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Formaliz. Math. 27 (**3**)

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FORMALIZED MATHEMATICS
Vol. 27, No. 3, Pages 223–228, 2019
DOI: 10.2478/forma-2019-0021

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On the Intersection of Fields F with $F[X]$

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Summary. This is the third part of a four-article series containing a Mizar [3], [1], [2] formalization of Kronecker's construction about roots of polynomials in field extensions, i.e. that for every field F and every polynomial $p \in F[X] \setminus F$ there exists a field extension E of F such that p has a root over E. The formalization follows Kronecker's classical proof using $F[X]/{<}p{>}$ as the desired field extension E [6], [4], [5].

In the first part we show that an irreducible polynomial $p \in F[X] \setminus F$ has a root over $F[X]/{<}p{>}$. Note, however, that this statement cannot be true in a rigid formal sense: We do not have $F \subseteq F[X]/{<}p{>}$ as sets, so F is not a subfield of $F[X]/{<}p{>}$, and hence formally p is not even a polynomial over $F[X]/{<}p{>}$. Consequently, we translate p along the canonical monomorphism $\phi : F \longrightarrow F[X]/{<}p{>}$ and show that the translated polynomial $\phi(p)$ has a root over $F[X]/{<}p{>}$.

Because F is not a subfield of $F[X]/{<}p{>}$ we construct in the second part the field $(E \setminus \phi F) \cup F$ for a given monomorphism $\phi : F \longrightarrow E$ and show that this field both is isomorphic to F and includes F as a subfield. In the literature this part of the proof usually consists of saying that "one can identify F with its image ϕF in $F[X]/{<}p{>}$ and therefore consider F as a subfield of $F[X]/{<}p{>}$". Interestingly, to do so we need to assume that $F \cap E = \emptyset$, in particular Kronecker's construction can be formalized for fields F with $F \cap F[X] = \emptyset$.

Surprisingly, as we show in this third part, this condition is not automatically true for arbitrary fields F: With the exception of $\mathbb{Z}_2$ we construct for every field F an isomorphic copy F' of F with $F' \cap F'[X] \neq \emptyset$. We also prove that for Mizar's representations of $\mathbb{Z}_n$, $\mathbb{Q}$ and $\mathbb{R}$ we have $\mathbb{Z}_n \cap \mathbb{Z}_n[X] = \emptyset$, $\mathbb{Q} \cap \mathbb{Q}[X] = \emptyset$ and $\mathbb{R} \cap \mathbb{R}[X] = \emptyset$, respectively.

In the fourth part we finally define field extensions: E is a field extension of F iff F is a subfield of E. Note, that in this case we have $F \subseteq E$ as sets, and thus a polynomial p over F is also a polynomial over E. We then apply the construction of the second part to $F[X]/{<}p{>}$ with the canonical monomorphism

ISSN 1426–2630(Print), 1898-9934(Online)

$\phi : F \longrightarrow F[X]/{<}p{>}$. Together with the first part this gives – for fields F with $F \cap F[X] = \emptyset$ – a field extension E of F in which $p \in F[X] \setminus F$ has a root.

MSC: 12E05 12F05 68T99 03B35

Keywords: roots of polynomials; field extensions; Kronecker's construction

MML identifier: FIELD_3, version: 8.1.09 5.59.1363

1. Preliminaries

Now we state the propositions:

(1) Let us consider a natural number n, and an object x. If $n = \{x\}$, then $x = 0$.

(2) Let us consider a natural number n, and objects x, y. If $n = \{x, y\}$ and $x \neq y$, then $x = 0$ and $y = 1$ or $x = 1$ and $y = 0$.

(3) Let us consider a natural number n. If $1 < n$, then $0_{\mathbb{Z}/n} = 0$.

(4) $1_{\mathbb{Z}/2} + 1_{\mathbb{Z}/2} = 0_{\mathbb{Z}/2}$. The theorem is a consequence of (3).

(5) Let us consider a ring R, and a non zero natural number n. Then $\mathrm{power}_R(0_R, n) = 0_R$.

One can verify that $\mathbb{Z}/3$ is non degenerated and almost left invertible and there exists a field which is finite and there exists a field which is infinite.

Let L be a non empty double loop structure. We say that L is almost trivial if and only if

(Def. 1) for every element a of L, $a = 1_L$ or $a = 0_L$.

Observe that every ring which is degenerated is also almost trivial and there exists a field which is non almost trivial.

Now we state the proposition:

(6) Let us consider a ring R. Then R is almost trivial if and only if R is degenerated or R and $\mathbb{Z}/2$ are isomorphic. The theorem is a consequence of (4).

Let R be a ring and a be an element of R. We say that a is trivial if and only if

(Def. 2) $a = 1_R$ or $a = 0_R$.

Let R be a non almost trivial ring. One can verify that there exists an element of R which is non trivial.

Let R be a ring. We say that R is polynomial-disjoint if and only if

(Def. 3) $\Omega_R \cap \Omega_{\mathrm{PolyRing}(R)} = \emptyset$.

2. Some Negative Results

Let R be a non almost trivial ring, x be a non trivial element of R, and o be an object. The functor carr(x, o) yielding a non empty set is defined by the term

(Def. 4) $\Omega_R \setminus \{x\} \cup \{o\}$.

Let a, b be elements of carr(x, o). The functor addR(a, b) yielding an element of carr(x, o) is defined by the term

$$\text{(Def. 5)}\quad \begin{cases} (\text{the addition of } R)(x, x), & \textbf{if } a = o \text{ and } b = o \text{ and} \\ & \quad (\text{the addition of } R)(x, x) \neq x, \\ (\text{the addition of } R)(a, x), & \textbf{if } a \neq o \text{ and } b = o \text{ and} \\ & \quad (\text{the addition of } R)(a, x) \neq x, \\ (\text{the addition of } R)(x, b), & \textbf{if } a = o \text{ and } b \neq o \text{ and} \\ & \quad (\text{the addition of } R)(x, b) \neq x, \\ (\text{the addition of } R)(a, b), & \textbf{if } a \neq o \text{ and } b \neq o \text{ and} \\ & \quad (\text{the addition of } R)(a, b) \neq x, \\ o, & \textbf{otherwise}. \end{cases}$$

The functor addR(x, o) yielding a binary operation on carr(x, o) is defined by

(Def. 6) for every elements a, b of carr(x, o), $it(a, b) = \text{addR}(a, b)$.

Let a, b be elements of carr(x, o). The functor multR(a, b) yielding an element of carr(x, o) is defined by the term

$$\text{(Def. 7)}\quad \begin{cases} (\text{the multiplication of } R)(x, x), & \textbf{if } a = o \text{ and } b = o \text{ and} \\ & \quad (\text{the multiplication of } R)(x, x) \neq x, \\ (\text{the multiplication of } R)(a, x), & \textbf{if } a \neq o \text{ and } b = o \text{ and} \\ & \quad (\text{the multiplication of } R)(a, x) \neq x, \\ (\text{the multiplication of } R)(x, b), & \textbf{if } a = o \text{ and } b \neq o \text{ and} \\ & \quad (\text{the multiplication of } R)(x, b) \neq x, \\ (\text{the multiplication of } R)(a, b), & \textbf{if } a \neq o \text{ and } b \neq o \text{ and} \\ & \quad (\text{the multiplication of } R)(a, b) \neq x, \\ o, & \textbf{otherwise}. \end{cases}$$

The functor multR(x, o) yielding a binary operation on carr(x, o) is defined by

(Def. 8) for every elements a, b of carr(x, o), $it(a, b) = \text{multR}(a, b)$.

Let F be a non almost trivial field and x be a non trivial element of F. The functor ExField(x, o) yielding a strict double loop structure is defined by

(Def. 9) the carrier of it = carr(x, o) and the addition of it = addR(x, o) and the multiplication of it = multR(x, o) and the one of $it = 1_F$ and the zero of $it = 0_F$.

One can check that $\mathrm{ExField}(x, o)$ is non degenerated and $\mathrm{ExField}(x, o)$ is Abelian.

From now on o denotes an object, F denotes a non almost trivial field, and x, a denote elements of F.

Let us consider a non trivial element x of F and an object o. Now we state the propositions:

(7) If $o \notin \Omega_F$, then $\mathrm{ExField}(x, o)$ is right zeroed and right complementable.

(8) If $o \notin \Omega_F$, then $\mathrm{ExField}(x, o)$ is add-associative.

Let F be a non almost trivial field, x be a non trivial element of F, and o be an object. One can verify that $\mathrm{ExField}(x, o)$ is commutative.

Let us consider a non trivial element x of F and an object o. Now we state the propositions:

(9) If $o \notin \Omega_F$, then $\mathrm{ExField}(x, o)$ is well unital.

(10) If $o \notin \Omega_F$, then $\mathrm{ExField}(x, o)$ is associative.

(11) If $o \notin \Omega_F$, then $\mathrm{ExField}(x, o)$ is distributive.

(12) If $o \notin \Omega_F$, then $\mathrm{ExField}(x, o)$ is almost left invertible.

(13) Let us consider a non trivial element x of F, and a ring P. Suppose $P = \mathrm{ExField}(x, \langle 0_F, 1_F \rangle)$. Then $\langle 0_F, 1_F \rangle \in \Omega_P \cap \Omega_{\mathrm{PolyRing}(P)}$.

(14) There exists a field K such that $\Omega_K \cap \Omega_{\mathrm{PolyRing}(K)} \neq \emptyset$. The theorem is a consequence of (7), (8), (10), (9), (12), (11), and (13).

In the sequel n denotes a non zero natural number.

(15) There exists a field K and there exists a polynomial p over K such that $\deg p = n$ and $p \in \Omega_K \cap \Omega_{\mathrm{PolyRing}(K)}$. The theorem is a consequence of (7), (8), (10), (9), (12), (11), and (5).

(16) There exists a field K and there exists an object x such that $x \notin \mathrm{rng}$(the canonical homomorphism of K into quotient field) and $x \in \Omega_K \cap \Omega_{\mathrm{PolyRing}(K)}$. The theorem is a consequence of (7), (8), (10), (9), (12), (11), and (13).

Let us note that there exists a field which is non polynomial-disjoint.

Let F be a non almost trivial field, x be a non trivial element of F, and o be an object. The functor $\mathrm{isoR}(x, o)$ yielding a function from F into $\mathrm{ExField}(x, o)$ is defined by

(Def. 10) $it(x) = o$ and for every element a of F such that $a \neq x$ holds $it(a) = a$.

One can check that $\mathrm{isoR}(x, o)$ is onto.

Now we state the propositions:

(17) Let us consider a non trivial element x of F, and an object o. If $o \notin \Omega_F$, then $\mathrm{isoR}(x, o)$ is one-to-one.

(18) Let us consider a non trivial element x of F, and an object u. Suppose $u \notin \Omega_F$. Then $\mathrm{isoR}(x, u)$ is additive, multiplicative, and unity-preserving. The theorem is a consequence of (7), (10), (8), (9), and (11).

Let us consider a non almost trivial field F. Now we state the propositions:

(19) There exists a non polynomial-disjoint field K such that K and F are isomorphic. The theorem is a consequence of (7), (8), (9), (10), (11), (12), (13), and (18).

(20) There exists a non polynomial-disjoint field K and there exists a polynomial p over K such that K and F are isomorphic and $\deg p = n$ and $p \in \Omega_K \cap \Omega_{\mathrm{PolyRing}(K)}$. The theorem is a consequence of (7), (8), (10), (9), (12), (11), (5), and (18).

3. An Intuitive "Solution"

Let R be a ring. We say that R is flat if and only if

(Def. 11) for every elements a, b of R, $\mathrm{rk}(a) = \mathrm{rk}(b)$.

One can check that there exists a ring which is flat.

Now we state the proposition:

(21) Let us consider a flat ring R, and a polynomial p over R. Then $p \notin \Omega_R$.

Note that every flat ring is polynomial-disjoint.

(22) Let us consider a non degenerated ring R. Suppose $0 \in$ the carrier of R. Then R is not flat.

One can check that $\mathbb{Z}^{\mathrm{R}}$ is non flat and $\mathbb{F}_{\mathbb{Q}}$ is non flat and $\mathbb{R}_{\mathrm{F}}$ is non flat.

Let n be a non trivial natural number. One can verify that $\mathbb{Z}/n$ is non flat.

4. Some Positive Results

Now we state the proposition:

(23) Let us consider a ring R, a polynomial p over R, and a natural number n. Then $p \neq n$.

Let n be a non trivial natural number. Let us observe that $\mathbb{Z}/n$ is polynomial-disjoint and there exists a finite field which is polynomial-disjoint.

(24) Let us consider a ring R, a polynomial p over R, and an integer i. Then $p \neq i$. The theorem is a consequence of (23).

One can verify that $\mathbb{Z}^{\mathrm{R}}$ is polynomial-disjoint.

(25) Let us consider a ring R, a polynomial p over R, and a rational number r. Then $p \neq r$.

