be a non zero natural number. Let us note that $\sqrt[n]{a^n}$ reduces to a and $(\sqrt[n]{a})^n$ reduces to a.

Now we state the propositions:

(11) If $a \nmid b$, then $a \cdot c \nmid b$.

(12) Let us consider non negative real numbers a, b, and a positive natural number n. Then $a^n = b^n$ if and only if $a = b$.

Let a be a real number and n be an even natural number. One can verify that a^n is non negative.

Let a be a negative real number and n be an odd natural number. One can verify that a^n is negative.

Now we state the propositions:

(13) Let us consider real numbers a, b, and an odd natural number n. Then $a^n = b^n$ if and only if $a = b$. The theorem is a consequence of (12).

(14) If a and b are relatively prime, then for every non zero natural number n, $a \cdot b = c^n$ iff $\sqrt[n]{a}, \sqrt[n]{b} \in \mathbb{N}$ and $c = \sqrt[n]{a} \cdot \sqrt[n]{b}$.
Proof: If $a \cdot b = c^n$, then $\sqrt[n]{a}, \sqrt[n]{b} \in \mathbb{N}$ and $c = \sqrt[n]{a} \cdot \sqrt[n]{b}$ by [14, (30)], [11, (11)], [1, (14)]. □

(15) Let us consider a non zero natural number n, an integer a, and an integer b. Then $b^n \mid a^n$ if and only if $b \mid a$.
Proof: If $b^n \mid a^n$, then $b \mid a$ by [10, (1)], [14, (3)], (4), [5, (3)]. □

(16) Let us consider an integer a, and natural numbers m, n. If $m \geqslant n$, then $a^n \mid a^m$.

(17) Let us consider integers a, b. If $a \mid b$ and $b^m \mid c$, then $a^m \mid c$. The theorem is a consequence of (4).

(18) Let us consider integers a, p. If $p^{2 \cdot n + k} \mid a^2$, then $p^n \mid a$. The theorem is a consequence of (16), (4), and (12).

(19) Let us consider odd, square elements a, b of $\mathbb{N}$. Then $8 \mid a - b$.

Let us consider odd natural numbers a, b. Now we state the propositions:

(20) If $4 \mid a - b$, then $4 \nmid a^n + b^n$.

(21) If $4 \mid a^n + b^n$, then $4 \nmid a^{2 \cdot n} + b^{2 \cdot n}$.

(22) If $4 \mid a^n - b^n$, then $4 \nmid a^{2 \cdot n} + b^{2 \cdot n}$.

(23) Let us consider odd natural numbers a, b. If $2^m \mid a^n - b^n$, then $2^{m+1} \mid a^{2 \cdot n} - b^{2 \cdot n}$.

(24) ${a_1}^3 - {b_1}^3 = (a_1 - b_1) \cdot ({a_1}^2 + {b_1}^2 + a_1 \cdot b_1)$. The theorem is a consequence of (2).

(25) Let us consider an odd natural number n. Then $3 \mid a^n + b^n$ if and only if $3 \mid a + b$.

PROOF: Consider k such that $n = 2 \cdot k + 1$. If $3 \mid a^n + b^n$, then $3 \mid a + b$ by [14, (173)], [5, (4)], [8, (1), (10)]. □

(26) Let us consider an integer c. If $c \mid a - b$, then $c \mid a^n - b^n$.

(27) Let us consider an odd natural number n. Then $3 \mid a^n - b^n$ if and only if $3 \mid a - b$.

PROOF: Consider k such that $n = 2 \cdot k + 1$. If $3 \mid a^n - b^n$, then $3 \mid a - b$ by [14, (173)], [8, (10)], [5, (4)], [8, (1)]. □

(28) Let us consider a natural number n. Then $a^n \equiv (a-b)^n \pmod{b}$.

(29) Let us consider a non trivial natural number a. Then there exists a prime natural number n such that $n \mid a$.

(30) Let us consider a prime natural number p. If $p \mid (p+(k+1)) \cdot (p-(k+1))$, then $k + 1 \geqslant p$.

(31) Let us consider a prime natural number p, and a non zero natural number k. If $k < p$, then $p \nmid p^2 - k^2$. The theorem is a consequence of (30).

(32) Let us consider integers a, b, and an odd, prime natural number p. If $p \nmid b$, then if $p \mid a - b$, then $p \nmid a + b$.

(33) Let us consider a non zero, square element a of $\mathbb{N}$, and a prime natural number p. If $p \mid a$, then $a + p$ is not square.

(34) Let us consider a non zero, square element a of $\mathbb{N}$, and a prime natural number p. If $a + p$ is square, then $p = 2 \cdot \sqrt{a} + 1$.

(35) Let us consider integers a, b, c. Suppose a and b are relatively prime. Then $\gcd(c, a \cdot b) = (\gcd(c, a)) \cdot (\gcd(c, b))$.

(36) Let us consider a prime natural number p. If $a \mid p^n$, then there exists k such that $a = p^k$.

Let us consider non zero natural numbers a, b and a prime natural number p. Now we state the propositions:

(37) If $a + b = p$, then a and b are relatively prime.

(38) If $a^n + b^n = p^n$, then a and b are relatively prime.

(39) Let us consider non zero natural numbers a, b. If $c \geqslant a + b$, then $c^{k+1} \cdot (a + b) > a^{k+2} + b^{k+2}$.

(40) Let us consider natural numbers a, c, and a non zero natural number b. If $a \cdot b < c < a \cdot (b+1)$, then $a \nmid c$ and $c \nmid a$.

(41) Let us consider real numbers a, b. Then $a + b = \min(a, b) + \max(a, b)$.

(42) Let us consider non negative real numbers a, b. Then

(i) $\max(a^n, b^n) = (\max(a, b))^n$, and

(ii) $\min(a^n, b^n) = (\min(a, b))^n$.

(43) Let us consider a prime natural number p. Suppose $a \cdot b = p^n$. Then there exist natural numbers k, l such that

(i) $a = p^k$, and

(ii) $b = p^l$, and

(iii) $k + l = n$.

(44) Let us consider non trivial natural numbers a, b. If a and b are relatively prime, then $a \nmid b$ and $b \nmid a$.

(45) Let us consider a non trivial natural number a, and a prime natural number p. If $p > a$, then $p \nmid a$ and $a \nmid p$. The theorem is a consequence of (44).

(46) Let us consider a prime natural number p. Then

(i) $\gcd(a, p) = 1$, or

(ii) $\gcd(a, p) = p$.

(47) Let us consider a non trivial natural number a, and a prime natural number p. If $a \mid p^n$, then $p \mid a$. The theorem is a consequence of (46).

(48) Let us consider odd natural numbers a, b, and an even natural number m. Then $2\text{-count}(a^m + b^m) = 1$.

(49) Let us consider a non zero natural number a. Then there exists an odd natural number k such that $a = 2^{2\text{-count}(a)} \cdot k$.

(50) Let us consider a non zero natural number b. Suppose $a > b$. Then there exists a prime natural number p such that $p\text{-count}(a) > p\text{-count}(b)$.
Proof: If for every prime natural number p, $p\text{-count}(a) \leqslant p\text{-count}(b)$, then $a \leqslant b$ by [12, (20)], [1, (14)]. $\square$

(51) Let us consider natural numbers a, b, c. Suppose $a \neq 1$ and $b \neq 0$ and $c \neq 0$ and $b > a\text{-count}(c)$. Then $a^b \nmid c$. The theorem is a consequence of (11).

Let us consider a non zero integer b and an integer a. Now we state the propositions:

(52) If $|a| \neq 1$, then $a^{|a|\text{-count}(|b|)} \mid b$ and $a^{(|a|\text{-count}(|b|))+1} \nmid b$.

(53) If $|a| \neq 1$, then if $a^n \mid b$ and $a^{n+1} \nmid b$, then $n = |a|\text{-count}(|b|)$.

(54) Let us consider a non zero natural number b, and a non trivial natural number a. Then $a \mid b$ if and only if a-count$(\gcd(a,b)) = 1$.
PROOF: If $a \mid b$, then a-count$(\gcd(a,b)) = 1$ by [14, (3)], [6, (22)]. □

(55) Let us consider non zero natural numbers b, n, and a non trivial natural number a. Then a-count$(\gcd(a,b)) = 1$ if and only if a^n-count$((\gcd(a,b))^n) = 1$. The theorem is a consequence of (15), (54), and (4).

(56) Let us consider a non zero natural number b, and a non trivial natural number a. Then a-count$(\gcd(a,b)) = 0$ if and only if a-count$(\gcd(a,b)) \neq 1$. The theorem is a consequence of (54).

Let a, b be integers. The functor a-count(b) yielding a natural number is defined by the term

(Def. 6) $|a|$-count$(|b|)$.

Let a be an integer. Assume $|a| \neq 1$. Let b be a non zero integer. One can check that the functor a-count(b) is defined by

(Def. 7) $a^{it} \mid b$ and $a^{it+1} \nmid b$.

Now we state the propositions:

(57) Let us consider a prime natural number p, and non zero integers a, b. Then p-count$(a \cdot b) = (p$-count$(a)) + (p$-count$(b))$.

(58) Let us consider a non trivial natural number a, and a non zero natural number b. Then $a^{a\text{-count}(b)} \leqslant b$.

(59) Let us consider a non trivial natural number a, and a non zero integer b. Then $a^n \mid b$ if and only if $n \leqslant a$-count(b).
PROOF: If $a^n \mid b$, then $n \leqslant a$-count(b) by [8, (9)], [7, (89)], [1, (13)]. If $a^n \nmid b$, then a-count$(b) < n$ by [8, (9)], [7, (89)]. □

(60) Let us consider a non trivial natural number a, a non zero integer b, and a non zero natural number n. Then $n \cdot (a$-count$(b)) \leqslant a$-count$(b^n) < n \cdot ((a$-count$(b)) + 1)$. The theorem is a consequence of (4) and (59).

(61) Let us consider a non trivial natural number a, and non zero natural numbers b, n. If $b < a$, then a-count$(b^n) < n$. The theorem is a consequence of (60).

(62) Let us consider a non trivial natural number a, and a non zero natural number b. If $b < a^n$, then a-count$(b) < n$. The theorem is a consequence of (59).

(63) Let us consider non zero natural numbers a, b, and a non trivial natural number n. Then $a + b$-count$(a^n + b^n) < n$. The theorem is a consequence of (62).

(64) Let us consider non zero natural numbers a, b. Then $\gcd(a,b) = 1$ if and only if for every non trivial natural number c, $(c$-count$(a)) \cdot (c$-count$(b)) = 0$.

Proof: If $\gcd(a, b) = 1$, then for every non trivial natural number c, $(c\text{-count}(a)) \cdot (c\text{-count}(b)) = 0$ by [6, (27)]. If for every prime natural number c, $(c\text{-count}(a)) \cdot (c\text{-count}(b)) = 0$, then $\gcd(a, b) = 1$ by [6, (27)]. □

Let us consider a non zero, even natural number m and odd natural numbers a, b. Now we state the propositions:

(65) If $a \neq b$, then $2\text{-count}(a^{2 \cdot m} - b^{2 \cdot m}) \geqslant (2\text{-count}(a^m - b^m)) + 1$. The theorem is a consequence of (12), (23), and (59).

(66) If $a \neq b$, then $2\text{-count}(a^{2 \cdot m} - b^{2 \cdot m}) = (2\text{-count}(a^m - b^m)) + 1$. The theorem is a consequence of (12), (57), and (48).

Let us consider a prime natural number p and integers a, b. Now we state the propositions:

(67) If $|a| \neq |b|$, then $p\text{-count}(a^2 - b^2) = (p\text{-count}(a - b)) + (p\text{-count}(a + b))$.

(68) If $|a| \neq |b|$, then $p\text{-count}(a^3 - b^3) = (p\text{-count}(a - b)) + (p\text{-count}(a^2 + a \cdot b + b^2))$. The theorem is a consequence of (24).

(69) Let us consider non zero natural numbers a, b. Then $\frac{a}{\gcd(a,b)} = \frac{\operatorname{lcm}(a,b)}{b}$.

Let us consider a non zero natural number b. Now we state the propositions:

(70) $\operatorname{lcm}(a, a \cdot n + b) = ((\frac{a \cdot n}{b}) + 1) \cdot \operatorname{lcm}(a, b)$. The theorem is a consequence of (69).

