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Chebyshev Distance

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Summary. In [21], Marco Riccardi formalized that $\mathbb{R}$N-basis n is a basis (in the algebraic sense defined in [26]) of $\mathcal{E}_T^n$ and in [20] he has formalized that $\mathcal{E}_T^n$ is second-countable, we build (in the topological sense defined in [23]) a denumerable base of $\mathcal{E}_T^n$.

Then we introduce the n-dimensional intervals (interval in n-dimensional Euclidean space, *pavé (borné) de* $\mathbb{R}^n$ [16], *semi-intervalle (borné) de* $\mathbb{R}^n$ [22]).

We conclude with the definition of Chebyshev distance [11].

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1. Preliminaries

From now on n denotes a natural number, r, s denote real numbers, x, y denote elements of $\mathcal{R}^n$, p, q denote points of $\mathcal{E}_{\mathrm{T}}^n$, and e denotes a point of $\mathcal{E}^n$.

Now we state the propositions:

(1) $|x - y| = |y - x|$.

(2) Let us consider a natural number i. If $i \in \mathrm{Seg}\, n$, then $|x|(i) \in \mathbb{R}$.

(3) Let us consider elements x, y of $\mathbb{R}$, and extended reals x_1, y_1. If $x \leqslant x_1$ and $y \leqslant y_1$, then $x + y \leqslant x_1 + y_1$.

(4) Let us consider real numbers a, c, and an extended real number b. Suppose $a < b$ and $[a, b[\subseteq [a, c[$. Then b is a real number.

(5) Let us consider a non empty set D, and a non empty subset D_1 of D. Then ${D_1}^n \subseteq D^n$.

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(6) Let us consider a non empty set X, and a function f. Suppose $f = \operatorname{Seg} n \longmapsto X$. Then f is a non-empty, n-element finite sequence.

Let n be a natural number. The functor $\mathbb{R}(n)$ yielding a non-empty, n-element finite sequence is defined by the term

(Def. 1) $\operatorname{Seg} n \longmapsto \mathbb{R}$.

Now we state the propositions:

(7) $\mathbb{R}(n) = \operatorname{Seg} n \longmapsto$ the carrier of $\mathbb{R}^{\mathbf{1}}$.

(8) $\prod(\operatorname{Seg} n \longmapsto \mathbb{R}) = \mathcal{R}^n$.

(9) $\prod \mathbb{R}(n) = \mathcal{R}^n$.

(10) Let us consider a set X. Then $\prod(\operatorname{Seg} n \longmapsto X) = X^n$.

(11) Let us consider a non empty set D, and an n-tuple x of D. Then $x \in D^{\operatorname{Seg} n}$.

(12) Let us consider a subset O_1 of $\mathcal{E}^n_{\mathrm{T}}$, and a subset O_2 of $(\mathcal{E}^n)_{\mathrm{top}}$. If $O_1 = O_2$, then O_1 is open iff O_2 is open.

(13) Suppose $e = p$. Then the set of all $\mathrm{OpenHypercube}(e, \frac{1}{m})$ where m is a non zero element of $\mathbb{N}$ = the set of all $\mathrm{OpenHypercube}(p, \frac{1}{m})$ where m is a non zero element of $\mathbb{N}$.

(14) If $q \in \mathrm{OpenHypercube}(p, r)$, then $p \in \mathrm{OpenHypercube}(q, r)$.

(15) If $q \in \mathrm{OpenHypercube}(p, \frac{r}{2})$, then $\mathrm{OpenHypercube}(q, \frac{r}{2}) \subseteq \mathrm{OpenHypercube}(p, r)$.

Let x be an element of $\mathbb{R} \times \mathbb{R}$. The functors: $(x)_{\mathbf{1}}$ and $(x)_{\mathbf{2}}$ yield elements of $\mathbb{R}$. Let n be a natural number and x be an element of $\mathcal{R}^n \times \mathcal{R}^n$. The functors: $(x)_{\mathbf{1}}$ and $(x)_{\mathbf{2}}$ yield elements of $\mathcal{R}^n$. Now we state the proposition:

(16) Let us consider an n-element finite sequence f of elements of $\mathbb{R} \times \mathbb{R}$. Then there exists an element x of $\mathcal{R}^n \times \mathcal{R}^n$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $(x)_{\mathbf{1}}(i) = (f_i)_{\mathbf{1}}$ and $(x)_{\mathbf{2}}(i) = (f_i)_{\mathbf{2}}$.

2. The Set of n-Tuples of Rational Numbers

Let us consider n. The functor $\mathcal{Q}^n$ yielding a set of finite sequences of $\mathbb{Q}$ is defined by the term

(Def. 2) $\mathbb{Q}^n$.

Now we state the proposition:

(17) $\mathcal{Q}^0 = \{0\}$.

One can check that $\mathcal{Q}^0$ is trivial.

Let us consider n. One can check that $\mathcal{Q}^n$ is non empty and every element of $\mathcal{Q}^n$ is n-element and $\mathcal{Q}^n$ is countable.

Let n be a positive natural number. Let us note that $\mathcal{Q}^n$ is infinite and $\mathcal{Q}^n$ is denumerable.

Now we state the proposition:

(18) $\mathcal{Q}^n$ is a dense subset of $\mathcal{E}^n_{\mathrm{T}}$.

PROOF: $\mathcal{Q}^n$ is a subset of $\mathcal{R}^n$. Reconsider $R = \mathcal{Q}^n$ as a subset of $\mathcal{E}^n_{\mathrm{T}}$. For every subset Q of $\mathcal{E}^n_{\mathrm{T}}$ such that $Q \neq \emptyset$ and Q is open holds R meets Q by [10, (67)], (12), [15, (23)], [13, (39)]. □

Let us consider n. One can check that $\mathcal{Q}^n$ is countable and dense as a subset of $\mathcal{E}^n_{\mathrm{T}}$.

3. A Countable Base of an n-Dimensional Euclidean Space

(VERSION 1: [20]):

Let n be a natural number. Let us observe that there exists a basis of $\mathcal{E}^n_{\mathrm{T}}$ which is countable.

Let us consider n and e. Note that OpenHypercubes e is countable.

The functor OpenHypercubes-$\mathbb{Q}(n)$ yielding a non empty set is defined by the term

(Def. 3) $\{\text{OpenHypercubes}\, q$, where q is a point of $\mathcal{E}^n : q \in \mathcal{Q}^n\}$.

Let q be an element of $\mathcal{Q}^n$. The functor ${}^{@}q$ yielding a point of $\mathcal{E}^n$ is defined by the term

(Def. 4) q.

Let q be an object. Assume $q \in \mathcal{Q}^n$. The functor object2$\mathbb{Q}(q, n)$ yielding an element of $\mathcal{Q}^n$ is defined by the term

(Def. 5) q.

Let us note that OpenHypercubes-$\mathbb{Q}(n)$ is countable and $\bigcup$ OpenHypercubes-$\mathbb{Q}(n)$ is countable.