Observe that $\mathbb{F}_{\mathbb{Q}}$ is polynomial-disjoint. Now we state the proposition:

(26) Let us consider a ring R, a polynomial p over R, and a real number r. Then $p \neq r$.

Note that $\mathbb{R}_{\mathrm{F}}$ is polynomial-disjoint and there exists an infinite field which is polynomial-disjoint.

Let R be a polynomial-disjoint ring. Let us observe that $\mathrm{PolyRing}(R)$ is polynomial-disjoint.

Let F be a field and p be an element of $\Omega_{\mathrm{PolyRing}(F)}$. One can check that $\frac{\mathrm{PolyRing}(F)}{\{p\}\text{-ideal}}$ is polynomial-disjoint.

Let F be a polynomial-disjoint field and p be a non constant element of the carrier of $\mathrm{PolyRing}(F)$. One can check that $\mathrm{PolyRing}(p)$ is polynomial-disjoint.

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Accepted August 29, 2019

FORMALIZED MATHEMATICS
Vol. 27, No. 3, Pages 229–235, 2019
DOI: 10.2478/forma-2019-0022

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Field Extensions and Kronecker's Construction

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Summary. This is the fourth part of a four-article series containing a Mizar [3], [2], [1] formalization of Kronecker's construction about roots of polynomials in field extensions, i.e. that for every field F and every polynomial $p \in F[X]\backslash F$ there exists a field extension E of F such that p has a root over E. The formalization follows Kronecker's classical proof using $F[X]/{<}p{>}$ as the desired field extension E [6], [4], [5].

In the first part we show that an irreducible polynomial $p \in F[X]\backslash F$ has a root over $F[X]/{<}p{>}$. Note, however, that this statement cannot be true in a rigid formal sense: We do not have $F \subseteq F[X]/{<}p{>}$ as sets, so F is not a subfield of $F[X]/{<}p{>}$, and hence formally p is not even a polynomial over $F[X]/{<}p{>}$. Consequently, we translate p along the canonical monomorphism $\phi : F \longrightarrow F[X]/{<}p{>}$ and show that the translated polynomial $\phi(p)$ has a root over $F[X]/{<}p{>}$.

Because F is not a subfield of $F[X]/{<}p{>}$ we construct in the second part the field $(E \setminus \phi F) \cup F$ for a given monomorphism $\phi : F \longrightarrow E$ and show that this field both is isomorphic to F and includes F as a subfield. In the literature this part of the proof usually consists of saying that "one can identify F with its image ϕF in $F[X]/{<}p{>}$ and therefore consider F as a subfield of $F[X]/{<}p{>}$". Interestingly, to do so we need to assume that $F \cap E = \emptyset$, in particular Kronecker's construction can be formalized for fields F with $F \cap F[X] = \emptyset$.

Surprisingly, as we show in the third part, this condition is not automatically true for arbitrary fields F: With the exception of $\mathbb{Z}_2$ we construct for every field F an isomorphic copy F' of F with $F' \cap F'[X] \neq \emptyset$. We also prove that for Mizar's representations of $\mathbb{Z}_n$, $\mathbb{Q}$ and $\mathbb{R}$ we have $\mathbb{Z}_n \cap \mathbb{Z}_n[X] = \emptyset$, $\mathbb{Q} \cap \mathbb{Q}[X] = \emptyset$ and $\mathbb{R} \cap \mathbb{R}[X] = \emptyset$, respectively.

In this fourth part we finally define field extensions: E is a field extension of F iff F is a subfield of E. Note, that in this case we have $F \subseteq E$ as sets, and thus a polynomial p over F is also a polynomial over E. We then apply the

ISSN 1426–2630(Print), 1898-9934(Online)

construction of the second part to $F[X]/{<}p{>}$ with the canonical monomorphism $\phi : F \longrightarrow F[X]/{<}p{>}$. Together with the first part this gives – for fields F with $F \cap F[X] = \emptyset$ – a field extension E of F in which $p \in F[X] \setminus F$ has a root.

MSC: 12E05 12F05 68T99 03B35

Keywords: roots of polynomials; field extensions; Kronecker's construction

MML identifier: FIELD_4, version: 8.1.09 5.59.1363

1. Preliminaries

From now on K, F, E denote fields and R, S denote rings.

Now we state the proposition:

(1) K is a subfield of K.

Let R be a non degenerated ring. One can verify that every subring of R is non degenerated.

Let R be a commutative ring. Note that every subring of R is commutative.

Let R be an integral domain. Let us observe that every subring of R is integral domain-like.

Now we state the proposition:

(2) Let us consider a subring S of R, a finite sequence F of elements of R, and a finite sequence G of elements of S. If $F = G$, then $\sum F = \sum G$.

2. Ring and Field Extensions

Let R, S be rings. We say that S is R-extending if and only if

(Def. 1) R is a subring of S.

Let R be a ring. Note that there exists a ring which is R-extending.

Let R be a commutative ring. One can check that there exists a commutative ring which is R-extending.

Let R be an integral domain. One can verify that there exists an integral domain which is R-extending.

Let F be a field. Let us observe that there exists a field which is F-extending.

Let R be a ring.

A ring extension of R is an R-extending ring. Let R be a commutative ring.

A commutative ring extension of R is an R-extending commutative ring. Let R be an integral domain.

A domain ring extension of R is an R-extending integral domain. Let F be a field.

An extension of F is an F-extending field. Now we state the propositions:

(3) R is a ring extension of R.

(4) Every commutative ring is a commutative ring extension of R.

(5) Every integral domain is a domain ring extension of R.

(6) F is an extension of F.

(7) E is an extension of F if and only if F is a subfield of E.

One can check that $\mathbb{C}_{\mathrm{F}}$ is $(\mathbb{R}_{\mathrm{F}})$-extending and $\mathbb{R}_{\mathrm{F}}$ is $(\mathbb{F}_{\mathbb{Q}})$-extending and $\mathbb{F}_{\mathbb{Q}}$ is $(\mathbb{Z}^{\mathrm{R}})$-extending.

Let R be a ring and S be a ring extension of R. One can check that every ring extension of S is R-extending.

Let R be a commutative ring and S be a commutative ring extension of R. One can verify that every commutative ring extension of S is R-extending.

Let R be an integral domain and S be a domain ring extension of R. Let us observe that every domain ring extension of S is R-extending.

Let F be a field and E be an extension of F. Observe that every extension of E is F-extending.

Let R be a non degenerated ring. Observe that every ring extension of R is non degenerated.

3. Extensions of Polynomial Rings

Now we state the propositions:

(8) Let us consider a ring extension S of R. Then every polynomial over R is a polynomial over S.

(9) Let us consider a subring R of S. Then every polynomial over R is a polynomial over S.

(10) Let us consider a ring extension S of R. Then the carrier of $\mathrm{PolyRing}(R) \subseteq$ the carrier of $\mathrm{PolyRing}(S)$. The theorem is a consequence of (8).

(11) If S is a ring extension of R, then $0_{\mathrm{PolyRing}(S)} = 0_{\mathrm{PolyRing}(R)}$.

(12) If S is a ring extension of R, then $\mathbf{0}.S = \mathbf{0}.R$. The theorem is a consequence of (11).

(13) If S is a ring extension of R, then $1_{\mathrm{PolyRing}(S)} = 1_{\mathrm{PolyRing}(R)}$. The theorem is a consequence of (12).

(14) Let us consider a ring extension S of R. Then $\mathbf{1}.S = \mathbf{1}.R$. The theorem is a consequence of (13).

(15) Let us consider a ring extension S of R, polynomials p, q over R, and polynomials p_1, q_1 over S. If $p = p_1$ and $q = q_1$, then $p + q = p_1 + q_1$.

(16) Let us consider a ring extension S of R. Then the addition of PolyRing

(R) = (the addition of PolyRing(S)) $\restriction$ (the carrier of PolyRing(R)). The theorem is a consequence of (10) and (15).

(17) Let us consider a ring extension S of R, polynomials p, q over R, and polynomials p_1, q_1 over S. If $p = p_1$ and $q = q_1$, then $p * q = p_1 * q_1$. The theorem is a consequence of (2).

(18) Suppose S is a ring extension of R. Then the multiplication of PolyRing (R) = (the multiplication of PolyRing(S)) $\restriction$ (the carrier of PolyRing(R)). The theorem is a consequence of (10) and (17).

Let R be a ring and S be a ring extension of R. One can verify that PolyRing(S) is (PolyRing(R))-extending. Now we state the propositions:

(19) Let us consider a ring R, and a ring extension S of R. Then PolyRing(S) is a ring extension of PolyRing(R).

(20) Let us consider a ring extension S of R, an element p of the carrier of PolyRing(R), and an element q of the carrier of PolyRing(S). If $p = q$, then $\deg p = \deg q$. The theorem is a consequence of (11).

(21) Let us consider a non degenerated ring R, a ring extension S of R, an element a of R, and an element b of S. If $a = b$, then $\mathrm{rpoly}(1, a) = \mathrm{rpoly}(1, b)$. The theorem is a consequence of (10).

4. Evaluation of Polynomials in Ring Extensions

Now we state the propositions:

(22) Let us consider an element a of S. Suppose S is a ring extension of R. Then $\mathrm{ExtEval}(\mathbf{0}.R, a) = 0_S$.

(23) Let us consider a non degenerated ring R, a ring extension S of R, and an element a of S. Then $\mathrm{ExtEval}(\mathbf{1}.R, a) = 1_S$.

(24) Let us consider a ring extension S of R, an element a of S, and polynomials p, q over R. Then $\mathrm{ExtEval}(p+q, a) = \mathrm{ExtEval}(p, a) + \mathrm{ExtEval}(q, a)$.

(25) Let us consider a commutative ring R, a commutative ring extension S of R, an element a of S, and polynomials p, q over R. Then $\mathrm{ExtEval}(p*q, a) = \mathrm{ExtEval}(p, a) \cdot \mathrm{ExtEval}(q, a)$.

(26) Let us consider a ring extension S of R, an element p of the carrier of PolyRing(R), an element q of the carrier of PolyRing(S), and an element a of S. If $p = q$, then $\mathrm{ExtEval}(p, a) = \mathrm{eval}(q, a)$. The theorem is a consequence of (11).

(27) Let us consider a ring extension S of R, an element p of the carrier of PolyRing(R), an element q of the carrier of PolyRing(S), an element a of

R, and an element b of S. If $q = p$ and $b = a$, then $\mathrm{eval}(q, b) = \mathrm{eval}(p, a)$. The theorem is a consequence of (26).

Let R be a ring, S be a ring extension of R, p be an element of the carrier of PolyRing(R), and a be an element of S. We say that a is a root of p in S if and only if

(Def. 2) $\mathrm{ExtEval}(p, a) = 0_S$.

We say that p has a root in S if and only if

(Def. 3) there exists an element a of S such that a is a root of p in S.

The functor Roots(S, p) yielding a subset of S is defined by the term

(Def. 4) $\{a$, where a is an element of S : a is a root of p in $S\}$.

Now we state the proposition:

(28) Let us consider a ring extension S of R, and an element p of the carrier of PolyRing(R). Then $\mathrm{Roots}(p) \subseteq \mathrm{Roots}(S, p)$.

Let R be a ring, S be a non degenerated ring, and p be a polynomial over R. We say that p splits in S if and only if

(Def. 5) there exists a non zero element a of S and there exists a product of linear polynomials q of S such that $p = a \cdot q$.

Now we state the proposition:

(29) Let us consider a field F, and a polynomial p over F. If $\deg p = 1$, then p splits in F.

5. The Degree of Field Extensions

Let R be a ring and S be a ring extension of R. The functor VecSp(S, R) yielding a strict vector space structure over R is defined by

(Def. 6) the carrier of *it* = the carrier of S and the addition of *it* = the addition of S and the zero of *it* $= 0_S$ and the left multiplication of *it* = (the multiplication of S)$\restriction$((the carrier of R) $\times$ (the carrier of S)).

Observe that VecSp(S, R) is non empty and VecSp(S, R) is Abelian, add-associative, right zeroed, and right complementable and VecSp(S, R) is scalar distributive, scalar associative, scalar unital, and vector distributive.

Now we state the proposition:

(30) Let us consider a ring extension S of R. Then VecSp(S, R) is a vector space over R.

Let F be a field and E be an extension of F. The functor $\deg(E, F)$ yielding an integer is defined by the term

(Def. 7) $\begin{cases} \dim(\mathrm{VecSp}(E, F)), & \textbf{if } \mathrm{VecSp}(E, F) \text{ is finite dimensional,} \\ -1, & \textbf{otherwise}. \end{cases}$

Let us note that $\deg(E, F)$ is a dim-like.

We say that E is F-finite if and only if

(Def. 8) $\mathrm{VecSp}(E, F)$ is finite dimensional.

Observe that there exists an extension of F which is F-finite.

Let E be an F-finite extension of F. One can verify that $\deg(E, F)$ is natural.

6. Kronecker's Construction

Let F be a field and p be a non constant element of the carrier of $\mathrm{PolyRing}(F)$. Let us note that the carrier of $\mathrm{PolyRing}(p)$ is F-polynomial membered and $\mathrm{PolyRing}(p)$ is F-polynomial membered.