(71) $\operatorname{lcm}(a, (n \cdot a + 1) \cdot b) = (n \cdot a + 1) \cdot \operatorname{lcm}(a, b)$. The theorem is a consequence of (70).

(72) Let us consider a non trivial natural number a, and non zero natural numbers n, b. Then $a\text{-count}(b) \geqslant n \cdot (a^n\text{-count}(b))$. The theorem is a consequence of (51).

Let us consider odd integers a, b. Now we state the propositions:

(73) $4 \mid a - b$ if and only if $4 \nmid a + b$.

(74) $2\text{-count}(a^2 + b^2) = 1$. The theorem is a consequence of (5) and (73).

(75) Let us consider a prime natural number p, and natural numbers a, b. Suppose $a \neq b$. Then $p\text{-count}(a + b) \geqslant p\text{-count}(\gcd(a, b))$.

(76) Let us consider a non zero integer a, a non trivial natural number b, and an integer c. If $a = b^{b\text{-count}(a)} \cdot c$, then $b \nmid c$.

Let a be a non zero integer and b be a non trivial natural number. Let us note that $\frac{a}{b^{b\text{-count}(a)}}$ is integer and $\frac{a}{2^{2\text{-count}(a)}}$ is integer and $\frac{a}{2^{2\text{-count}(a)}}$ is odd.

Now we state the proposition:

(77) Let us consider a non zero integer a, and a non trivial natural number b. Then $b\text{-count}(a) = 0$ if and only if $b \nmid a$.

Let a be an odd integer. Observe that $2\text{-count}(a)$ is zero.

Observe that $\frac{a}{2^{2\text{-count}(a)}}$ reduces to a.

Now we state the propositions:

(78) Let us consider a prime natural number a, a non zero integer b, and a natural number c. Then $a\text{-count}(b^c) = c \cdot (a\text{-count}(b))$.

(79) Let us consider non zero natural numbers a, b, and an odd natural number n. Then $\frac{a^{n+2}+b^{n+2}}{a+b} = a^{n+1} + b^{n+1} - a \cdot b \cdot (\frac{a^n+b^n}{a+b})$. The theorem is a consequence of (3).

(80) Let us consider odd integers a, b, and a natural number n. Then $2\text{-count}(a^{2\cdot n+1} - b^{2\cdot n+1}) = 2\text{-count}(a - b)$. The theorem is a consequence of (13), (2), and (57).

(81) Let us consider odd integers a, b, and an odd natural number m. Then $2\text{-count}(a^m + b^m) = 2\text{-count}(a + b)$. The theorem is a consequence of (80).

(82) Let us consider odd natural numbers a, b. Suppose $a \neq b$. Then $1 = \min(2\text{-count}(a - b), 2\text{-count}(a + b))$.

Let us consider a non trivial natural number a and non zero integers b, c. Now we state the propositions:

(83) If $a\text{-count}(b) > a\text{-count}(c)$, then $a^{a\text{-count}(c)} \mid b$ and $a^{a\text{-count}(b)} \nmid c$.

(84) If $a^{a\text{-count}(b)} \mid c$ and $a^{a\text{-count}(c)} \mid b$, then $a\text{-count}(b) = a\text{-count}(c)$. The theorem is a consequence of (83).

(85) Let us consider integers a, b, and natural numbers m, n. If $a^n \mid b$ and $a^m \nmid b$, then $m > n$. The theorem is a consequence of (16).

Let us consider a non trivial natural number a and non zero integers b, c. Now we state the propositions:

(86) If $a\text{-count}(b) = a\text{-count}(c)$ and $a^n \mid b$, then $a^n \mid c$. The theorem is a consequence of (85).

(87) $a\text{-count}(b) = a\text{-count}(c)$ if and only if for every natural number n, $a^n \mid b$ iff $a^n \mid c$.

Proof: If $a\text{-count}(b) \neq a\text{-count}(c)$, then there exists a natural number n such that $a^n \mid b$ and $a^n \nmid c$ or $a^n \mid c$ and $a^n \nmid b$ by (83), [1, (13)], [7, (89)], [8, (9)]. □

(88) Let us consider odd integers a, b. Suppose $|a| \neq |b|$. Then

(i) $2\text{-count}((a - b)^2) \neq 2\text{-count}((a + b)^2)$, and

(ii) $2\text{-count}((a - b)^2) \neq (2\text{-count}(a^2)) - b^2$.

The theorem is a consequence of (78), (73), and (87).

(89) Let us consider a non trivial natural number b, and a non zero integer a. Then $b\text{-count}(a) \neq 0$ if and only if $b \mid a$.

Proof: $b\text{-count}(|a|) \neq 0$ iff $b \mid |a|$ by [6, (27)]. □

(90) Let us consider a non trivial natural number b, and a non zero natural number a. Then b-count$(a) = 0$ if and only if $a \bmod b \neq 0$. The theorem is a consequence of (89).

(91) Let us consider a prime natural number p, and a non trivial natural number a. Then a-count$(p) \leqslant 1$.

(92) Let us consider non trivial natural numbers a, b, and a non zero natural number c. Then $a^{(a\text{-count}(b))\cdot(b\text{-count}(c))} \leqslant c$. The theorem is a consequence of (58).

(93) Let us consider a prime natural number p, a non trivial natural number a, and a non zero natural number b. Then a-count$(p^b) \leqslant b$. The theorem is a consequence of (89) and (59).

(94) Let us consider a prime natural number p, and a non trivial natural number a. Then $(p\text{-count}(a))\cdot(a\text{-count}(p^n)) \leqslant n$. The theorem is a consequence of (92).

(95) Let us consider non trivial natural numbers a, b, and a non zero natural number c. Then $(a\text{-count}(b)) \cdot (b\text{-count}(c)) \leqslant a\text{-count}(c)$. The theorem is a consequence of (17).

(96) Let us consider a non zero natural number a, and an odd natural number b. Then $2\text{-count}(a \cdot b) = 2\text{-count}(a)$.

Let us consider a non trivial natural number a. Now we state the propositions:

(97) $a^{n+1} + a^n < a^{n+2}$.

(98) $(a+1)^n + (a+1)^n < (a+1)^{n+1}$.

(99) Let us consider a non trivial, odd natural number a. Then $a^n + a^n < a^{n+1}$. The theorem is a consequence of (98).

(100) Let us consider a non trivial natural number p. If $a \nmid b$, then $(p^a)^c \neq p^b$.

(101) Let us consider non zero integers a, b, and a non zero natural number n. Suppose there exists a prime natural number p such that $n \nmid p\text{-count}(a)$. Then $a \neq b^n$.

(102) Let us consider non zero integers a, b, and a non zero natural number n. Suppose $a = b^n$. Let us consider a prime natural number p. Then $n \mid p\text{-count}(a)$.

(103) Let us consider positive real numbers a, b, and a non trivial natural number n. Then $(a+b)^n > a^n + b^n$. The theorem is a consequence of (42) and (41).

(104) Let us consider non zero integers a, b, and an odd, prime natural number p. Suppose $|a| \neq |b|$ and $p \nmid b$. Then $p\text{-count}(a^2 - b^2) = \max(p\text{-count}(a - b), p\text{-count}(a+b))$. The theorem is a consequence of (32), (77), and (57).

(105) Let us consider a non trivial natural number a, and a non zero integer b. Then a-count$(a^n \cdot b) = n + (a$-count$(b))$.

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Riemann-Stieltjes Integral[1]

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Summary. In this article, the definitions and basic properties of Riemann-Stieltjes integral are formalized in Mizar [1]. In the first section, we showed the preliminary definition. We proved also some properties of finite sequences of real numbers. In Sec. 2, we defined variation. Using the definition, we also defined bounded variation and total variation, and proved theorems about related properties.

In Sec. 3, we defined Riemann-Stieltjes integral. Referring to the way of the article [7], we described the definitions. In the last section, we proved theorems about linearity of Riemann-Stieltjes integral. Because there are two types of linearity in Riemann-Stieltjes integral, we proved linearity in two ways. We showed the proof of theorems based on the description of the article [7]. These formalizations are based on [8], [5], [3], and [4].

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1. Properties of Real Finite Sequences

Let A be a subset of $\mathbb{R}$ and ϱ be a real-valued function. The functor $\mathrm{vol}(A, \varrho)$ yielding a real number is defined by the term

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(Def. 1) $\begin{cases} 0, & \textbf{if } A \text{ is empty,} \\ \varrho(\sup A) - \varrho(\inf A), & \textbf{otherwise}. \end{cases}$

Now we state the propositions:

(1) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a partition D of A, a function ϱ from A into $\mathbb{R}$, a non empty, closed interval subset B of $\mathbb{R}$, and a finite sequence v of elements of $\mathbb{R}$. Suppose $B \subseteq A$ and $\operatorname{len} D = \operatorname{len} v$ and for every natural number i such that $i \in \operatorname{dom} v$ holds $v(i) = \operatorname{vol}(B \cap \operatorname{divset}(D, i), \varrho)$. Then $\sum v = \operatorname{vol}(B, \varrho)$.

(2) Let us consider natural numbers n, m, a function a from $\operatorname{Seg} n \times \operatorname{Seg} m$ into $\mathbb{R}$, and finite sequences p, q of elements of $\mathbb{R}$. Suppose $\operatorname{dom} p = \operatorname{Seg} n$ and for every natural number i such that $i \in \operatorname{dom} p$ there exists a finite sequence r of elements of $\mathbb{R}$ such that $\operatorname{dom} r = \operatorname{Seg} m$ and $p(i) = \sum r$ and for every natural number j such that $j \in \operatorname{dom} r$ holds $r(j) = a(i, j)$ and $\operatorname{dom} q = \operatorname{Seg} m$ and for every natural number j such that $j \in \operatorname{dom} q$ there exists a finite sequence s of elements of $\mathbb{R}$ such that $\operatorname{dom} s = \operatorname{Seg} n$ and $q(j) = \sum s$ and for every natural number i such that $i \in \operatorname{dom} s$ holds $s(i) = a(i, j)$. Then $\sum p = \sum q$.

2. The Definitions of Bounded Variation

Let A be a non empty, closed interval subset of $\mathbb{R}$, ϱ be a real-valued function, and t be a partition of A. A var-volume of ϱ and t is a finite sequence of elements of $\mathbb{R}$ and is defined by

(Def. 2) $\operatorname{len} it = \operatorname{len} t$ and for every natural number k such that $k \in \operatorname{dom} t$ holds $it(k) = |\operatorname{vol}(\operatorname{divset}(t, k), \varrho)|$.

Now we state the propositions:

(3) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a function ϱ from A into $\mathbb{R}$, a partition t of A, a var-volume F of ϱ and t, and a natural number k. If $k \in \operatorname{dom} F$, then $0 \leqslant F(k)$.

(4) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a function ϱ from A into $\mathbb{R}$, a partition t of A, and a var-volume F of ϱ and t. Then $0 \leqslant \sum F$. The theorem is a consequence of (3).

Let A be a non empty, closed interval subset of $\mathbb{R}$ and ϱ be a function from A into $\mathbb{R}$. We say that ϱ is bounded-variation if and only if

(Def. 3) there exists a real number d such that $0 < d$ and for every partition t of A and for every var-volume F of ϱ and t, $\sum F \leqslant d$.

Assume ϱ is bounded-variation. The functor $\operatorname{TotalVD}(\varrho)$ yielding a real number is defined by

(Def. 4) there exists a non empty subset V of $\mathbb{R}$ such that V is upper bounded and $V = \{r$, where r is a real number : there exists a partition t of A and there exists a var-volume F of ϱ and t such that $r = \sum F\}$ and $it = \sup V$.

Now we state the propositions:

(5) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a function ϱ from A into $\mathbb{R}$, and a partition T of A. Suppose ϱ is bounded-variation. Let us consider a var-volume F of ϱ and T. Then $\sum F \leqslant \mathrm{TotalVD}(\varrho)$.

(6) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, and a function ϱ from A into $\mathbb{R}$. If ϱ is bounded-variation, then $0 \leqslant \mathrm{TotalVD}(\varrho)$. The theorem is a consequence of (4).

3. The Definitions of Riemann-Stieltjes Integral

Let A be a non empty, closed interval subset of $\mathbb{R}$, ϱ be a function from A into $\mathbb{R}$, and u be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Assume ϱ is bounded-variation and $\mathrm{dom}\, u = A$. Let t be a partition of A.