Now we state the propositions:

(19) $\bigcup$ OpenHypercubes-$\mathbb{Q}(n)$ is an open family of subsets of $\mathcal{E}^n_{\mathrm{T}}$. The theorem is a consequence of (12).

(20) Let us consider a non empty, open subset O of $\mathcal{E}^n_{\mathrm{T}}$. Then there exists an element p of $\mathcal{Q}^n$ such that $p \in O$. The theorem is a consequence of (18).

(21) Let us consider a family $\mathcal{B}$ of subsets of $\mathcal{E}^n_{\mathrm{T}}$.
Suppose $\mathcal{B} = \bigcup$ OpenHypercubes-$\mathbb{Q}(n)$. Then $\mathcal{B}$ is quasi basis.
PROOF: F is quasi basis by (12), [15, (23)], [10, (67)], (20). □

Let us consider n. Observe that $\bigcup$ OpenHypercubes-$\mathbb{Q}(n)$ is non empty.

The functor OpenHypercubesℚUnion(n) yielding a countable, open family of subsets of $\mathcal{E}^n_{\mathrm{T}}$ is defined by the term

(Def. 6) $\bigcup$OpenHypercubes-ℚ(n).

Now we state the proposition:

(22) OpenHypercubesℚUnion(n) $=$ {OpenHypercube($q, \frac{1}{m}$), where q is a point of $\mathcal{E}^n$, m is a positive natural number : $q \in \mathcal{Q}^n$}.

(Version 2):

Let n be a natural number. Observe that there exists a basis of $\mathcal{E}^n_{\mathrm{T}}$ which is countable.

Now we state the propositions:

(23) OpenHypercubesℚUnion(n) is a countable basis of $\mathcal{E}^n_{\mathrm{T}}$.

(24) Let us consider an open subset O of $\mathcal{E}^n_{\mathrm{T}}$. Then there exists a subset Y of OpenHypercubesℚUnion(n) such that

(i) Y is countable, and

(ii) $O = \bigcup Y$.

The theorem is a consequence of (21).

Let us consider an open, non empty subset O of $\mathcal{E}^n_{\mathrm{T}}$. Now we state the propositions:

(25) There exists a subset Y of OpenHypercubesℚUnion(n) such that

(i) Y is not empty, and

(ii) $O = \bigcup Y$, and

(iii) there exists a function g from $\mathbb{N}$ into Y such that for every object x, $x \in O$ iff there exists an object y such that $y \in \mathbb{N}$ and $x \in g(y)$.

The theorem is a consequence of (24).

(26) There exists a sequence s of OpenHypercubesℚUnion(n) such that for every object x, $x \in O$ iff there exists an object y such that $y \in \mathbb{N}$ and $x \in s(y)$. The theorem is a consequence of (25).

(27) There exists a sequence s of OpenHypercubesℚUnion(n) such that $O = \bigcup s$. The theorem is a consequence of (26).

4. The Set of All Left Open Real Bounded Intervals

The set of all left open real bounded intervals yielding a family of subsets of $\mathbb{R}$ is defined by the term

(Def. 7) the set of all $]a, b]$ where a, b are real numbers.

Let us note that the set of all left open real bounded intervals is non empty.

Now we state the propositions:

(28) The set of all left open real bounded intervals $\subseteq \{I$, where I is a subset of $\mathbb{R}$: I is left open interval$\}$.

(29) The set of all left open real bounded intervals is $\cap$-closed and $\setminus_{fp}$-closed and has the empty element and countable cover.

(30) The set of all left open real bounded intervals is a semiring of $\mathbb{R}$.

5. The Set of All Right Open Real Bounded Intervals

The set of all right open real bounded intervals yielding a family of subsets of $\mathbb{R}$ is defined by the term

(Def. 8) the set of all $[a, b[$ where a, b are real numbers.

Observe that the set of all right open real bounded intervals is non empty.

Now we state the propositions:

(31) The set of all right open real bounded intervals $\subseteq \{I$, where I is a subset of $\mathbb{R}$: I is right open interval$\}$.

(32) The set of all right open real bounded intervals has the empty element.

(33) (i) the set of all right open real bounded intervals is $\cap$-closed, and

(ii) the set of all right open real bounded intervals is $\setminus_{fp}$-closed and has the empty element.

The theorem is a consequence of (31), (32), and (4).

(34) The set of all right open real bounded intervals has countable cover.

Proof: Define $\mathcal{F}[\text{object}, \text{object}] \equiv \$_1$ is an element of $\mathbb{N}$ and $\$_2 \in$ the set of all right open real bounded intervals and there exists a real number x such that $x = \$_1$ and $\$_2 = [-x, x[$. For every object x such that $x \in \mathbb{N}$ there exists an object y such that $y \in$ the set of all right open real bounded intervals and $\mathcal{F}[x, y]$. Consider f being a function such that $\operatorname{dom} f = \mathbb{N}$ and $\operatorname{rng} f \subseteq$ the set of all right open real bounded intervals and for every object x such that $x \in \mathbb{N}$ holds $\mathcal{F}[x, f(x)]$ from [7, Sch. 6]. $\operatorname{rng} f$ is countable by [27, (4)], [14, (58)]. $\operatorname{rng} f$ is a cover of $\mathbb{R}$ by [2, (2)], [12, (8)], [3, (13)], [17, (45)]. $\square$

(35) The set of all right open real bounded intervals is a semiring of $\mathbb{R}$.

6. Finite Product of Left Open Intervals

In the sequel n denotes a non zero natural number.

Let n be a non zero natural number. The functor LeftOpenIntervals(n) yielding a classical semiring family of $\mathbb{R}(n)$ is defined by the term

(Def. 9) $\mathrm{Seg}\, n \longmapsto$ (the set of all left open real bounded intervals).

Now we state the propositions:

(36) LeftOpenIntervals(n) $= \mathrm{Seg}\, n \longmapsto$ the set of all $]a, b]$ where a, b are real numbers.

(37) MeasurableRectangle LeftOpenIntervals(n) is a semiring of $\mathcal{R}^n$. The theorem is a consequence of (8).

Let us consider an object x.

Let us assume that $x \in$ MeasurableRectangle LeftOpenIntervals(n). Now we state the propositions:

(38) There exists a subset y of $\mathcal{R}^n$ such that

(i) $x = y$, and

(ii) for every natural number i such that $i \in \mathrm{Seg}\, n$ there exist real numbers a, b such that for every element t of $\mathcal{R}^n$ such that $t \in y$ holds $t(i) \in]a, b]$.

The theorem is a consequence of (37).

(39) There exists a subset y of $\mathcal{R}^n$ and there exists an n-element finite sequence f of elements of $\mathbb{R} \times \mathbb{R}$ such that $x = y$ and for every element t of $\mathcal{R}^n$, $t \in y$ iff for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $t(i) \in](f_i)_{\mathbf{1}}, (f_i)_{\mathbf{2}}]$.

