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# Parity as a Property of Integers

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**Summary.** Even and odd numbers appear early in history of mathematics [9], as they serve to describe the property of objects easily noticeable by human eye [7]. Although the use of parity allowed to discover irrational numbers [6], there is a common opinion that this property is "not rich enough to become the main content focus of any particular research" [9].

On the other hand, due to the use of decimal system, divisibility by 2 is often regarded as the property of the last digit of a number (similarly to divisibility by 5, but not to divisibility by any other primes), which probably restricts its use for any advanced purposes.

The article aims to extend the definition of parity towards its notion in binary representation of integers, thus making an alternative to the articles grouped in [5], [4], and [3] branches, formalized in Mizar [1], [2].

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Let $a$ be an integer. One can check that $a \bmod a$ is zero and $a \bmod 2$ is natural.

Let $a$, $b$ be integers. Observe that $\gcd(a \cdot b, |a|)$ reduces to $|a|$.

Let $a$ be an odd natural number. Note that $a \bmod 2$ is non zero.

Let $a$ be an even integer. One can check that $a \bmod 2$ is zero.

Note that $a + 1 \bmod 2$ reduces to 1.

Let $a$, $b$ be real numbers. Let us observe that $\max(a, b) - \min(a, b)$ is non negative.

Let $a$ be a natural number and $b$ be a non zero natural number. Note that $a \bmod (a + b)$ reduces to $a$. One can check that $a \operatorname{div} (a + b)$ is zero.




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Let $a$ be a non trivial natural number. Let us observe that $a$-count$(1)$ is zero and $a$-count$(-1)$ is zero.

Let $b$ be a natural number. One can check that $a$-count$(a^b)$ reduces to $b$ and $a$-count$(-a^b)$ reduces to $b$.

Now we state the proposition:

(1) Let us consider integers $a$, $b$. If $a \mid b$, then $\frac{b}{a}$ is integer.

Note that there exists an even integer which is non zero and every natural number which is non zero and trivial is also odd and there exists an odd natural number which is non trivial.

Let $a$ be an integer and $b$ be an even integer. One can verify that $\operatorname{lcm}(a, b)$ is even.

Let $a$, $b$ be odd integers. Let us observe that $\operatorname{lcm}(a, b)$ is odd.

Let $a$, $b$ be integers. Observe that $\frac{a+b}{\gcd(a,b)}$ is integer and $\frac{a-b}{\gcd(a,b)}$ is integer.

Let us consider real numbers $a$, $b$. Now we state the propositions:

(2) (i) $|a + b| = |a| + |b|$, or

(ii) $|a - b| = |a| + |b|$.

(3) (i) $||a| - |b|| = |a + b|$, or

(ii) $||a| - |b|| = |a - b|$.

(4) $||a| - |b|| = |a + b|$ if and only if $|a - b| = |a| + |b|$.

(5) $|a + b| = |a| + |b|$ if and only if $|a - b| = ||a| - |b||$. The theorem is a consequence of (4).

(6) Let us consider non zero real numbers $a$, $b$. Then $||a| - |b|| = |a + b|$ and $|a - b| = |a| + |b|$ if and only if it is not true that $||a| - |b|| = |a - b|$ and $|a + b| = |a| + |b|$.

Proof: $||a| - |b|| = |a + b|$ iff $|a - b| = |a| + |b|$. $||a| - |b|| = |a - b|$ iff $|a + b| = |a| + |b|$. $|a + b| = |a| + |b|$ iff $|a - b| \neq |a| + |b|$. $\square$

Let us consider positive real numbers $a$, $b$ and a natural number $n$. Now we state the propositions:

(7) $\min(a^n, b^n) = (\min(a, b))^n$.

(8) $\max(a^n, b^n) = (\max(a, b))^n$.

Let us consider a non zero natural number $a$ and natural numbers $m$, $n$. Now we state the propositions:

(9) $\min(a^n, a^m) = a^{\min(n,m)}$.

(10) $\max(a^n, a^m) = a^{\max(n,m)}$.