A middle volume of ϱ, u and t is a finite sequence of elements of $\mathbb{R}$ and is defined by

(Def. 5) $\mathrm{len}\, it = \mathrm{len}\, t$ and for every natural number k such that $k \in \mathrm{dom}\, t$ there exists a real number r such that $r \in \mathrm{rng}(u \restriction \mathrm{divset}(t, k))$ and $it(k) = r \cdot \mathrm{vol}(\mathrm{divset}(t, k), \varrho)$.

Let T be a division sequence of A. A middle volume sequence of ϱ, u and T is a sequence of $\mathbb{R}^*$ and is defined by

(Def. 6) for every element k of $\mathbb{N}$, $it(k)$ is a middle volume of ϱ, u and $T(k)$.

Let S be a middle volume sequence of ϱ, u and T and k be a natural number. One can check that the functor $S(k)$ yields a middle volume of ϱ, u and $T(k)$. From now on A denotes a non empty, closed interval subset of $\mathbb{R}$, ϱ denotes a function from A into $\mathbb{R}$, u denotes a partial function from $\mathbb{R}$ to $\mathbb{R}$, T denotes a division sequence of A, S denotes a middle volume sequence of ϱ, u and T, and k denotes a natural number.

Let A be a non empty, closed interval subset of $\mathbb{R}$, ϱ be a function from A into $\mathbb{R}$, u be a partial function from $\mathbb{R}$ to $\mathbb{R}$, T be a division sequence of A, and S be a middle volume sequence of ϱ, u and T. The functor middle-sum(S) yielding a sequence of real numbers is defined by

(Def. 7) for every natural number i, $it(i) = \sum(S(i))$.

We say that u is Riemann-Stieltjes integrable with ϱ if and only if

(Def. 8) there exists a real number I such that for every division sequence T of A for every middle volume sequence S of ϱ, u and T such that δ_T is convergent and $\lim \delta_T = 0$ holds middle-sum(S) is convergent and $\lim$ middle-sum$(S) = I$.

Assume ϱ is bounded-variation and $\operatorname{dom} u = A$ and u is Riemann-Stieltjes integrable with ϱ. The functor $\int_{\varrho} u(x)dx$ yielding a real number is defined by

(Def. 9) for every division sequence T of A and for every middle volume sequence S of ϱ, u and T such that δ_T is convergent and $\lim \delta_T = 0$ holds middle-sum(S) is convergent and $\lim$ middle-sum$(S) = it$.

4. Linearity of Riemann-Stieltjes Integral

Now we state the propositions:

(7) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a real number r, a function ϱ from A into $\mathbb{R}$, and partial functions u, w from $\mathbb{R}$ to $\mathbb{R}$. Suppose ϱ is bounded-variation and $\operatorname{dom} u = A$ and $\operatorname{dom} w = A$ and $w = r \cdot u$ and u is Riemann-Stieltjes integrable with ϱ. Then

(i) w is Riemann-Stieltjes integrable with ϱ, and

(ii) $\int_{\varrho} w(x)dx = r \cdot \int_{\varrho} u(x)dx$.

(8) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a function ϱ from A into $\mathbb{R}$, and partial functions u, w from $\mathbb{R}$ to $\mathbb{R}$. Suppose ϱ is bounded-variation and $\operatorname{dom} u = A$ and $\operatorname{dom} w = A$ and $w = -u$ and u is Riemann-Stieltjes integrable with ϱ. Then

(i) w is Riemann-Stieltjes integrable with ϱ, and

(ii) $\int_{\varrho} w(x)dx = -\int_{\varrho} u(x)dx$.

The theorem is a consequence of (7).

Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a function ϱ from A into $\mathbb{R}$, and partial functions u, v, w from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(9) Suppose ϱ is bounded-variation and $\operatorname{dom} u = A$ and $\operatorname{dom} v = A$ and $\operatorname{dom} w = A$ and $w = u + v$ and u is Riemann-Stieltjes integrable with ϱ and v is Riemann-Stieltjes integrable with ϱ. Then

(i) w is Riemann-Stieltjes integrable with ϱ, and

(ii) $\int\limits_{\varrho} w(x)dx = \int\limits_{\varrho} u(x)dx + \int\limits_{\varrho} v(x)dx$.

(10) Suppose ϱ is bounded-variation and $\operatorname{dom} u = A$ and $\operatorname{dom} v = A$ and $\operatorname{dom} w = A$ and $w = u - v$ and u is Riemann-Stieltjes integrable with ϱ and v is Riemann-Stieltjes integrable with ϱ. Then

(i) w is Riemann-Stieltjes integrable with ϱ, and

(ii) $\int\limits_{\varrho} w(x)dx = \int\limits_{\varrho} u(x)dx - \int\limits_{\varrho} v(x)dx$.

The theorem is a consequence of (8) and (9).

(11) Let us consider non empty, closed interval subsets A, B of $\mathbb{R}$, a real number r, and functions ϱ, ϱ_1 from A into $\mathbb{R}$. Suppose $B \subseteq A$ and $\varrho = r \cdot \varrho_1$. Then $\operatorname{vol}(B, \varrho) = r \cdot \operatorname{vol}(B, \varrho_1)$.

Proof: Set $x_1 = \sup B$. Set $x_2 = \inf B$. $|x_2 - x_1| = x_1 - x_2$ by [6, (11)], [2, (44)]. □

(12) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a real number r, functions ϱ, ϱ_1 from A into $\mathbb{R}$, and a partial function u from $\mathbb{R}$ to $\mathbb{R}$. Suppose ϱ is bounded-variation and ϱ_1 is bounded-variation and $\operatorname{dom} u = A$ and $\varrho = r \cdot \varrho_1$ and u is Riemann-Stieltjes integrable with ϱ_1. Then

(i) u is Riemann-Stieltjes integrable with ϱ, and

(ii) $\int\limits_{\varrho} u(x)dx = r \cdot \int\limits_{\varrho_1} u(x)dx$.

The theorem is a consequence of (11).

(13) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, functions ϱ, ϱ_1 from A into $\mathbb{R}$, and a partial function u from $\mathbb{R}$ to $\mathbb{R}$. Suppose ϱ is bounded-variation and ϱ_1 is bounded-variation and $\operatorname{dom} u = A$ and $\varrho = -\varrho_1$ and u is Riemann-Stieltjes integrable with ϱ_1. Then

(i) u is Riemann-Stieltjes integrable with ϱ, and

(ii) $\int\limits_{\varrho} u(x)dx = -\int\limits_{\varrho_1} u(x)dx$.

The theorem is a consequence of (12).

(14) Let us consider non empty, closed interval subsets A, B of $\mathbb{R}$, and functions ϱ, ϱ_1, ϱ_2 from A into $\mathbb{R}$. Suppose $B \subseteq A$ and $\varrho = \varrho_1 + \varrho_2$. Then $\operatorname{vol}(B, \varrho) = \operatorname{vol}(B, \varrho_1) + \operatorname{vol}(B, \varrho_2)$.

Proof: Set $x_1 = \sup B$. Set $x_2 = \inf B$. $|x_2 - x_1| = x_1 - x_2$ by [6, (11)], [2, (44)]. □

(15) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, functions ϱ, ϱ_1, ϱ_2 from A into $\mathbb{R}$, and a partial function u from $\mathbb{R}$ to $\mathbb{R}$. Suppose ϱ is bounded-variation and ϱ_1 is bounded-variation and ϱ_2 is bounded-variation and $\operatorname{dom} u = A$ and $\varrho = \varrho_1 + \varrho_2$ and u is Riemann-Stieltjes integrable with ϱ_1 and u is Riemann-Stieltjes integrable with ϱ_2. Then

(i) u is Riemann-Stieltjes integrable with ϱ, and

(ii) $\int\limits_{\varrho} u(x)dx = \int\limits_{\varrho_1} u(x)dx + \int\limits_{\varrho_2} u(x)dx$.

The theorem is a consequence of (14).

(16) Let us consider non empty, closed interval subsets A, B of $\mathbb{R}$, and functions ϱ, ϱ_1, ϱ_2 from A into $\mathbb{R}$. Suppose $B \subseteq A$ and $\varrho = \varrho_1 - \varrho_2$. Then $\operatorname{vol}(B, \varrho) = \operatorname{vol}(B, \varrho_1) - \operatorname{vol}(B, \varrho_2)$. The theorem is a consequence of (14).

(17) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, functions ϱ, ϱ_1, ϱ_2 from A into $\mathbb{R}$, and a partial function u from $\mathbb{R}$ to $\mathbb{R}$. Suppose ϱ is bounded-variation and ϱ_1 is bounded-variation and ϱ_2 is bounded-variation and $\operatorname{dom} u = A$ and $\varrho = \varrho_1 - \varrho_2$ and u is Riemann-Stieltjes integrable with ϱ_1 and u is Riemann-Stieltjes integrable with ϱ_2. Then

(i) u is Riemann-Stieltjes integrable with ϱ, and

(ii) $\int\limits_{\varrho} u(x)dx = \int\limits_{\varrho_1} u(x)dx - \int\limits_{\varrho_2} u(x)dx$.

The theorem is a consequence of (16).

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Quasi-uniform Space

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Summary. In this article, using mostly Pervin [9], Kunzi [6], [8], [7], Williams [11] and Bourbaki [3] works, we formalize in Mizar [2] the notions of quasi-uniform space, semi-uniform space and locally uniform space.

We define the topology induced by a quasi-uniform space. Finally we formalize from the sets of the form $((X \setminus \Omega) \times X) \cup (X \times \Omega)$, the Csaszar-Pervin quasi-uniform space induced by a topological space.

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1. Preliminaries

From now on X denotes a set, A denotes a subset of X, and R, S denote binary relations on X.

Now we state the propositions:

(1) $(X \setminus A) \times X \cup X \times A \subseteq X \times X$.

(2) $(X \setminus A) \times X \cup X \times A = A \times A \cup (X \setminus A) \times X$.

Proof: $(X \setminus A) \times X \cup X \times A \subseteq A \times A \cup (X \setminus A) \times X$ by (1), [4, (87)]. □

(3) $R \cdot S = \{\langle x, y \rangle$, where x, y are elements of X : there exists an element z of X such that $\langle x, z \rangle \in R$ and $\langle z, y \rangle \in S\}$.

Proof: $R \cdot S \subseteq \{\langle x, y \rangle$, where x, y are elements of X : there exists an element z of X such that $\langle x, z \rangle \in R$ and $\langle z, y \rangle \in S\}$ by [4, (87)]. $\{\langle x, y \rangle$, where x, y are elements of X : there exists an element z of X such that $\langle x, z \rangle \in R$ and $\langle z, y \rangle \in S\} \subseteq R \cdot S$. □

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Let X be a set and $\mathcal{B}$ be a family of subsets of X. One can check that $[\mathcal{B}]$ is non empty.

Let $\mathcal{B}$ be a family of subsets of $X \times X$. Note that every element of $\mathcal{B}$ is relation-like.

Let B be an element of $\mathcal{B}$. We introduce the notation $B\,[\sim]$ as a synonym of $B^{\smile}$.

Let us observe that the functor $B\,[\sim]$ yields a subset of $X \times X$. Let B_1, B_2 be elements of $\mathcal{B}$. We introduce the notation $B_1 \otimes B_2$ as a synonym of $B_1 \cdot B_2$.

One can verify that the functor $B_1 \otimes B_2$ yields a subset of $X \times X$. Now we state the propositions:

(4) Let us consider a set X, and a family G of subsets of X. If G is upper, then $\mathrm{FinMeetCl}(G)$ is upper.

(5) If R is symmetric in X, then $R^{\smile}$ is symmetric in X.

2. Uniform Space Structure

We consider uniform space structures which extend 1-sorted structures and are systems

$$\langle \text{a carrier}, \text{entourages} \rangle$$

where the carrier is a set, the entourages constitute a family of subsets of (the carrier) $\times$ (the carrier).

Let U be a uniform space structure. We say that U is void if and only if

(Def. 1) the entourages of U is empty.

Let X be a set. The functor $\mathrm{UniformSpace}(X)$ yielding a strict uniform space structure is defined by the term

(Def. 2) $\langle X, \emptyset_{2^{X \times X}} \rangle$.

