

Product Pre-Measure

Noboru Endou
Gifu National College of Technology
Gifu, Japan

Summary. In this article we formalize in Mizar [5] product pre-measure on product sets of measurable sets. Although there are some approaches to construct product measure [22], [6], [9], [21], [25], we start it from σ -measure because existence of σ -measure on any semialgebras has been proved in [15]. In this approach, we use some theorems for integrals.

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1. PRELIMINARIES

Now we state the proposition:

- (1) Let us consider non empty sets A, A_1, A_2, B, B_1, B_2 . Then $A_1 \times B_1$ misses $A_2 \times B_2$ and $A \times B = A_1 \times B_1 \cup A_2 \times B_2$ if and only if A_1 misses A_2 and $A = A_1 \cup A_2$ and $B = B_1$ and $B = B_2$ or B_1 misses B_2 and $B = B_1 \cup B_2$ and $A = A_1$ and $A = A_2$.

Let C, D be non empty sets, F be a sequence of D^C , and n be a natural number. One can check that the functor $F(n)$ yields a function from C into D .

- (2) Let us consider sets X, Y, A, B , and objects x, y . Suppose $x \in X$ and $y \in Y$. Then $\chi_{A,X}(x) \cdot \chi_{B,Y}(y) = \chi_{A \times B, X \times Y}(x, y)$.

Let A, B be sets. One can verify that $\chi_{A,B}$ is non-negative.

- (3) Let us consider a non empty set X , a semialgebra S of sets of X , a pre-measure P of S , an induced measure m of S and P , and an induced σ -measure M of S and m . Then $\text{COM}(M)$ is complete on $\text{COM}(\sigma(\text{the field generated by } S), M)$.

The functor $\text{Intervals}_{\mathbb{R}}$ yielding a semialgebra of sets of \mathbb{R} is defined by the term

(Def. 1) the set of all I where I is an interval.

Now we state the propositions:

(4) $\text{Halflines} \subseteq \text{Intervals}_{\mathbb{R}}$.

(5) Let us consider a subset I of \mathbb{R} . If I is an interval, then $I \in$ the Borel sets.

(6) (i) $\sigma(\text{Intervals}_{\mathbb{R}}) =$ the Borel sets, and

(ii) $\sigma(\text{the field generated by } \text{Intervals}_{\mathbb{R}}) =$ the Borel sets.

The theorem is a consequence of (4) and (5).

2. FAMILY OF SEMIALGEBRAS, FIELDS AND MEASURES

Now we state the propositions:

(7) Let us consider sets X_1, X_2 , a non empty family S_1 of subsets of X_1 , and a non empty family S_2 of subsets of X_2 . Then the set of all $a \times b$ where a is an element of S_1 , b is an element of S_2 is a non empty family of subsets of $X_1 \times X_2$.

(8) Let us consider sets X, Y , a family M of subsets of X with the empty element, and a family N of subsets of Y with the empty element. Then the set of all $A \times B$ where A is an element of M , B is an element of N is a family of subsets of $X \times Y$ with the empty element. The theorem is a consequence of (7).

(9) Let us consider a set X , and disjoint valued finite sequences O, T of elements of X . Suppose $\bigcup \text{rng } O$ misses $\bigcup \text{rng } T$. Then $O \cap T$ is a disjoint valued finite sequence of elements of X .

(10) Let us consider sets X_1, X_2 , a semiring S_1 of X_1 , and a semiring S_2 of X_2 . Then the set of all $A \times B$ where A is an element of S_1 , B is an element of S_2 is a semiring of $X_1 \times X_2$.

(11) Let us consider sets X_1, X_2 , a semialgebra S_1 of sets of X_1 , and a semialgebra S_2 of sets of X_2 . Then the set of all $A \times B$ where A is an element of S_1 , B is an element of S_2 is a semialgebra of sets of $X_1 \times X_2$. The theorem is a consequence of (10).

