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On Multiset Ordering

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Summary. Formalization of a part of [11]. Unfortunately, not all is possible to be formalized. Namely, in the paper there is a mistake in the proof of Lemma 3. It states that there exists $x \in M_1$ such that $M_1(x) > N_1(x)$ and $(\forall y \in N_1) x \not< y$. It should be $M_1(x) \geqslant N_1(x)$. Nevertheless we do not know whether $x \in N_1$ or not and cannot prove the contradiction. In the article we referred to [8], [9] and [10].

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1. Preliminaries

Now we state the propositions:

(1) Let us consider natural numbers m, n. Then $n = m -' (m -' n) + (n -' m)$.

(2) Let us consider natural numbers n, m. Then $m -' n \geqslant m - n$.

Let us consider natural numbers m, n, x, y. Now we state the propositions:

(3) If $n = m -' x + y$, then $m -' n \leqslant x$ and $n -' m \leqslant y$. The theorem is a consequence of (2).

(4) If $x \leqslant m$ and $n = m -' x + y$, then $x -' (m -' n) = y -' (n -' m)$. The theorem is a consequence of (3).

Now we state the propositions:

(5) Let us consider natural numbers k, x_1, x_2, y_1, y_2. Suppose $x_2 \leqslant k$ and $x_1 \leqslant k -' x_2 + y_2$. Then

(i) $x_2 + (x_1 -' y_2) \leqslant k$, and

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(ii) $k -' x_2 + y_2 -' x_1 + y_1 = k -' (x_2 + (x_1 -' y_2)) + (y_2 -' x_1 + y_1)$.

PROOF: $x_2 + (x_1 -' y_2) \leqslant k$ by [12, (8)]. $\square$

(6) Let us consider natural numbers x, y. If $x + y > 0$, then $x > 0$ or $y > 0$.

From now on a, b denote objects and I, J denote sets.

Let us consider I. Let J be a non empty set. Let us note that every function from I into J is total and there exists a relational structure which is asymmetric, transitive, and non empty.

Let us consider I. One can verify that there exists a binary relation on I which is asymmetric and transitive.

Let R be a transitive relational structure. Observe that the internal relation of R is transitive.

Let R be an asymmetric relational structure. Let us observe that the internal relation of R is asymmetric.

Let us consider I. Let p, q be I-valued finite sequences. Let us observe that $p \frown q$ is I-valued.

Now we state the proposition:

(7) Let us consider finite sequences p, q. Suppose $p \frown q$ is I-valued. Then

(i) p is I-valued, and

(ii) q is I-valued.

Let us consider I. Let f be an I-valued finite sequence and n be a natural number. Let us note that $f \restriction n$ is I-valued.

Now we state the propositions:

(8) Let us consider a finite sequence p. Suppose $a \in \operatorname{rng} p$. Then there exist finite sequences q, r such that $p = (q \frown \langle a \rangle) \frown r$.

(9) Let us consider finite sequences p, q. Then $p \subset q$ if and only if $\operatorname{len} p < \operatorname{len} q$ and for every natural number i such that $i \in \operatorname{dom} p$ holds $p(i) = q(i)$.

(10) Let us consider finite sequences p, q, r. Then $r \frown p \subset r \frown q$ if and only if $p \subset q$.

PROOF: If $r \frown p \subset r \frown q$, then $p \subset q$ by [4, (22)], (9), [15, (30)], [4, (28)]. $\square$

Let R be an asymmetric, non empty relational structure and x, y be elements of R. Let us observe that the predicate $x \leqslant y$ is asymmetric.

Now we state the proposition:

(11) Let us consider an asymmetric, non empty relational structure R, and elements x, y of R. Then $x \leqslant y$ if and only if $x < y$.

2. Relational Extension

Let us consider I.

A multiset of I is an element of $I^{\otimes}$. Observe that every multiset of I is I-defined and natural-valued and every multiset of I is total.

Let m be a natural-valued function. Let us note that the functor support m is defined by the term

(Def. 1) $m^{-1}(\mathbb{N} \setminus \{0\})$.

Let us consider I. One can check that every multiset of I is finite-support.

Now we state the propositions:

(12) a is a multiset of I if and only if a is a bag of I.

(13) $1_{I^{\otimes}} = \mathrm{EmptyBag}\, I$.