The functors: the trivial uniform space and the non empty trivial uniform space yielding strict uniform space structures are defined by terms

(Def. 3) $\langle \emptyset, 2_*^{\emptyset \times \emptyset} \rangle$,

(Def. 4) there exists a family S_1 of subsets of $\{\emptyset\} \times \{\emptyset\}$ such that $S_1 = \{\{\emptyset\} \times \{\emptyset\}\}$ and the non empty trivial uniform space $= \langle \{\emptyset\}, S_1 \rangle$,

respectively. Let X be an empty set. One can verify that $\mathrm{UniformSpace}(X)$ is empty.

Let X be a non empty set. One can check that $\mathrm{UniformSpace}(X)$ is non empty.

Let X be a set. Note that $\mathrm{UniformSpace}(X)$ is void and the trivial uniform space is empty and non void and the non empty trivial uniform space is non empty and non void and there exists a uniform space structure which is empty, strict, and void and there exists a uniform space structure which is empty, strict, and non void and there exists a uniform space structure which is non empty, strict, and void and there exists a uniform space structure which is non empty, strict, and non void.

Let X be a set and S_1 be a family of subsets of $X \times X$. The functor $S_1\,[\sim]$ yielding a family of subsets of $X \times X$ is defined by the term

(Def. 5) the set of all $S\,[\sim]$ where S is an element of S_1.

Let U be a uniform space structure. The functor $U\,[\sim]$ yielding a uniform space structure is defined by the term

(Def. 6) $\langle$the carrier of U, (the entourages of U) $[\sim]\,\rangle$.

Let U be a non empty uniform space structure. One can verify that $U\,[\sim]$ is non empty.

3. Axioms

Let U be a uniform space structure. We say that U is upper if and only if

(Def. 7) the entourages of U is upper.

We say that U is $\cap$-closed if and only if

(Def. 8) the entourages of U is $\cap$-closed.

We say that U satisfies axiom U1 if and only if

(Def. 9) for every element S of the entourages of U, $\mathrm{id}_\alpha \subseteq S$, where α is the carrier of U.

We say that U satisfies axiom U2 if and only if

(Def. 10) for every element S of the entourages of U, $S\,[\sim] \in$ the entourages of U.

We say that U satisfies axiom U3 if and only if

(Def. 11) for every element S of the entourages of U, there exists an element W of the entourages of U such that $W \otimes W \subseteq S$.

Let us consider a non void uniform space structure U. Now we state the propositions:

(6) U satisfies axiom U1 if and only if for every element S of the entourages of U, there exists a binary relation R on the carrier of U such that $R = S$ and R is reflexive in the carrier of U.

(7) U satisfies axiom U1 if and only if for every element S of the entourages of U, there exists a total, reflexive binary relation R on the carrier of U such that $R = S$. The theorem is a consequence of (6).

Note that every uniform space structure which is void does not satisfy also axiom U2.

Now we state the proposition:

(8) Let us consider a uniform space structure U. Suppose U satisfies axiom U2. Let us consider an element S of the entourages of U, and elements x, y of U. Suppose $\langle x, y\rangle \in S$. Then $\langle y, x\rangle \in \bigcup$(the entourages of U).

Let us consider a non void uniform space structure U. Now we state the propositions:

(9) Suppose for every element S of the entourages of U, there exists a binary relation R on the carrier of U such that $S = R$ and R is symmetric in the carrier of U. Then U satisfies axiom U2. The theorem is a consequence of (5).

(10) Suppose for every element S of the entourages of U, there exists a binary relation R on the carrier of U such that $S = R$ and R is symmetric. Then U satisfies axiom U2. The theorem is a consequence of (9).

(11) If for every element S of the entourages of U, there exists a tolerance R of the carrier of U such that $S = R$, then U satisfies axiom U1 and axiom U2. The theorem is a consequence of (7) and (10).

Let X be an empty set. Observe that UniformSpace(X) is upper and $\cap$-closed and satisfies axiom U1 and axiom U3 and does not satisfy axiom U2 and UniformSpace($\{\emptyset\}$) is upper and $\cap$-closed and does not satisfy axiom U2 and the trivial uniform space is upper and $\cap$-closed and satisfies axiom U1, axiom U2, and axiom U3 and the non empty trivial uniform space is upper and $\cap$-closed and satisfies axiom U1, axiom U2, and axiom U3.

There exists a uniform space structure which is strict, empty, non void, upper, and $\cap$-closed and satisfies axiom U1, axiom U2, and axiom U3 and every strict uniform space structure which is empty satisfies also axiom U1 and there exists a uniform space structure which is strict, non empty, non void, upper, and $\cap$-closed and satisfies axiom U1, axiom U2, and axiom U3.

Let S_4 be a non empty uniform space structure satisfying axiom U1, x be an element of S_4, and V be an element of the entourages of S_4. The functor $\mathrm{Nbh}(V, x)$ yielding a non empty subset of S_4 is defined by the term

(Def. 12) $\{y$, where y is an element of $S_4 : \langle x, y\rangle \in V\}$.

Now we state the proposition:

(12) Let us consider a non empty uniform space structure U satisfying axiom U1, an element x of the carrier of U, and an element V of the entourages of U. Then $x \in \mathrm{Nbh}(V, x)$.

Let U be a $\cap$-closed uniform space structure and V_1, V_2 be elements of the entourages of U. One can check that the functor $V_1 \cap V_2$ yields an element of the entourages of U. Now we state the proposition:

(13) Let us consider a non empty, $\cap$-closed uniform space structure U satisfying axiom U1, an element x of U, and elements V, W of the entourages

of U. Then $\mathrm{Nbh}(V, x) \cap \mathrm{Nbh}(W, x) = \mathrm{Nbh}(V \cap W, x)$.

Let U be a non empty uniform space structure satisfying axiom U1. Let us observe that the entourages of U has non empty elements and the entourages of U is non empty.

Let x be a point of U. The functor Neighborhood x yielding a family of subsets of U is defined by the term

(Def. 13) the set of all $\mathrm{Nbh}(V, x)$ where V is an element of the entourages of U.

Let us note that Neighborhood x is non empty.

Now we state the proposition:

(14) Let us consider a non empty uniform space structure S_4 satisfying axiom U1, an element x of the carrier of S_4, and an element V of the entourages of S_4. Then

(i) $\mathrm{Nbh}(V, x) = V^{\circ}\{x\}$, and

(ii) $\mathrm{Nbh}(V, x) = \mathrm{rng}(V \restriction \{x\})$, and

(iii) $\mathrm{Nbh}(V, x) = V^{\circ}x$, and

(iv) $\mathrm{Nbh}(V, x) = [x]_V$, and

(v) $\mathrm{Nbh}(V, x) = \mathrm{neighbourhood}(x, V)$.

Proof: $\mathrm{Nbh}(V, x) = V^{\circ}\{x\}$ by [4, (87)]. $\square$

Let U be a non empty uniform space structure satisfying axiom U1. The functor Neighborhood U yielding a function from the carrier of U into $2^{2^{(\text{the carrier of } U)}}$ is defined by

(Def. 14) for every element x of U, $it(x) = $ Neighborhood x.

We say that U is topological if and only if

(Def. 15) $\langle$the carrier of U, Neighborhood $U\rangle$ is a topology from neighbourhoods.

4. Quasi-Uniform Space

A quasi-uniform space is an upper, $\cap$-closed uniform space structure satisfying axiom U1 and axiom U3. In the sequel Q denotes a quasi-uniform space.

Now we state the propositions:

(15) If the entourages of Q is empty, then the entourages of $Q\,[\sim] = \{\emptyset\}$.

(16) Suppose the entourages of $Q\,[\sim] = \{\emptyset\}$ and the entourages of $Q\,[\sim]$ is upper. Then the carrier of Q is empty.

Let Q be a non void quasi-uniform space. One can check that $Q\,[\sim]$ is upper and $\cap$-closed and satisfies axiom U1 and axiom U3.

Let X be a set and $\mathcal{B}$ be a family of subsets of $X \times X$. We say that $\mathcal{B}$ satisfies axiom UP1 if and only if

(Def. 16) for every element B of $\mathcal{B}$, $\mathrm{id}_X \subseteq B$.

We say that $\mathcal{B}$ satisfies axiom UP3 if and only if

(Def. 17) for every element B_1 of $\mathcal{B}$, there exists an element B_2 of $\mathcal{B}$ such that $B_2 \otimes B_2 \subseteq B_1$.

Now we state the propositions:

(17) Let us consider a non empty set X, and an empty family $\mathcal{B}$ of subsets of $X \times X$. Then $\mathcal{B}$ does not satisfy axiom UP1.

(18) Let us consider a set X, and a family $\mathcal{B}$ of subsets of $X \times X$. Suppose $\mathcal{B}$ is quasi-basis and satisfies axiom UP1 and axiom UP3. Then $\langle X, [\mathcal{B}]\rangle$ is a quasi-uniform space.

5. Semi-Uniform Space

A semi-uniform space is an upper, $\cap$-closed uniform space structure satisfying axiom U1 and axiom U2. From now on S_4 denotes a semi-uniform space.

Let us observe that every semi-uniform space is non void.

Now we state the proposition:

(19) If S_4 is empty, then $\emptyset \in$ the entourages of S_4.

Let S_4 be an empty semi-uniform space. One can verify that the entourages of S_4 has the empty element.

6. Locally Uniform Space

Let S_4 be a non empty semi-uniform space. We say that S_4 satisfies axiom UL if and only if

(Def. 18) for every element S of the entourages of S_4 and for every element x of S_4, there exists an element W of the entourages of S_4 such that $\{y$, where y is an element of $S_4 : \langle x, y\rangle \in W \otimes W\} \subseteq \mathrm{Nbh}(S, x)$.

One can verify that every non empty semi-uniform space which satisfies axiom U3 satisfies also axiom UL and there exists a non empty semi-uniform space which satisfies axiom UL.

A locally uniform space is a non empty semi-uniform space satisfying axiom UL. Now we state the propositions:

(20) Let us consider a non empty, upper uniform space structure U satisfying axiom U1, and an element x of the carrier of U. Then Neighborhood x is upper.

(21) Let us consider a non empty uniform space structure U satisfying axiom U1, an element x of U, and an element V of the entourages of U. Then $x \in \mathrm{Nbh}(V, x)$.

(22) Let us consider a non empty, $\cap$-closed uniform space structure U satisfying axiom U1, and an element x of U. Then Neighborhood x is $\cap$-closed. The theorem is a consequence of (13).

(23) Let us consider a non empty, upper, $\cap$-closed uniform space structure U satisfying axiom U1, and an element x of U. Then Neighborhood x is a filter of the carrier of U. The theorem is a consequence of (22) and (20).

Let us observe that every locally uniform space is topological.

7. Topological Space Induced by a Uniform Space Structure

Let U be a topological, non empty uniform space structure satisfying axiom U1. The FMT induced by U yielding a non empty, strict topology from neighbourhoods is defined by the term

(Def. 19) $\langle$the carrier of U, Neighborhood $U\rangle$.

The topological space induced by U yielding a topological space is defined by the term

(Def. 20) FMT2TopSpace(the FMT induced by U).

8. The Quasi-Uniform Pervin Space Induced by a Topological Space

Let X be a set and A be a subset of X. The functor qBlock(A) yielding a subset of $X \times X$ is defined by the term

(Def. 21) $(X \setminus A) \times X \cup X \times A$.

Now we state the proposition:

(24) (i) $\mathrm{id}_X \subseteq \mathrm{qBlock}(A)$, and

(ii) $\mathrm{qBlock}(A) \cdot \mathrm{qBlock}(A) \subseteq \mathrm{qBlock}(A)$.

Proof: $\mathrm{id}_X \subseteq \mathrm{qBlock}(A)$ by [4, (96)]. □

Let T be a topological space. The functor qBlocks(T) yielding a family of subsets of (the carrier of T) $\times$ (the carrier of T) is defined by the term

(Def. 22) the set of all qBlock(O) where O is an element of the topology of T.

Let T be a non empty topological space. One can check that qBlocks(T) is non empty.

Let T be a topological space. The functor FMCqBlocks(T) yielding a family of subsets of (the carrier of T) $\times$ (the carrier of T) is defined by the term

(Def. 23) FinMeetCl(qBlocks(T)).

Let X be a set. One can check that every non empty family of subsets of $X \times X$ which is $\cap$-closed is also quasi-basis.

In the sequel T denotes a topological space.