Let p be an irreducible element of the carrier of $\mathrm{PolyRing}(F)$. The functor $\mathrm{KroneckerIso}(p)$ yielding a function from the carrier of $\mathrm{PolyRing}(p)$ into the carrier of $\mathrm{KroneckerField}(F, p)$ is defined by

(Def. 9) for every element q of the carrier of $\mathrm{PolyRing}(p)$, $it(q) = [q]_{\mathrm{EqRel}(\mathrm{PolyRing}(F), \{p\}-\mathrm{ideal})}$.

Observe that $\mathrm{KroneckerIso}(p)$ is additive, multiplicative, unity-preserving, one-to-one, and onto and $\mathrm{KroneckerField}(F, p)$ is $(\mathrm{PolyRing}(p))$-homomorphic, $(\mathrm{PolyRing}(p))$-monomorphic, and $(\mathrm{PolyRing}(p))$-isomorphic.

$\mathrm{PolyRing}(p)$ is $(\mathrm{KroneckerField}(F, p))$-homomorphic, $(\mathrm{KroneckerField}(F, p))$-monomorphic, and $(\mathrm{KroneckerField}(F, p))$-isomorphic and $\mathrm{PolyRing}(p)$ is F-homomorphic and F-monomorphic.

Now we state the proposition:

(31) Let us consider a polynomial-disjoint field F, and a non constant element f of the carrier of $\mathrm{PolyRing}(F)$. Then there exists an extension E of F such that f has a root in E.

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Accepted August 29, 2019

FORMALIZED MATHEMATICS
Vol. 27, No. 3, Pages 237–259, 2019
DOI: 10.2478/forma-2019-0023

sciendo
https://www.sciendo.com/

Underlying Simple Graphs

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Summary. In this article the notion of the underlying simple graph of a graph (as defined in [8]) is formalized in the Mizar system [5], along with some convenient variants. The property of a graph to be without decorators (as introduced in [7]) is formalized as well to serve as the base of graph enumerations in the future.

MSC: 68T99 03B35 05C76

Keywords: graph operations; underlying simple graph

MML identifier: GLIB_009, version: 8.1.09 5.59.1363

0. Introduction

In the Mizar Mathematical Library [2] there are several formalizations of graphs with a varying degree of generality, see [1], [6], [10], [8], [9]. The GLIB_ series (starting with [8]) formalizes general digraphs (that is, digraphs with loops and parallel edges allowed) in Mizar [5] and provides a rich notation so that any digraph in Mizar can be seen as an undirected graph simply by ignoring the direction of the edges (although they are always there). In conclusion, there is no need for another formalization of undirected graphs, in contrast to how it is typically done in the literature (cf. [12], [3]), and the underlying (undirected) graph of a digraph (in the sense of [8]) is itself. For undirected graphs or digraphs possibly containing loops and multiple parallel edges, the underlying (simple) graph or digraph is derived by removing the loops and replacing each

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ISSN 1426–2630(Print), 1898-9934(Online)

set of parallel edges with a single edge. That concept requires formalization and this article provides subgraph modes that respectively remove loops, (directed) parallel edges or both from a given (di)graph. "Much of graph theory is concerned with the study of simple graphs" [4, p. 3] which results in many books only studying simple graphs, even when graphs are more generally introduced in the respective book (for example [11]).

The rather extensive preliminaries contain many theorems that would fit well into earlier articles of the `GLIB` series, for example:

- The source and target of a directed edge in a graph are uniquely determined.
- A walk in a graph is uniquely determined by its vertex and edge sequence.
- Adding vertices to a graph doesn't change adjacencies.

The next section introduces plain graphs. Graphs, as defined in [8], can arbitrarily be expanded with decorators as done in [7]. Therefore for any non empty set S the set containing all graphs with vertex and edge sets contained in S does not exist because of possible decorators, even if S only contains a single element. A graph is called `plain` if it does not contain additional decorators, and then the set of all plain graphs with vertex and edge sets contained in S can be constructed, which will be needed for graph enumeration at a later point in time.

In the section after that the set of all loops of a graph is introduced as well as a graph operator removing all loops from a given graph as a special case of removing edges.

At the start of the following section, two equivalence relations are defined on the edge set, where two edges are equivalent iff they are (directed) parallel. Then modes are introduced to pick one edge out of each set of (directed) parallel edges. Using such representative edge selections, the graphs with parallel edges removed can be defined as induced subgraphs. While the directed and undirected variants are formalized along each other, there are also some theorems focusing on how they interact with each other.

This trend is continued in the last section, where the underlying simple graphs are introduced as induced subgraphs on the representative edge selections with the loops removed. Naturally, these subgraphs can also be constructed by removing loops and then parallel edges from a graph or vice versa.

1. PRELIMINARIES

Now we state the propositions:

(1) Let us consider sets X, Y. If $Y \subseteq X$, then $X \setminus (X \setminus Y) = Y$.

(2) Let us consider a binary relation R, and a set X. Then

  (i) $(R \restriction X)^\smile = X \upharpoonleft R^\smile$, and

  (ii) $(X \upharpoonleft R)^\smile = R^\smile \restriction X$.

Let us consider a function f and a set Y. Now we state the propositions:

(3) $\mathrm{dom}(Y \upharpoonleft f) = f^{-1}(Y)$.
PROOF: For every object x, $x \in \mathrm{dom}(Y \upharpoonleft f)$ iff $x \in f^{-1}(Y)$. $\square$

(4) $Y \upharpoonleft f = f \restriction \mathrm{dom}(Y \upharpoonleft f)$. The theorem is a consequence of (3).

(5) Let us consider a one-to-one function f, and a set X. Then

  (i) $(f \restriction X)^{-1} = X \upharpoonleft f^{-1}$, and

  (ii) $(X \upharpoonleft f)^{-1} = f^{-1} \restriction X$.

The theorem is a consequence of (2).

(6) Let us consider a graph G, and objects e, x_1, y_1, x_2, y_2. Suppose e joins x_1 to y_1 in G and e joins x_2 to y_2 in G. Then

  (i) $x_1 = x_2$, and

  (ii) $y_1 = y_2$.

Let G be a trivial graph. Let us observe that the vertices of G is trivial and every graph which is trivial and non-directed-multi is also non-multi.

Let G be a trivial, non-directed-multi graph. Let us observe that the edges of G is trivial.

Now we state the propositions:

(7) Let us consider a graph G, sets X, Y, and objects e, x, y. Suppose e joins x to y in G and $x \in X$ and $y \in Y$. Then e joins a vertex from X to a vertex from Y in G.

(8) Let us consider a trivial graph G, and a graph H. Suppose the vertices of $H \subseteq$ the vertices of G and the edges of $H \subseteq$ the edges of G. Then H is trivial and subgraph of G.

(9) Let us consider a graph G. Then $G \approx G \restriction$(the graph selectors).

Let us consider a graph G, sets X, Y, and an object e. Now we state the propositions:

(10) e joins a vertex from X and a vertex from Y in G if and only if e joins a vertex from Y and a vertex from X in G.

(11) e joins a vertex from X and a vertex from Y in G if and only if e joins a vertex from X to a vertex from Y in G or e joins a vertex from Y to a vertex from X in G.

Let us consider a graph G and objects e, v, w. Now we state the propositions:

(12) If e joins a vertex from $\{v\}$ and a vertex from $\{w\}$ in G, then e joins v and w in G.

(13) If e joins a vertex from $\{v\}$ to a vertex from $\{w\}$ in G, then e joins v to w in G.

(14) Let us consider a graph G, and objects v, w. Suppose $v \neq w$. Then

(i) G.edgesDBetween($\{v\}, \{w\}$) misses G.edgesDBetween($\{w\}, \{v\}$), and

(ii) G.edgesBetween($\{v\}, \{w\}$) $=$ G.edgesDBetween($\{v\}, \{w\}$)$\cup$ G.edgesDBetween($\{w\}, \{v\}$).

The theorem is a consequence of (11).

(15) Let us consider a graph G, and a set X. Then G.edgesBetween(X, X) $=$ G.edgesDBetween(X, X). The theorem is a consequence of (11).

(16) Let us consider a graph G, and sets X, Y. Then G.edgesBetween(X, Y) $=$ G.edgesBetween(Y, X). The theorem is a consequence of (10).

Let us consider a graph G. Now we state the propositions:

(17) G is loopless if and only if for every object v, there exists no object e such that e joins v to v in G.

PROOF: For every object v, there exists no object e such that e joins v and v in G. $\square$

(18) G is loopless if and only if for every object v, there exists no object e such that e joins a vertex from $\{v\}$ and a vertex from $\{v\}$ in G.

PROOF: For every object v, there exists no object e such that e joins v and v in G. $\square$

(19) G is loopless if and only if for every object v, there exists no object e such that e joins a vertex from $\{v\}$ to a vertex from $\{v\}$ in G. The theorem is a consequence of (11) and (18).

(20) G is loopless if and only if for every object v, G.edgesBetween($\{v\}, \{v\}$) $=$ $\emptyset$. The theorem is a consequence of (18).

(21) G is loopless if and only if for every object v, G.edgesDBetween($\{v\}, \{v\}$) $=$ $\emptyset$. The theorem is a consequence of (19).

Let G be a loopless graph and v be an object. One can verify that G.edgesBetween($\{v\}, \{v\}$) is empty and G.edgesDBetween($\{v\}, \{v\}$) is empty.

(22) Let us consider a graph G. Then G is non-multi if and only if for every objects v, w, G.edgesBetween$(\{v\}, \{w\})$ is trivial. The theorem is a consequence of (12).

Let G be a non-multi graph and v, w be objects. One can verify that G.edgesBetween$(\{v\}, \{w\})$ is trivial. Now we state the proposition:

(23) Let us consider a graph G. Then G is non-directed-multi if and only if for every objects v, w, G.edgesDBetween$(\{v\}, \{w\})$ is trivial. The theorem is a consequence of (13) and (7).

Let G be a non-directed-multi graph and v, w be objects. One can check that G.edgesDBetween$(\{v\}, \{w\})$ is trivial.

Let G be a non trivial graph. Let us note that every subgraph of G which is spanning is also non trivial.

Let G be a graph. One can check that every vertex of G which is isolated is also non endvertex.

Let us consider a graph G and a vertex v of G. Now we state the propositions:

(24) $(G.\mathrm{walkOf}(v)).\mathrm{edgeSeq}() = \varepsilon_\alpha$, where α is the edges of G.

(25) $(G.\mathrm{walkOf}(v)).\mathrm{edges}() = \emptyset$. The theorem is a consequence of (24).

Let G be a graph and W be a trivial walk of G. Note that W.edges() is empty and trivial.

Let W be a walk of G. Note that W.vertices() is non empty.

Now we state the propositions:

(26) Let us consider graphs G_1, G_2, a walk W_1 of G_1, and a walk W_2 of G_2. Suppose $W_1.\mathrm{vertexSeq}() = W_2.\mathrm{vertexSeq}()$ and $W_1.\mathrm{edgeSeq}() = W_2.\mathrm{edgeSeq}()$. Then $W_1 = W_2$.

Proof: For every natural number n such that $1 \leqslant n \leqslant \mathrm{len}\, W_1$ holds $W_1(n) = W_2(n)$. □

(27) Let us consider a graph G, a finite sequence p of elements of the vertices of G, and a finite sequence q of elements of the edges of G. Suppose $\mathrm{len}\, p = 1 + \mathrm{len}\, q$ and for every element n of $\mathbb{N}$ such that $1 \leqslant n$ and $n + 1 \leqslant \mathrm{len}\, p$ holds $q(n)$ joins $p(n)$ and $p(n + 1)$ in G. Then there exists a walk W of G such that

(i) $W.\mathrm{vertexSeq}() = p$, and

(ii) $W.\mathrm{edgeSeq}() = q$.

Proof: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a natural number m such that $m = \$_1$ and if m is odd, then $\$_2 = p(m+1 \operatorname{div} 2)$ and if m is even, then $\$_2 = q(m \operatorname{div} 2)$. For every natural number k such that $k \in \mathrm{Seg}(\mathrm{len}\, p+\mathrm{len}\, q)$ there exists an element x of (the vertices of G)$\cup$(the edges of G) such that $\mathcal{P}[k, x]$. Consider W being a finite sequence of elements of (the vertices of

G) $\cup$ (the edges of G) such that $\operatorname{dom} W = \operatorname{Seg}(\operatorname{len} p + \operatorname{len} q)$ and for every natural number k such that $k \in \operatorname{Seg}(\operatorname{len} p + \operatorname{len} q)$ holds $\mathcal{P}[k, W(k)]$. $W(1) \in$ the vertices of G. For every odd element n of $\mathbb{N}$ such that $n < \operatorname{len} W$ holds $W(n+1)$ joins $W(n)$ and $W(n+2)$ in G. For every natural number k such that $1 \leqslant k \leqslant \operatorname{len} p$ holds $p(k) = (W.\mathrm{vertexSeq}())(k)$. For every natural number k such that $1 \leqslant k \leqslant \operatorname{len} q$ holds $q(k) = (W.\mathrm{edgeSeq}())(k)$. □

(28) Let us consider a graph G, and a walk W of G. Then $\operatorname{len}(W.\mathrm{vertexSeq}()) = W.\mathrm{length}() + 1$.