(11) Let us consider natural numbers $a$, $b$. Then $a \bmod b \leqslant a$.

Let us consider a natural number $a$ and non zero natural numbers $b$, $c$. Now we state the propositions:

(12) $(a \bmod c) + (b \bmod c) \geqslant a + b \bmod c$. The theorem is a consequence of (11).

(13) $(a \bmod c) \cdot (b \bmod c) \geqslant a \cdot b \bmod c$. The theorem is a consequence of (11).

Let us consider a natural number $a$ and non zero natural numbers $b$, $n$. Now we state the propositions:

(14) $(a \bmod b)^n \geqslant a^n \bmod b$. The theorem is a consequence of (11).

(15) If $a \bmod b = 1$, then $a^n \bmod b = 1$.

(16) Let us consider natural numbers $a$, $b$, and a non zero natural number $c$. Then $(a \bmod c) \cdot (b \bmod c) < c$ if and only if $a \cdot b \bmod c = (a \bmod c) \cdot (b \bmod c)$.

(17) Let us consider natural numbers $a$, $b$, $c$. Suppose $(a \bmod c) \cdot (b \bmod c) = c$. Then $a \cdot b \bmod c = 0$.

(18) Let us consider natural numbers $a$, $b$, and a non zero natural number $c$. Suppose $(a \bmod c) \cdot (b \bmod c) \geqslant c$. Then $a \bmod c > 1$.

(19) Let us consider integers $a$, $b$, and a non zero natural number $c$. Then

(i) if $a + b \bmod c = b \bmod c$, then $a \bmod c = 0$, and

(ii) if $a + b \bmod c \neq b \bmod c$, then $a \bmod c > 0$.

Proof: If $a + b \bmod c = b \bmod c$, then $a \bmod c = 0$ by [8, (7)]. □

(20) Let us consider a natural number $a$, and non zero natural numbers $b$, $c$. Suppose $a \cdot b \bmod c = b$. Then $a \cdot (\gcd(b, c)) \bmod c = \gcd(b, c)$.

(21) Let us consider integers $a$, $b$. Then $a \equiv b \pmod{\gcd(a, b)}$.

Let us consider odd, a square integers $k$, $l$. Now we state the propositions:

(22) $k - l \bmod 8 = 0$.

(23) $k + l \bmod 8 = 2$. The theorem is a consequence of (22).

Let $a$ be an integer. The functor parity$(a)$ yielding a trivial natural number is defined by the term

(Def. 1) $a \bmod 2$.

Note that the functor parity$(a)$ yields a trivial natural number and is defined by the term

(Def. 2) $2 - (\gcd(a, 2))$.

Let $a$ be an even integer. Let us observe that parity$(a)$ is zero.

Let $a$ be an odd integer. One can check that parity$(a)$ is non zero.

Let $a$ be an integer. The functor Parity$(a)$ yielding a natural number is defined by the term

(Def. 3) $\begin{cases} 0, & \textbf{if } a = 0, \\ 2^{2\text{-count}(a)}, & \textbf{otherwise}. \end{cases}$

Let $a$ be a non zero integer. Observe that $\text{Parity}(a)$ is non zero.

Let $a$ be a non zero, even integer. One can verify that $\text{Parity}(a)$ is non trivial and $\text{Parity}(a)$ is even.

Let $a$ be an even integer. Observe that $\text{Parity}(a)$ is even and $\text{Parity}(a+1)$ is odd.

Let $a$ be an odd integer. Note that $\text{Parity}(a)$ is trivial.

Let $n$ be a natural number. Observe that $\text{Parity}(2^n)$ reduces to $2^n$.

Note that $\text{Parity}(1)$ reduces to 1 and $\text{Parity}(2)$ reduces to 2.

Now we state the propositions:

(24) Let us consider an integer $a$. Then $\text{Parity}(a) \mid a$.

(25) Let us consider integers $a$, $b$. Then $\text{Parity}(a \cdot b) = (\text{Parity}(a)) \cdot (\text{Parity}(b))$.