Proof: MeasurableRectangle LeftOpenIntervals(n) is a family of subsets of $\mathcal{R}^n$. Reconsider $y = x$ as a subset of $\mathcal{R}^n$. Consider g being a function such that $x = \prod g$ and $g \in \prod$ LeftOpenIntervals(n). Define $\mathcal{P}$[natural number, set] $\equiv$ there exists an element x of $\mathbb{R} \times \mathbb{R}$ such that $\$_2 = x$ and $g(\$_1) =](x)_{\mathbf{1}}, (x)_{\mathbf{2}}]$. For every natural number i such that $i \in \mathrm{Seg}\, n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$. There exists a finite sequence f_1 of elements of $\mathbb{R} \times \mathbb{R}$ such that $\mathrm{len}\, f_1 = n$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $\mathcal{P}[i, {f_1}_i]$ from [25, Sch. 1]. Consider f_1 being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that $\mathrm{len}\, f_1 = n$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ there exists an element x of $\mathbb{R} \times \mathbb{R}$ such that ${f_1}_i = x$ and $g(i) =](x)_{\mathbf{1}}, (x)_{\mathbf{2}}]$. For every natural number i such that $i \in \mathrm{Seg}\, n$ holds $g(i) =]({f_1}_i)_{\mathbf{1}}, ({f_1}_i)_{\mathbf{2}}]$. For every element t of $\mathcal{R}^n$ such that $t \in y$ for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $t(i) \in]({f_1}_i)_{\mathbf{1}}, ({f_1}_i)_{\mathbf{2}}]$. For every element t of $\mathcal{R}^n$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in](f1_i)_{\mathbf{1}}, (f1_i)_{\mathbf{2}}]$ holds $t \in y$ by [6, (93)]. □

(40) There exists a subset y of $\mathcal{R}^n$ and there exist elements a, b of $\mathcal{R}^n$ such that $x = y$ and for every object s, $s \in y$ iff there exists an element t of $\mathcal{R}^n$ such that $s = t$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in]a(i), b(i)]$. The theorem is a consequence of (39) and (16).

Now we state the proposition:

(41) Let us consider a set x. Suppose $x \in$ MeasurableRectangle LeftOpenIntervals(n). Then there exist elements a, b of $\mathcal{R}^n$ such that for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in]a(i), b(i)]$. The theorem is a consequence of (39) and (16).

7. Finite Product of Right Open Intervals

Let n be a non zero natural number. The functor RightOpenIntervals(n) yielding a classical semiring family of $\mathbb{R}(n)$ is defined by the term

(Def. 10) $\operatorname{Seg} n \longmapsto$ (the set of all right open real bounded intervals).

From now on n denotes a non zero natural number.

Now we state the propositions:

(42) RightOpenIntervals(n) $=$ $\operatorname{Seg} n \longmapsto$ the set of all $[a, b[$ where a, b are real numbers.

(43) MeasurableRectangle RightOpenIntervals(n) is a semiring of $\mathcal{R}^n$. The theorem is a consequence of (8).

Let us consider an object x.

Let us assume that $x \in$ MeasurableRectangle RightOpenIntervals(n). Now we state the propositions:

(44) There exists a subset y of $\mathcal{R}^n$ such that

(i) $x = y$, and

(ii) for every natural number i such that $i \in \operatorname{Seg} n$ there exist real numbers a, b such that for every element t of $\mathcal{R}^n$ such that $t \in y$ holds $t(i) \in [a, b[$.

The theorem is a consequence of (43).

(45) There exists a subset y of $\mathcal{R}^n$ and there exists an n-element finite sequence f of elements of $\mathbb{R} \times \mathbb{R}$ such that $x = y$ and for every element t of $\mathcal{R}^n$, $t \in y$ iff for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in [(f_i)_{\mathbf{1}}, (f_i)_{\mathbf{2}}[$.

PROOF: MeasurableRectangle RightOpenIntervals(n) is a family of subsets of $\mathcal{R}^n$. Reconsider $y = x$ as a subset of $\mathcal{R}^n$. Consider g being a function such that $x = \prod g$ and $g \in \prod \mathrm{RightOpenIntervals}(n)$. Define $\mathcal{P}[\text{natural number}, \text{set}] \equiv$ there exists an element x of $\mathbb{R} \times \mathbb{R}$ such that $\$_2 = x$ and $g(\$_1) = [(x)_{\mathbf{1}}, (x)_{\mathbf{2}}[$. For every natural number i such that $i \in \mathrm{Seg}\, n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$. There exists a finite sequence f_1 of elements of $\mathbb{R} \times \mathbb{R}$ such that $\mathrm{len}\, f_1 = n$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $\mathcal{P}[i, f_{1i}]$ from [25, Sch. 1]. Consider f_1 being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that $\mathrm{len}\, f_1 = n$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ there exists an element x of $\mathbb{R} \times \mathbb{R}$ such that $f_{1i} = x$ and $g(i) = [(x)_{\mathbf{1}}, (x)_{\mathbf{2}}[$. For every natural number i such that $i \in \mathrm{Seg}\, n$ holds $g(i) = [(f_{1i})_{\mathbf{1}}, (f_{1i})_{\mathbf{2}}[$. For every element t of $\mathcal{R}^n$ such that $t \in y$ for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $t(i) \in [(f_{1i})_{\mathbf{1}}, (f_{1i})_{\mathbf{2}}[$. For every element t of $\mathcal{R}^n$ such that for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $t(i) \in [(f_{1i})_{\mathbf{1}}, (f_{1i})_{\mathbf{2}}[$ holds $t \in y$ by [6, (93)]. $\square$

(46) There exists a subset y of $\mathcal{R}^n$ and there exist elements a, b of $\mathcal{R}^n$ such that $x = y$ and for every object s, $s \in y$ iff there exists an element t of $\mathcal{R}^n$ such that $s = t$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $t(i) \in [a(i), b(i)[$. The theorem is a consequence of (45) and (16).

Now we state the proposition:

(47) Let us consider a set x. Suppose $x \in \mathrm{MeasurableRectangle}\, \mathrm{RightOpenIntervals}(n)$. Then there exist elements a, b of $\mathcal{R}^n$ such that for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $t(i) \in [a(i), b(i)[$. The theorem is a consequence of (45) and (16).

8. n-Dimensional Product of Subset Family

In the sequel n denotes a natural number, X denotes a set, and S denotes a family of subsets of X.

Let us consider n and X. The functor $\mathrm{Product}(n, X)$ yielding a set is defined by

(Def. 11) for every object f, $f \in it$ iff there exists a function g such that $f = \prod g$ and $g \in \prod(\mathrm{Seg}\, n \longmapsto X)$.

Now we state the propositions:

(48) $\mathrm{Product}(n, X) \subseteq 2^{(\bigcup \bigcup (\mathrm{Seg}\, n \longmapsto X))^{\mathrm{dom}(\mathrm{Seg}\, n \longmapsto X)}}$.

(49) $\mathrm{Product}(n, S)$ is a family of subsets of $\prod(\mathrm{Seg}\, n \longmapsto X)$.