(29) Let us consider graphs G_1, G_2, a walk W_1 of G_1, a walk W_2 of G_2, and an odd natural number n. If $W_1.\mathrm{vertexSeq}() = W_2.\mathrm{vertexSeq}()$, then $W_1(n) = W_2(n)$.

Let us consider graphs G_1, G_2, a walk W_1 of G_1, and a walk W_2 of G_2. Now we state the propositions:

(30) Suppose $W_1.\mathrm{vertexSeq}() = W_2.\mathrm{vertexSeq}()$. Then

(i) $\operatorname{len} W_1 = \operatorname{len} W_2$, and

(ii) $W_1.\mathrm{length}() = W_2.\mathrm{length}()$, and

(iii) $W_1.\mathrm{first}() = W_2.\mathrm{first}()$, and

(iv) $W_1.\mathrm{last}() = W_2.\mathrm{last}()$, and

(v) W_2 is walk from $W_1.\mathrm{first}()$ to $W_1.\mathrm{last}()$.

The theorem is a consequence of (29).

(31) If $W_1.\mathrm{vertexSeq}() = W_2.\mathrm{vertexSeq}()$, then if W_1 is not trivial, then W_2 is not trivial and if W_1 is closed, then W_2 is closed. The theorem is a consequence of (30).

(32) Suppose $W_1.\mathrm{vertexSeq}() = W_2.\mathrm{vertexSeq}()$ and $\operatorname{len} W_1 \neq 5$. Then

(i) if W_1 is path-like, then W_2 is path-like, and

(ii) if W_1 is cycle-like, then W_2 is cycle-like.

PROOF: If W_1 is path-like, then W_2 is path-like. □

The scheme *IndWalk* deals with a graph $\mathcal{G}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 1) For every walk W of $\mathcal{G}$, $\mathcal{P}[W]$

provided

- for every trivial walk W of $\mathcal{G}$, $\mathcal{P}[W]$ and
- for every walk W of $\mathcal{G}$ and for every object e such that $e \in W.\mathrm{last}().\mathrm{edgesInOut}()$ and $\mathcal{P}[W]$ holds $\mathcal{P}[W.\mathrm{addEdge}(e)]$.

The scheme *IndDWalk* deals with a graph $\mathcal{G}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 2) For every dwalk W of $\mathcal{G}$, $\mathcal{P}[W]$

provided

- for every trivial dwalk W of $\mathcal{G}$, $\mathcal{P}[W]$ and
- for every dwalk W of $\mathcal{G}$ and for every object e such that $e \in W$.last().edgesOut() and $\mathcal{P}[W]$ holds $\mathcal{P}[W$.addEdge$(e)]$.

Now we state the propositions:

(33) Let us consider a graph G_1, a subset E of the edges of G_1, and a subgraph G_2 of G_1 induced by the vertices of G_1 and E. If G_2 is connected, then G_1 is connected.

(34) Let us consider a graph G_1, a set E, and a subgraph G_2 of G_1 with edges E removed. If G_2 is connected, then G_1 is connected.

Let G_1 be a non connected graph and E be a set. One can check that every subgraph of G_1 with edges E removed is non connected.

(35) Let us consider a graph G_1, and a subgraph G_2 of G_1. Suppose for every walk W_1 of G_1, there exists a walk W_2 of G_2 such that W_2 is walk from W_1.first() to W_1.last(). Let us consider a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then G_1.reachableFrom$(v_1) = G_2$.reachableFrom(v_2).

(36) Let us consider a graph G_1, and a subgraph G_2 of G_1. Suppose for every walk W_1 of G_1, there exists a walk W_2 of G_2 such that W_2 is walk from W_1.first() to W_1.last(). If G_1 is connected, then G_2 is connected.

Let us consider a graph G_1 and a spanning subgraph G_2 of G_1. Now we state the propositions:

(37) Suppose for every vertex v_1 of G_1 and for every vertex v_2 of G_2 such that $v_1 = v_2$ holds G_1.reachableFrom$(v_1) = G_2$.reachableFrom(v_2). Then G_1.componentSet() $= G_2$.componentSet().

(38) Suppose for every vertex v_1 of G_1 and for every vertex v_2 of G_2 such that $v_1 = v_2$ holds G_1.reachableFrom$(v_1) = G_2$.reachableFrom(v_2). Then G_1.numComponents() $= G_2$.numComponents(). The theorem is a consequence of (37).

(39) Let us consider a graph G. Then G is loopless if and only if for every vertex v of G, v and v are not adjacent.

Let G be a non complete graph. One can check that every subgraph of G which is spanning is also non complete.

Now we state the propositions:

(40) Let us consider graphs G_2, G_3, and a supergraph G_1 of G_3. If $G_1 \approx G_2$, then G_2 is a supergraph of G_3.

(41) Let us consider a graph G_2, a set V, a supergraph G_1 of G_2 extended by the vertices from V, sets x, y, and an object e. Then

(i) e joins x and y in G_1 iff e joins x and y in G_2, and

(ii) e joins x to y in G_1 iff e joins x to y in G_2, and

(iii) e joins a vertex from x and a vertex from y in G_1 iff e joins a vertex from x and a vertex from y in G_2, and

(iv) e joins a vertex from x to a vertex from y in G_1 iff e joins a vertex from x to a vertex from y in G_2.

(42) Let us consider graphs G_1, G_2. Suppose $G_1 \approx G_2$. Then G_2 is a graph given by reversing directions of the edges $\emptyset$ of G_1.

(43) Every graph is a graph given by reversing directions of the edges $\emptyset$ of G.

2. Plain Graphs

Let G be a graph. We say that G is plain if and only if

(Def. 1) $\operatorname{dom} G =$ the graph selectors.

Note that $G{\restriction}$(the graph selectors) is plain.

Let V be a non empty set, E be a set, and S, T be functions from E into V. Let us observe that createGraph(V, E, S, T) is plain.

Let G be a graph and X be a set. Note that G.set(WeightSelector, X) is non plain and G.set(ELabelSelector, X) is non plain and G.set(VLabelSelector, X) is non plain and there exists a graph which is plain.

Now we state the proposition:

(44) Let us consider plain graphs G_1, G_2. If $G_1 \approx G_2$, then $G_1 = G_2$.

Let G be a graph. Note that there exists a subgraph of G which is plain.

Let V be a set. One can check that there exists a subgraph of G with vertices V removed which is plain.

Let E be a set. Let us note that there exists a subgraph of G induced by V and E which is plain and there exists a subgraph of G with edges E removed which is plain and there exists a graph given by reversing directions of the edges E of G which is plain.

Let v be a set. One can verify that there exists a subgraph of G with vertex v removed which is plain.

Let e be a set. One can verify that there exists a subgraph of G with edge e removed which is plain and there exists a supergraph of G which is plain.

Let V be a set. Let us note that there exists a supergraph of G extended by the vertices from V which is plain.

Let v, e, w be objects. One can check that there exists a supergraph of G extended by e between vertices v and w which is plain and there exists a supergraph of G extended by v, w and e between them which is plain.

Let v be an object and V be a set. Let us note that there exists a supergraph of G extended by vertex v and edges from V of G to v which is plain and there exists a supergraph of G extended by vertex v and edges from v to V of G which is plain and there exists a supergraph of G extended by vertex v and edges between v and V of G which is plain.

3. Graphs with Loops Removed

Let G be a graph. The functor G.loops() yielding a subset of the edges of G is defined by

(Def. 2) for every object e, $e \in it$ iff there exists an object v such that e joins v and v in G.

Now we state the propositions:

(45) Let us consider a graph G, and an object e. Then $e \in G$.loops() if and only if there exists an object v such that e joins v to v in G.

(46) Let us consider a graph G, and objects e, v, w. If e joins v and w in G and $v \neq w$, then $e \notin G$.loops().

(47) Let us consider a graph G. Then G is loopless if and only if G.loops() $= \emptyset$.

Let G be a loopless graph. Let us observe that G.loops() is empty.

Let G be a non loopless graph. Let us observe that G.loops() is non empty.

Now we state the propositions:

(48) Let us consider a graph G_1, and a subgraph G_2 of G_1. Then G_2.loops() $\subseteq$ G_1.loops(). The theorem is a consequence of (45).

(49) Let us consider a graph G_2, and a supergraph G_1 of G_2. Then G_2.loops() $\subseteq$ G_1.loops(). The theorem is a consequence of (48).

(50) Let us consider graphs G_1, G_2. If $G_1 \approx G_2$, then G_1.loops() $=$ G_2.loops(). The theorem is a consequence of (48).

(51) Let us consider a graph G_1, a set E, and a graph G_2 given by reversing directions of the edges E of G_1. Then G_1.loops() $=$ G_2.loops().

(52) Let us consider a graph G_2, a set V, and a supergraph G_1 of G_2 extended by the vertices from V. Then G_1.loops() $=$ G_2.loops(). The theorem is a consequence of (41) and (49).

(53) Let us consider a graph G_2, objects v_1, e, v_2, and a supergraph G_1 of G_2 extended by e between vertices v_1 and v_2. If $v_1 \neq v_2$, then G_1.loops() = G_2.loops(). The theorem is a consequence of (50) and (49).

(54) Let us consider a graph G_2, a vertex v of G_2, an object e, and a supergraph G_1 of G_2 extended by e between vertices v and v. Suppose $e \notin$ the edges of G_2. Then G_1.loops() = G_2.loops() $\cup \{e\}$. The theorem is a consequence of (45) and (49).

(55) Let us consider a graph G_2, objects v_1, e, v_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Then G_1.loops() = G_2.loops(). The theorem is a consequence of (49) and (50).

(56) Let us consider a graph G_2, an object v, a set V, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Then G_1.loops() = G_2.loops(). The theorem is a consequence of (49) and (50).

(57) Let us consider a graph G, and a path P of G. Then

  (i) P.edges() misses G.loops(), or

  (ii) there exist objects v, e such that e joins v and v in G and $P = G.\mathrm{walkOf}(v, e, v)$.

Let G be a graph. A subgraph of G with loops removed is a subgraph of G with edges G.loops() removed. Now we state the proposition:

(58) Let us consider a loopless graph G_1, and a graph G_2. Then $G_1 \approx G_2$ if and only if G_2 is a subgraph of G_1 with loops removed.

Let us consider graphs G_1, G_2 and a subgraph G_3 of G_1 with loops removed.

(59) $G_2 \approx G_3$ if and only if G_2 is a subgraph of G_1 with loops removed.

(60) If $G_1 \approx G_2$, then G_3 is a subgraph of G_2 with loops removed. The theorem is a consequence of (50).

Let G be a graph. Observe that every subgraph of G with loops removed is loopless and there exists a subgraph of G with loops removed which is plain.

Let G be a non-multi graph. Observe that every subgraph of G with loops removed is simple.

Let G be a non-directed-multi graph. One can check that every subgraph of G with loops removed is directed-simple.

Let G be a complete graph. Observe that every subgraph of G with loops removed is complete.

Now we state the propositions:

(61) Let us consider a graph G_1, a subgraph G_2 of G_1 with loops removed, and a walk W_1 of G_1. Then there exists a walk W_2 of G_2 such that W_2 is walk from W_1.first() to W_1.last(). The theorem is a consequence of (57).

(62) Let us consider a graph G_1, a subgraph G_2 of G_1 with loops removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then G_1.reachableFrom$(v_1) = G_2$.reachableFrom(v_2). The theorem is a consequence of (61) and (35).

Let G be a connected graph. Observe that every subgraph of G with loops removed is connected. Let G be a non connected graph. Observe that every subgraph of G with loops removed is non connected. Let us consider a graph G_1 and a subgraph G_2 of G_1 with loops removed. Now we state the propositions:

(63) G_1.componentSet() $= G_2$.componentSet(). The theorem is a consequence of (62) and (37).

(64) G_1.numComponents() $= G_2$.numComponents(). The theorem is a consequence of (62) and (38).

(65) G_1 is chordal if and only if G_2 is chordal. The theorem is a consequence of (46) and (57).

Let G be a chordal graph. Let us observe that every subgraph of G with loops removed is chordal. Now we state the proposition:

(66) Let us consider a graph G_1, a set v, a subgraph G_2 of G_1 with loops removed, and a subgraph G_3 of G_1 with vertex v removed. Then every subgraph of G_2 with vertex v removed is a subgraph of G_3 with loops removed. The theorem is a consequence of (1), (48), (59), and (60).

Let us consider a graph G_1, a subgraph G_2 of G_1 with loops removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. Now we state the propositions:

(67) If $v_1 = v_2$, then v_1 is cut-vertex iff v_2 is cut-vertex. The theorem is a consequence of (66) and (64).

(68) If $v_1 = v_2$ and v_1 is endvertex, then v_2 is endvertex. The theorem is a consequence of (46).