Let $a$ be an integer. The functor $\text{Oddity}(a)$ yielding an integer is defined by the term

(Def. 4) $\frac{a}{\text{Parity}(a)}$.

Now we state the proposition:

(26) Let us consider a non zero integer $a$. Then $\frac{a}{\text{Parity}(a)} = a \operatorname{div} \text{Parity}(a)$.
The theorem is a consequence of (24).

Let $a$ be an integer. One can check that $(\text{Parity}(a)) \cdot (\text{Oddity}(a))$ reduces to $a$ and $\text{Parity}(\text{Parity}(a))$ reduces to $\text{Parity}(a)$ and $\text{Oddity}(\text{Oddity}(a))$ reduces to $\text{Oddity}(a)$. Observe that $\text{Parity}(\text{Oddity}(a))$ is trivial and $a + \text{Parity}(a)$ is even and $a - \text{Parity}(a)$ is even and $\frac{a}{\text{Parity}(a)}$ is integer.

Now we state the propositions:

(27) Let us consider a non zero integer $a$. Then $\text{Oddity}(\text{Parity}(a)) = 1$.

(28) Let us consider integers $a$, $b$. Then $\text{Oddity}(a \cdot b) = (\text{Oddity}(a)) \cdot (\text{Oddity}(b))$.
The theorem is a consequence of (25).

Let $a$ be a non zero integer. Observe that $\frac{a}{\text{Parity}(a)}$ is odd and $a \operatorname{div} \text{Parity}(a)$ is odd.

Now we state the proposition:

(29) Let us consider integers $a$, $b$. Then

(i) $\text{Parity}(a) \mid \text{Parity}(b)$, or

(ii) $\text{Parity}(b) \mid \text{Parity}(a)$.

Let us consider non zero integers $a$, $b$. Now we state the propositions:

(30) $\text{Parity}(a) \mid \text{Parity}(b)$ if and only if $\text{Parity}(b) \geqslant \text{Parity}(a)$.
PROOF: If $\text{Parity}(b) \geqslant \text{Parity}(a)$, then $\text{Parity}(a) \mid \text{Parity}(b)$. $\square$

(31) If $\text{Parity}(a) > \text{Parity}(b)$, then $2 \cdot (\text{Parity}(b)) \mid \text{Parity}(a)$.

Let us consider an integer $a$. Now we state the propositions:

(32) $\text{Parity}(a) = \text{Parity}(-a)$.

(33) $\text{Parity}(a) = \text{Parity}(|a|)$. The theorem is a consequence of (32).

(34) $\text{Parity}(a) \leqslant |a|$. The theorem is a consequence of (24) and (33).

(35) Let us consider integers $a$, $b$. If $a$ and $b$ are relatively prime, then $a$ is odd or $b$ is odd.

Let us consider odd integers $a$, $b$. Now we state the propositions:

(36) If $|a| \neq |b|$, then $\min(\text{Parity}(a - b), \text{Parity}(a + b)) = 2$. The theorem is a consequence of (33), (9), (2), and (4).

(37) $\min(\text{Parity}(a - b), \text{Parity}(a + b)) \leqslant 2$. The theorem is a consequence of (3), (33), and (36).

(38) Let us consider integers $a$, $b$. Suppose $a$ and $b$ are relatively prime. Then $\min(\text{Parity}(a - b), \text{Parity}(a + b)) \leqslant 2$. The theorem is a consequence of (35) and (37).

(39) Let us consider non zero integers $a$, $b$, and a non trivial natural number $c$. Then $c\text{-count}(\gcd(a, b)) = \min(c\text{-count}(a), c\text{-count}(b))$.

(40) Let us consider non zero integers $a$, $b$.
Then $\text{Parity}(\gcd(a, b)) = \min(\text{Parity}(a), \text{Parity}(b))$. The theorem is a consequence of (39) and (9).