Proof: Reconsider $S_1 = \mathrm{Product}(n, S)$ as a subset of $2^{(\bigcup \bigcup (\mathrm{Seg}\, n \longmapsto S))^{\mathrm{dom}(\mathrm{Seg}\, n \longmapsto S)}}$. $S_1 \subseteq 2^{\prod(\mathrm{Seg}\, n \longmapsto X)}$ by [1, (9)], [24, (13), (7)], [9, (77), (81)]. $\square$

(50) Let us consider a non empty family S of subsets of X. Then $\mathrm{Product}(n, S) =$ the set of all $\prod f$ where f is an n-tuple of S.
PROOF: $\mathrm{Product}(n, S) \subseteq$ the set of all $\prod f$ where f is an n-tuple of S by (10), [6, (131)]. the set of all $\prod f$ where f is an n-tuple of $S \subseteq \mathrm{Product}(n, S)$ by [6, (131)], (10). $\square$

(51) Let us consider a non zero natural number n. Then $\mathrm{Product}(n, X) \subseteq 2^{(\bigcup X)^{\mathrm{Seg}\, n}}$.

Let us consider a non zero natural number n, a non empty set X, and a non empty family S of subsets of X.

Let us assume that $S \neq \{\emptyset\}$. Now we state the propositions:

(52) $\mathrm{Product}(n, S) \subseteq 2^{X^{\mathrm{Seg}\, n}}$. The theorem is a consequence of (51) and (5).

(53) $\bigcup \mathrm{Product}(n, S) \subseteq X^{\mathrm{Seg}\, n}$. The theorem is a consequence of (52).

Let n be a natural number and X be a non empty set. Let us note that $\mathrm{Product}(n, X)$ is non empty.

Now we state the proposition:

(54) Let us consider a non empty set X, a non empty family S of subsets of X, and a subset x of $X^{\mathrm{Seg}\, n}$. Then x is an element of $\mathrm{Product}(n, S)$ if and only if there exists an n-tuple s of S such that for every element t of $X^{\mathrm{Seg}\, n}$, for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $t(i) \in s(i)$ iff $t \in x$.

9. The Set of All Closed Real Bounded Intervals

The set of all closed real bounded intervals yielding a family of subsets of $\mathbb{R}$ is defined by the term

(Def. 12) the set of all $[a, b]$ where a, b are real numbers.

Now we state the proposition:

(55) The set of all closed real bounded intervals $= \{I$, where I is a subset of $\mathbb{R} : I$ is closed interval$\}$.

Let us note that the set of all closed real bounded intervals is non empty.

Now we state the propositions:

(56) The set of all closed real bounded intervals is $\cap$-closed and has the empty element and countable cover.
PROOF: The set of all closed real bounded intervals is $\cap$-closed. There exists a countable subset X of the set of all closed real bounded intervals such that X is a cover of $\mathbb{R}$ by [27, (4)], [14, (58)], [2, (2)], [12, (8)]. $\square$

(57) Let us consider a natural number n. Then $\mathrm{Seg}\, n \longmapsto$ (the set of all closed real bounded intervals) is an n-element finite sequence.

10. The Set of All Open Real Bounded Intervals

The set of all open real bounded intervals yielding a family of subsets of $\mathbb{R}$ is defined by the term

(Def. 13) the set of all $]a, b[$ where a, b are real numbers.

Now we state the proposition:

(58) The set of all open real bounded intervals $\subseteq \{I$, where I is a subset of $\mathbb{R} : I$ is open interval$\}$.

Let us observe that the set of all open real bounded intervals is non empty.
Now we state the propositions:

(59) The set of all open real bounded intervals is $\cap$-closed and has the empty element and countable cover.
PROOF: The set of all open real bounded intervals is $\cap$-closed. There exists a countable subset X of the set of all open real bounded intervals such that X is a cover of $\mathbb{R}$ by [27, (4)], [14, (58)], [2, (2)], [12, (8)]. $\square$

(60) Let us consider a natural number n. Then $\operatorname{Seg} n \longmapsto$ (the set of all open real bounded intervals) is an n-element finite sequence.

11. n-Dimensional Subset Family of $\mathbb{R}$

From now on n denotes a natural number and S denotes a family of subsets of $\mathbb{R}$.

Now we state the proposition:

(61) $\operatorname{Product}(n, S)$ is a family of subsets of $\mathcal{R}^n$. The theorem is a consequence of (49) and (8).

Let us consider n and S. One can check that the functor $\operatorname{Product}(n, S)$ yields a family of subsets of $\mathcal{R}^n$. Now we state the propositions:

(62) Let us consider a non empty family S of subsets of $\mathbb{R}$, and a subset x of $\mathcal{R}^n$. Then x is an element of $\operatorname{Product}(n, S)$ if and only if there exists an n-tuple s of S such that for every element t of $\mathcal{R}^n$, for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in s(i)$ iff $t \in x$.
PROOF: If x is an element of $\operatorname{Product}(n, S)$, then there exists an n-tuple s of S such that for every element t of $\mathcal{R}^n$, for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in s(i)$ iff $t \in x$ by [6, (93)]. If there exists an n-tuple s of S such that for every element t of $\mathcal{R}^n$, for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in s(i)$ iff $t \in x$, then x is an element of $\operatorname{Product}(n, S)$ by [6, (93)]. $\square$

(63) Let us consider a non zero natural number n, and an n-tuple s of the set of all closed real bounded intervals. Then there exist elements a, b of $\mathcal{R}^n$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $s(i) = [a(i), b(i)]$.

PROOF: $s \in$ (the set of all closed real bounded intervals)$^{\operatorname{Seg} n}$. Consider f being a function such that $s = f$ and $\operatorname{dom} f = \operatorname{Seg} n$ and $\operatorname{rng} f \subseteq$ the set of all closed real bounded intervals. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an element f of $\mathbb{R} \times \mathbb{R}$ such that $f = \$_2$ and $s(\$_1) = [(f)_{\mathbf{1}}, (f)_{\mathbf{2}}]$. For every natural number i such that $i \in \operatorname{Seg} n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$ by [7, (3)]. Consider f being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that $\operatorname{len} f = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, f_i]$ from [25, Sch. 1]. Consider z being an element of $\mathcal{R}^n \times \mathcal{R}^n$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $(z)_{\mathbf{1}}(i) = (f_i)_{\mathbf{1}}$ and $(z)_{\mathbf{2}}(i) = (f_i)_{\mathbf{2}}$. Reconsider $a = (z)_{\mathbf{1}}$, $b = (z)_{\mathbf{2}}$ as an element of $\mathcal{R}^n$. For every natural number i such that $i \in \operatorname{Seg} n$ holds $s(i) = [a(i), b(i)]$. $\square$

(64) Let us consider a non zero natural number n, and an element x of $\operatorname{Product}(n, \text{the set of all closed real bounded intervals})$. Then there exist elements a, b of $\mathcal{R}^n$ such that for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in [a(i), b(i)]$. The theorem is a consequence of (62) and (63).