(12) Let us consider sets X_1, X_2 , a field O of subsets of X_1 , and a field T of subsets of X_2 . Then the set of all $A \times B$ where A is an element of O , B is an element of T is a semialgebra of sets of $X_1 \times X_2$. The theorem is a consequence of (11).

Let n be a non zero natural number and X be a non-empty, n -element finite sequence.

A family of semialgebras of X is an n -element finite sequence and is defined by

(Def. 2) for every natural number i such that $i \in \text{Seg } n$ holds $it(i)$ is a semialgebra of sets of $X(i)$.

Let us observe that a family of semialgebras of X is a \cap -closed yielding family of semirings of X . Now we state the proposition:

(13) Let us consider a non zero natural number n , a non-empty, n -element finite sequence X , a family S of semialgebras of X , and a natural number i . If $i \in \text{Seg } n$, then $X(i) \in S(i)$.

Let us consider a non-empty, 1-element finite sequence X and a family S of semialgebras of X . Now we state the propositions:

(14) the set of all $\prod \langle s \rangle$ where s is an element of $S(1)$ is a semialgebra of sets of the set of all $\langle x \rangle$ where x is an element of $X(1)$. The theorem is a consequence of (13).

(15) $\text{SemiringProduct}(S)$ is a semialgebra of sets of $\prod X$. The theorem is a consequence of (14).

(16) Let us consider sets X_1, X_2 , a semialgebra S_1 of sets of X_1 , and a semialgebra S_2 of sets of X_2 . Then the set of all $s_1 \times s_2$ where s_1 is an element of S_1 , s_2 is an element of S_2 is a semialgebra of sets of $X_1 \times X_2$.

(17) Let us consider a non zero natural number n , a non-empty, n -element finite sequence X , and a family S of semialgebras of X . Then $\text{SemiringProduct}(S)$ is a semialgebra of sets of $\prod X$.

PROOF: Define $\mathcal{P}[\text{non zero natural number}] \equiv$ for every non-empty, \mathbb{N} -element finite sequence X for every family S of semialgebras of X , $\text{SemiringProduct}(S)$ is a semialgebra of sets of $\prod X$. $\mathcal{P}[1]$. For every non zero natural number k , $\mathcal{P}[k]$ from [3, Sch. 10]. \square

(18) Let us consider a non zero natural number n , a non-empty, n -element finite sequence X_8 , a non-empty, 1-element finite sequence X_1 , a family S_4 of semialgebras of X_8 , and a family S_1 of semialgebras of X_1 . Then $\text{SemiringProduct}(S_4 \cap S_1)$ is a semialgebra of sets of $\prod(X_8 \cap X_1)$. The theorem is a consequence of (17), (16), and (13).

Let n be a non zero natural number and X be a non-empty, n -element finite sequence.

A family of fields of X is an n -element finite sequence and is defined by

(Def. 3) for every natural number i such that $i \in \text{Seg } n$ holds $it(i)$ is a field of subsets of $X(i)$.

Let S be a family of fields of X and i be a natural number. Assume $i \in \text{Seg } n$. Observe that the functor $S(i)$ yields a field of subsets of $X(i)$.

Observe that a family of fields of X is a family of semialgebras of X .

Let us consider a non-empty, 1-element finite sequence X and a family S of fields of X . Now we state the propositions:

- (19) the set of all $\prod \langle s \rangle$ where s is an element of $S(1)$ is a field of subsets of the set of all $\langle x \rangle$ where x is an element of $X(1)$. The theorem is a consequence of (14).
- (20) $\text{SemiringProduct}(S)$ is a field of subsets of $\prod X$. The theorem is a consequence of (19).

Let n be a non zero natural number, X be a non-empty, n -element finite sequence, and S be a family of fields of X .

A family of measures of S is an n -element finite sequence and is defined by

- (Def. 4) for every natural number i such that $i \in \text{Seg } n$ holds $it(i)$ is a measure on $S(i)$.