(41) Let us consider integers $a$, $b$. Then $\gcd(\text{Parity}(a), \text{Parity}(b)) = \text{Parity}(\gcd(a, b))$. The theorem is a consequence of (33), (29), and (40).

(42) Let us consider a natural number $a$. Then $\text{Parity}(2 \cdot a) = 2 \cdot (\text{Parity}(a))$. The theorem is a consequence of (25).

(43) Let us consider integers $a$, $b$. Then $\text{lcm}(\text{Parity}(a), \text{Parity}(b)) = \text{Parity}(\text{lcm}(a, b))$. The theorem is a consequence of (25), (33), and (41).

(44) Let us consider non zero integers $a$, $b$.
Then $\text{Parity}(\text{lcm}(a, b)) = \max(\text{Parity}(a), \text{Parity}(b))$. The theorem is a consequence of (41), (40), and (43).

(45) Let us consider integers $a$, $b$. Then $\text{Parity}(a + b) = (\text{Parity}(\gcd(a, b))) \cdot (\text{Parity}(\frac{a+b}{\gcd(a,b)}))$. The theorem is a consequence of (25).

(46) Let us consider an integer $a$, and a natural number $n$. Then $\text{Parity}(a^n) = (\text{Parity}(a))^n$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv \text{Parity}(a^{\$_1}) = (\text{Parity}(a))^{\$_1}$. $\mathcal{P}[0]$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$. For every natural number $x$, $\mathcal{P}[x]$. $\square$

(47) Let us consider non zero integers $a$, $b$, and a natural number $n$. Then $\min(\text{Parity}(a^n), \text{Parity}(b^n)) = (\min(\text{Parity}(a), \text{Parity}(b)))^n$. The theorem is a consequence of (40) and (46).

Let $a$ be an odd integer. We identify $\text{parity}(a)$ with $\text{Parity}(a)$. We identify $\text{Parity}(a)$ with $\text{parity}(a)$. Let us observe that $a^{\text{parity}(a)}$ reduces to $a$.

Let $a$ be an even integer. Let us observe that $a^{\mathrm{parity}(a)}$ is trivial and non zero.

Let $a$ be an integer. One can check that $\mathrm{parity}(\mathrm{parity}(a))$ reduces to $\mathrm{parity}(a)$ and $\mathrm{Parity}(\mathrm{parity}(a))$ reduces to $\mathrm{parity}(a)$.

Now we state the proposition:

(48) Let us consider an integer $a$. Then

- (i) $a$ is even iff $\mathrm{parity}(a)$ is even, and
- (ii) $\mathrm{parity}(a)$ is even iff $\mathrm{Parity}(a)$ is even.

Let $a$ be an integer. Note that $\mathrm{parity}(a) + \mathrm{Parity}(a)$ is even and $\mathrm{Parity}(a) - \mathrm{parity}(a)$ is even and $\mathrm{Parity}(a) - \mathrm{parity}(a)$ is natural and $a + \mathrm{parity}(a)$ is even and $a - \mathrm{parity}(a)$ is even.

Let us consider an integer $a$. Now we state the propositions:

(49) $\mathrm{parity}(\mathrm{Parity}(a)) = \mathrm{parity}(a)$.

(50) $\mathrm{parity}(a) = \mathrm{parity}(-a)$.

Let us consider integers $a$, $b$. Now we state the propositions:

(51) $\mathrm{parity}(a - b) = |\,\mathrm{parity}(a) - \mathrm{parity}(b)|$.

(52) $\mathrm{parity}(a + b) = \mathrm{parity}(\mathrm{parity}(a) + \mathrm{parity}(b))$.

(53) $\mathrm{parity}(a + b) = \mathrm{parity}(a - b)$. The theorem is a consequence of (50).

(54) $\mathrm{parity}(a + b) = |\,\mathrm{parity}(a) - \mathrm{parity}(b)|$. The theorem is a consequence of (53) and (51).