Let us consider a non zero natural number n, a subset x of $\mathcal{R}^n$, and elements a, b of $\mathcal{R}^n$. Now we state the propositions:

(65) Suppose for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in [a(i), b(i)]$. Then x is an element of $\operatorname{Product}(n, \text{the set of all closed real bounded intervals})$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a natural number n such that $\$_1 = n$ and $\$_2 = [a(n), b(n)]$. For every natural number i such that $i \in \operatorname{Seg} n$ there exists an element d of the set of all closed real bounded intervals such that $\mathcal{P}[i, d]$. There exists a finite sequence g of elements of the set of all closed real bounded intervals such that $\operatorname{len} g = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, g_i]$ from [25, Sch. 1]. Consider g being a finite sequence of elements of the set of all closed real bounded intervals such that $\operatorname{len} g = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, g_i]$. For every natural number i such that $i \in \operatorname{Seg} n$ holds $g(i) = [a(i), b(i)]$. There exists a function g such that $x = \prod g$ and $g \in \prod(\operatorname{Seg} n \longmapsto (\text{the set of all closed real bounded intervals}))$ by [4, (89)], [24, (13), (7)], [1, (9)]. $\square$

(66) Suppose for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number

i such that $i \in \operatorname{Seg} n$ holds $t(i) \in]a(i), b(i)]$. Then x is an element of Product(n, the set of all left open real bounded intervals).

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a natural number n such that $\$_1 = n$ and $\$_2 =]a(n), b(n)]$. For every natural number i such that $i \in \operatorname{Seg} n$ there exists an element d of the set of all left open real bounded intervals such that $\mathcal{P}[i, d]$. There exists a finite sequence g of elements of the set of all left open real bounded intervals such that $\operatorname{len} g = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, g_i]$ from [25, Sch. 1]. Consider g being a finite sequence of elements of the set of all left open real bounded intervals such that $\operatorname{len} g = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, g_i]$. For every natural number i such that $i \in \operatorname{Seg} n$ holds $g(i) =]a(i), b(i)]$. There exists a function g such that $x = \prod g$ and $g \in \prod(\operatorname{Seg} n \longmapsto$ (the set of all left open real bounded intervals)) by [4, (89)], [24, (13), (7)], [1, (9)]. □

(67) Suppose for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in [a(i), b(i)[$. Then x is an element of Product(n, the set of all right open real bounded intervals).

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a natural number n such that $\$_1 = n$ and $\$_2 = [a(n), b(n)[$. For every natural number i such that $i \in \operatorname{Seg} n$ there exists an element d of the set of all right open real bounded intervals such that $\mathcal{P}[i, d]$. There exists a finite sequence g of elements of the set of all right open real bounded intervals such that $\operatorname{len} g = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, g_i]$ from [25, Sch. 1]. Consider g being a finite sequence of elements of the set of all right open real bounded intervals such that $\operatorname{len} g = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, g_i]$. For every natural number i such that $i \in \operatorname{Seg} n$ holds $g(i) = [a(i), b(i)[$. There exists a function g such that $x = \prod g$ and $g \in \prod(\operatorname{Seg} n \longmapsto$ (the set of all right open real bounded intervals)) by [4, (89)], [24, (13), (7)], [1, (9)]. □

Now we state the propositions:

(68) Let us consider a non zero natural number n, and an n-tuple s of the set of all left open real bounded intervals. Then there exist elements a, b of $\mathcal{R}^n$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $s(i) =]a(i), b(i)]$.

PROOF: $s \in$ (the set of all left open real bounded intervals)$^{\operatorname{Seg} n}$. Consider f being a function such that $s = f$ and $\operatorname{dom} f = \operatorname{Seg} n$ and $\operatorname{rng} f \subseteq$ the set of all left open real bounded intervals. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an element f of $\mathbb{R} \times \mathbb{R}$ such that $f = \$_2$ and $s(\$_1) =](f)_{\mathbf{1}}, (f)_{\mathbf{2}}]$. For every natural number i such that $i \in \operatorname{Seg} n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$ by [7, (3)]. Consider f being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that $\operatorname{len} f = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, f_i]$ from [25, Sch. 1]. Consider z being an element of $\mathcal{R}^n \times \mathcal{R}^n$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $(z)_{\mathbf{1}}(i) = (f_i)_{\mathbf{1}}$ and $(z)_{\mathbf{2}}(i) = (f_i)_{\mathbf{2}}$. Reconsider $a = (z)_{\mathbf{1}}$, $b = (z)_{\mathbf{2}}$ as an element of $\mathcal{R}^n$. For every natural number i such that $i \in \operatorname{Seg} n$ holds $s(i) =]a(i), b(i)]$. $\square$

(69) Let us consider a non zero natural number n, and an element x of $\operatorname{Product}(n,$ the set of all left open real bounded intervals$)$. Then there exist elements a, b of $\mathcal{R}^n$ such that for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in]a(i), b(i)]$. The theorem is a consequence of (62) and (68).

(70) Let us consider a non zero natural number n, and an n-tuple s of the set of all right open real bounded intervals. Then there exist elements a, b of $\mathcal{R}^n$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $s(i) = [a(i), b(i)[$.

PROOF: $s \in$ (the set of all right open real bounded intervals)$^{\operatorname{Seg} n}$. Consider f being a function such that $s = f$ and $\operatorname{dom} f = \operatorname{Seg} n$ and $\operatorname{rng} f \subseteq$ the set of all right open real bounded intervals. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an element f of $\mathbb{R} \times \mathbb{R}$ such that $f = \$_2$ and $s(\$_1) = [(f)_{\mathbf{1}}, (f)_{\mathbf{2}}[$. For every natural number i such that $i \in \operatorname{Seg} n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$ by [7, (3)]. Consider f being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that $\operatorname{len} f = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, f_i]$ from [25, Sch. 1]. Consider z being an element of $\mathcal{R}^n \times \mathcal{R}^n$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $(z)_{\mathbf{1}}(i) = (f_i)_{\mathbf{1}}$ and $(z)_{\mathbf{2}}(i) = (f_i)_{\mathbf{2}}$. Reconsider $a = (z)_{\mathbf{1}}$, $b = (z)_{\mathbf{2}}$ as an element of $\mathcal{R}^n$. For every natural number i such that $i \in \operatorname{Seg} n$ holds $s(i) = [a(i), b(i)[$. $\square$

(71) Let us consider a non zero natural number n, and an element x of $\operatorname{Product}(n,$ the set of all right open real bounded intervals$)$. Then there exist elements a, b of $\mathcal{R}^n$ such that for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in [a(i), b(i)[$. The theorem is a consequence of (62) and (70).

12. Closed/Open/Left-Open/Right-Open – Hyper Interval

From now on n denotes a natural number and a, b, c, d denote elements of $\mathcal{R}^n$.

Let us consider n, a, and b. The functor ClosedHyperInterval(a, b) yielding a subset of $\mathcal{R}^n$ is defined by

(Def. 14) for every object x, $x \in it$ iff there exists an element y of $\mathcal{R}^n$ such that $x = y$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $y(i) \in [a(i), b(i)]$.

The functor OpenHyperInterval(a, b) yielding a subset of $\mathcal{R}^n$ is defined by

(Def. 15) for every object x, $x \in it$ iff there exists an element y of $\mathcal{R}^n$ such that $x = y$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $y(i) \in]a(i), b(i)[$.