Let R be a relational structure and x, y be elements of R. We say that $x \equiv y$ if and only if

(Def. 2) $x \not\leqslant y$ and $y \not\leqslant x$.

Observe that the predicate is symmetric.

We consider relational multiplicative magmas which extend multiplicative magmas and relational structures and are systems

$$\langle \text{a carrier}, \text{a multiplication}, \text{an internal relation} \rangle$$

where the carrier is a set, the multiplication is a binary operation on the carrier, the internal relation is a binary relation on the carrier.

We consider relational monoids which extend multiplicative loop structures and relational structures and are systems

$$\langle \text{a carrier}, \text{a multiplication}, \text{a one}, \text{an internal relation} \rangle$$

where the carrier is a set, the multiplication is a binary operation on the carrier, the one is an element of the carrier, the internal relation is a binary relation on the carrier.

Let M be a multiplicative loop structure.

A relational extension of M is a relational monoid and is defined by

(Def. 3) the multiplicative loop structure of *it* = the multiplicative loop structure of M.

Let M be a non empty multiplicative loop structure. Let us observe that every relational extension of M is non empty.

Let M be a multiplicative loop structure. One can check that there exists a relational extension of M which is strict.

Let us consider a multiplicative loop structure N and a relational extension M of N. Now we state the propositions:

(14) a is an element of M if and only if a is an element of N.

(15) $1_N = 1_M$.

Let us consider I. Let M be a relational extension of $I^{\otimes}$. Let us observe that every element of M is function-like and relation-like and every element of M is I-defined, natural-valued, and finite-support and every element of M is total.

Now we state the proposition:

(16) Let us consider a relational extension M of $I^{\otimes}$. Then the carrier of $M = \operatorname{Bags} I$. The theorem is a consequence of (12) and (14).

The scheme *RelEx* deals with a non empty multiplicative loop structure $\mathcal{M}$ and a binary predicate $\mathcal{R}$ and states that

(Sch. 1) There exists a strict relational extension N of $\mathcal{M}$ such that for every elements x, y of N, $x \leqslant y$ iff $\mathcal{R}[x, y]$.

Now we state the proposition:

(17) Let us consider a multiplicative loop structure N, and strict relational extensions M_1, M_2 of N. Suppose for every elements m, n of M_1 for every elements x, y of M_2 such that $m = x$ and $n = y$ holds $m \leqslant n$ iff $x \leqslant y$. Then $M_1 = M_2$.

PROOF: The internal relation of M_1 = the internal relation of M_2 by [7, (87)]. $\square$

3. Dershowitz-Manna Order

Let R be a non empty relational structure. The Dershowitz-Manna order R yielding a strict relational extension of (the carrier of $R)^{\otimes}$ is defined by

(Def. 4) for every elements m, n of *it*, $m \leqslant n$ iff there exist elements x, y of *it* such that $1_{it} \neq x \mid n$ and $m = n -' x + y$ and for every element b of R such that $y(b) > 0$ there exists an element a of R such that $x(a) > 0$ and $b \leqslant a$.

Now we state the proposition:

(18) Let us consider bags m, n of I. Then $n = m -' (m -' n) + (n -' m)$. The theorem is a consequence of (1).

Let us consider bags m, n, x, y of I. Now we state the propositions:

(19) If $n = m -' x + y$, then $m -' n \mid x$ and $n -' m \mid y$. The theorem is a consequence of (3).

(20) If $x \mid m$ and $n = m -' x + y$, then $x -' (m -' n) = y -' (n -' m)$. The theorem is a consequence of (4).

Now we state the propositions:

(21) Let us consider bags m, x, y of I. If $x \mid m$ and $x \neq y$, then $m \neq m -' x + y$.

(22) Let us consider a non empty set I, a binary relation R on I, and a reduction sequence r w.r.t. R. If $\operatorname{len} r > 1$, then $r(\operatorname{len} r) \in I$.

(23) Let us consider an asymmetric, transitive binary relation R on I. Then every reduction sequence w.r.t. R is one-to-one.

Proof: For every natural numbers i, j such that $i > j$ and $i, j \in \operatorname{dom} r$ holds $r(i) \neq r(j)$ by [1, (13)], [13, (22)], [1, (11)], [15, (25)]. □

(24) Let us consider an asymmetric, transitive, non empty relational structure R, and a set X. Suppose X is finite and there exists an element x of R such that $x \in X$. Then there exists an element x of R such that x is maximal in X.