(55) Let us consider natural numbers $a$, $b$. Then

- (i) if $\mathrm{parity}(a + b) = \mathrm{parity}(b)$, then $\mathrm{parity}(a) = 0$, and
- (ii) if $\mathrm{parity}(a + b) \neq \mathrm{parity}(b)$, then $\mathrm{parity}(a) = 1$.

The theorem is a consequence of (19).

Let us consider integers $a$, $b$. Now we state the propositions:

(56)
- (i) $\mathrm{parity}(a + b) = \mathrm{parity}(a) + \mathrm{parity}(b) - 2 \cdot (\mathrm{parity}(a)) \cdot (\mathrm{parity}(b))$, and
- (ii) $\mathrm{parity}(a) - \mathrm{parity}(b) = \mathrm{parity}(a + b) - 2 \cdot (\mathrm{parity}(a + b)) \cdot (\mathrm{parity}(b))$, and
- (iii) $\mathrm{parity}(a) - \mathrm{parity}(b) = 2 \cdot (\mathrm{parity}(a)) \cdot (\mathrm{parity}(a + b)) - \mathrm{parity}(a + b)$.

(57) $a + b$ is even if and only if $\mathrm{parity}(a) = \mathrm{parity}(b)$. The theorem is a consequence of (54).

(58) $\mathrm{parity}(a \cdot b) = (\mathrm{parity}(a)) \cdot (\mathrm{parity}(b))$.

(59) $\mathrm{parity}(\mathrm{lcm}(a, b)) = \mathrm{parity}(a \cdot b)$.

(60) $\mathrm{parity}(\gcd(a, b)) = \max(\mathrm{parity}(a), \mathrm{parity}(b))$.

(61) $\mathrm{parity}(a \cdot b) = \min(\mathrm{parity}(a), \mathrm{parity}(b))$.

(62) Let us consider an integer $a$, and a non zero natural number $n$. Then $\mathrm{parity}(a^n) = \mathrm{parity}(a)$.

(63) Let us consider non zero integers $a$, $b$. Suppose $\mathrm{Parity}(a+b) \geqslant \mathrm{Parity}(a) + \mathrm{Parity}(b)$. Then $\mathrm{Parity}(a) = \mathrm{Parity}(b)$.

(64) Let us consider integers $a$, $b$. Suppose $\mathrm{Parity}(a+b) > \mathrm{Parity}(a) + \mathrm{Parity}(b)$. Then $\mathrm{Parity}(a) = \mathrm{Parity}(b)$. The theorem is a consequence of (63).

(65) Let us consider odd integers $a$, $b$, and an odd natural number $m$. Then $\mathrm{Parity}(a^m + b^m) = \mathrm{Parity}(a+b)$.

(66) Let us consider odd integers $a$, $b$, and an even natural number $m$. Then $\mathrm{Parity}(a^m + b^m) = 2$.

Let us consider non zero integers $a$, $b$. Now we state the propositions:

(67) If $a+b \neq 0$, then if $\mathrm{Parity}(a) = \mathrm{Parity}(b)$, then $\mathrm{Parity}(a+b) \geqslant \mathrm{Parity}(a) + \mathrm{Parity}(b)$.

(68) $\mathrm{Parity}(a+b) = \mathrm{Parity}(b)$ if and only if $\mathrm{Parity}(a) > \mathrm{Parity}(b)$. The theorem is a consequence of (67).

(69) Let us consider non zero natural numbers $a$, $b$. Suppose $\mathrm{Parity}(a+b) < \mathrm{Parity}(a) + \mathrm{Parity}(b)$. Then $\mathrm{Parity}(a+b) = \min(\mathrm{Parity}(a), \mathrm{Parity}(b))$. The theorem is a consequence of (67).

(70) Let us consider non zero integers $a$, $b$. Suppose $a+b \neq 0$. If $\mathrm{Parity}(a+b) = \mathrm{Parity}(a)$, then $\mathrm{Parity}(a) < \mathrm{Parity}(b)$. The theorem is a consequence of (67).