The functor LeftOpenHyperInterval(a, b) yielding a subset of $\mathcal{R}^n$ is defined by

(Def. 16) for every object x, $x \in it$ iff there exists an element y of $\mathcal{R}^n$ such that $x = y$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $y(i) \in]a(i), b(i)]$.

The functor RightOpenHyperInterval(a, b) yielding a subset of $\mathcal{R}^n$ is defined by

(Def. 17) for every object x, $x \in it$ iff there exists an element y of $\mathcal{R}^n$ such that $x = y$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $y(i) \in [a(i), b(i)[$.

Now we state the proposition:

(72) ClosedHyperInterval$(a, a) = \{a\}$.
PROOF: ClosedHyperInterval$(a, a) \subseteq \{a\}$ by [6, (124)].
$\{a\} \subseteq$ ClosedHyperInterval(a, a). □

Let us consider n and a. Let us observe that ClosedHyperInterval(a, a) is trivial.

Now we state the proposition:

(73) (i) OpenHyperInterval$(a, b) \subseteq$ LeftOpenHyperInterval(a, b), and

(ii) OpenHyperInterval$(a, b) \subseteq$ RightOpenHyperInterval(a, b), and

(iii) LeftOpenHyperInterval$(a, b) \subseteq$ ClosedHyperInterval(a, b), and

(iv) RightOpenHyperInterval$(a, b) \subseteq$ ClosedHyperInterval(a, b).

Let us consider n, a, and b. We say that $a \leqslant b$ if and only if

(Def. 18) for every natural number i such that $i \in \operatorname{Seg} n$ holds $a(i) \leqslant b(i)$.

One can verify that the predicate is reflexive.

Now we state the propositions:

(74) If $a \leqslant b \leqslant c$, then $a \leqslant c$.

(75) If $a \leqslant c$ and $d \leqslant b$, then ClosedHyperInterval$(c, d) \subseteq$ ClosedHyperInterval(a, b).

(76) If $a \leqslant b$, then ClosedHyperInterval(a, b) is not empty. The theorem is a consequence of (75) and (72).

Let us consider n, a, and b. We say that $a < b$ if and only if

(Def. 19) for every natural number i such that $i \in \operatorname{Seg} n$ holds $a(i) < b(i)$.

Now we state the propositions:

(77) If $a < b < c$, then $a < c$.

(78) If $b < a$ and n is not zero, then ClosedHyperInterval(a, b) is empty.

(79) $n \mapsto r$ is an element of $\mathcal{R}^n$.

Proof: Set $f = n \mapsto r$. $f \in \mathbb{R}^{\operatorname{Seg} n}$ by [6, (112), (93)]. □

Let us consider n and r. Note that the functor $n \mapsto r$ yields an element of $\mathcal{R}^n$. One can check that the functor $\langle r \rangle$ yields an element of $\mathcal{R}^1$. Now we state the propositions:

(80) Let us consider a non zero natural number n, and a point e of $\mathcal{E}^n$. Then there exists an element a of $\mathcal{R}^n$ such that

(i) $a = e$, and

(ii) OpenHypercube(e, r) = OpenHyperInterval$(a - n \mapsto r, a + n \mapsto r)$.

Proof: Reconsider $a = e$ as an element of $\mathcal{R}^n$. Reconsider $p = e$ as a point of $\mathcal{E}^n_{\mathrm{T}}$. Consider e_0 being a point of $\mathcal{E}^n$ such that $p = e_0$ and OpenHypercube(e_0, r) = OpenHypercube(p, r). OpenHypercube$(e, r) \subseteq$ OpenHyperInterval$(a - n \mapsto r, a + n \mapsto r)$ by [8, (27)], [6, (57)], [8, (11)], [18, (4)]. OpenHyperInterval$(a-n \mapsto r, a+n \mapsto r) \subseteq$ OpenHypercube(e, r) by [10, (22)], [8, (27)], [6, (57)], [8, (11)]. □

(81) Let us consider a point p of $\mathcal{E}^n_{\mathrm{T}}$. Then there exists an element a of $\mathcal{R}^n$ such that

(i) $a = p$, and

(ii) ClosedHypercube(p, b) = ClosedHyperInterval$(a - b, a + b)$.

Proof: Reconsider $a = p$ as an element of $\mathcal{R}^n$. ClosedHypercube$(p, b) \subseteq$ ClosedHyperInterval$(a-b, a+b)$ by [10, (22)], [8, (11), (27)]. ClosedHyperInterval$(a - b, a + b) \subseteq$ ClosedHypercube(p, b) by [10, (22)], [8, (11), (27)]. □

13. Correspondance between Interval and 1-Dimensional Hyper Interval

Let us consider a real number x. Now we state the propositions:

(82) $x \in [r, s]$ if and only if $1 \mapsto x \in$ ClosedHyperInterval$(\langle r \rangle, \langle s \rangle)$.

Proof: Set $a_1 = \langle r \rangle$. Set $b_1 = \langle s \rangle$. For every real number x such that $x \in [r, s]$ holds $1 \mapsto x \in$ ClosedHyperInterval(a_1, b_1) by [4, (2)], [24, (7)]. For every real number x such that $1 \mapsto x \in$ ClosedHyperInterval(a_1, b_1) holds $x \in [r, s]$ by [24, (7)]. □

(83) $x \in]r, s[$ if and only if $1 \mapsto x \in \mathrm{OpenHyperInterval}(\langle r\rangle, \langle s\rangle)$.
PROOF: Set $a_1 = \langle r\rangle$. Set $b_1 = \langle s\rangle$. For every real number x such that $x \in]r, s[$ holds $1 \mapsto x \in \mathrm{OpenHyperInterval}(a_1, b_1)$ by [4, (2)], [24, (7)]. For every real number x such that $1 \mapsto x \in \mathrm{OpenHyperInterval}(a_1, b_1)$ holds $x \in]r, s[$ by [24, (7)]. □

(84) $x \in]r, s]$ if and only if $1 \mapsto x \in \mathrm{LeftOpenHyperInterval}(\langle r\rangle, \langle s\rangle)$.
PROOF: Set $a_1 = \langle r\rangle$. Set $b_1 = \langle s\rangle$. For every real number x such that $x \in]r, s]$ holds $1 \mapsto x \in \mathrm{LeftOpenHyperInterval}(a_1, b_1)$ by [4, (2)], [24, (7)]. For every real number x such that $1 \mapsto x \in \mathrm{LeftOpenHyperInterval}(a_1, b_1)$ holds $x \in]r, s]$ by [24, (7)]. □

(85) $x \in [r, s[$ if and only if $1 \mapsto x \in \mathrm{RightOpenHyperInterval}(\langle r\rangle, \langle s\rangle)$.
PROOF: Set $a_1 = \langle r\rangle$. Set $b_1 = \langle s\rangle$. For every real number x such that $x \in [r, s[$ holds $1 \mapsto x \in \mathrm{RightOpenHyperInterval}(a_1, b_1)$ by [4, (2)], [24, (7)]. For every real number x such that $1 \mapsto x \in \mathrm{RightOpenHyperInterval}(a_1, b_1)$ holds $x \in [r, s[$ by [24, (7)]. □

14. Correspondance between Measurable Rectangle and Product

From now on n denotes a non zero natural number.