Let us consider T. One can check that FMCqBlocks(T) is $\cap$-closed and FMCqBlocks(T) is quasi-basis and FMCqBlocks(T) satisfies axiom UP1 and FMCqBlocks(T) satisfies axiom UP3.

Let T be a topological space. The Pervin quasi-uniformity of T yielding a strict quasi-uniform space is defined by the term

(Def. 24) $\langle$the carrier of T, [FMCqBlocks(T)]$\rangle$.

Now we state the propositions:

(25) Let us consider a non empty topological space T, and an element V of the entourages of the Pervin quasi-uniformity of T. Then there exists an element b of FinMeetCl(qBlocks(T)) such that $b \subseteq V$.

(26) Let us consider a non empty topological space T, and a subset V of (the carrier of T) $\times$ (the carrier of T). Suppose there exists an element b of FinMeetCl(qBlocks(T)) such that $b \subseteq V$. Then V is an element of the entourages of the Pervin quasi-uniformity of T.

(27) qBlocks(T) $\subseteq$ the entourages of the Pervin quasi-uniformity of T.

Let us consider a non void quasi-uniform space Q. Now we state the propositions:

(28) (The carrier of Q) $\times$ (the carrier of Q) $\in$ the entourages of Q.

(29) Suppose the carrier of T = the carrier of Q and qBlocks(T) $\subseteq$ the entourages of Q. Then the entourages of the Pervin quasi-uniformity of T $\subseteq$ the entourages of Q.

Proof: The entourages of the Pervin quasi-uniformity of T $\subseteq$ the entourages of Q by (28), [1, (1)]. □

Let T be a non empty topological space. One can check that the Pervin quasi-uniformity of T is non empty and the Pervin quasi-uniformity of T is topological.

Now we state the propositions:

(30) Let us consider a non empty topological space T, an element x of qBlocks (T), and an element y of the Pervin quasi-uniformity of T. Then $\{z$, where z is an element of the Pervin quasi-uniformity of $T : \langle y, z\rangle \in x\} \in$ the topology of T.

(31) Let us consider a non empty topological space T, an element x of the carrier of the Pervin quasi-uniformity of T, and an element b of FinMeetCl (qBlocks(T)). Then $\{y$, where y is an element of $T : \langle x, y\rangle \in b\} \in$ the to-

pology of T. The theorem is a consequence of (30).

(32) Let us consider a non empty, strict quasi-uniform space U, a non empty, strict formal topological space F, and an element x of F. Suppose $F = \langle$the carrier of U, Neighborhood $U\rangle$. Then there exists an element y of U such that

(i) $x = y$, and

(ii) $U_F(x) =$ Neighborhood y.

(33) Let us consider a non empty topological space T. Then the open set family of the FMT induced by the Pervin quasi-uniformity of $T =$ the topology of T.

Proof: The open set family of the FMT induced by the Pervin quasi-uniformity of $T \subseteq$ the topology of T by (32), [5, (18)], (31), [12, (25)]. The topology of $T \subseteq$ the open set family of the FMT induced by the Pervin quasi-uniformity of T by (32), [10, (4)], [5, (18)], [4, (87)]. □

(34) Let us consider a non empty, strict topological space T. Then the topological space induced by the Pervin quasi-uniformity of $T = T$. The theorem is a consequence of (33).

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Uniform Space

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Summary. In this article, we formalize in Mizar [1] the notion of uniform space introduced by André Weil using the concepts of entourages [2].

We present some results between uniform space and pseudo metric space. We introduce the concepts of left-uniformity and right-uniformity of a topological group.

Next, we define the concept of the partition topology. Following the Vlach's works [11, 10], we define the semi-uniform space induced by a tolerance and the uniform space induced by an equivalence relation.

Finally, using mostly Gehrke, Grigorieff and Pin [4] works, a Pervin uniform space defined from the sets of the form $((X \setminus A) \times (X \setminus A)) \cup (A \times A)$ is presented.

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1. Preliminaries

From now on X denotes a set, D denotes a partition of X, T denotes a non empty topological group, and A denotes a subset of X.

Now we state the propositions:

(1) $A \times A \cup (X \setminus A) \times (X \setminus A) \subseteq (X \setminus A) \times X \cup X \times A$.

(2) $\{1, 2, 3\} \setminus \{1\} = \{2, 3\}$.

(3) Suppose $X = \{1, 2, 3\}$ and $A = \{1\}$. Then

(i) $\langle 2, 1 \rangle \in (X \setminus A) \times X \cup X \times A$, and

(ii) $\langle 2, 1 \rangle \notin A \times A \cup (X \setminus A) \times (X \setminus A)$.

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The theorem is a consequence of (2).

(4) Let us consider a subset A of X. Then $(A\times A\cup(X\setminus A)\times(X\setminus A))^{\smile} = A\times A\cup(X\setminus A)\times(X\setminus A)$.

(5) Let us consider subsets P_1, P_2 of D. If $\bigcup P_1 = \bigcup P_2$, then $P_1 = P_2$.

(6) Let us consider a subset P of D. Then $\bigcup(D\setminus P) = \bigcup D \setminus \bigcup P$.

(7) Let us consider an upper family S_1 of subsets of X, and an element S of S_1. Then $\bigcap S_1 \subseteq S$.

(8) Let us consider an additive group G, and subsets A, B, C, D of G. If $A \subseteq B$ and $C \subseteq D$, then $A + C \subseteq B + D$.

Let us consider an element e of T and a neighbourhood V of $\mathbf{1}_T$. Now we state the propositions:

(9) $\{e\}\cdot V$ is a neighbourhood of e.

(10) $V\cdot\{e\}$ is a neighbourhood of e.

(11) Let us consider a neighbourhood V of $\mathbf{1}_T$. Then V^{-1} is a neighbourhood of $\mathbf{1}_T$.

2. Uniform Space

A uniform space is an upper, $\cap$-closed uniform space structure satisfying axiom U1, axiom U2, and axiom U3. From now on Q denotes a uniform space.

Now we state the propositions:

(12) Q is a quasi-uniform space.

(13) Q is a semi-uniform space.

Let X be a set and $\mathcal{B}$ be a family of subsets of $X\times X$. We say that $\mathcal{B}$ satisfies axiom UP2 if and only if

(Def. 1) for every element B_1 of $\mathcal{B}$, there exists an element B_2 of $\mathcal{B}$ such that $B_2 \subseteq {B_1}^{\smile}$.

Now we state the proposition:

(14) Let us consider an empty set X. Then every empty family of subsets of $X\times X$ is quasi-basis and satisfies axiom UP1, axiom UP2, and axiom UP3.

One can verify that there exists a uniform space which is strict.

Now we state the proposition:

(15) Let us consider a set X, and a family S_1 of subsets of $X\times X$. Suppose $X=\{\emptyset\}$ and $S_1=\{X\times X\}$. Then $\langle X, S_1\rangle$ is a uniform space.

Let us observe that there exists a strict uniform space which is non empty.

Now we state the proposition:

(16) Let us consider a set X, and a family $\mathcal{B}$ of subsets of $X \times X$. Suppose $\mathcal{B}$ is quasi-basis and satisfies axiom UP1, axiom UP2, and axiom UP3. Then there exists a strict uniform space Q such that

(i) the carrier of $Q = X$, and

(ii) the entourages $Q = [\mathcal{B}]$.

3. Open Set and Uniform Space

Now we state the propositions:

(17) Let us consider a non empty uniform space Q. Then

(i) the carrier of the topological space induced by $Q =$ the carrier of Q, and

(ii) the topology of the topological space induced by $Q =$ the open set family of the FMTinduced by Q.

(18) Let us consider a non empty uniform space Q, and a subset S of the FMT-induced by Q. Then S is open if and only if for every element x of Q such that $x \in S$ holds $S \in \text{Neighborhood}\, x$.

(19) Let us consider a non empty uniform space Q. Then the open set family of the FMTinduced by $Q =$ the set of all O where O is an open subset of the FMTinduced by Q.

Let us consider a non empty uniform space Q and a subset S of the FMTinduced by Q. Now we state the propositions:

(20) S is open if and only if $S \in$ the open set family of the FMTinduced by Q.

(21) $S \in$ the open set family of the FMTinduced by Q if and only if for every element x of Q such that $x \in S$ holds $S \in \text{Neighborhood}\, x$.

4. Pseudo Metric Space and Uniform Space

Let M be a non empty metric structure and r be a positive real number. The functor $\text{ent}(M, r)$ yielding a subset of (the carrier of M) $\times$ (the carrier of M) is defined by the term

(Def. 2) $\{\langle x, y \rangle$, where x, y are elements of $M : \rho(x, y) \leqslant r\}$.

Let M be a non empty, reflexive metric structure. Let us observe that $\text{ent}(M, r)$ is non empty.

Let M be a non empty metric structure. The functor $\text{ENT}(M)$ yielding a non empty family of subsets of (the carrier of M) $\times$ (the carrier of M) is defined by the term

(Def. 3) the set of all $\operatorname{ent}(M, r)$ where r is a positive real number.

The uniformity induced by M yielding a uniform space structure is defined by the term

(Def. 4) $\langle$the carrier of $M, [\operatorname{ENT}(M)]\rangle$.

Let M be a pseudo metric space. The uniformity induced by M yielding a non empty, strict uniform space is defined by the term

(Def. 5) $\langle$the carrier of $M, [\operatorname{ENT}(M)]\rangle$.

Let us consider a pseudo metric space M. Now we state the propositions:

(22) The open set family of the FMTinduced by the uniformity induced by $M =$ the open set family of M.
PROOF: Set $X =$ the open set family of the FMTinduced by the uniformity induced by M. Set $Y =$ the open set family of M. $X \subseteq Y$ by (18), (20), [5, (11)]. Reconsider $t_1 = t$ as a subset of M. For every element x of the uniformity induced by M such that $x \in t_1$ holds $t_1 \in \operatorname{Neighborhood} x$ by [5, (11)]. $\square$

(23) The topological space induced by the uniformity induced by $M = M_{\mathrm{top}}$. The theorem is a consequence of (22).

5. Uniform Space and Topological Group

Let G be a topological group and Q be a neighbourhood of $\mathbf{1}_G$. The functor $\operatorname{leftU}(Q)$ yielding a subset of (the carrier of G) $\times$ (the carrier of G) is defined by the term

(Def. 6) $\{\langle x, y\rangle$, where x is an element of G, y is an element of $G : x^{-1} \cdot y \in Q\}$.

Let T be a non empty topological group. The functor $\operatorname{SleftU}(T)$ yielding a non empty family of subsets of (the carrier of T) $\times$ (the carrier of T) is defined by the term

(Def. 7) the set of all $\operatorname{leftU}(Q)$ where Q is a neighbourhood of $\mathbf{1}_T$.

The left-uniformity T yielding a non empty uniform space is defined by the term

(Def. 8) $\langle$the carrier of $T, [\operatorname{SleftU}(T)]\rangle$.

Let G be a topological group and Q be a neighbourhood of $\mathbf{1}_G$. The functor $\operatorname{rightU}(Q)$ yielding a subset of (the carrier of G) $\times$ (the carrier of G) is defined by the term

(Def. 9) $\{\langle x, y\rangle$, where x is an element of G, y is an element of $G : y \cdot x^{-1} \in Q\}$.

Let T be a non empty topological group. The functor $\operatorname{SrightU}(T)$ yielding a non empty family of subsets of (the carrier of T) $\times$ (the carrier of T) is defined by the term

(Def. 10) the set of all rightU(Q) where Q is a neighbourhood of $\mathbf{1}_T$.

The right-uniformity T yielding a non empty uniform space is defined by the term

(Def. 11) $\langle$the carrier of T, [SrightU(T)]$\rangle$.

Now we state the propositions:

(24) Let us consider a non empty, commutative topological group T, and a neighbourhood Q of $\mathbf{1}_T$. Then leftU(Q) = rightU(Q).

(25) Let us consider a non empty, commutative topological group T. Then the left-uniformity T = the right-uniformity T. The theorem is a consequence of (24).

Let G be a semi additive topological group and Q be a neighbourhood of 0_G. The functor leftU(Q) yielding a subset of (the carrier of G) $\times$ (the carrier of G) is defined by the term

(Def. 12) $\{\langle x, y\rangle$, where x is an element of G, y is an element of $G : -x + y \in Q\}$.