Proof: Reconsider $X_1 = X$ as a finite set. Set $Y = \{r$, where r is an element of ${X_1}^* : r$ is a reduction sequence w.r.t. the internal relation of $R\}$. Define $\mathcal{P}[\text{natural number}] \equiv$ there exists a reduction sequence r w.r.t. the internal relation of R such that $r \in Y$ and $\operatorname{len} r = \$_1$. For every natural number k such that $\mathcal{P}[k]$ holds $k \leqslant \overline{\overline{X_1}}$ by (23), [1, (43)]. $\mathcal{P}[1]$ by [2, (6)], [4, (74), (39)]. Consider k being a natural number such that $\mathcal{P}[k]$ and for every natural number n such that $\mathcal{P}[n]$ holds $n \leqslant k$ from [1, Sch. 6]. Consider r being a reduction sequence w.r.t. the internal relation of R such that $r \in Y$ and $\operatorname{len} r = k$. Consider q being an element of ${X_1}^*$ such that $r = q$ and q is a reduction sequence w.r.t. the internal relation of R. □

(25) Let us consider bags m, n of I. Then $m -' n \mid m$.

Let us consider I. Note that every element of $\operatorname{Bags} I$ is function-like and relation-like.

Now we state the proposition:

(26) Let us consider bags m, n of I. Then

(i) $m -' n \neq \operatorname{EmptyBag} I$, or

(ii) $m = n$, or

(iii) $n -' m \neq \operatorname{EmptyBag} I$.

Let R be an asymmetric, transitive, non empty relational structure. Let us observe that the Dershowitz-Manna order R is defined by

(Def. 5) for every elements m, n of *it*, $m \leqslant n$ iff $m \neq n$ and for every element a of R such that $m(a) > n(a)$ there exists an element b of R such that $a \leqslant b$ and $m(b) < n(b)$.

Now we state the proposition:

(27) Let us consider bags k, x_1, x_2, y_1, y_2 of I. Suppose $x_2 \mid k$ and $x_1 \mid k -' x_2 + y_2$. Then

(i) $x_2 + (x_1 -' y_2) \mid k$, and

(ii) $k -' x_2 + y_2 -' x_1 + y_1 = k -' (x_2 + (x_1 -' y_2)) + (y_2 -' x_1 + y_1)$.

The theorem is a consequence of (5).

Let R be an asymmetric, transitive, non empty relational structure. Let us observe that the Dershowitz-Manna order R is asymmetric and transitive.

Let us consider I. The functor DivOrder(I) yielding a binary relation on Bags I is defined by

(Def. 6) for every bags b_1, b_2 of I, $\langle b_1, b_2 \rangle \in it$ iff $b_1 \neq b_2$ and $b_1 \mid b_2$.

Now we state the proposition:

(28) Let us consider bags a, b, c of I. If $a \mid b \mid c$, then $a \mid c$.

Let us consider I. Note that DivOrder(I) is asymmetric and transitive.

Let us consider an asymmetric, transitive, non empty relational structure R. Now we state the propositions:

(29) DivOrder(the carrier of R) $\subseteq$ the internal relation of the Dershowitz-Manna order R. The theorem is a consequence of (12) and (14).

(30) Suppose the internal relation of R is empty. Then the internal relation of the Dershowitz-Manna order R = DivOrder(the carrier of R). The theorem is a consequence of (29).

Now we state the proposition:

(31) Let us consider asymmetric, transitive, non empty relational structures R_1, R_2. Suppose the carrier of R_1 = the carrier of R_2 and the internal relation of $R_1 \subseteq$ the internal relation of R_2. Then the internal relation of the Dershowitz-Manna order $R_1 \subseteq$ the internal relation of the Dershowitz-Manna order R_2. The theorem is a consequence of (12) and (14).

4. Monoidal Order

Let us consider I. Let f be a (Bags I)-valued finite sequence. The functor $\sum f$ yielding a bag of I is defined by

(Def. 7) there exists a function F from $\mathbb{N}$ into Bags I such that $it = F(\operatorname{len} f)$ and $F(0) = \text{EmptyBag}\, I$ and for every natural number i and for every bag b of I such that $i < \operatorname{len} f$ and $b = f(i+1)$ holds $F(i+1) = F(i) + b$.