Now we state the propositions:

(86) Let us consider an n-tuple s of the set of all open real bounded intervals. Then there exist elements a, b of $\mathcal{R}^n$ such that for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $s(i) =]a(i), b(i)[$.
PROOF: $s \in (\text{the set of all open real bounded intervals})^{\mathrm{Seg}\, n}$. Consider f being a function such that $s = f$ and $\mathrm{dom}\, f = \mathrm{Seg}\, n$ and $\mathrm{rng}\, f \subseteq$ the set of all open real bounded intervals. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an element f of $\mathbb{R} \times \mathbb{R}$ such that $f = \$_2$ and $s(\$_1) =](f)_{\mathbf{1}}, (f)_{\mathbf{2}}[$. For every natural number i such that $i \in \mathrm{Seg}\, n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$ by [7, (3)]. Consider f being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that $\mathrm{len}\, f = n$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $\mathcal{P}[i, f_i]$ from [25, Sch. 1]. Consider z being an element of $\mathcal{R}^n \times \mathcal{R}^n$ such that for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $(z)_{\mathbf{1}}(i) = (f_i)_{\mathbf{1}}$ and $(z)_{\mathbf{2}}(i) = (f_i)_{\mathbf{2}}$. Reconsider $a = (z)_{\mathbf{1}}$, $b = (z)_{\mathbf{2}}$ as an element of $\mathcal{R}^n$. For every natural number i such that $i \in \mathrm{Seg}\, n$ holds $s(i) =]a(i), b(i)[$. □

(87) Let us consider an element x of $\mathrm{Product}(n, \text{the set of all open real bounded intervals})$. Then there exist elements a, b of $\mathcal{R}^n$ such that for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number i such that

$i \in \operatorname{Seg} n$ holds $t(i) \in]a(i), b(i)[$. The theorem is a consequence of (62) and (86).

(88) Let us consider an n-tuple s of the set of all left open real bounded intervals. Then there exist elements a, b of $\mathcal{R}^n$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $s(i) =]a(i), b(i)]$.

Proof: $s \in (\text{the set of all left open real bounded intervals})^{\operatorname{Seg} n}$. Consider f being a function such that $s = f$ and $\operatorname{dom} f = \operatorname{Seg} n$ and $\operatorname{rng} f \subseteq$ the set of all left open real bounded intervals. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an element f of $\mathbb{R} \times \mathbb{R}$ such that $f = \$_2$ and $s(\$_1) =](f)_{\mathbf{1}}, (f)_{\mathbf{2}}]$. For every natural number i such that $i \in \operatorname{Seg} n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$ by [7, (3)]. Consider f being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that $\operatorname{len} f = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, f_i]$ from [25, Sch. 1]. Consider z being an element of $\mathcal{R}^n \times \mathcal{R}^n$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $(z)_{\mathbf{1}}(i) = (f_i)_{\mathbf{1}}$ and $(z)_{\mathbf{2}}(i) = (f_i)_{\mathbf{2}}$. Reconsider $a = (z)_{\mathbf{1}}$, $b = (z)_{\mathbf{2}}$ as an element of $\mathcal{R}^n$. For every natural number i such that $i \in \operatorname{Seg} n$ holds $s(i) =]a(i), b(i)]$. $\square$

(89) Let us consider an element x of $\operatorname{Product}(n, \text{the set of all left open real bounded intervals})$. Then there exist elements a, b of $\mathcal{R}^n$ such that for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in]a(i), b(i)]$. The theorem is a consequence of (62) and (88).

(90) Let us consider an n-tuple s of the set of all right open real bounded intervals. Then there exist elements a, b of $\mathcal{R}^n$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $s(i) = [a(i), b(i)[$.

Proof: $s \in (\text{the set of all right open real bounded intervals})^{\operatorname{Seg} n}$. Consider f being a function such that $s = f$ and $\operatorname{dom} f = \operatorname{Seg} n$ and $\operatorname{rng} f \subseteq$ the set of all right open real bounded intervals. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an element f of $\mathbb{R} \times \mathbb{R}$ such that $f = \$_2$ and $s(\$_1) = [(f)_{\mathbf{1}}, (f)_{\mathbf{2}}[$. For every natural number i such that $i \in \operatorname{Seg} n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$ by [7, (3)]. Consider f being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that $\operatorname{len} f = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, f_i]$ from [25, Sch. 1]. Consider z being an element of $\mathcal{R}^n \times \mathcal{R}^n$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $(z)_{\mathbf{1}}(i) = (f_i)_{\mathbf{1}}$ and $(z)_{\mathbf{2}}(i) = (f_i)_{\mathbf{2}}$. Reconsider $a = (z)_{\mathbf{1}}$, $b = (z)_{\mathbf{2}}$ as an element of $\mathcal{R}^n$. For every natural number i such that $i \in \operatorname{Seg} n$ holds $s(i) = [a(i), b(i)[$. $\square$

(91) Let us consider an element x of $\operatorname{Product}(n, \text{the set of all right open real bounded intervals})$. Then there exist elements a, b of $\mathcal{R}^n$ such that

for every element t of $\mathcal{R}^n$, $t \in x$ iff for every natural number i such that $i \in \operatorname{Seg} n$ holds $t(i) \in [a(i), b(i)[$. The theorem is a consequence of (62) and (90).

(92) MeasurableRectangle LeftOpenIntervals(n) = Product$(n,$ the set of all left open real bounded intervals$)$. The theorem is a consequence of (40) and (66).

(93) MeasurableRectangle RightOpenIntervals(n) = Product$(n,$ the set of all right open real bounded intervals$)$. The theorem is a consequence of (46) and (67).

15. Chebyshev Distance

In the sequel n denotes a non zero natural number and x, y, z denote elements of $\mathcal{R}^n$.

Let us consider n. The functor $D^n_{\mathrm{Chebyshev}}$ yielding a function from $\mathcal{R}^n \times \mathcal{R}^n$ into $\mathbb{R}$ is defined by

(Def. 20) for every elements x, y of $\mathcal{R}^n$, $it(x, y) = \sup \operatorname{rng}|x - y|$.

Now we state the propositions:

(94) (i) the set of all $|x(i) - y(i)|$ where i is an element of $\operatorname{Seg} n$ is real-membered, and

(ii) the set of all $|x(i) - y(i)|$ where i is an element of $\operatorname{Seg} n = \operatorname{rng}|x - y|$.

Proof: Set $S =$ the set of all $|x(i) - y(i)|$ where i is an element of $\operatorname{Seg} n$. $S \subseteq \operatorname{rng}|x - y|$ by [8, (27)], [6, (124)]. For every object t such that $t \in \operatorname{rng}|x - y|$ holds $t \in S$ by [6, (124)], [8, (27)]. □

(95) There exists an extended real-membered set S such that

(i) $S =$ the set of all $|x(i) - y(i)|$ where i is an element of $\operatorname{Seg} n$, and

(ii) $(D^n_{\mathrm{Chebyshev}})(x, y) = \sup S$.