3. PRODUCT OF TWO MEASURES

Let X_1, X_2 be sets, S_1 be a field of subsets of X_1 , and S_2 be a field of subsets of X_2 . The functor $\text{MeasRect}(S_1, S_2)$ yielding a semialgebra of sets of $X_1 \times X_2$ is defined by the term

- (Def. 5) the set of all $A \times B$ where A is an element of S_1 , B is an element of S_2 .

Now we state the proposition:

- (21) Let us consider a set X , and a field F of subsets of X . Then there exists a semialgebra S of sets of X such that
- (i) $F = S$, and
- (ii) F = the field generated by S .

Let X_1, X_2 be sets, S_1 be a field of subsets of X_1 , S_2 be a field of subsets of X_2 , m_1 be a measure on S_1 , and m_2 be a measure on S_2 . The functor $\text{ProdpreMeas}(m_1, m_2)$ yielding a non-negative, zeroed function from $\text{MeasRect}(S_1, S_2)$ into $\overline{\mathbb{R}}$ is defined by

- (Def. 6) for every element C of $\text{MeasRect}(S_1, S_2)$, there exists an element A of S_1 and there exists an element B of S_2 such that $C = A \times B$ and $it(C) = m_1(A) \cdot m_2(B)$.

Now we state the propositions:

- (22) Let us consider sets X_1, X_2 , a field S_1 of subsets of X_1 , a field S_2 of subsets of X_2 , a measure m_1 on S_1 , a measure m_2 on S_2 , and sets A, B .

Suppose $A \in S_1$ and $B \in S_2$. Then $(\text{ProdpreMeas}(m_1, m_2))(A \times B) = m_1(A) \cdot m_2(B)$.

- (23) Let us consider sets X_1, X_2 , a non empty family S_1 of subsets of X_1 , a non empty family S_2 of subsets of X_2 , a non empty family S of subsets of $X_1 \times X_2$, and a finite sequence H of elements of S . Suppose $S =$ the set of all $A \times B$ where A is an element of S_1, B is an element of S_2 . Then there exists a finite sequence F of elements of S_1 and there exists a finite sequence G of elements of S_2 such that $\text{len } H = \text{len } F$ and $\text{len } H = \text{len } G$ and for every natural number k such that $k \in \text{dom } H$ and $H(k) \neq \emptyset$ holds $H(k) = F(k) \times G(k)$.

PROOF: For every natural number k such that $k \in \text{dom } H$ there exists an element A of S_1 and there exists an element B of S_2 such that $H(k) = A \times B$. Define $\mathcal{P}[\text{natural number, set}] \equiv$ there exists an element B of S_2 such that $H(\$1) = \$2 \times B$. Consider F being a finite sequence of elements of S_1 such that $\text{dom } F = \text{Seg len } H$ and for every natural number k such that $k \in \text{Seg len } H$ holds $\mathcal{P}[k, F(k)]$ from [4, Sch. 5]. Define $\mathcal{Q}[\text{natural number, set}] \equiv$ there exists an element A of S_1 such that $H(\$1) = A \times \2 . For every natural number k such that $k \in \text{Seg len } H$ there exists an element B of S_2 such that $\mathcal{Q}[k, B]$. Consider G being a finite sequence of elements of S_2 such that $\text{dom } G = \text{Seg len } H$ and for every natural number k such that $k \in \text{Seg len } H$ holds $\mathcal{Q}[k, G(k)]$ from [4, Sch. 5]. \square

- (24) Let us consider a set X , a non empty, semi-diff-closed, \cap -closed family S of subsets of X , and elements E_1, E_2 of S . Then there exist disjoint valued finite sequences O, T, F of elements of S such that
- (i) $\bigcup \text{rng } O = E_1 \setminus E_2$, and
 - (ii) $\bigcup \text{rng } T = E_2 \setminus E_1$, and
 - (iii) $\bigcup \text{rng } F = E_1 \cap E_2$, and
 - (iv) $(O \cap T) \cap F$ is a disjoint valued finite sequence of elements of S .