4. Graphs with Parallel Edges Removed

Let G be a graph. The functors: EdgeParEqRel(G) and DEdgeParEqRel(G) yielding equivalence relations of the edges of G are defined by conditions

(Def. 3) for all objects e_1, e_2, $\langle e_1, e_2\rangle \in$ EdgeParEqRel(G) iff there exist objects v_1, v_2 such that e_1 joins v_1 and v_2 in G and e_2 joins v_1 and v_2 in G,

(Def. 4) for all objects e_1, e_2, $\langle e_1, e_2\rangle \in$ DEdgeParEqRel(G) iff there exist objects v_1, v_2 such that e_1 joins v_1 to v_2 in G and e_2 joins v_1 to v_2 in G,

respectively.

Let us consider a graph G. Now we state the propositions:

(69) DEdgeParEqRel$(G) \subseteq$ EdgeParEqRel(G).

(70) G is non-multi if and only if EdgeParEqRel$(G) = \mathrm{id}_\alpha$, where α is the edges of G.

(71) G is non-directed-multi if and only if DEdgeParEqRel$(G) = \mathrm{id}_\alpha$, where α is the edges of G.

Let G be an edgeless graph. One can verify that EdgeParEqRel(G) is empty and DEdgeParEqRel(G) is empty.

Let G be a non edgeless graph. Observe that EdgeParEqRel(G) is non empty and DEdgeParEqRel(G) is non empty.

Let G be a graph.

A representative selection of the parallel edges of G is a subset of the edges of G defined by

(Def. 5) for every objects v, w, e_0 such that e_0 joins v and w in G there exists an object e such that e joins v and w in G and $e \in it$ and for every object e' such that e' joins v and w in G and $e' \in it$ holds $e' = e$.

A representative selection of the directed-parallel edges of G is a subset of the edges of G defined by

(Def. 6) for every objects v, w, e_0 such that e_0 joins v to w in G there exists an object e such that e joins v to w in G and $e \in it$ and for every object e' such that e' joins v to w in G and $e' \in it$ holds $e' = e$.

Let G be an edgeless graph. Let us observe that every representative selection of the parallel edges of G is empty and every representative selection of the directed-parallel edges of G is empty.

Let G be a non edgeless graph. Let us observe that every representative selection of the parallel edges of G is non empty and every representative selection of the directed-parallel edges of G is non empty.

Now we state the propositions:

(72) Let us consider a graph G, and a representative selection of the directed-parallel edges E_1 of G. Then there exists a representative selection of the parallel edges E_2 of G such that $E_2 \subseteq E_1$.

PROOF: Set $A = \{\{e$, where e is an element of the edges of $G : e$ joins v_1 and v_2 in G and $e \in E_1\}$, where v_1, v_2 are vertices of $G :$ there exists an object e_0 such that e_0 joins v_1 and v_2 in $G\}$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a non empty set S such that $\$_1 = S$ and $\$_2 =$ the element of S. For every object x such that $x \in A$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = A$ and for every object x such that $x \in A$ holds $\mathcal{P}[x, f(x)]$. For every object e such that $e \in \operatorname{rng} f$ holds $e \in E_1$. Reconsider $E_2 = \operatorname{rng} f$ as a subset of the edges of G. For every objects v, w, e_0 such that e_0 joins v and w in G there exists an object e such that e joins v and w in G and $e \in E_2$ and for every object

e' such that e' joins v and w in G and $e' \in E_2$ holds $e' = e$. □

(73) Let us consider a graph G, and a representative selection of the parallel edges E_2 of G. Then there exists a representative selection of the directed-parallel edges E_1 of G such that $E_2 \subseteq E_1$.

Proof: Set $A = \{\{e$, where e is an element of the edges of $G : e$ joins v_1 to v_2 in $G\}$, where v_1, v_2 are vertices of $G :$ there exists an object e_0 such that e_0 joins v_1 to v_2 in G and for every object e_0 such that e_0 joins v_1 to v_2 in G holds $e_0 \notin E_2\}$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a non empty set S such that $\$_1 = S$ and $\$_2 =$ the element of S. For every object x such that $x \in A$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = A$ and for every object x such that $x \in A$ holds $\mathcal{P}[x, f(x)]$. For every object e such that $e \in \operatorname{rng} f$ holds $e \in$ the edges of G. Reconsider $E_1 = E_2 \cup \operatorname{rng} f$ as a subset of the edges of G. For every objects v, w, e_0 such that e_0 joins v to w in G there exists an object e such that e joins v to w in G and $e \in E_1$ and for every object e' such that e' joins v to w in G and $e' \in E_1$ holds $e' = e$. □

(74) Let us consider a non-multi graph G, and a representative selection of the parallel edges E of G. Then $E =$ the edges of G.

(75) Let us consider a graph G. Suppose there exists a representative selection of the parallel edges E of G such that $E =$ the edges of G. Then G is non-multi.

(76) Let us consider a non-directed-multi graph G, and a representative selection of the directed-parallel edges E of G. Then $E =$ the edges of G.

(77) Let us consider a graph G. Suppose there exists a representative selection of the directed-parallel edges E of G such that $E =$ the edges of G. Then G is non-directed-multi.

(78) Let us consider a graph G_1, a subgraph G_2 of G_1, and a representative selection of the parallel edges E of G_1. Suppose $E \subseteq$ the edges of G_2. Then E is a representative selection of the parallel edges of G_2.

(79) Let us consider a graph G_1, a subgraph G_2 of G_1, and a representative selection of the directed-parallel edges E of G_1. Suppose $E \subseteq$ the edges of G_2. Then E is a representative selection of the directed-parallel edges of G_2.

(80) Let us consider a graph G_1, a subgraph G_2 of G_1, and a representative selection of the parallel edges E_2 of G_2. Then there exists a representative selection of the parallel edges E_1 of G_1 such that $E_2 = E_1 \cap$ (the edges of G_2).

Proof: Set $A = \{\{e$, where e is an element of the edges of $G_1 : e$ joins v_1

and v_2 in G_1}, where v_1, v_2 are vertices of G_1 : there exists an object e_0 such that e_0 joins v_1 and v_2 in G_1 and for every object e_0 such that e_0 joins v_1 and v_2 in G_1 holds $e_0 \notin E_2\}$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a non empty set S such that $\$_1 = S$ and $\$_2 =$ the element of S. For every object x such that $x \in A$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = A$ and for every object x such that $x \in A$ holds $\mathcal{P}[x, f(x)]$. For every object e such that $e \in \operatorname{rng} f$ holds $e \in$ the edges of G_1. Reconsider $E_1 = E_2 \cup \operatorname{rng} f$ as a subset of the edges of G_1. For every objects v, w, e_0 such that e_0 joins v and w in G_1 there exists an object e such that e joins v and w in G_1 and $e \in E_1$ and for every object e' such that e' joins v and w in G_1 and $e' \in E_1$ holds $e' = e$. For every object x, $x \in E_2$ iff $x \in E_1$ and $x \in$ the edges of G_2. $\square$

(81) Let us consider a graph G_1, a subgraph G_2 of G_1, and a representative selection of the directed-parallel edges E_2 of G_2. Then there exists a representative selection of the directed-parallel edges E_1 of G_1 such that $E_2 = E_1 \cap (\text{the edges of } G_2)$.

Proof: Set $A = \{\{e$, where e is an element of the edges of G_1 : e joins v_1 to v_2 in G_1}, where v_1, v_2 are vertices of G_1 : there exists an object e_0 such that e_0 joins v_1 to v_2 in G_1 and for every object e_0 such that e_0 joins v_1 to v_2 in G_1 holds $e_0 \notin E_2\}$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a non empty set S such that $\$_1 = S$ and $\$_2 =$ the element of S. For every object x such that $x \in A$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = A$ and for every object x such that $x \in A$ holds $\mathcal{P}[x, f(x)]$. For every object e such that $e \in \operatorname{rng} f$ holds $e \in$ the edges of G_1. Reconsider $E_1 = E_2 \cup \operatorname{rng} f$ as a subset of the edges of G_1. For every objects v, w, e_0 such that e_0 joins v to w in G_1 there exists an object e such that e joins v to w in G_1 and $e \in E_1$ and for every object e' such that e' joins v to w in G_1 and $e' \in E_1$ holds $e' = e$. For every object x, $x \in E_2$ iff $x \in E_1$ and $x \in$ the edges of G_2. $\square$

(82) Let us consider a graph G_1, a representative selection of the parallel edges E_1 of G_1, a subgraph G_2 of G_1 induced by the vertices of G_1 and E_1, and a representative selection of the parallel edges E_2 of G_2. Then $E_1 = E_2$.

Proof: For every object e such that $e \in E_1$ holds $e \in E_2$. $\square$

(83) Let us consider a graph G_1, a representative selection of the directed-parallel edges E_1 of G_1, a subgraph G_2 of G_1 induced by the vertices of G_1 and E_1, and a representative selection of the directed-parallel edges E_2 of G_2. Then $E_1 = E_2$.

Proof: For every object e such that $e \in E_1$ holds $e \in E_2$. $\square$

(84) Let us consider a graph G_1, a representative selection of the directed-parallel edges E_1 of G_1, a subgraph G_2 of G_1 induced by the vertices of G_1 and E_1, and a representative selection of the parallel edges E_2 of G_2. Then

(i) $E_2 \subseteq E_1$, and

(ii) E_2 is a representative selection of the parallel edges of G_1.

Let us consider a graph G and representative selections of the parallel edges E_1, E_2 of G. Now we state the propositions:

(85) There exists a one-to-one function f such that

(i) $\operatorname{dom} f = E_1$, and

(ii) $\operatorname{rng} f = E_2$, and

(iii) for every objects e, v, w such that $e \in E_1$ holds e joins v and w in G iff $f(e)$ joins v and w in G.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv \$_2 \in E_2$ and there exist objects v, w such that $\$_1$ joins v and w in G and $\$_2$ joins v and w in G. For every objects x, y_1, y_2 such that $x \in E_1$ and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. For every object x such that $x \in E_1$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = E_1$ and for every object x such that $x \in E_1$ holds $\mathcal{P}[x, f(x)]$. Consider v_0, w_0 being objects such that e joins v_0 and w_0 in G and $f(e)$ joins v_0 and w_0 in G. $\square$

(86) $\overline{\overline{E_1}} = \overline{\overline{E_2}}$. The theorem is a consequence of (85).

Let us consider a graph G and representative selections of the directed-parallel edges E_1, E_2 of G. Now we state the propositions:

(87) There exists a one-to-one function f such that

(i) $\operatorname{dom} f = E_1$, and

(ii) $\operatorname{rng} f = E_2$, and

(iii) for every objects e, v, w such that $e \in E_1$ holds e joins v to w in G iff $f(e)$ joins v to w in G.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv \$_2 \in E_2$ and there exist objects v, w such that $\$_1$ joins v to w in G and $\$_2$ joins v to w in G. For every objects x, y_1, y_2 such that $x \in E_1$ and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. For every object x such that $x \in E_1$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = E_1$ and for every object x such that $x \in E_1$ holds $\mathcal{P}[x, f(x)]$. Consider v_0, w_0 being objects such that e joins v_0 to w_0 in G and $f(e)$ joins v_0 to w_0 in G. $v_0 = v$ and $w_0 = w$. $\square$

(88) $\overline{\overline{E_1}} = \overline{\overline{E_2}}$. The theorem is a consequence of (87).

Let G be a graph.

A subgraph of G with parallel edges removed is a subgraph of G defined by

(Def. 7) there exists a representative selection of the parallel edges E of G such that *it* is a subgraph of G induced by the vertices of G and E.

A subgraph of G with directed-parallel edges removed is a subgraph of G defined by

(Def. 8) there exists a representative selection of the directed-parallel edges E of G such that *it* is a subgraph of G induced by the vertices of G and E.

Observe that every subgraph of G with parallel edges removed is spanning and non-multi and every subgraph of G with directed-parallel edges removed is spanning and non-directed-multi and there exists a subgraph of G with parallel edges removed which is plain and there exists a subgraph of G with directed-parallel edges removed which is plain.

Let G be a loopless graph. Let us observe that every subgraph of G with parallel edges removed is simple and every subgraph of G with directed-parallel edges removed is directed-simple.

Now we state the propositions:

(89) Let us consider a non-multi graph G_1, and a graph G_2. Then $G_1 \approx G_2$ if and only if G_2 is a subgraph of G_1 with parallel edges removed. The theorem is a consequence of (74).

(90) Let us consider a non-directed-multi graph G_1, and a graph G_2. Then $G_1 \approx G_2$ if and only if G_2 is a subgraph of G_1 with directed-parallel edges removed. The theorem is a consequence of (76).

(91) Let us consider graphs G_1, G_2, and a subgraph G_3 of G_1 with parallel edges removed. If $G_1 \approx G_2$, then G_3 is a subgraph of G_2 with parallel edges removed. The theorem is a consequence of (78).