The theorem is a consequence of (94).

(96) $(D^n_{\mathrm{Chebyshev}})(x, y) = |x - y|(\text{max-diff-index}(x, y))$.

Proof: $(D^n_{\mathrm{Chebyshev}})(x, y) \leqslant |x - y|(\text{max-diff-index}(x, y))$ by [15, (5)]. □

(97) $(D^n_{\mathrm{Chebyshev}})(x, y) = 0$ if and only if $x = y$.

Proof: Consider S being an extended real-membered set such that $S =$ the set of all $|x(i) - y(i)|$ where i is an element of $\operatorname{Seg} n$ and $(D^n_{\mathrm{Chebyshev}})(x, y) = \sup S$. $S = \{0\}$ by [19, (2)], [3, (53)], [4, (1)]. □

(98) $(D^n_{\mathrm{Chebyshev}})(x, y) = (D^n_{\mathrm{Chebyshev}})(y, x)$. The theorem is a consequence of (1).

(99) $(D^n_{\rm Chebyshev})(x,y) \leqslant (D^n_{\rm Chebyshev})(x,z) + (D^n_{\rm Chebyshev})(z,y)$.
PROOF: Reconsider $s_1 = \sup \operatorname{rng}|x-y|$, $s_2 = \sup \operatorname{rng}|x-z|$, $s_3 = \sup \operatorname{rng}|z-y|$ as a real number. $s_1 \leqslant s_2 + s_3$ by [8, (27)], [5, (56)], [6, (124)], (2). $\square$

(100) $D^n_{\rm Chebyshev}$ is a metric of $\mathcal{R}^n$. The theorem is a consequence of (97), (98), and (99).

(101) $\rho^2([0,0],[1,1]) = \sqrt{2}$.

(102) $(D^2_{\rm Chebyshev})([0,0],[1,1]) = 1$.
PROOF: Consider S being an extended real-membered set such that $S =$ the set of all $|[0,0](i) - [1,1](i)|$ where i is an element of $\operatorname{Seg} 2$ and $(D^2_{\rm Chebyshev})([0,0],[1,1]) = \sup S$. $S = \{|0-1|\}$ by [4, (2), (44)]. $\square$

Let us consider elements x, y of $\mathcal{R}^1$. Now we state the propositions:

(103) $(D^1_{\rm Chebyshev})(x,y) = |x(1) - y(1)|$.
PROOF: Consider S being an extended real-membered set such that $S =$ the set of all $|x(i) - y(i)|$ where i is an element of $\operatorname{Seg} 1$ and $(D^1_{\rm Chebyshev})(x,y) = \sup S$. $S = \{|x(1) - y(1)|\}$ by [4, (2)]. $\square$

(104) $\rho^1(x,y) = |x(1) - y(1)|$.

Now we state the propositions:

(105) $\rho^1 = D^1_{\rm Chebyshev}$. The theorem is a consequence of (104) and (103).

(106) $\rho^2 \neq D^2_{\rm Chebyshev}$. The theorem is a consequence of (101) and (102).

Let n be a non zero natural number. The functor $\mathsf{L}_\infty(n)$ yielding a strict metric space is defined by the term

(Def. 21) $\langle \mathcal{R}^n, D^n_{\rm Chebyshev} \rangle$.

Let us observe that $\mathsf{L}_\infty(n)$ is non empty.

The functor $\mathcal{E}^n_\infty(n)$ yielding a strict real linear topological structure is defined by

(Def. 22) the topological structure of $it = (\mathsf{L}_\infty(n))_{\rm top}$ and the RLS structure of $it = \mathbb{R}^{\operatorname{Seg} n}_{\mathbb{R}}$.

Now we state the proposition:

(107) The RLS structure of $\mathcal{E}^n_{\rm T}$ = the RLS structure of $\mathcal{E}^n_\infty(n)$.

Let n be a non zero natural number. Let us note that $\mathcal{E}^n_\infty(n)$ is non empty.

Now we state the propositions:

(108) Let us consider an element x of $\mathcal{R}^0$. Then

(i) $\operatorname{Intervals}(x,r)$ is empty, and

(ii) $\prod \operatorname{Intervals}(x,r) = \{\emptyset\}$.

(109) If $r \leqslant 0$, then $\prod \operatorname{Intervals}(x,r)$ is empty.

In the sequel p denotes an element of $\mathsf{L}_\infty(n)$.

Let n be a non zero natural number and p be an element of $\mathsf{L}_\infty(n)$. The functor ${}^{@}p$ yielding an element of $\mathcal{R}^n$ is defined by the term

(Def. 23) p.

Now we state the propositions:

(110) $\mathrm{Ball}(p,r) = \prod \mathrm{Intervals}({}^{@}p, r)$. The theorem is a consequence of (109), (95), and (96).

(111) Let us consider a point e of $\mathcal{E}^n$. If $e = p$, then $\mathrm{Ball}(p,r) = \mathrm{OpenHypercube}(e,r)$. The theorem is a consequence of (110).

Let n be a non zero natural number, p be an element of $\mathsf{L}_\infty(n)$, and r be a negative real number. Let us note that $\overline{\mathrm{Ball}}(p,r)$ is empty.

Now we state the propositions:

(112) Let us consider an object t. Then $t \in \overline{\mathrm{Ball}}(p,r)$ if and only if there exists a function f such that $t = f$ and $\mathrm{dom}\, f = \mathrm{Seg}\, n$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $f(i) \in [({}^{@}p)(i) - r, ({}^{@}p)(i) + r]$. The theorem is a consequence of (95).

(113) Let us consider a point p of $\mathcal{E}^n_{\mathrm{T}}$, and an element q of $\mathsf{L}_\infty(n)$. Suppose $q = p$. Then $\overline{\mathrm{Ball}}(q,r) = \mathrm{ClosedHypercube}(p, n \mapsto r)$.
PROOF: For every object x such that $x \in \overline{\mathrm{Ball}}(q,r)$ holds $x \in \mathrm{ClosedHypercube}(p, n \mapsto r)$ by (112), [6, (57), (93)], [10, (22)]. For every object x such that $x \in \mathrm{ClosedHypercube}(p, n \mapsto r)$ holds $x \in \overline{\mathrm{Ball}}(q,r)$ by [10, (22)], [6, (131), (124), (57)]. $\square$

(114) $\mathrm{Ball}(p,r) = \mathrm{OpenHyperInterval}({}^{@}p - n \mapsto r, {}^{@}p + n \mapsto r)$. The theorem is a consequence of (80) and (110).

(115) $\overline{\mathrm{Ball}}(p,r) = \mathrm{ClosedHyperInterval}({}^{@}p - n \mapsto r, {}^{@}p + n \mapsto r)$. The theorem is a consequence of (81) and (113).

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