The theorem is a consequence of (9).

- (25) Let us consider sets X_1, X_2 , a field S_1 of subsets of X_1 , a field S_2 of subsets of X_2 , a measure m_1 on S_1 , a measure m_2 on S_2 , and elements E_1, E_2 of $\text{MeasRect}(S_1, S_2)$. Suppose E_1 misses E_2 and $E_1 \cup E_2 \in \text{MeasRect}(S_1, S_2)$. Then $(\text{ProdpreMeas}(m_1, m_2))(E_1 \cup E_2) = (\text{ProdpreMeas}(m_1, m_2))(E_1) + (\text{ProdpreMeas}(m_1, m_2))(E_2)$. The theorem is a consequence of (1) and (22).
- (26) Let us consider a non empty set X , a non empty family S of subsets of X , a function f from \mathbb{N} into S , and a sequence F of partial functions from X into $\overline{\mathbb{R}}$. Suppose f is disjoint valued and for every natural number

n , $F(n) = \chi_{f(n), X}$. Let us consider an object x . Suppose $x \in X$. Then $\chi_{\bigcup_{f, X}}(x) = (\lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})(x)$.

- (27) Let us consider a non empty set X , a σ -field S of subsets of X , a σ -measure M on S , a partial function f from X to $\overline{\mathbb{R}}$, and a real number r . Suppose $\text{dom } f \in S$ and $0 \leq r$ and for every object x such that $x \in \text{dom } f$ holds $f(x) = r$. Then $\int f \, dM = r \cdot M(\text{dom } f)$.

Let us consider a non empty set X , a σ -field S of subsets of X , a σ -measure M on S , a partial function f from X to $\overline{\mathbb{R}}$, and an element A of S . Now we state the propositions:

- (28) Suppose there exists an element E of S such that $E = \text{dom } f$ and f is measurable on E and for every object x such that $x \in \text{dom } f \setminus A$ holds $f(x) = 0$ and f is non-negative. Then $\int f \, dM = \int f \upharpoonright A \, dM$. The theorem is a consequence of (27).
- (29) If f is integrable on M and for every object x such that $x \in \text{dom } f \setminus A$ holds $f(x) = 0$, then $\int f \, dM = \int f \upharpoonright A \, dM$. The theorem is a consequence of (27).
- (30) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_2 on S_2 , a function D from \mathbb{N} into S_1 , a function E from \mathbb{N} into S_2 , an element A of S_1 , an element B of S_2 , a sequence F of partial functions from X_2 into $\overline{\mathbb{R}}$, a sequence R of \mathbb{R}^{X_1} , and an element x of X_1 . Suppose for every natural number n , $R(n) = \chi_{D(n), X_1}$ and for every natural number n , $F(n) = R(n)(x) \cdot \chi_{E(n), X_2}$ and for every natural number n , $E(n) \subseteq B$. Then there exists a sequence I of extended reals such that

- (i) for every natural number n , $I(n) = M_2(E(n)) \cdot \chi_{D(n), X_1}(x)$, and
- (ii) I is summable, and
- (iii) $\int \lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \, dM_2 = \sum I$.

PROOF: For every natural number n , $\text{dom}(F(n)) = X_2$. Reconsider $S_3 = X_2$ as an element of S_2 . For every natural number n and for every set y such that $y \in E(n)$ holds $F(n)(y) = 0$ or $F(n)(y) = 1$ by [10, (3)], [18, (1)], [12, (39)]. For every natural number n and for every set y such that $y \notin E(n)$ holds $F(n)(y) = 0$. For every natural number n , $F(n)$ is non-negative and $F(n)$ is measurable on B by [8, (51)], [17, (37)], [18, (29)]. For every element y of X_2 such that $y \in B$ holds $F \# y$ is summable by [8, (51), (39)], [19, (16)], [29, (37)].