Now we state the proposition:

(32) $\sum \varepsilon_{\text{Bags}\, I} = \text{EmptyBag}\, I$.

Let us consider I. Let b be a bag of I. One can verify that $\langle b \rangle$ is (Bags I)-valued as a finite sequence.

Now we state the proposition:

(33) Let us consider a $(\mathrm{Bags}\,I)$-valued finite sequence p, and a bag b of I. Then $\sum(p \frown \langle b \rangle) = \sum p + b$.

PROOF: Set $f = p \frown \langle b \rangle$. Consider F being a function from $\mathbb{N}$ into $\mathrm{Bags}\,I$ such that $\sum f = F(\mathrm{len}\, f)$ and $F(0) = \mathrm{EmptyBag}\,I$ and for every natural number i and for every bag b of I such that $i < \mathrm{len}\, f$ and $b = f(i+1)$ holds $F(i+1) = F(i)+b$. Consider F_1 being a function from $\mathbb{N}$ into $\mathrm{Bags}\,I$ such that $\sum p = F_1(\mathrm{len}\, p)$ and $F_1(0) = \mathrm{EmptyBag}\,I$ and for every natural number i and for every bag b of I such that $i < \mathrm{len}\, p$ and $b = p(i+1)$ holds $F_1(i+1) = F_1(i)+b$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant \mathrm{len}\, p$, then $F(\$_1) = F_1(\$_1)$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [5, (16)], [1, (13), (11)], [15, (25)]. For every natural number i, $\mathcal{P}[i]$ from [1, Sch. 2]. □

From now on b denotes a bag of I.

Now we state the propositions:

(34) $\sum\langle b \rangle = b$. The theorem is a consequence of (33) and (32).

(35) Let us consider $(\mathrm{Bags}\,I)$-valued finite sequences p, q. Then $\sum(p \frown q) = \sum p + \sum q$.

PROOF: Set $f = p \frown q$. Consider F being a function from $\mathbb{N}$ into $\mathrm{Bags}\,I$ such that $\sum f = F(\mathrm{len}\, f)$ and $F(0) = \mathrm{EmptyBag}\,I$ and for every natural number i and for every bag b of I such that $i < \mathrm{len}\, f$ and $b = f(i+1)$ holds $F(i+1) = F(i)+b$. Consider F_1 being a function from $\mathbb{N}$ into $\mathrm{Bags}\,I$ such that $\sum p = F_1(\mathrm{len}\, p)$ and $F_1(0) = \mathrm{EmptyBag}\,I$ and for every natural number i and for every bag b of I such that $i < \mathrm{len}\, p$ and $b = p(i+1)$ holds $F_1(i+1) = F_1(i) + b$. Consider F_2 being a function from $\mathbb{N}$ into $\mathrm{Bags}\,I$ such that $\sum q = F_2(\mathrm{len}\, q)$ and $F_2(0) = \mathrm{EmptyBag}\,I$ and for every natural number i and for every bag b of I such that $i < \mathrm{len}\, q$ and $b = q(i+1)$ holds $F_2(i+1) = F_2(i) + b$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant \mathrm{len}\, p$, then $F(\$_1) = F_1(\$_1)$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [4, (22)], [1, (11), (13)], [15, (25)]. For every natural number i, $\mathcal{P}[i]$ from [1, Sch. 2]. Define $\mathcal{Q}[\text{natural number}] \equiv$ if $\$_1 \leqslant \mathrm{len}\, q$, then $F(\mathrm{len}\, p + \$_1) = \sum p + F_2(\$_1)$. For every natural number i such that $\mathcal{Q}[i]$ holds $\mathcal{Q}[i+1]$ by [4, (22)], [1, (13), (11)], [15, (25)]. For every natural number i, $\mathcal{Q}[i]$ from [1, Sch. 2]. □

Let us consider a $(\mathrm{Bags}\,I)$-valued finite sequence p. Now we state the propositions:

(36) $\sum(\langle b \rangle \frown p) = b + \sum p$. The theorem is a consequence of (35) and (34).

(37) If $b \in \mathrm{rng}\, p$, then $b \mid \sum p$. The theorem is a consequence of (8), (7), (33), and (35).