(92) Let us consider graphs G_1, G_2, and a subgraph G_3 of G_1 with directed-parallel edges removed. Suppose $G_1 \approx G_2$. Then G_3 is a subgraph of G_2 with directed-parallel edges removed. The theorem is a consequence of (79).

(93) Let us consider graphs G_1, G_2, and a subgraph G_3 of G_1 with parallel edges removed. If $G_2 \approx G_3$, then G_2 is a subgraph of G_1 with parallel edges removed.

(94) Let us consider graphs G_1, G_2, and a subgraph G_3 of G_1 with directed-parallel edges removed. Suppose $G_2 \approx G_3$. Then G_2 is a subgraph of G_1 with directed-parallel edges removed.

Let us consider a graph G_1 and a subgraph G_2 of G_1 with directed-parallel edges removed. Now we state the propositions:

(95) Every subgraph of G_2 with parallel edges removed is a subgraph of G_1 with parallel edges removed. The theorem is a consequence of (84).

(96) There exists a subgraph G_3 of G_1 with parallel edges removed such that G_3 is a subgraph of G_2 with parallel edges removed. The theorem is a consequence of (72) and (78).

(97) Let us consider a graph G_1, and a subgraph G_3 of G_1 with parallel edges removed. Then there exists a subgraph G_2 of G_1 with directed-parallel edges removed such that G_3 is a subgraph of G_2 with parallel edges removed. The theorem is a consequence of (73) and (78).

Let G be a complete graph. Let us observe that every subgraph of G with parallel edges removed is complete and every subgraph of G with directed-parallel edges removed is complete.

Now we state the propositions:

(98) Let us consider a graph G_1, a subgraph G_2 of G_1 with parallel edges removed, and a walk W_1 of G_1. Then there exists a walk W_2 of G_2 such that W_1.vertexSeq() = W_2.vertexSeq().

PROOF: Define $\mathcal{P}$[walk of G_1] $\equiv$ there exists a walk W_2 of G_2 such that $\$_1$.vertexSeq() = W_2.vertexSeq(). For every trivial walk W of G_1, $\mathcal{P}[W]$. For every walk W of G_1 and for every object e such that $e \in W$.last().edgesInOut() and $\mathcal{P}[W]$ holds $\mathcal{P}[W$.addEdge(e)]. For every walk W_1 of G_1, $\mathcal{P}[W_1]$. $\square$

(99) Let us consider a graph G_1, a subgraph G_2 of G_1 with directed-parallel edges removed, and a walk W_1 of G_1. Then there exists a walk W_2 of G_2 such that W_1.vertexSeq() = W_2.vertexSeq(). The theorem is a consequence of (95) and (98).

(100) Let us consider a graph G_1, a subgraph G_2 of G_1 with parallel edges removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then G_1.reachableFrom(v_1) = G_2.reachableFrom(v_2). The theorem is a consequence of (35).

(101) Let us consider a graph G_1, a subgraph G_2 of G_1 with directed-parallel edges removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then G_1.reachableFrom(v_1) = G_2.reachableFrom(v_2). The theorem is a consequence of (35).

Let G be a connected graph. Note that every subgraph of G with parallel edges removed is connected and every subgraph of G with directed-parallel edges removed is connected.

Let G be a non connected graph. One can verify that every subgraph of G with parallel edges removed is non connected and every subgraph of G with directed-parallel edges removed is non connected.

Now we state the propositions:

(102) Let us consider a graph G_1, and a subgraph G_2 of G_1 with parallel edges removed. Then G_1.componentSet() $= G_2$.componentSet(). The theorem is a consequence of (100) and (37).

(103) Let us consider a graph G_1, and a subgraph G_2 of G_1 with directed-parallel edges removed. Then G_1.componentSet() $= G_2$.componentSet(). The theorem is a consequence of (101) and (37).

(104) Let us consider a graph G_1, and a subgraph G_2 of G_1 with parallel edges removed. Then G_1.numComponents() $= G_2$.numComponents(). The theorem is a consequence of (100) and (38).

(105) Let us consider a graph G_1, and a subgraph G_2 of G_1 with directed-parallel edges removed. Then G_1.numComponents() $=$ G_2.numComponents(). The theorem is a consequence of (101) and (38).

(106) Let us consider a graph G_1, and a subgraph G_2 of G_1 with parallel edges removed. Then G_1 is chordal if and only if G_2 is chordal. The theorem is a consequence of (98), (30), (32), and (29).

(107) Let us consider a graph G_1, and a subgraph G_2 of G_1 with directed-parallel edges removed. Then G_1 is chordal if and only if G_2 is chordal. The theorem is a consequence of (95) and (106).

Let G be a chordal graph. Note that every subgraph of G with parallel edges removed is chordal and every subgraph of G with directed-parallel edges removed is chordal.

Now we state the propositions:

(108) Let us consider a graph G_1, a set v, a subgraph G_2 of G_1 with parallel edges removed, and a subgraph G_3 of G_1 with vertex v removed. Then every subgraph of G_2 with vertex v removed is a subgraph of G_3 with parallel edges removed. The theorem is a consequence of (93) and (91).

(109) Let us consider a graph G_1, a subgraph G_2 of G_1 with parallel edges removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then v_1 is cut-vertex iff v_2 is cut-vertex. The theorem is a consequence of (108) and (104).

(110) Let us consider a graph G_1, a subgraph G_2 of G_1 with directed-parallel edges removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then v_1 is cut-vertex iff v_2 is cut-vertex. The theorem is a consequence of (95) and (109).

(111) Let us consider a graph G_1, a subgraph G_2 of G_1 with parallel edges removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then v_1 is isolated iff v_2 is isolated.

PROOF: v_1.edgesInOut() $= \emptyset$. □

(112) Let us consider a graph G_1, a subgraph G_2 of G_1 with directed-parallel edges removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then v_1 is isolated iff v_2 is isolated. The theorem is a consequence of (95) and (111).

(113) Let us consider a graph G_1, a subgraph G_2 of G_1 with parallel edges removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$ and v_1 is endvertex, then v_2 is endvertex. The theorem is a consequence of (111).

(114) Let us consider a graph G_1, a subgraph G_2 of G_1 with directed-parallel edges removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$ and v_1 is endvertex, then v_2 is endvertex. The theorem is a consequence of (112).

Let G be a graph. A simple graph of G is a subgraph of G defined by

(Def. 9) there exists a representative selection of the parallel edges E of G such that *it* is a subgraph of G induced by the vertices of G and $E \setminus (G$.loops()).

A directed-simple graph of G is a subgraph of G defined by

(Def. 10) there exists a representative selection of the directed-parallel edges E of G such that *it* is a subgraph of G induced by the vertices of G and $E \setminus (G$.loops()).

Now we state the propositions:

(115) Let us consider a graph G_1, and a subgraph G_2 of G_1 with parallel edges removed. Then every subgraph of G_2 with loops removed is a simple graph of G_1. The theorem is a consequence of (48).

(116) Let us consider a graph G_1, and a subgraph G_2 of G_1 with directed-parallel edges removed. Then every subgraph of G_2 with loops removed is a directed-simple graph of G_1. The theorem is a consequence of (48).

Let us consider a graph G_1 and a subgraph G_2 of G_1 with loops removed. Now we state the propositions:

(117) Every subgraph of G_2 with parallel edges removed is a simple graph of G_1. The theorem is a consequence of (80).

(118) Every subgraph of G_2 with directed-parallel edges removed is a directed-simple graph of G_1. The theorem is a consequence of (81).

(119) Let us consider a graph G_1, and a simple graph G_3 of G_1. Then there exists a subgraph G_2 of G_1 with parallel edges removed such that G_3 is a subgraph of G_2 with loops removed.

PROOF: Consider E being a representative selection of the parallel edges of G_1 such that G_3 is a subgraph of G_1 induced by the vertices of G_1 and $E \setminus (G_1$.loops()). Set $G_2 =$ the subgraph of G_1 induced by the vertices of

G_1 and E. For every object e, $e \in$ the edges of G_3 iff $e \in$ (the edges of G_2) $\setminus$ (G_2.loops()). $\square$

(120) Let us consider a graph G_1, and a directed-simple graph G_3 of G_1. Then there exists a subgraph G_2 of G_1 with directed-parallel edges removed such that G_3 is a subgraph of G_2 with loops removed.

PROOF: Consider E being a representative selection of the directed-parallel edges of G_1 such that G_3 is a subgraph of G_1 induced by the vertices of G_1 and $E \setminus (G_1$.loops()). Set G_2 = the subgraph of G_1 induced by the vertices of G_1 and E. For every object e, $e \in$ the edges of G_3 iff $e \in$ (the edges of G_2) $\setminus$ (G_2.loops()). $\square$

Let us consider a graph G_1 and a subgraph G_2 of G_1 with loops removed. Now we state the propositions:

(121) Every simple graph of G_1 is a subgraph of G_2 with parallel edges removed.

(122) Every directed-simple graph of G_1 is a subgraph of G_2 with directed-parallel edges removed. The theorem is a consequence of (45) and (6).

Let us consider a loopless graph G_1 and a graph G_2. Now we state the propositions:

(123) G_2 is a simple graph of G_1 if and only if G_2 is a subgraph of G_1 with parallel edges removed.

(124) G_2 is a directed-simple graph of G_1 if and only if G_2 is a subgraph of G_1 with directed-parallel edges removed.

(125) Let us consider a non-multi graph G_1, and a graph G_2. Then G_2 is a simple graph of G_1 if and only if G_2 is a subgraph of G_1 with loops removed. The theorem is a consequence of (74).

(126) Let us consider a non-directed-multi graph G_1, and a graph G_2. Then G_2 is a directed-simple graph of G_1 if and only if G_2 is a subgraph of G_1 with loops removed. The theorem is a consequence of (76).

Let G be a graph. Note that every simple graph of G is spanning, loopless, non-multi, and simple and every directed-simple graph of G is spanning, loopless, non-directed-multi, and directed-simple and there exists a simple graph of G which is plain and there exists a directed-simple graph of G which is plain.

Now we state the propositions:

(127) Let us consider a simple graph G_1, and a graph G_2. Then $G_1 \approx G_2$ if and only if G_2 is a simple graph of G_1. The theorem is a consequence of (74).

(128) Let us consider a directed-simple graph G_1, and a graph G_2. Then $G_1 \approx G_2$ if and only if G_2 is a directed-simple graph of G_1. The theorem is

a consequence of (76).

(129) Let us consider graphs G_1, G_2, and a simple graph G_3 of G_1. If $G_1 \approx G_2$, then G_3 is a simple graph of G_2. The theorem is a consequence of (50) and (78).

(130) Let us consider graphs G_1, G_2, and a directed-simple graph G_3 of G_1. If $G_1 \approx G_2$, then G_3 is a directed-simple graph of G_2. The theorem is a consequence of (50) and (79).

(131) Let us consider graphs G_1, G_2, and a simple graph G_3 of G_1. If $G_2 \approx G_3$, then G_2 is a simple graph of G_1.

(132) Let us consider graphs G_1, G_2, and a directed-simple graph G_3 of G_1. If $G_2 \approx G_3$, then G_2 is a directed-simple graph of G_1.

Let us consider a graph G_1 and a directed-simple graph G_2 of G_1. Now we state the propositions:

(133) Every simple graph of G_2 is a simple graph of G_1. The theorem is a consequence of (122), (123), (95), and (117).

(134) There exists a simple graph G_3 of G_1 such that G_3 is a simple graph of G_2. The theorem is a consequence of (122), (96), (117), and (123).

(135) Let us consider a graph G_1, and a simple graph G_3 of G_1. Then there exists a directed-simple graph G_2 of G_1 such that G_3 is a simple graph of G_2. The theorem is a consequence of (121), (97), (118), and (123).

Let G be a complete graph. Observe that every simple graph of G is complete and every directed-simple graph of G is complete.

Now we state the propositions:

(136) Let us consider a graph G_1, a simple graph G_2 of G_1, and a walk W_1 of G_1. Then there exists a walk W_2 of G_2 such that W_2 is walk from W_1.first() to W_1.last(). The theorem is a consequence of (119) and (61).

(137) Let us consider a graph G_1, a directed-simple graph G_2 of G_1, and a walk W_1 of G_1. Then there exists a walk W_2 of G_2 such that W_2 is walk from W_1.first() to W_1.last(). The theorem is a consequence of (133) and (136).

(138) Let us consider a graph G_1, a simple graph G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then G_1.reachableFrom(v_1) = G_2.reachableFrom(v_2). The theorem is a consequence of (136) and (35).

(139) Let us consider a graph G_1, a directed-simple graph G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then G_1.reachableFrom(v_1) = G_2.reachableFrom(v_2). The theorem is a consequence of (137) and (35).

Let G be a connected graph. Observe that every simple graph of G is connected and every directed-simple graph of G is connected.

Let G be a non connected graph. One can verify that every simple graph of G is non connected and every directed-simple graph of G is non connected.

Now we state the propositions:

(140) Let us consider a graph G_1, and a simple graph G_2 of G_1. Then G_1.componentSet() $= G_2$.componentSet(). The theorem is a consequence of (138) and (37).