Consider I being a sequence of extended reals such that for every natural number n , $I(n) = \int F(n) \upharpoonright B \, dM_2$ and I is summable and $\int \lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \upharpoonright B \, dM_2 = \sum I$. For every natural number n , $I(n) =$

$M_2(E(n)) \cdot \chi_{D(n), X_1}(x)$ by [28, (61)], [10, (47), (49)], [18, (29)]. For every natural number n , $F(n)$ is measurable on S_3 by [18, (29)], [17, (37)]. For every natural number n , $F(n)$ is without $-\infty$. For every element y of X_2 such that $y \in S_3$ holds $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \# y$ is convergent by [19, (38)]. For every object y such that $y \in \text{dom} \lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \setminus B$ holds $(\lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})(y) = 0$ by [19, (43)], [16, (52)]. For every object y such that $y \in \text{dom} \lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}$ holds $(\lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})(y) \geq 0$ by [19, (36)], [8, (51)], [19, (10), (38)]. \square

- (31) Let us consider a non empty set X , a σ -field S of subsets of X , an element A of S , and an extended real number p . Then $X \mapsto p$ is measurable on A .
 PROOF: For every real number r , $A \cap \text{GTE-dom}(X \mapsto p, r) \in S$ by [26, (7)], [7, (7)]. \square

Let A, X be sets. The functor $\bar{\chi}_{A, X}$ yielding a function from X into $\bar{\mathbb{R}}$ is defined by

- (Def. 7) for every object x such that $x \in X$ holds if $x \in A$, then $it(x) = +\infty$ and if $x \notin A$, then $it(x) = 0$.

Now we state the proposition:

- (32) Let us consider a non empty set X , a σ -field S of subsets of X , and elements A, B of S . Then $\bar{\chi}_{A, X}$ is measurable on B .

Let X, A be sets. Let us observe that $\bar{\chi}_{A, X}$ is non-negative.

- (33) Let us consider a non empty set X , a σ -field S of subsets of X , a σ -measure M on S , and an element A of S . Then

- (i) if $M(A) \neq 0$, then $\int \bar{\chi}_{A, X} dM = +\infty$, and
- (ii) if $M(A) = 0$, then $\int \bar{\chi}_{A, X} dM = 0$.

PROOF: Reconsider $X_3 = X$ as an element of S . Reconsider $X_2 = X_3 \setminus A$ as an element of S . Reconsider $F = \bar{\chi}_{A, X} \upharpoonright A$ as a partial function from X to $\bar{\mathbb{R}}$. Reconsider $O = \bar{\chi}_{A, X} \upharpoonright X_2$ as a partial function from X to $\bar{\mathbb{R}}$. Reconsider $T = \bar{\chi}_{A, X} \upharpoonright (X_2 \cup A)$ as a partial function from X to $\bar{\mathbb{R}}$. $\int F dM = 0$. O is measurable on X_2 . For every element x of X such that $x \in \text{dom}(\bar{\chi}_{A, X} \upharpoonright X_2)$ holds $(\bar{\chi}_{A, X} \upharpoonright X_2)(x) = 0$ by [10, (47)]. $\int T dM = \int O dM + 0$. \square

- (34) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , and a disjoint valued function K from \mathbb{N} into $\text{MeasRect}(S_1, S_2)$. Suppose $\bigcup K \in \text{MeasRect}(S_1, S_2)$. Then $(\text{ProdpreMeas}(M_1, M_2))(\bigcup K) = \overline{\sum}(\text{ProdpreMeas}(M_1, M_2) \cdot K)$.