Let us consider an integer $a$. Now we state the propositions:

(71) (i) $\mathrm{Parity}(a + \mathrm{Parity}(a)) = (\mathrm{Parity}(\mathrm{Oddity}(a) + 1)) \cdot (\mathrm{Parity}(a))$, and

(ii) $\mathrm{Parity}(a - \mathrm{Parity}(a)) = (\mathrm{Parity}(\mathrm{Oddity}(a) - 1)) \cdot (\mathrm{Parity}(a))$.

The theorem is a consequence of (25).

(72) (i) $2 \cdot (\mathrm{Parity}(a)) \mid \mathrm{Parity}(a + \mathrm{Parity}(a))$, and

(ii) $2 \cdot (\mathrm{Parity}(a)) \mid \mathrm{Parity}(a - \mathrm{Parity}(a))$.

The theorem is a consequence of (71).

(73) Let us consider integers $a$, $b$. Suppose $\mathrm{Parity}(a) = \mathrm{Parity}(b)$. Then $\mathrm{Parity}(a+b) = \mathrm{Parity}(a + \mathrm{Parity}(a) + (b - \mathrm{Parity}(b)))$.

Let us consider a natural number $a$. Now we state the propositions:

(74) $\mathrm{Parity}(a + \mathrm{Parity}(a)) \geqslant 2 \cdot (\mathrm{Parity}(a))$. The theorem is a consequence of (72).

(75) (i) $\mathrm{Parity}(a - \mathrm{Parity}(a)) \geqslant 2 \cdot (\mathrm{Parity}(a))$, or

(ii) $a = \mathrm{Parity}(a)$.

The theorem is a consequence of (71).

Let us consider odd integers $a$, $b$. Now we state the propositions:

(76) $\text{Parity}(a+b) \neq \text{Parity}(a-b)$. The theorem is a consequence of (25).

(77) If $\text{Parity}(a+1) = \text{Parity}(b-1)$, then $a \neq b$. The theorem is a consequence of (76).

(78) Let us consider an odd natural number $a$, and a non trivial, odd natural number $b$. Then

(i) $\text{Parity}(a+b) = \min(\text{Parity}(a+1), \text{Parity}(b-1))$, or

(ii) $\text{Parity}(a+b) \geqslant 2 \cdot (\text{Parity}(a+1))$.

The theorem is a consequence of (67).

Let us consider non zero integers $a$, $b$. Now we state the propositions:

(79) If $\text{Parity}(a) > \text{Parity}(b)$, then $a \operatorname{div} \text{Parity}(b)$ is even. The theorem is a consequence of (31) and (24).

(80) $\text{Parity}(a) > \text{Parity}(b)$ if and only if $\text{Parity}(a) \operatorname{div} \text{Parity}(b)$ is non zero and even. The theorem is a consequence of (31).

(81) Let us consider an odd natural number $a$. Then $\text{Parity}(a-1) = 2 \cdot (\text{Parity}(a \operatorname{div} 2))$. The theorem is a consequence of (25).

(82) Let us consider non zero integers $a$, $b$. Then

(i) $\min(\text{Parity}(a), \text{Parity}(b)) \mid a$, and

(ii) $\min(\text{Parity}(a), \text{Parity}(b)) \mid b$.

The theorem is a consequence of (30) and (24).

Let $a$, $b$ be non zero integers. Note that $\frac{a+b}{\min(\text{Parity}(a),\text{Parity}(b))}$ is integer.

Let $p$ be a non square integer and $n$ be an odd natural number. Let us note that $p^n$ is non square.

Let $a$ be an integer and $n$ be an even natural number. Let us note that $a^n$ is a square.

Let $p$ be a prime natural number and $a$ be a non zero, a square integer. Let us observe that $p$-count$(a)$ is even.

Let $a$ be an odd integer. Note that $2 \cdot a$ is non square.

Let $a$ be square integer. One can check that $\text{Parity}(a)$ is a square and $\text{Oddity}(a)$ is a square.

Let $a$ be a non zero, a square integer. One can check that 2-count$(a)$ is even.