(141) Let us consider a graph G_1, and a directed-simple graph G_2 of G_1. Then G_1.componentSet() $= G_2$.componentSet(). The theorem is a consequence of (139) and (37).

(142) Let us consider a graph G_1, and a simple graph G_2 of G_1. Then G_1.numComponents() $= G_2$.numComponents(). The theorem is a consequence of (138) and (38).

(143) Let us consider a graph G_1, and a directed-simple graph G_2 of G_1. Then G_1.numComponents() $= G_2$.numComponents(). The theorem is a consequence of (139) and (38).

(144) Let us consider a graph G_1, and a simple graph G_2 of G_1. Then G_1 is chordal if and only if G_2 is chordal. The theorem is a consequence of (119), (65), and (106).

(145) Let us consider a graph G_1, and a directed-simple graph G_2 of G_1. Then G_1 is chordal if and only if G_2 is chordal. The theorem is a consequence of (120), (65), and (107).

Let G be a chordal graph. One can check that every simple graph of G is chordal and every directed-simple graph of G is chordal.

Now we state the propositions:

(146) Let us consider a graph G_1, a simple graph G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then v_1 is cut-vertex iff v_2 is cut-vertex. The theorem is a consequence of (119), (67), and (109).

(147) Let us consider a graph G_1, a directed-simple graph G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then v_1 is cut-vertex iff v_2 is cut-vertex. The theorem is a consequence of (120), (67), and (110).

(148) Let us consider a loopless graph G_1, a simple graph G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then v_1 is isolated iff v_2 is isolated. The theorem is a consequence of (119), (58), and (111).

(149) Let us consider a loopless graph G_1, a directed-simple graph G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then v_1 is isolated iff v_2 is isolated. The theorem is a consequence of (120), (58), and (112).

(150) Let us consider a graph G_1, a simple graph G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$ and v_1 is endvertex, then v_2 is endvertex.

The theorem is a consequence of (119), (113), and (68).

(151) Let us consider a graph G_1, a directed-simple graph G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$ and v_1 is endvertex, then v_2 is endvertex. The theorem is a consequence of (120), (114), and (68).

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Accepted August 29, 2019

FORMALIZED MATHEMATICS
Vol. 27, No. 3, Pages 261–301, 2019
DOI: 10.2478/forma-2019-0024

https://www.sciendo.com/

About Graph Mappings

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Summary. In this articles adjacency-preserving mappings from a graph to another are formalized in the Mizar system [7], [2]. The generality of the approach seems to be largely unpreceeded in the literature to the best of the author's knowledge. However, the most important property defined in the article is that of two graphs being isomorphic, which has been extensively studied. Another graph decorator is introduced as well.

MSC: 05C60 68T99 03B35

Keywords: graph homomorphism; graph isomorphism

MML identifier: GLIB_010, version: 8.1.09 5.59.1363

0. Introduction

Writing this article has been rather challenging. "Much of graph theory is concerned with the study of simple graphs" [3, p. 3], so most graph theory books are only concerned with graph homomorphisms between simple graphs, if they are concerned with anything more general than isomorphisms at all. [3] writes about general graphs; isomorphisms are done in the first chapter while homomorphisms are only looked at in the context of vertex colorings in chapter 14. The book "Graphs and homomorphisms" [8] only handles (di)graphs without multiple parallel edges. The book "Graph coloring problems" [10] notes homomorphisms between loopless graphs, but doesn't elaborate. [6] only handles homomorphisms between simple graphs. [14] shortly describes homomorphisms between undirected graphs. [9] handles homomorphisms between

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digraphs without parallel edges. [16] writes about general graphs but, like most graph books, only about isomorphisms. The best source so far has been [11], where graph homomorphisms are introduced for digraphs possibly containing loops and multiple parallel edges (just like graphs are formalized in [15]) but the focus is almost immediately shifted to homomorphisms between simple graphs. So a quick overview of the formalized notation seems in order.

A *graph* G consists of a non empty set $V(G)$ called vertices of G, a set $E(G)$ called edges of G and two functions $s(G), t(G) : E(G) \to V(G)$, the source and target of G. For $e \in E(G), v, w \in V(G)$ we write *e joins v to w* if $s(G)(e) = v$ and $t(G)(e) = w$, and we write *e joins v and w* if e joins v to w or e joins w to v. Let G_1, G_2 be graphs. A *partial graph mapping* from G_1 to G_2 is an ordered pair $F = \langle F_{\mathbb{V}}, F_{\mathbb{E}} \rangle$ with the following properties:

- $F_{\mathbb{V}}$ is a partial function from $V(G_1)$ to $V(G_2)$.
- $F_{\mathbb{E}}$ is a partial function from $E(G_1)$ to $E(G_2)$.
- For any $e \in \operatorname{dom} F_{\mathbb{E}}$ holds $s(G)(e), t(G)(e) \in \operatorname{dom} F_{\mathbb{V}}$.
- For any $e \in \operatorname{dom} F_{\mathbb{E}}$ and $v, w \in \operatorname{dom} F_{\mathbb{V}}$ such that e joins v and w holds $F_{\mathbb{E}}(e)$ joins $F_{\mathbb{V}}(v)$ and $F_{\mathbb{V}}(w)$.

Note that $\langle f, \emptyset \rangle$ is a valid partial graph mapping for any partial function $f : V(G_1) \to V(G_2)$, especially for $f = \emptyset$. Now define the following attributes:

- F is *empty* if $\operatorname{dom} F_{\mathbb{V}} = \emptyset$.
- F is *total* (or a homomorphism) if $\operatorname{dom} F_{\mathbb{V}} = V(G_1)$ and $\operatorname{dom} F_{\mathbb{E}} = E(G_1)$.
- F is *onto* (or surjective) if $\operatorname{rng} F_{\mathbb{V}} = V(G_2)$ and $\operatorname{rng} F_{\mathbb{E}} = E(G_2)$.
- F is *one-to-one* (or injective) if $F_{\mathbb{V}}$ and $F_{\mathbb{E}}$ are.
- F is *semi-continuous* if for any $e \in \operatorname{dom} F_{\mathbb{E}}$ and $v, w \in \operatorname{dom} F_{\mathbb{V}}$ such that $F_{\mathbb{E}}(e)$ joins $F_{\mathbb{V}}(v)$ and $F_{\mathbb{V}}(w)$ holds e joins v and w.
- F is *continuous* if for any $\tilde{e} \in E(G_2)$ and $v, w \in \operatorname{dom} F_{\mathbb{V}}$ such that $\tilde{e}$ joins $F_{\mathbb{V}}(v)$ and $F_{\mathbb{V}}(w)$ exists an $e \in \operatorname{dom} F_{\mathbb{E}}$ such that $F_{\mathbb{E}}(e) = \tilde{e}$ and e joins v and w.
- F is a *weak subgraph-embedding* if it is total and one-to-one.
- F is a *strong subgraph-embedding* if it is total, one-to-one and continuous.
- F is an *isomorphism* if it is total, one-to-one and onto.

Because modes in Mizar must always be inhabitated, partial graph mappings are the chosen foundation rather than homomorphisms, which may not exist between two graphs. The attributes *total*, *onto* and *one-to-one* were named like their function analogons from [4] and [5]. The *continuous* attribute was inspired by the continuous vertex mappings of [11] and is in fact sometimes different from *semi-continuous*. *Semi-continuous* seemed like the natural generalization of *continuous* for graph mappings instead of vertex mappings, but that turned out to be false. Still, a semi-continuous graph mapping already carries a lot of properties from G_1 to G_2, so the definition was kept. Corresponding attributes for directed graph mappings are given in this article as well.

If F is a weak subgraph-embedding, then G_1 is isomorphic to a subgraph of G_2. If F is a strong subgraph-embedding, then G_1 is isomorphic to an induced subgraph of G_2. The short term *embedding* was desperately avoided to be available for embeddings of graphs into the plane and other surfaces. If F is one-to-one, it is also semi-continuous. If F is semi-continuous and onto, it is also continuous.

Originally, only an article about graph isomorphisms was planned, but it was changed to provide a solid foundation of general graph mappings. Now this article also includes the restriction of F to subgraphs of G_1 or G_2, the domain and range of F defined as the plain subgraphs of G_1 and G_2 induced by $\operatorname{dom} F_{\mathbb{V}}$, $\operatorname{dom} F_{\mathbb{E}}$ and $\operatorname{rng} F_{\mathbb{V}}$, $\operatorname{rng} F_{\mathbb{E}}$ respectively, and the images of walks under F. Of course the inverse of F and the composition of two graph mappings are included as well.

Additionally, the ordering of a graph, which is just an enumeration of its vertices, has been introduced as yet another graph decorator. This decorator is planned as a tool to identify graphs with trees from [1]. Attributes describing if F preserves the weights, edge labels, vertex labels or the ordering have been added as well.

1. PRELIMINARIES

Now we state the propositions:

(1) Let us consider functions A, B, C, D. Suppose $D \cdot A = C \restriction \operatorname{dom} A$. Then $(D \restriction \operatorname{dom} B) \cdot A = C \restriction \operatorname{dom}(B \cdot A)$.

PROOF: Set $f = (D \restriction \operatorname{dom} B) \cdot A$. Set $g = C \restriction \operatorname{dom}(B \cdot A)$. For every object x such that $x \in \operatorname{dom} f$ holds $f(x) = g(x)$. □

(2) Let us consider a one-to-one function A, and functions C, D. Suppose $D \cdot A = C \restriction \operatorname{dom} A$. Then $C \cdot (A^{-1}) = D \restriction \operatorname{dom}(A^{-1})$.

PROOF: For every object y, $y \in \operatorname{dom}(C \cdot (A^{-1}))$ iff $y \in \operatorname{dom}(D \restriction \operatorname{dom}(A^{-1}))$. For every object y such that $y \in \operatorname{dom}(C \cdot (A^{-1}))$ holds $(C \cdot (A^{-1}))(y) = (D \restriction \operatorname{dom}(A^{-1}))(y)$. □

Let G be a non finite graph and X be a set. One can verify that

G.set(WeightSelector, X) is non finite and G.set(ELabelSelector, X) is non finite and G.set(VLabelSelector, X) is non finite.

Let G be a non loopless graph. One can check that G.set(WeightSelector, X) is non loopless and G.set(ELabelSelector, X) is non loopless and

G.set(VLabelSelector, X) is non loopless.

Let G be a non non-multi graph. Note that G.set(WeightSelector, X) is non non-multi and G.set(ELabelSelector, X) is non non-multi and

G.set(VLabelSelector, X) is non non-multi. Let G be a non non-directed-multi graph. Let us note that G.set(WeightSelector, X) is non non-directed-multi and G.set(ELabelSelector, X) is non non-directed-multi and

G.set(VLabelSelector, X) is non non-directed-multi.

Let G be a non connected graph. Observe that G.set(WeightSelector, X) is non connected and G.set(ELabelSelector, X) is non connected and

G.set(VLabelSelector, X) is non connected.

Let G be a non acyclic graph. Let us observe that G.set(WeightSelector, X) is non acyclic and G.set(ELabelSelector, X) is non acyclic and

G.set(VLabelSelector, X) is non acyclic. Let G be a graph. We say that G is elabel-full if and only if

(Def. 1) ELabelSelector $\in \operatorname{dom} G$ and there exists a many sorted set f indexed by the edges of G such that G(ELabelSelector) $= f$.

We say that G is vlabel-full if and only if

(Def. 2) VLabelSelector $\in \operatorname{dom} G$ and there exists a many sorted set f indexed by the vertices of G such that G(VLabelSelector) $= f$.

Let us observe that every graph which is elabel-full is also elabeled and every graph which is vlabel-full is also vlabeled.

Let G be an e-graph. We say that G is elabel-distinct if and only if

(Def. 3) the elabel of G is one-to-one.

Let G be a v-graph. We say that G is vlabel-distinct if and only if

(Def. 4) the vlabel of G is one-to-one.

Let G be a graph. Observe that G.set(ELabelSelector, $\mathrm{id}_{\text{the edges of } G}$) is elabel-full and elabel-distinct and G.set(VLabelSelector, $\mathrm{id}_{\text{the vertices of } G}$) is vlabel-full and vlabel-distinct and there exists an e-graph which is elabel-distinct and elabel-full and there exists a v-graph which is vlabel-distinct and vlabel-full.

Let G be an elabel-full graph. Let us observe that the elabel of G yields a many sorted set indexed by the edges of G. Let G be a vlabel-full graph. Observe that the vlabel of G yields a many sorted set indexed by the vertices of G. Let G be an elabel-distinct e-graph. Let us note that the elabel of G is one-to-one.

Let G be a vlabel-distinct v-graph. Observe that the vlabel of G is one-to-one. Let G be an elabel-full graph and X be a set. One can verify that

G.set(WeightSelector, X) is elabel-full and G.set(VLabelSelector, X) is elabel-full. Let G be a vlabel-full graph. One can check that G.set(WeightSelector, X) is vlabel-full and G.set(ELabelSelector, X) is vlabel-full.