PROOF: Consider A being an element of S_1 , B being an element of S_2 such that $\bigcup K = A \times B$. Consider P being an element of S_1 , Q being an element of S_2 such that $\bigcup K = P \times Q$ and $(\text{ProdpreMeas}(M_1, M_2))(\bigcup K) = M_1(P) \cdot$

$M_2(Q)$. Define $\mathcal{F}(\text{object}) = \chi_{K(\$_1), X_1 \times X_2}$. Consider X_6 being a sequence of partial functions from $X_1 \times X_2$ into $\overline{\mathbb{R}}$ such that for every natural number n , $X_6(n) = \mathcal{F}(n)$ from [24, Sch. 1]. Define $\mathcal{P}[\text{natural number, object}] \equiv \$_2 = \pi_1(K(\$_1))$. For every element i of \mathbb{N} , there exists an element A of S_1 such that $\mathcal{P}[i, A]$ by [2, (9)], [7, (7)]. Consider D being a function from \mathbb{N} into S_1 such that for every element i of \mathbb{N} , $\mathcal{P}[i, D(i)]$ from [11, Sch. 3]. Define $\mathcal{Q}[\text{natural number, object}] \equiv \$_2 = \pi_2(K(\$_1))$. For every element i of \mathbb{N} , there exists an element B of S_2 such that $\mathcal{Q}[i, B]$ by [2, (9)], [7, (7)].

Consider E being a function from \mathbb{N} into S_2 such that for every element i of \mathbb{N} , $\mathcal{Q}[i, E(i)]$ from [11, Sch. 3]. Define $\mathcal{O}(\text{object}) = \chi_{D(\$_1), X_1}$. Consider X_7 being a sequence of partial functions from X_1 into $\overline{\mathbb{R}}$ such that for every natural number n , $X_7(n) = \mathcal{O}(n)$ from [24, Sch. 1]. Define $\mathcal{T}(\text{object}) = \chi_{E(\$_1), X_2}$. Consider X_4 being a sequence of partial functions from X_2 into $\overline{\mathbb{R}}$ such that for every natural number n , $X_4(n) = \mathcal{T}(n)$ from [24, Sch. 1]. For every natural number n and for every objects x, y such that $x \in X_1$ and $y \in X_2$ holds $X_6(n)(x, y) = X_7(n)(x) \cdot X_4(n)(y)$ by [14, (87)], [2, (9)], (2). (ProdpreMeas(M_1, M_2))($\cup K$) = $M_1(A) \cdot M_2(B)$ by [14, (110)]. Reconsider $C_1 = \chi_{A \times B, X_1 \times X_2}$ as a function from $X_1 \times X_2$ into $\overline{\mathbb{R}}$. For every element x of X_1 , $M_2(B) \cdot \chi_{A, X_1}(x) = \int \text{curry}(C_1, x) dM_2$ by (2), [13, (5)], [19, (14)], [23, (4)]. For every object n such that $n \in \mathbb{N}$ holds $X_7(n) \in \mathbb{R}^{X_1}$ by [12, (39)]. Reconsider $R_1 = X_7$ as a sequence of \mathbb{R}^{X_1} . For every natural number n , $D(n) \subseteq A$ and $E(n) \subseteq B$ by [2, (10)], [1, (1)]. For every element x of X_1 , there exists a sequence X_5 of partial functions from X_2 into $\overline{\mathbb{R}}$ and there exists a sequence I of extended reals such that for every natural number n , $X_5(n) = R_1(n)(x) \cdot \chi_{E(n), X_2}$ and for every natural number n , $I(n) = M_2(E(n)) \cdot \chi_{D(n), X_1}(x)$ and I is summable and $\int \lim(\sum_{\alpha=0}^{\kappa} X_5(\alpha))_{\kappa \in \mathbb{N}} dM_2 = \sum I$ by [13, (45)], (30).