Let T be a non empty semi additive topological group. The functor SleftU(T) yielding a non empty family of subsets of (the carrier of T) $\times$ (the carrier of T) is defined by the term

(Def. 13) the set of all leftU(Q) where Q is a neighbourhood of 0_T.

Let T be a non empty topological additive group. The left-uniformity T yielding a non empty uniform space is defined by the term

(Def. 14) $\langle$the carrier of T, [SleftU(T)]$\rangle$.

Let G be a semi additive topological group and Q be a neighbourhood of 0_G. The functor rightU(Q) yielding a subset of (the carrier of G) $\times$ (the carrier of G) is defined by the term

(Def. 15) $\{\langle x, y\rangle$, where x is an element of G, y is an element of $G : y + -x \in Q\}$.

Let T be a non empty semi additive topological group. The functor SrightU(T) yielding a non empty family of subsets of (the carrier of T) $\times$ (the carrier of T) is defined by the term

(Def. 16) the set of all rightU(Q) where Q is a neighbourhood of 0_T.

Let T be a non empty topological additive group. The right-uniformity T yielding a non empty uniform space is defined by the term

(Def. 17) $\langle$the carrier of T, [SrightU(T)]$\rangle$.

Now we state the propositions:

(26) Let us consider an Abelian semi additive topological group T, and a neighbourhood Q of 0_T. Then leftU(Q) = rightU(Q).

(27) Let us consider a non empty topological additive group T. Suppose T

is Abelian. Then the left-uniformity T = the right-uniformity T. The theorem is a consequence of (26).

(28) The topology of the topological space induced by the left-uniformity T = the topology of T.

PROOF: Set X = the topology of FMT2TopSpace(the FMTinduced by the left-uniformity T). Set Y = the topology of T. $X \subseteq Y$ by (9), [6, (7)]. $Y \subseteq X$ by [9, (3)], [6, (6)], [8, (6)]. $\square$

(29) The topology of the topological space induced by the right-uniformity T = the topology of T.

PROOF: Set X = the topology of FMT2TopSpace(the FMTinduced by the right-uniformity T). Set Y = the topology of T. $X \subseteq Y$ by (10), [6, (7)]. $Y \subseteq X$ by [9, (3)], [6, (6)], [8, (6)]. $\square$

6. Function Uniformly Continuous

Let Q_1, Q_2 be uniform space structures and f be a function from Q_1 into Q_2. We say that f is uniformly continuous if and only if

(Def. 18) for every element V of the entourages Q_2, there exists an element Q of the entourages Q_1 such that for every objects x, y such that $\langle x, y\rangle \in Q$ holds $\langle f(x), f(y)\rangle \in V$.

Let Q_1, Q_2 be non empty uniform space structures satisfying axiom U1. One can check that there exists a function from Q_1 into Q_2 which is uniformly continuous.

7. Partition Topology

Now we state the propositions:

(30) the set of all $\bigcup P$ where P is a subset of D = UniCl(D).

(31) $X \in$ UniCl(D). The theorem is a consequence of (30).

(32) If $D = \emptyset$, then X is empty and UniCl(D) = $\{\emptyset\}$.

Let X be a set and D be a partition of X. Let us note that UniCl(D) is $\cap$-closed and UniCl(D) is union-closed and every family of subsets of X which is union-closed is also $\cup$-closed.

Let D be a partition of X. Let us note that UniCl(D) is closed for complement operator and UniCl(D) is $\cup$-closed and $\setminus$-closed.

Now we state the proposition:

(33) UniCl(D) is a ring of sets. The theorem is a consequence of (30).

Let us consider X and D. One can verify that $\mathrm{UniCl}(D)$ has the empty element.

Let X be a set and D be a partition of X. Let us observe that $\mathrm{UniCl}(D)$ is non empty.

Now we state the proposition:

(34) $\mathrm{UniCl}(D)$ is a field of subsets of X.

Let X be a set and D be a partition of X. Observe that $\mathrm{UniCl}(D)$ is σ-additive and $\mathrm{UniCl}(D)$ is σ-multiplicative.

Now we state the proposition:

(35) $\mathrm{UniCl}(D)$ is a σ-field of subsets of X.

Let X be a set and D be a partition of X. Observe that $\mathrm{UniCl}(D)$ is closed for countable unions and closed for countable meets.

Now we state the proposition:

(36) Let us consider a non empty set Ω, and a partition D of Ω. Then $\mathrm{UniCl}(D)$ is a Dynkin system of Ω.

Let X be a set and D be a partition of X. The partition topology D yielding a topological space is defined by the term

(Def. 19) $\langle X, \mathrm{UniCl}(D)\rangle$.

Now we state the propositions:

(37) Every open subset of the partition topology D is closed.

(38) Every closed subset of the partition topology D is open.

(39) Let us consider a subset S of the partition topology D. Then S is open if and only if S is closed.

Let X be a non empty set and D be a partition of X. Observe that the partition topology D is non empty.

Let us consider a non empty set X and a partition D of X. Now we state the propositions:

(40) LC(the partition topology D) $= \mathrm{UniCl}(D)$. The theorem is a consequence of (38) and (31).

(41) OpenClosedSet(the partition topology D) $=$ the topology of the partition topology D. The theorem is a consequence of (37).

8. Uniform Space and Partition Topology

In the sequel R denotes a binary relation on X.

Let X be a set and R be a binary relation on X. The functor $\rho(R)$ yielding a non empty family of subsets of $X \times X$ is defined by the term

(Def. 20) $\{S$, where S is a subset of $X \times X : R \subseteq S\}$.

Now we state the propositions:

(42) $[\rho(R)] = \rho(R)$.

(43) $[\{R\}] = \rho(R)$.

(44) $\rho(R)$ is upper and $\cap$-closed.

Let us consider X and R. Observe that $\rho(R)$ is quasi-basis.

Now we state the propositions:

(45) Let us consider a total, reflexive binary relation R on X. Then $\rho(R)$ satisfies axiom UP1.

(46) Let us consider a symmetric binary relation R on X. Then $\rho(R)$ satisfies axiom UP2.

(47) Let us consider a total, transitive binary relation R on X. Then $\rho(R)$ satisfies axiom UP3.

Let X be a set and R be a binary relation on X. The uniformity induced by R yielding an upper, $\cap$-closed, strict uniform space structure is defined by the term

(Def. 21) $\langle X, \rho(R)\rangle$.

Now we state the propositions:

(48) Let us consider a set X, and a total, reflexive binary relation R on X. Then the uniformity induced by R satisfies axiom U1. The theorem is a consequence of (45).

(49) Let us consider a set X, and a symmetric binary relation R on X. Then the uniformity induced by R satisfies axiom U2. The theorem is a consequence of (46).

(50) Let us consider a set X, and a total, transitive binary relation R on X. Then the uniformity induced by R satisfies axiom U3. The theorem is a consequence of (47).

Let X be a set and R be a tolerance of X. Note that the uniformity induced by R yields a strict semi-uniform space. Now we state the proposition:

(51) Let us consider a set X, and an equivalence relation R of X. Then the uniformity induced by R is a uniform space.

Let X be a set and R be an equivalence relation of X. Observe that the uniformity induced by R yields a strict uniform space. Let X be a non empty set and R be a tolerance of X. Let us note that the uniformity induced by R is non empty and every non empty uniform space is topological.

Let Q be a non empty uniform space. The functor ${}^{@}Q$ yielding a topological, non empty uniform space structure satisfying axiom U1 is defined by the term

(Def. 22) Q.

Now we state the proposition:

(52) Let us consider a non empty set X, and an equivalence relation R of X. Then the topological space induced by $^{@}$(the uniformity induced by R) = the partition topology Classes R. The theorem is a consequence of (30) and (18).

9. Uniformity Induced by a Tolerance or by an Equivalence

Now we state the proposition:

(53) Let us consider an upper uniform space structure Q. Suppose $\bigcap$(the entourages Q) $\in$ the entourages Q. Then there exists a binary relation R on the carrier of Q such that

(i) $\bigcap$(the entourages Q) $= R$, and

(ii) the entourages $Q = \rho(R)$.

PROOF: Reconsider $R = \bigcap$(the entourages Q) as a binary relation on the carrier of Q. $\rho(R) \subseteq$ the entourages Q. The entourages $Q \subseteq \rho(R)$ by [7, (3)]. □

Let Q be a uniform space structure. The functor Uniformity2InternalRel(Q) yielding a binary relation on the carrier of Q is defined by the term

(Def. 23) $\bigcap$(the entourages Q).

The functor UniformSpaceStr2RelStr(Q) yielding a relational structure is defined by the term

(Def. 24) $\langle$the carrier of Q, Uniformity2InternalRel(Q)$\rangle$.

Let R_1 be a relational structure. The functor InternalRel2Uniformity(R_1) yielding a family of subsets of (the carrier of R_1) $\times$ (the carrier of R_1) is defined by the term

(Def. 25) $\{R$, where R is a binary relation on the carrier of R_1 : the internal relation of $R_1 \subseteq R\}$.

The functor RelStr2UniformSpaceStr(R_1) yielding a strict uniform space structure is defined by the term

(Def. 26) $\langle$the carrier of R_1, InternalRel2Uniformity(R_1)$\rangle$.

The functor InternalRel2Element(R_1) yielding an element of the entourages RelStr2UniformSpaceStr(R_1) is defined by the term

(Def. 27) the internal relation of R_1.

Now we state the propositions:

(54) Let us consider a binary relation R on X. Then $\bigcap \rho(R) = R$.

(55) Let us consider a strict relational structure R_1. Then UniformSpaceStr2-RelStr(RelStr2UniformSpaceStr(R_1)) $= R_1$. The theorem is a consequence of (54).

(56) Let us consider a uniform space structure Q. Then

(i) the carrier of RelStr2UniformSpaceStr(UniformSpaceStr2RelStr(Q)) $=$ the carrier of Q, and

(ii) the entourages RelStr2UniformSpaceStr(UniformSpaceStr2RelStr (Q)) $= \rho(\bigcap$(the entourages Q)).

(57) Let us consider a family S_1 of subsets of $X \times X$, and a binary relation R on X. If $S_1 = \rho(R)$, then $S_1 \subseteq \rho(\bigcap S_1)$.

(58) Let us consider an upper family S_1 of subsets of $X \times X$. If $\bigcap S_1 \in S_1$, then $\rho(\bigcap S_1) \subseteq S_1$.

(59) Let us consider an upper family S_1 of subsets of $X \times X$, and a binary relation R on X. Suppose $R \in S_1$ and $S_1 = \rho(R)$ and $\bigcap S_1 \in S_1$. Then $\rho(\bigcap S_1) = S_1$.

(60) Let us consider an upper uniform space structure Q. Suppose there exists a binary relation R on the carrier of Q such that the entourages $Q = \rho(R)$ and $\bigcap$(the entourages Q) $\in$ the entourages Q. Then the entourages $Q = \rho(\bigcap$(the entourages Q)). The theorem is a consequence of (57) and (58).

(61) Let us consider an upper uniform space structure Q, and a binary relation R on the carrier of Q. Suppose the entourages $Q = \rho(R)$ and $\bigcap$(the entourages Q) $\in$ the entourages Q.
Then the entourages $Q = \rho(\bigcap$(the entourages Q)).

Let us consider a tolerance R of X. Now we state the propositions:

(62) (i) the uniformity induced by R is a semi-uniform space, and

(ii) the entourages the uniformity induced by $R = \rho(R)$, and

(iii) $\bigcap$(the entourages the uniformity induced by R) $= R$.

(63) RelStr2UniformSpaceStr(UniformSpaceStr2RelStr(the uniformity induced by R)) $=$ the uniformity induced by R. The theorem is a consequence of (54).

(64) Let us consider an equivalence relation R of X. Then RelStr2UniformSpaceStr(UniformSpaceStr2RelStr(the uniformity induced by R)) $=$ the uniformity induced by R. The theorem is a consequence of (54).

10. Uniform Pervin Space

Let X be a set, S_1 be a family of subsets of X, and A be an element of S_1. The functor Block(A) yielding a subset of $X \times X$ is defined by the term

(Def. 28) $(X \setminus A) \times (X \setminus A) \cup A \times A$.

From now on S_1 denotes a family of subsets of X and A denotes an element of S_1.

Now we state the propositions:

(65) If $A = \emptyset$, then $\text{Block}(A) = X \times X$.