Now we state the propositions:

(83) Let us consider non negative real numbers $a$, $b$. Then $\max(a, b) - \min(a, b) = |a - b|$.

(84) Let us consider an even integer $a$. If $4 \nmid a$, then $a$ is not square.

PROOF: $2 \nmid \frac{a}{2}$ by [10, (2)]. $\square$

(85) Let us consider odd integers $a$, $b$. If $a - b$ is a square, then $a + b$ is not a square. The theorem is a consequence of (2), (5), (83), (84), and (4).

Let us consider non zero integers $a$, $b$. Now we state the propositions:

(86) $\mathrm{Parity}(a+b) = (\min(\mathrm{Parity}(a), \mathrm{Parity}(b))) \cdot (\mathrm{Parity}(\frac{a+b}{\min(\mathrm{Parity}(a), \mathrm{Parity}(b))}))$. The theorem is a consequence of (30) and (25).

(87) (i) $\mathrm{Parity}(a)$ and $\mathrm{Oddity}(b)$ are relatively prime, and

(ii) $\gcd(\mathrm{Parity}(a), \mathrm{Oddity}(b)) = 1$.

(88) Let us consider an integer $a$. Then $|\,\mathrm{Oddity}(a)| = \mathrm{Oddity}(|a|)$. The theorem is a consequence of (33).

(89) Let us consider integers $a$, $b$.
Then $\gcd(\mathrm{Oddity}(a), \mathrm{Oddity}(b)) = \mathrm{Oddity}(\gcd(a, b))$. The theorem is a consequence of (87), (28), (41), (27), and (88).

(90) Let us consider non zero integers $a$, $b$.
Then $\gcd(a, b) = (\gcd(\mathrm{Parity}(a), \mathrm{Parity}(b))) \cdot (\gcd(\mathrm{Oddity}(a), \mathrm{Oddity}(b)))$. The theorem is a consequence of (87).

(91) Let us consider an odd natural number $a$. Then $\mathrm{Parity}(a + 1) = 2$ if and only if $\mathrm{parity}(a \operatorname{div} 2) = 0$. The theorem is a consequence of (78), (76), and (25).

(92) Let us consider an even integer $a$. Then $a \operatorname{div} 2 = a + 1 \operatorname{div} 2$.

(93) Let us consider integers $a$, $b$. Then $a + b = 2 \cdot ((a \operatorname{div} 2) + (b \operatorname{div} 2)) + \mathrm{parity}(a) + \mathrm{parity}(b)$.

Let us consider odd integers $a$, $b$. Now we state the propositions:

(94) $\mathrm{Parity}(a + b) = 2 \cdot (\mathrm{Parity}((a \operatorname{div} 2) + (b \operatorname{div} 2) + 1))$. The theorem is a consequence of (93) and (25).

(95) $\mathrm{Parity}(a + b) = 2$ if and only if $\mathrm{parity}(a \operatorname{div} 2) = \mathrm{parity}(b \operatorname{div} 2)$. The theorem is a consequence of (94) and (57).

Let us consider non zero integers $a$, $b$. Now we state the propositions:

(96) $\mathrm{Parity}(a+b) = \mathrm{Parity}(a) + \mathrm{Parity}(b)$ if and only if $\mathrm{Parity}(a) = \mathrm{Parity}(b)$ and $\mathrm{parity}(\mathrm{Oddity}(a) \operatorname{div} 2) = \mathrm{parity}(\mathrm{Oddity}(b) \operatorname{div} 2)$. The theorem is a consequence of (63), (25), and (95).

(97) Suppose $a+b \neq 0$ and $\mathrm{Parity}(a) = \mathrm{Parity}(b)$ and $\mathrm{parity}(\mathrm{Oddity}(a) \operatorname{div} 2) \neq \mathrm{parity}(\mathrm{Oddity}(b) \operatorname{div} 2)$. Then $\mathrm{Parity}(a + b) > \mathrm{Parity}(a) + \mathrm{Parity}(b)$. The theorem is a consequence of (67) and (96).

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