Reconsider $L_1 = \lim(\sum_{\alpha=0}^{\kappa} X_6(\alpha))_{\kappa \in \mathbb{N}}$ as a function from $X_1 \times X_2$ into $\overline{\mathbb{R}}$. For every element x of X_1 and for every element y of X_2 , $(\text{curry}(C_1, x))(y) = (\text{curry}(L_1, x))(y)$. For every element x of X_1 , $\text{curry}(C_1, x) = \text{curry}(L_1, x)$. For every element x of X_1 , $M_2(B) \cdot \chi_{A, X_1}(x) = \int \text{curry}(L_1, x) dM_2$. For every element x of X_1 , there exists a sequence I of extended reals such that for every natural number n , $I(n) = M_2(E(n)) \cdot \chi_{D(n), X_1}(x)$ and $M_2(B) \cdot \chi_{A, X_1}(x) = \sum I$ by [8, (51)], [19, (38), (29), (30)]. Define $\mathcal{R}[\text{natural number, object}] \equiv$ if $M_2(E(\$_1)) = +\infty$, then $\$_2 = \bar{\chi}_{D(\$_1), X_1}$ and if $M_2(E(\$_1)) \neq +\infty$, then there exists a real number m_2 such that $m_2 = M_2(E(\$_1))$ and $\$_2 = m_2 \cdot \chi_{D(\$_1), X_1}$. For every element n of \mathbb{N} , there exists an element y of $X_1 \dot{\rightarrow} \overline{\mathbb{R}}$ such that $\mathcal{R}[n, y]$ by [13, (45)], [8, (51)]. Consider F_1 being a function from \mathbb{N} into $X_1 \dot{\rightarrow} \overline{\mathbb{R}}$ such that for every element n of \mathbb{N} , $\mathcal{R}[n, F_1(n)]$ from [11, Sch. 3]. For every natural number

n , $\text{dom}(F_1(n)) = X_1$. For every natural number n , $F_1(n)$ is non-negative by [8, (51)]. For every natural numbers n, m , $\text{dom}(F_1(n)) = \text{dom}(F_1(m))$.

Reconsider $X_3 = X_1$ as an element of S_1 . For every natural number n , $F_1(n)$ is non-negative and $F_1(n)$ is measurable on A and $F_1(n)$ is measurable on X_3 by (32), [18, (29)], [17, (37)]. For every element x of X_1 such that $x \in A$ holds $F_1 \# x$ is summable by [8, (51), (39)], [20, (2)]. Consider J being a sequence of extended reals such that for every natural number n , $J(n) = \int F_1(n) \upharpoonright A \, dM_1$ and J is summable and $\int \lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}} \upharpoonright A \, dM_1 = \sum J$. For every natural number n , $J(n) = \int F_1(n) \, dM_1$. Reconsider $X_3 = X_1$ as an element of S_1 . For every element n of \mathbb{N} , $J(n) = (\text{ProdpreMeas}(M_1, M_2) \cdot K)(n)$ by (33), [8, (51)], [18, (29)], [16, (86), (88)]. For every element x of X_1 , $(\lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}})(x) \geq 0$ by [19, (38)], [29, (37), (23)], [8, (51)]. For every natural number n , $F_1(n)$ is measurable on X_3 and $F_1(n)$ is without $-\infty$. For every object x such that $x \in \text{dom} \lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}} \setminus A$ holds $(\lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}})(x) = 0$ by [19, (30), (32)], [16, (52)]. $\int \lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}} \, dM_1 = \int \lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}} \upharpoonright A \, dM_1$. $\int \lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}} \, dM_1 = M_1(A) \cdot M_2(B)$ by [11, (63)], [19, (30), (32)], [8, (51)]. \square

- (35) Let us consider a without $-\infty$ finite sequence f of elements of $\overline{\mathbb{R}}$, and a without $-\infty$ sequence s of extended reals. Suppose for every object n such that $n \in \text{dom} f$ holds $f(n) = s(n)$.

Then $\sum f + s(0) = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\text{len} f)$.