Now we state the proposition:

(38) Let us consider a (Bags I)-valued finite sequence p, and an object i. Suppose $i \in \operatorname{support} \sum p$. Then there exists b such that

(i) $b \in \operatorname{rng} p$, and

(ii) $i \in \operatorname{support} b$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every (Bags I)-valued finite sequence p such that $\operatorname{len} p = \$_1$ for every object i such that $i \in \operatorname{support} \sum p$ there exists b such that $b \in \operatorname{rng} p$ and $i \in \operatorname{support} b$. $\mathcal{P}[0]$. For every natural number j such that $\mathcal{P}[j]$ holds $\mathcal{P}[j+1]$ by [3, (3)], (7), [4, (40)], [15, (25)]. For every natural number j, $\mathcal{P}[j]$ from [1, Sch. 2]. $\square$

Let us consider I and b.

A partition of b is a (Bags I)-valued finite sequence and is defined by

(Def. 8) $b = \sum it$.

Observe that the functor $\langle b \rangle$ yields a partition of b. Let R be a relational structure, M be a relational extension of (the carrier of R)$^{\otimes}$, b be an element of M, and p be a partition of b. We say that p is co-ordered if and only if

(Def. 9) for every natural number i such that i, $i+1 \in \operatorname{dom} p$ for every elements b_1, b_2 of M such that $b_1 = p(i)$ and $b_2 = p(i+1)$ holds $b_2 \leqslant b_1$.

Let R be a non empty relational structure and b be a bag of the carrier of R. We say that p is ordered if and only if

(Def. 10) for every bag m of the carrier of R such that $m \in \operatorname{rng} p$ for every element x of R such that $m(x) > 0$ holds $m(x) = b(x)$ and for every bag m of the carrier of R such that $m \in \operatorname{rng} p$ for every elements x, y of R such that $m(x) > 0$ and $m(y) > 0$ and $x \neq y$ holds $x \equiv y$ and for every bag m of the carrier of R such that $m \in \operatorname{rng} p$ holds $m \neq$ EmptyBag(the carrier of R) and for every natural number i such that i, $i+1 \in \operatorname{dom} p$ for every element x of R such that $p_{i+1}(x) > 0$ there exists an element y of R such that $p_i(y) > 0$ and $x \leqslant y$.

In the sequel R denotes an asymmetric, transitive, non empty relational structure, a, b, c denote bags of the carrier of R, and x, y, z denote elements of R.

Now we state the propositions:

(39) $\langle a \rangle$ is ordered if and only if $a \neq$ EmptyBag(the carrier of R) and for every x and y such that $a(x) > 0$ and $a(y) > 0$ and $x \neq y$ holds $x \equiv y$.

(40) Let us consider a (Bags I)-valued finite sequence p, and bags a, b of I. Then $\langle a \rangle \frown p$ is a partition of b if and only if $a \mid b$ and p is a partition of $b -' a$. The theorem is a consequence of (36).

From now on p denotes a partition of $b -' a$ and q denotes a partition of b.

Now we state the proposition:

(41) If $q = \langle a \rangle \frown p$ and q is ordered, then p is ordered. The theorem is a consequence of (37) and (25).

Let us consider I. Let m be a bag of I and J be a set. The functor $m{\restriction}J$ yielding a bag of I is defined by

(Def. 11) for every object i such that $i \in I$ holds if $i \in J$, then $it(i) = m(i)$ and if $i \notin J$, then $it(i) = 0$.

From now on J denotes a set and m denotes a bag of I.

Now we state the propositions:

(42) $\mathrm{support}(m{\restriction}J) = J \cap \mathrm{support}\, m$.

(43) $m{\restriction}J + m{\restriction}(I \setminus J) = m$.

(44) $m{\restriction}J \mid m$.

(45) If $\mathrm{support}\, m \subseteq J$, then $m{\restriction}J = m$.

(46) $\mathrm{support}(m -' m{\restriction}J) = \mathrm{support}\, m \setminus J$.

(47) If q is ordered and $q = \langle a \rangle \frown p$ and $a(x) > 0$, then $a(x) = b(x)$.

(48) If q is ordered and $q = \langle a \rangle \frown p$ and $a(x) > 0$ and $a(y) > 0$ and $x \neq y$, then $x \equiv y$.