(66) Suppose X is not empty. Then $\text{Block}(A) = \{\langle x, y\rangle$, where x, y are elements of $X : x \in A$ iff $y \in A\}$.
Proof: Set $S = \{\langle x, y\rangle$, where x, y are elements of $X : x \in A$ iff $y \in A\}$. $\text{Block}(A) \subseteq S$ by [3, (87)]. $S \subseteq \text{Block}(A)$ by [3, (87)]. □

(67) (i) $\text{id}_X \subseteq \text{Block}(A)$, and

(ii) $\text{Block}(A) \cdot \text{Block}(A) \subseteq \text{Block}(A)$.

Let X be a set and S_1 be a family of subsets of X. The functor Blocks(S_1) yielding a family of subsets of $X \times X$ is defined by the term

(Def. 29) the set of all Block(A) where A is an element of S_1.

Let us observe that Blocks(S_1) is non empty.

The functor FMCBlocks(S_1) yielding a family of subsets of $X \times X$ is defined by the term

(Def. 30) FinMeetCl(Blocks(S_1)).

Now we state the propositions:

(68) FMCBlocks(S_1) is $\cap$-closed.

(69) FMCBlocks(S_1) is quasi-basis. The theorem is a consequence of (68).

(70) FMCBlocks(S_1) satisfies axiom UP1.

(71) Let us consider an element A of S_1, and a binary relation R on X. If $R = \text{Block}(A)$, then $R^{\smile} = \text{Block}(A)$. The theorem is a consequence of (65) and (4).

(72) Let us consider a binary relation R on X. Suppose R is an element of Blocks(S_1). Then $R^{\smile}$ is an element of Blocks(S_1). The theorem is a consequence of (71).

Let us consider a non empty family Y of subsets of $X \times X$. Now we state the propositions:

(73) If $Y \subseteq \text{Blocks}(S_1)$, then $Y[\sim] = Y$. The theorem is a consequence of (71).

(74) If $Y \subseteq \mathrm{Blocks}(S_1)$, then $(\bigcap Y)^\smile = \bigcap Y\,[\sim]$. The theorem is a consequence of (73) and (71).

(75) If $Y \subseteq \mathrm{Blocks}(S_1)$, then $\bigcap Y = (\bigcap Y)^\smile$. The theorem is a consequence of (73) and (74).

(76) $\mathrm{FMCBlocks}(S_1)$ satisfies axiom UP2. The theorem is a consequence of (73) and (75).

(77) $\mathrm{FMCBlocks}(S_1)$ satisfies axiom UP3. The theorem is a consequence of (67).

Let X be a set and S_1 be a family of subsets of X. The Pervin uniform space of S_1 yielding a strict uniform space is defined by the term

(Def. 31) $\langle X, [\mathrm{FMCBlocks}(S_1)]\rangle$.

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Some Algebraic Properties of Polynomial Rings

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Summary. In this article we extend the algebraic theory of polynomial rings, formalized in Mizar [1], based on [2], [3]. After introducing constant and monic polynomials we present the canonical embedding of R into $R[X]$ and deal with both unit and irreducible elements. We also define polynomial GCDs and show that for fields F and irreducible polynomials p the field $F[X]/\langle p \rangle$ is isomorphic to the field of polynomials with degree smaller than the one of p.

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1. Preliminaries

Let R be a non empty double loop structure and a be an element of R. Observe that the functor $\{a\}$ yields a subset of R. Observe that every ring which is almost left invertible and commutative is also almost right invertible and every ring which is almost right invertible and commutative is also almost left invertible and every ring which is almost left cancelable and commutative is also almost right cancelable and every ring which is almost right cancelable and commutative is also almost left cancelable.

Let L be a non empty zero structure and X be a set. We say that X is L-polynomial membered if and only if

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(Def. 1) for every object p such that $p \in X$ holds p is a polynomial over L.

Let X be a 1-sorted structure. We say that X is L-polynomial membered if and only if

(Def. 2) the carrier of X is L-polynomial membered.

Let us note that there exists a set which is non empty and L-polynomial membered and there exists a 1-sorted structure which is non empty and L-polynomial membered.

Let X be a non empty, L-polynomial membered 1-sorted structure. One can check that the carrier of X is L-polynomial membered.

Let L be an add-associative, right zeroed, right complementable, distributive, non empty double loop structure. Let us observe that Polynom-Ring(L) is L-polynomial membered.

Let L be a non empty zero structure and X be a non empty, L-polynomial membered set.

Observe that an element of X is a polynomial over L. Let R be a ring. One can verify that there exists an element of the carrier of Polynom-Ring(R) which is zero and there exists an element of Polynom-Ring(R) which is zero and there exists a polynomial over R which is zero.

Let R be a non degenerated ring. Let us note that there exists an element of the carrier of Polynom-Ring(R) which is non zero and there exists an element of Polynom-Ring(R) which is non zero.

Let L be an add-associative, right zeroed, right complementable, distributive, non empty double loop structure and p, q be polynomials over L. We say that $p \mid q$ if and only if

(Def. 3) there exist elements a, b of Polynom-Ring(L) such that $a = p$ and $b = q$ and $a \mid b$.

Now we state the proposition:

(1) Let us consider an add-associative, right zeroed, right complementable, distributive, non empty double loop structure L, and polynomials p, q over L. Then $p \mid q$ if and only if there exists a polynomial r over L such that $p * r = q$.

Let us consider a field F and polynomials p, q over F. Now we state the propositions:

(2) If $\deg p < \deg q$, then $p \bmod q = p$.

(3) $p \bmod q = \mathbf{0}.F$ if and only if $q \mid p$. The theorem is a consequence of (1).

(4) $p = (p \operatorname{div} q) * q + (p \bmod q)$.

Let us consider a field F, polynomials p, r over F, and a non zero polynomial q over F. Now we state the propositions:

(5) (i) $p + r \operatorname{div} q = (p \operatorname{div} q) + (r \operatorname{div} q)$, and

(ii) $p + r \bmod q = (p \bmod q) + (r \bmod q)$.

The theorem is a consequence of (4).

(6) $p * r \bmod q = (p \bmod q) * (r \bmod q) \bmod q$. The theorem is a consequence of (4), (5), (3), and (1).

Now we state the propositions:

(7) Let us consider a field F, polynomials r, q, u over F, and a non zero polynomial p over F. Then $(r * q \bmod p) * u \bmod p = (r * q) * u \bmod p$. The theorem is a consequence of (5), (3), and (1).

(8) Let us consider an add-associative, right zeroed, right complementable, left distributive, non empty double loop structure L, and a sequence p of L. Then $0_L \cdot p = \mathbf{0}. L$.

(9) Let us consider a left unital, non empty double loop structure L, and a sequence p of L. Then $1_L \cdot p = p$.

(10) Let us consider an add-associative, right zeroed, right complementable, right unital, distributive, associative, commutative, non empty double loop structure L, sequences p, q of L, and an element a of L. Then $a \cdot (p * q) = p * (a \cdot q)$.

(11) Let us consider an associative, non empty multiplicative magma L, a sequence p of L, and elements a, b of L. Then $(a \cdot b) \cdot p = a \cdot (b \cdot p)$.

(12) Let us consider an add-associative, right zeroed, right complementable, left distributive, left unital, non empty double loop structure L, and a sequence p of L. Then $\mathbf{1}. L * p = p$.

Let L be an add-associative, right zeroed, right complementable, well unital, distributive, non empty double loop structure. Let us observe that Polynom-Ring(L) is well unital.

2. Constant Polynomials

Let R be an add-associative, right zeroed, right complementable, distributive, non empty double loop structure and x be an element of the carrier of Polynom-Ring(R). We say that x is constant if and only if

(Def. 4) $\deg x \leqslant 0$.

Let R be a non degenerated ring. Observe that there exists an element of Polynom-Ring(R) which is non zero and constant and there exists an element of the carrier of Polynom-Ring(R) which is non zero and constant.

Let R be an integral domain. Let us observe that there exists an element of Polynom-Ring(R) which is non constant and there exists an element of the carrier of Polynom-Ring(R) which is non constant.

Let L be a non empty zero structure and a be an element of L. The functor $a\restriction L$ yielding a sequence of L is defined by the term

(Def. 5) $\mathbf{0}. L +\cdot (0, a)$.

Note that $a\restriction L$ is finite-Support and $a\restriction L$ is constant.

Let a be a non zero element of L. Let us note that $a\restriction L$ is non zero and there exists a polynomial over L which is non zero and constant.

Now we state the propositions:

(13) Let us consider a non empty zero structure L. Then $0_L\restriction L = \mathbf{0}. L$.

(14) Let us consider a non empty multiplicative loop with zero structure L. Then $1_L\restriction L = \mathbf{1}. L$.

Let L be a non empty zero structure. Observe that $0_L\restriction L$ is zero.

Let L be a non degenerated multiplicative loop with zero structure. Let us note that $1_L\restriction L$ is non zero.

Now we state the propositions:

(15) Let us consider an add-associative, right zeroed, right complementable, distributive, non empty double loop structure L, and an element p of the carrier of Polynom-Ring(L). Then p is non zero and constant if and only if $\deg p = 0$.

(16) Let us consider an add-associative, right zeroed, right complementable, right distributive, right unital, non empty double loop structure L, and an element a of L. Then $a\restriction L = a \cdot \mathbf{1}. L$.

Let us consider a ring R and elements a, b of R. Now we state the propositions:

(17) $a\restriction R + b\restriction R = (a + b)\restriction R$.

(18) $(a\restriction R) * (b\restriction R) = a \cdot b\restriction R$.

(19) $a\restriction R = b\restriction R$ if and only if $a = b$.

(20) Let us consider a ring R, and an element p of the carrier of Polynom-Ring-(R). Then p is constant if and only if there exists an element a of R such that $p = a\restriction R$.

(21) Let us consider a ring R, and an element a of R. Then $\deg(a\restriction R) = 0$ if and only if $a \neq 0_R$. The theorem is a consequence of (19).

3. Monic Polynomials

Let L be a non empty double loop structure and p be a polynomial over L. We introduce the notation p is monic as a synonym of p is normalized.

Let L be an add-associative, right zeroed, right complementable, distributive, non degenerated double loop structure. Let us observe that $\mathbf{1}.\,L$ is monic and $\mathbf{0}.\,L$ is non monic and there exists a polynomial over L which is monic and there exists a polynomial over L which is non monic and there exists an element of the carrier of Polynom-Ring(L) which is monic and there exists an element of the carrier of Polynom-Ring(L) which is non monic.

Let L be a well unital, non degenerated double loop structure and x be an element of L. One can verify that rpoly$(1, x)$ is monic.

Let L be a field and p be an element of the carrier of Polynom-Ring(L). Let us observe that the functor NormPolynomial p yields an element of the carrier of Polynom-Ring(L). Let F be a field and p be a non zero polynomial over F. Observe that NormPolynomial p is monic.

Let L be a field and p be a non zero element of the carrier of Polynom-Ring(L). Observe that NormPolynomial p is monic.

Now we state the proposition:

(22) Let us consider a field F. Then NormPolynomial $\mathbf{0}.\,F = \mathbf{0}.\,F$.

Let us consider a field F and a non zero element p of the carrier of Polynom-Ring(F). Now we state the propositions:

(23) NormPolynomial $p = (\mathrm{LC}\,p)^{-1} \cdot p$.

(24) p is monic if and only if NormPolynomial $p = p$. The theorem is a consequence of (23) and (9).

Let us consider a field F and elements p, q of the carrier of Polynom-Ring(F). Now we state the propositions:

(25) $q \mid p$ if and only if NormPolynomial $q \mid p$. The theorem is a consequence of (22), (1), (9), (11), (10), and (23).

(26) $q \mid p$ if and only if $q \mid$ NormPolynomial p. The theorem is a consequence of (22), (1), (23), (10), (9), and (11).

Let us consider a field F and an element p of the carrier of Polynom-Ring(F). Now we state the propositions:

(27) NormPolynomial p is associated to p. The theorem is a consequence of (1), (26), and (25).

(28) NormPolynomial p is irreducible if and only if p is irreducible. The theorem is a consequence of (27).

Now we state the propositions:

(29) Let us consider an integral domain R, and elements p, q of the carrier of Polynom-Ring(R). If p is associated to q, then $\deg p = \deg q$.