PROOF: Consider F being a sequence of $\overline{\mathbb{R}}$ such that $\sum f = F(\text{len} f)$ and $F(0) = 0$ and for every natural number i such that $i < \text{len} f$ holds $F(i+1) = F(i) + f(i+1)$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$1 \leq \text{len} f$, then $F(\$1) + s(0) = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\$1)$ and $F(\$1) \neq -\infty$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [3, (11)], [27, (25)], [16, (10)], [3, (13)]. For every natural number k , $\mathcal{P}[k]$ from [3, Sch. 2]. \square

- (36) Let us consider a non-negative finite sequence f of elements of $\overline{\mathbb{R}}$, and a sequence s of extended reals. Suppose for every object n such that $n \in \text{dom} f$ holds $f(n) = s(n)$ and for every element n of \mathbb{N} such that $n \notin \text{dom} f$ holds $s(n) = 0$. Then

(i) $\sum f = \sum s$, and

(ii) $\sum f = \overline{\sum} s$.

PROOF: For every object n such that $n \in \text{dom} s$ holds $0 \leq s(n)$ by [8, (51)]. $\sum f + s(0) = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\text{len} f)$. Define $\mathcal{P}[\text{natural number}] \equiv (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\text{len} f) = ((\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}} \upharpoonright \text{len} f)(\$1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [27, (25)]. For every natural number k , $\mathcal{P}[k]$ from [3, Sch. 2]. \square

- (37) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , and a disjoint valued finite sequence F of elements of $\text{MeasRect}(S_1, S_2)$. Suppose $\bigcup F \in \text{MeasRect}(S_1, S_2)$. Then $(\text{ProdpreMeas}(M_1, M_2))(\bigcup F) = \sum(\text{ProdpreMeas}(M_1, M_2) \cdot F)$.

PROOF: Set $S = \text{MeasRect}(S_1, S_2)$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv \text{if } \$1 \in \text{dom } F, \text{ then } \$2 = F(\$1) \text{ and if } \$1 \notin \text{dom } F, \text{ then } \$2 = \emptyset$. For every element n of \mathbb{N} , there exists an element y of S such that $\mathcal{P}[n, y]$ by [10, (3)]. Consider G being a function from \mathbb{N} into S such that for every element n of \mathbb{N} , $\mathcal{P}[n, G(n)]$ from [11, Sch. 3]. For every object x such that $x \notin \text{dom } F$ holds $G(x) = \emptyset$. For every objects x, y such that $x \neq y$ holds $G(x)$ misses $G(y)$. $(\text{ProdpreMeas}(M_1, M_2))(\bigcup F) = \overline{\sum}(\text{ProdpreMeas}(M_1, M_2) \cdot G)$. For every object n such that $n \in \text{dom}(\text{ProdpreMeas}(M_1, M_2) \cdot F)$ holds $(\text{ProdpreMeas}(M_1, M_2) \cdot F)(n) = (\text{ProdpreMeas}(M_1, M_2) \cdot G)(n)$ by [10, (11), (12), (13)]. For every element n of \mathbb{N} such that $n \notin \text{dom}(\text{ProdpreMeas}(M_1, M_2) \cdot F)$ holds $(\text{ProdpreMeas}(M_1, M_2) \cdot G)(n) = 0$ by [10, (3), (11), (13)]. \square

- (38) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , and a σ -measure M_2 on S_2 . Then $\text{ProdpreMeas}(M_1, M_2)$ is a pre-measure of $\text{MeasRect}(S_1, S_2)$. The theorem is a consequence of (37) and (34).

Let X_1, X_2 be non empty sets, S_1 be a σ -field of subsets of X_1 , S_2 be a σ -field of subsets of X_2 , M_1 be a σ -measure on S_1 , and M_2 be a σ -measure on S_2 . Let us observe that the functor $\text{ProdpreMeas}(M_1, M_2)$ yields a pre-measure of $\text{MeasRect}(S_1, S_2)$.

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