(49) If q is ordered and $q = \langle a \rangle \frown p$, then $a \neq \mathrm{EmptyBag}(\text{the carrier of } R)$.

(50) Let us consider a bag c of the carrier of R, and a (Bags(the carrier of R))-valued finite sequence r. Suppose q is ordered and $q = \langle a, c \rangle \frown r$ and $c(y) > 0$. Then there exists x such that

(i) $a(x) > 0$, and

(ii) $y \leqslant x$.

(51) If $x \in I$ and for every y such that $y \in I$ and $y \neq x$ holds $x \equiv y$, then x is maximal in I.

(52) If q is ordered and $q = \langle a \rangle \frown p$ and $c \in \mathrm{rng}\, p$ and $c(x) > 0$, then there exists y such that $a(y) > 0$ and $x \leqslant y$.

Proof: Consider i being an object such that $i \in \mathrm{dom}\, p$ and $c = p(i)$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \in \mathrm{dom}\, p$, then for every x such that $p_{\$_1}(x) > 0$ there exists y such that $a(y) > 0$ and $x \leqslant y$. $\mathcal{P}[1]$ by [4, (28)], [15, (25)], [4, (40)]. For every natural number i such that $i \geqslant 1$ and $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [1, (13)], [15, (25)], [4, (28)], [16, (3)]. For every natural number i such that $i \geqslant 1$ holds $\mathcal{P}[i]$ from [1, Sch. 8]. $\square$

Let us assume that q is ordered and $q = \langle a \rangle \frown p$. Now we state the propositions:

(53) x is maximal in $\mathrm{support}\, b$ if and only if $a(x) > 0$.

Proof: $a \mid \sum q = b$. There exists no y such that $y \in \mathrm{support}\, b$ and $x < y$ by (48), (38), [4, (31), (39)]. $\square$

(54) $a = b \restriction \{x : x$ is maximal in $\operatorname{support} b\}$. The theorem is a consequence of (53) and (47).

Now we state the propositions:

(55) Let us consider a (Bags I)-valued finite sequence p. Suppose $\sum p =$ EmptyBag I and for every bag a of I such that $a \in \operatorname{rng} p$ holds $a \neq$ EmptyBag I. Then $p = \emptyset$. The theorem is a consequence of (37).

(56) Let us consider bags a, b of I. If $a \neq$ EmptyBag I, then $a + b \neq$ EmptyBag I.

(57) Let us consider partitions p, q of b. If p is ordered and q is ordered, then $p = q$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every b and q such that $\operatorname{len} q = \$_1$ and q is ordered for every partition p of b such that p is ordered holds $q = p$. $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [5, (130)], (40), (49), (36). For every natural number i, $\mathcal{P}[i]$ from [1, Sch. 2]. $\square$

Let us consider I. Let a, b be bags of I. One can verify that the functor $\langle a, b\rangle$ yields an element of Bags $I \times$ Bags I. Now we state the proposition:

(58) Suppose $a \neq$ EmptyBag(the carrier of R). Then $\{x : x$ is maximal in $\operatorname{support} a\} \neq \emptyset$. The theorem is a consequence of (24).

Let us consider R and b. The ordered partition of b yielding a (Bags(the carrier of R))-valued finite sequence is defined by

(Def. 12) there exist functions F, G from $\mathbb{N}$ into Bags(the carrier of R) such that $F(0) = b$ and $G(0) =$ EmptyBag(the carrier of R) and for every natural number i, $G(i+1) = F(i) \restriction \{x : x$ is maximal in $\operatorname{support}(F(i))\}$ and $F(i+1) = F(i) -' G(i+1)$ and there exists a natural number i such that $F(i) =$ EmptyBag(the carrier of R) and $it = G \restriction \operatorname{Seg} i$ and for every natural number j such that $j < i$ holds $F(j) \neq$ EmptyBag(the carrier of R).

One can verify that the ordered partition of b yields a partition of b. Let us note that the ordered partition of b is ordered as a partition of b.

Now we state the proposition:

(59) $b =$ EmptyBag(the carrier of R) if and only if the ordered partition of $b = \emptyset$. The theorem is a consequence of (32).