(30) Let us consider an integral domain R, and monic elements p, q of the carrier of Polynom-Ring(R). Then p is associated to q if and only if $p = q$. The theorem is a consequence of (29), (20), (16), (10), and (12).

4. The Canonical Homomorphism from R into $R[X]$

Let R be a ring. The canonical homomorphism of R into quotient field yielding a function from R into Polynom-Ring(R) is defined by

(Def. 6) for every element x of R, $it(x) = x \restriction R$.

Note that the canonical homomorphism of R into quotient field is additive, multiplicative, and unity-preserving and the canonical homomorphism of R into quotient field is monomorphic and Polynom-Ring(R) is R-homomorphic and R-monomorphic.

Now we state the proposition:

(31) Let us consider a ring R. Then char(Polynom-Ring(R)) = char(R).

Let R be a non degenerated ring. Let us note that Polynom-Ring(R) is infinite and every ring with characteristic 0 is infinite.

Now we state the proposition:

(32) Let us consider a ring R. If char(R) = 0, then R is infinite.

Let n be a non trivial natural number.

One can verify that Polynom-Ring($\mathbb{Z}/n$) is infinite. Now we state the proposition:

(33) Let us consider a non trivial natural number n. Then char(Polynom-Ring($\mathbb{Z}/n$)) $\neq 0$.

Let n be a non trivial natural number. Observe that there exists a ring which is infinite and has characteristic n.

5. Units and Irreducible Polynomials

Let us note that there exists an integral domain which is non almost left invertible.

Let R be a non almost left invertible integral domain. One can verify that there exists a non-unit of R which is non zero and $\mathbb{Z}^{\mathrm{R}}$ is non almost left invertible.

Let R be an integral domain. Observe that Polynom-Ring(R) is non almost left invertible.

Now we state the propositions:

(34) Let us consider an integral domain R. Then R is a field if and only if for every non-unit a of R, $a = 0_R$.

(35) Let us consider an integral domain R, and an element a of R. Then $a \upharpoonright R$ is a unit of Polynom-Ring(R) if and only if a is a unit of R. The theorem is a consequence of (1), (20), (18), and (19).

(36) Let us consider an integral domain F, and an element p of the carrier of Polynom-Ring(F). If p is a unit of Polynom-Ring(F), then $\deg p = 0$. The theorem is a consequence of (1).

(37) Let us consider a field F, and an element p of the carrier of Polynom-Ring (F). Then p is a unit of Polynom-Ring(F) if and only if $\deg p = 0$. The theorem is a consequence of (1), (20), and (18).

(38) Let us consider an integral domain R, and an element p of the carrier of Polynom-Ring(R). Suppose p is a unit of Polynom-Ring(R). Then p is non zero and constant. The theorem is a consequence of (36) and (15).

(39) Let us consider a field F, and an element p of the carrier of Polynom-Ring (F). Then p is a unit of Polynom-Ring(F) if and only if p is non zero and constant. The theorem is a consequence of (37) and (15).

Let R be an integral domain. One can check that every element of Polynom-Ring(R) which is non constant is also non zero and non unital.

Let F be an integral domain. Let us observe that every element of the carrier of Polynom-Ring(F) which is non constant is also non zero and non unital.

Let F be a field. Observe that every element of Polynom-Ring(F) which is non zero and constant is also unital and every element of Polynom-Ring(F) which is unital is also non zero and constant and every element of the carrier of Polynom-Ring(F) which is non zero and constant is also unital and every element of the carrier of Polynom-Ring(F) which is unital is also non zero and constant.

Now we state the propositions:

(40) Let us consider an integral domain R, and an element p of the carrier of Polynom-Ring(R). Suppose there exists an element q of the carrier of Polynom-Ring(R) such that $q \mid p$ and $1 \leqslant \deg q < \deg p$. Then p is reducible. The theorem is a consequence of (36).

(41) Let us consider a field F, and an element p of the carrier of Polynom-Ring (F). Then p is reducible if and only if $p = \mathbf{0}.F$ or p is a unit of Polynom-Ring (F) or there exists an element q of the carrier of Polynom-Ring(F) such that $q \mid p$ and $1 \leqslant \deg q < \deg p$. The theorem is a consequence of (1), (37), and (40).

(42) Let us consider an integral domain R, and a monic element p of the carrier of Polynom-Ring(R). If $\deg p = 1$, then p is irreducible. The theorem

is a consequence of (36), (20), (16), (10), (12), and (35).

(43) There exists a non monic element p of the carrier of Polynom-Ring($\mathbb{Z}^{\mathrm{R}}$) such that

(i) $\deg p = 1$, and

(ii) p is reducible.

The theorem is a consequence of (16), (10), (12), (15), (35), and (36).

(44) Let us consider a field F, and an element p of the carrier of Polynom-Ring (F). If $\deg p = 1$, then p is irreducible. The theorem is a consequence of (36), (20), (21), and (35).

(45) Let us consider an algebraic closed field F, and an element p of the carrier of Polynom-Ring(F). Then p is irreducible if and only if $\deg p = 1$. The theorem is a consequence of (36) and (44).

(46) Let us consider a field F. Then F is algebraic closed if and only if for every monic element p of the carrier of Polynom-Ring(F), p is irreducible iff $\deg p = 1$. The theorem is a consequence of (37), (41), (28), and (45).

Let R be an integral domain. Note that there exists an element of Polynom-Ring(R) which is irreducible and there exists an element of the carrier of Polynom-Ring(R) which is irreducible.

Let R be a ring. Let us observe that there exists an element of Polynom-Ring (R) which is reducible and there exists an element of the carrier of Polynom-Ring(R) which is reducible. Let R be an integral domain.

Note that IRR(Polynom-Ring(R)) is non empty.

Let F be a field. Observe that every element of Polynom-Ring(F) which is constant is also reducible and every element of the carrier of Polynom-Ring(F) which is constant is also reducible and every element of Polynom-Ring(F) which is irreducible is also non constant and every element of the carrier of Polynom-Ring(F) which is irreducible is also non constant.

6. The Field $F[X]/<p>$

Let F be a field and p be an element of the carrier of Polynom-Ring(F). Let us note that $\frac{\text{Polynom-Ring}(F)}{\{p\}\text{–ideal}}$ is Abelian, add-associative, right zeroed, right complementable, commutative, associative, well unital, and distributive.

Let p be an irreducible element of the carrier of Polynom-Ring(F). Observe that $\frac{\text{Polynom-Ring}(F)}{\{p\}\text{–ideal}}$ is non degenerated and almost left invertible.

Let p be a polynomial over F. The functor PolyMultMod(p) yielding a binary operation on Polynom-Ring(F) is defined by

(Def. 7) for every polynomials r, q over F, $it(r, q) = r * q \bmod p$.

Let p be a non constant element of the carrier of Polynom-Ring(F). The functor Polynom-Ring(p) yielding a strict double loop structure is defined by

(Def. 8) the carrier of $it = \{q$, where q is a polynomial over $F : \deg q < \deg p\}$ and the addition of it = (the addition of Polynom-Ring(F)) $\restriction$ (the carrier of it) and the multiplication of it = PolyMultMod(p) $\restriction$ (the carrier of it) and the one of $it = \mathbf{1}.F$ and the zero of $it = \mathbf{0}.F$.

Observe that Polynom-Ring(p) is non degenerated and Polynom-Ring(p) is Abelian, add-associative, right zeroed, and right complementable and Polynom-Ring(p) is associative, well unital, and distributive.

The functor PolyMod(p) yielding a function from Polynom-Ring(F) into Polynom-Ring(p) is defined by

(Def. 9) for every polynomial q over F, $it(q) = q \bmod p$.

Observe that PolyMod(p) is additive, multiplicative, and unity-preserving and Polynom-Ring(p) is (Polynom-Ring(F))-homomorphic and PolyMod(p) is onto.

Let us consider a field F and a non constant element p of the carrier of Polynom-Ring(F). Now we state the propositions:

(47) ker PolyMod(p) = $\{p\}$–ideal. The theorem is a consequence of (1) and (3).

(48) $\frac{\text{Polynom-Ring}(F)}{\{p\}\text{–ideal}}$ and Polynom-Ring(p) are isomorphic. The theorem is a consequence of (47).

Let F be a field and p be a non constant element of the carrier of Polynom-Ring (F). Observe that Polynom-Ring(p) is commutative.

Let p be an irreducible element of the carrier of Polynom-Ring(F). Observe that Polynom-Ring(p) is almost left invertible.

7. Polynomial GCDs

Let L be a non empty multiplicative magma, x, y be elements of L, and z be an element of L. We say that z is x,y-GCD if and only if

(Def. 10) $z \mid x$ and $z \mid y$ and for every element r of L such that $r \mid x$ and $r \mid y$ holds $r \mid z$.

Let L be a GCD domain. Note that there exists an element of L which is x,y-GCD.

A GCD of x and y is an x,y-GCD element of L. Now we state the proposition:

(49) Let us consider a GCD domain L, elements x, y of L, and GCDs u, v of x and y. Then u is associated to v.

Let L be a GCD domain and x, y be elements of L. One can verify that every element of L which is x,y-GCD is also y,x-GCD.

Let F be a field and p, q be elements of the carrier of Polynom-Ring(F). The functor $\gcd(p, q)$ yielding an element of the carrier of Polynom-Ring(F) is defined by

(Def. 11) (i) *it* $= \mathbf{0}. F$, **if** $p = \mathbf{0}. F$ and $q = \mathbf{0}. F$,

(ii) *it* is GCD of p and q and monic, **otherwise**.

One can check that the functor $\gcd(p, q)$ is commutative.

Let p, q be elements of Polynom-Ring(F). Let us note that the functor $\gcd(p, q)$ is commutative.

Let p, q be elements of the carrier of Polynom-Ring(F). Let us observe that $\gcd(p, q)$ is p,q-GCD.

Let p, q be elements of Polynom-Ring(F). Observe that $\gcd(p, q)$ is p,q-GCD.

Let p be an element of the carrier of Polynom-Ring(F) and q be a non zero element of the carrier of Polynom-Ring(F). Note that $\gcd(p, q)$ is non zero and monic.

Let p be an element of Polynom-Ring(F) and q be a non zero element of Polynom-Ring(F). Let us observe that $\gcd(p, q)$ is non zero and monic.

Let p, q be zero elements of the carrier of Polynom-Ring(F). Let us note that $\gcd(p, q)$ is zero.

Let p, q be zero elements of Polynom-Ring(F). One can verify that $\gcd(p, q)$ is zero.

Now we state the propositions:

(50) Let us consider a field F, and elements p, q of the carrier of Polynom-Ring (F). Then

(i) $\gcd(p, q) \mid p$, and

(ii) $\gcd(p, q) \mid q$, and

(iii) for every element r of the carrier of Polynom-Ring(F) such that $r \mid p$ and $r \mid q$ holds $r \mid \gcd(p, q)$.

(51) Let us consider a field F, and elements p, q of Polynom-Ring(F). Then

(i) $\gcd(p, q) \mid p$, and

(ii) $\gcd(p, q) \mid q$, and

(iii) for every element r of Polynom-Ring(F) such that $r \mid p$ and $r \mid q$ holds $r \mid \gcd(p, q)$.

Let F be a field and p, q be polynomials over F. The functor $\gcd(p, q)$ yielding a polynomial over F is defined by

(Def. 12) there exist elements a, b of Polynom-Ring(F) such that $a = p$ and $b = q$ and $it = \gcd(a, b)$.

Observe that the functor $\gcd(p, q)$ is commutative.

Let p be a polynomial over F and q be a non zero polynomial over F. Let us note that $\gcd(p, q)$ is non zero and monic.

Let p, q be zero polynomials over F. One can verify that $\gcd(p, q)$ is zero.

Now we state the propositions:

(52) Let us consider a field F, and polynomials p, q over F. Then

(i) $\gcd(p, q) \mid p$, and

(ii) $\gcd(p, q) \mid q$, and

(iii) for every polynomial r over F such that $r \mid p$ and $r \mid q$ holds $r \mid \gcd(p, q)$.

The theorem is a consequence of (1).

(53) Let us consider a field F, a polynomial p over F, a non zero polynomial q over F, and a monic polynomial s over F. Then $s = \gcd(p, q)$ if and only if $s \mid p$ and $s \mid q$ and for every polynomial r over F such that $r \mid p$ and $r \mid q$ holds $r \mid s$. The theorem is a consequence of (52).

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