Let us consider R. The functor $\prec_{\mathcal{M}} R$ yielding a strict relational extension of (the carrier of $R)^{\otimes}$ is defined by

(Def. 13) for every elements m, n of it, $m \leqslant n$ iff $m \neq n$ and for every x such that $m(x) > 0$ holds $m(x) < n(x)$ or there exists y such that $n(y) > 0$ and $x \leqslant y$.

Let us note that $\prec_{\mathcal{M}} R$ is asymmetric and transitive.

Let us consider I. Let R be a relation between I and I.

The functor $\mathrm{LexOrder}(I, R)$ yielding a binary relation on I^* is defined by

(Def. 14) for every I-valued finite sequences p, q, $\langle p, q\rangle \in it$ iff $p \subset q$ or there exists a natural number k such that $k \in \mathrm{dom}\, p$ and $k \in \mathrm{dom}\, q$ and $\langle p(k), q(k)\rangle \in R$ and for every natural number n such that $1 \leqslant n < k$ holds $p(n) = q(n)$.

Let R be a transitive binary relation on I. One can verify that $\mathrm{LexOrder}(I, R)$ is transitive.

Let R be an asymmetric binary relation on I. Note that $\mathrm{LexOrder}(I, R)$ is asymmetric.

Now we state the proposition:

(60) Let us consider an asymmetric binary relation R on I, and I-valued finite sequences p, q, r. Then $\langle p, q\rangle \in \mathrm{LexOrder}(I, R)$ if and only if $\langle r ^\frown p, r ^\frown q\rangle \in \mathrm{LexOrder}(I, R)$. The theorem is a consequence of (10).

Let us consider R. The functor $\prec\!\prec_{\mathcal{M}} R$ yielding a strict relational extension of (the carrier of $R)^{\otimes}$ is defined by

(Def. 15) for every elements m, n of it, $m \leqslant n$ iff $\langle$the ordered partition of m, the ordered partition of $n\rangle \in \mathrm{LexOrder}(($the carrier of $\prec_{\mathcal{M}} R)$, (the internal relation of $\prec_{\mathcal{M}} R))$.

Observe that $\prec\!\prec_{\mathcal{M}} R$ is asymmetric and transitive.

Now we state the propositions:

(61) Let us consider elements a, b of the Dershowitz-Manna order R. Suppose $a \leqslant b$. Then $b \neq \mathrm{EmptyBag}($the carrier of $R)$. The theorem is a consequence of (29).

(62) Let us consider elements a, b, c, d of the Dershowitz-Manna order R, and a bag e of the carrier of R. Suppose $a \leqslant b$ and $e \mid a$ and $e \mid b$. If $c = a -' e$ and $d = b -' e$, then $c \leqslant d$.

(63) Let us consider a $(\mathrm{Bags}\, I)$-valued finite sequence p, and an object x. Suppose $x \in I$ and $(\sum p)(x) > 0$. Then there exists a natural number i such that

(i) $i \in \mathrm{dom}\, p$, and

(ii) $p_i(x) > 0$.

PROOF: Define $\mathcal{P}[\text{object}] \equiv$ for every $(\mathrm{Bags}\, I)$-valued finite sequence p such that $p = \$_1$ and $(\sum p)(x) > 0$ there exists a natural number i such that $i \in \mathrm{dom}\, p$ and $p_i(x) > 0$. $\mathcal{P}[\emptyset]$ by (32), [14, (7)]. For every finite sequence p and for every object a such that $\mathcal{P}[p]$ holds $\mathcal{P}[p ^\frown \langle a\rangle]$ by (7), [4, (40)], [15, (25)], [6, (102)]. For every finite sequence p, $\mathcal{P}[p]$ from [4, Sch. 3]. $\square$

(64) If q is ordered and $q_1(x) = 0$ and $b(x) > 0$, then there exists y such that $q_1(y) > 0$ and $x \leqslant y$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \in \operatorname{dom} q$, then for every x such that $q_{\$_1}(x) > 0$ there exists y such that $q_1(y) > 0$ and $x \leqslant y$. $\mathcal{P}[2]$ by [15, (25)]. For every natural number i such that $2 \leqslant i$ and $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [1, (11)], [15, (25)], [16, (3)]. For every natural number i such that $i \geqslant 2$ holds $\mathcal{P}[i]$ from [1, Sch. 8]. Consider i being a natural number such that $i \in \operatorname{dom} q$ and $q_i(x) > 0$. $\square$

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