Let G be an elabel-distinct e-graph. Note that G.set(WeightSelector, X) is elabel-distinct and G.set(VLabelSelector, X) is elabel-distinct.

Let G be a vlabel-distinct v-graph. Let us observe that G.set(WeightSelector, X) is vlabel-distinct and G.set(ELabelSelector, X) is vlabel-distinct and there exists an ev-graph which is elabel-full, elabel-distinct, vlabel-full, and vlabel-distinct.

Let G_1 be a w-graph, E be a set, and G_2 be a graph given by reversing directions of the edges E of G_1. Observe that G_2.set(WeightSelector, the weight of G_1) is weighted.

Let G_1 be an e-graph. One can verify that G_2.set(ELabelSelector, the elabel of G_1) is elabeled.

Let G_1 be a v-graph, V be a set, and G_2 be a graph given by reversing directions of the edges V of G_1. Observe that G_2.set(VLabelSelector, the vlabel of G_1) is vlabeled.

Let G_1 be an elabel-full graph, E be a set, and G_2 be a graph given by reversing directions of the edges E of G_1. Note that G_2.set(ELabelSelector, the elabel of G_1) is elabel-full.

Let G_1 be a vlabel-full graph, V be a set, and G_2 be a graph given by reversing directions of the edges V of G_1. Note that G_2.set(VLabelSelector, the vlabel of G_1) is vlabel-full. Let G_1 be an elabel-distinct e-graph, E be a set, and G_2 be a graph given by reversing directions of the edges E of G_1. Note that G_2.set(ELabelSelector, the elabel of G_1) is elabel-distinct. Let G_1 be a vlabel-distinct v-graph. Observe that G_2.set(VLabelSelector, the vlabel of G_1) is vlabel-distinct.

2. Ordering of a Graph

The functor OrderingSelector yielding an element of $\mathbb{N}$ is defined by the term

(Def. 5) 8.

Let G be a graph structure. We say that G is ordered if and only if

(Def. 6) OrderingSelector $\in \operatorname{dom} G$ and G(OrderingSelector) is an enumeration of the vertices of G.

Let G be a graph and X be a set. Note that G.set(OrderingSelector, X) is graph-like and G.set(OrderingSelector, X) is non plain.

Let G be a w-graph. One can verify that G.set(OrderingSelector, X) is weighted.

Let G be an e-graph. One can check that G.set(OrderingSelector, X) is elabeled.

Let G be a v-graph. Note that G.set(OrderingSelector, X) is vlabeled.

Let G be a graph and X be an enumeration of the vertices of G. Note that G.set(OrderingSelector, X) is ordered and there exists a graph structure which is graph-like, weighted, elabeled, vlabeled, and ordered.

Let G be an ordered graph. The ordering of G yielding an enumeration of the vertices of G is defined by the term

(Def. 7) G(OrderingSelector).

Now we state the proposition:

(3) Let us consider a graph G, and a set X. Then $G \approx G$.set(OrderingSelector, X).

Let G be an elabel-full graph and X be a set. Let us note that G.set(OrderingSelector, X) is elabel-full.

Let G be a vlabel-full graph. Let us note that G.set(OrderingSelector, X) is vlabel-full.

Let G be an elabel-distinct e-graph. Let us note that G.set(OrderingSelector, X) is elabel-distinct.

Let G be a vlabel-distinct v-graph. Observe that G.set(OrderingSelector, X) is vlabel-distinct.

Let G be a finite graph. Let us observe that G.set(OrderingSelector, X) is finite.

Let G be a non finite graph. Let us observe that G.set(OrderingSelector, X) is non finite.

Let G be a loopless graph. Let us observe that G.set(OrderingSelector, X) is loopless.

Let G be a non loopless graph. Let us observe that G.set(OrderingSelector, X) is non loopless.

Let G be a trivial graph. Let us observe that G.set(OrderingSelector, X) is trivial.

Let G be a non trivial graph. Let us observe that G.set(OrderingSelector, X) is non trivial.

Let G be a non-multi graph. Let us observe that G.set(OrderingSelector, X) is non-multi.

Let G be a non non-multi graph. Let us observe that G.set(OrderingSelector, X) is non non-multi.

Let G be a non-directed-multi graph. Let us observe that G.set(OrderingSelector, X) is non-directed-multi.

Let G be a non non-directed-multi graph. Let us observe that G.set(OrderingSelector, X) is non non-directed-multi.

Let G be a connected graph. Let us observe that G.set(OrderingSelector, X) is connected.

Let G be a non connected graph. Let us note that G.set(OrderingSelector, X) is non connected.

Let G be an acyclic graph. Let us note that G.set(OrderingSelector, X) is acyclic.

Let G be a non acyclic graph. One can check that G.set(OrderingSelector, X) is non acyclic.

Let G be an edgeless graph. One can check that G.set(OrderingSelector, X) is edgeless.

Let G be a non edgeless graph. Let us observe that G.set(OrderingSelector, X) is non edgeless.

Let G be an ordered graph. Let us observe that G.set(WeightSelector, X) is ordered and G.set(ELabelSelector, X) is ordered and G.set(VLabelSelector, X) is ordered.

Let G_1 be an ordered graph and G_2 be a spanning subgraph of G_1. Note that G_2.set(OrderingSelector, the ordering of G_1) is ordered.

Let E be a set and G_2 be a graph given by reversing directions of the edges E of G_1. Let us observe that G_2.set(OrderingSelector, the ordering of G_1) is ordered.

3. Graph Mappings

Let G_1, G_2 be graphs. A partial graph mapping from G_1 to G_2 is an object defined by

(Def. 8) there exist functions f, g such that $it = \langle f, g \rangle$ and $\operatorname{dom} f \subseteq$ the vertices of G_1 and $\operatorname{rng} f \subseteq$ the vertices of G_2 and $\operatorname{dom} g \subseteq$ the edges of G_1 and $\operatorname{rng} g \subseteq$ the edges of G_2 and for every object e such that $e \in \operatorname{dom} g$ holds (the source of G_1)(e), (the target of G_1)(e) $\in \operatorname{dom} f$ and for every objects e, v, w such that $e \in \operatorname{dom} g$ and v, $w \in \operatorname{dom} f$ holds if e joins v and w in G_1, then $g(e)$ joins $f(v)$ and $f(w)$ in G_2.

Let us observe that every partial graph mapping from G_1 to G_2 is pair.

Let F be a partial graph mapping from G_1 to G_2. We introduce the notation $F_{\mathbb{V}}$ as a synonym of $(F)_{\mathbf{1}}$ and $F_{\mathbb{E}}$ as a synonym of $(F)_{\mathbf{2}}$.

One can check that $\langle F_{\mathbb{V}}, F_{\mathbb{E}} \rangle$ reduces to F.

One can verify that $F_{\mathbb{V}}$ is function-like and relation-like as a set and $F_{\mathbb{E}}$ is function-like and relation-like as a set and $F_{\mathbb{V}}$ is (the vertices of G_1)-defined and (the vertices of G_2)-valued as a function and $F_{\mathbb{E}}$ is (the edges of G_1)-defined and (the edges of G_2)-valued as a function.

Note that the functor $F_{\mathbb{V}}$ yields a partial function from the vertices of G_1 to the vertices of G_2. Observe that the functor $F_{\mathbb{E}}$ yields a partial function from the edges of G_1 to the edges of G_2. Now we state the proposition:

(4) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and objects e, v, w. Suppose $e \in \operatorname{dom}(F_{\mathbb{E}})$ and $v, w \in \operatorname{dom}(F_{\mathbb{V}})$. If e joins v and w in G_1, then $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v)$ and $(F_{\mathbb{V}})(w)$ in G_2.

Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and an object e. Now we state the propositions:

(5) Suppose $e \in \operatorname{dom}(F_{\mathbb{E}})$. Then (the source of $G_1)(e)$, (the target of $G_1)(e) \in \operatorname{dom}(F_{\mathbb{V}})$.

(6) Suppose $e \in \operatorname{rng} F_{\mathbb{E}}$. Then (the source of $G_2)(e)$, (the target of $G_2)(e) \in \operatorname{rng} F_{\mathbb{V}}$. The theorem is a consequence of (5) and (4).

(7) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Then

(i) $\operatorname{dom}(F_{\mathbb{E}}) \subseteq G_1$.edgesBetween($\operatorname{dom}(F_{\mathbb{V}})$), and

(ii) $\operatorname{rng} F_{\mathbb{E}} \subseteq G_2$.edgesBetween($\operatorname{rng} F_{\mathbb{V}}$).

PROOF: For every object e such that $e \in \operatorname{dom}(F_{\mathbb{E}})$ holds $e \in G_1$.edgesBetween($\operatorname{dom}(F_{\mathbb{V}})$). For every object e such that $e \in \operatorname{rng} F_{\mathbb{E}}$ holds $e \in G_2$.edgesBetween($\operatorname{rng} F_{\mathbb{V}}$). $\square$

(8) Let us consider graphs G_1, G_2, a partial function f from the vertices of G_1 to the vertices of G_2, and a partial function g from the edges of G_1 to the edges of G_2. Suppose for every object e such that $e \in \operatorname{dom} g$ holds (the source of $G_1)(e)$, (the target of $G_1)(e) \in \operatorname{dom} f$ and for every objects e, v, w such that $e \in \operatorname{dom} g$ and $v, w \in \operatorname{dom} f$ holds if e joins v and w in G_1, then $g(e)$ joins $f(v)$ and $f(w)$ in G_2. Then $\langle f, g\rangle$ is a partial graph mapping from G_1 to G_2.

Let us consider graphs G_1, G_2, G_3, G_4 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(9) If $G_1 \approx G_3$ and $G_2 \approx G_4$, then F is a partial graph mapping from G_3 to G_4. The theorem is a consequence of (5), (4), and (8).

(10) Suppose there exist sets E_1, E_2 such that G_3 is a graph given by reversing directions of the edges E_1 of G_1 and G_4 is a graph given by reversing directions of the edges E_2 of G_2. Then F is a partial graph mapping from G_3 to G_4. The theorem is a consequence of (5), (4), and (8).

Let G be a graph. The functor id_G yielding a partial graph mapping from G to G is defined by the term

(Def. 9) $\langle \mathrm{id}_\alpha, \mathrm{id}_\beta\rangle$, where α is the vertices of G and β is the edges of G.

Now we state the propositions:

(11) Let us consider graphs G_1, G_2. Suppose $G_1 \approx G_2$. Then

(i) $\mathrm{id}_{G_1} = \mathrm{id}_{G_2}$, and

(ii) id_{G_1} is a partial graph mapping from G_1 to G_2.

The theorem is a consequence of (9).

(12) Let us consider a graph G_1, a set E, and a graph G_2 given by reversing directions of the edges E of G_1. Then

(i) $\mathrm{id}_{G_1} = \mathrm{id}_{G_2}$, and

(ii) id_{G_1} is a partial graph mapping from G_1 to G_2.

Proof: There exist sets E_1, E_2 such that G_1 is a graph given by reversing directions of the edges E_1 of G_1 and G_2 is a graph given by reversing directions of the edges E_2 of G_1. $\square$

Let G_1, G_2 be graphs and F be a partial graph mapping from G_1 to G_2. We say that F is empty if and only if

(Def. 10) $\mathrm{dom}(F_{\mathbb{V}})$ is empty.

We say that F is total if and only if

(Def. 11) $\mathrm{dom}(F_{\mathbb{V}}) =$ the vertices of G_1 and $\mathrm{dom}(F_{\mathbb{E}}) =$ the edges of G_1.

We say that F is onto if and only if

(Def. 12) $\mathrm{rng}\, F_{\mathbb{V}} =$ the vertices of G_2 and $\mathrm{rng}\, F_{\mathbb{E}} =$ the edges of G_2.

We say that F is one-to-one if and only if

(Def. 13) $F_{\mathbb{V}}$ is one-to-one and $F_{\mathbb{E}}$ is one-to-one.

We say that F is directed if and only if

(Def. 14) for every objects e, v, w such that $e \in \mathrm{dom}(F_{\mathbb{E}})$ and $v, w \in \mathrm{dom}(F_{\mathbb{V}})$ holds if e joins v to w in G_1, then $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v)$ to $(F_{\mathbb{V}})(w)$ in G_2.

We say that F is semi-continuous if and only if

(Def. 15) for every objects e, v, w such that $e \in \mathrm{dom}(F_{\mathbb{E}})$ and $v, w \in \mathrm{dom}(F_{\mathbb{V}})$ holds if $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v)$ and $(F_{\mathbb{V}})(w)$ in G_2, then e joins v and w in G_1.

We say that F is continuous if and only if

(Def. 16) for every objects $\tilde{e}$, v, w such that $v, w \in \mathrm{dom}(F_{\mathbb{V}})$ and $\tilde{e}$ joins $(F_{\mathbb{V}})(v)$ and $(F_{\mathbb{V}})(w)$ in G_2 there exists an object e such that e joins v and w in G_1 and $e \in \mathrm{dom}(F_{\mathbb{E}})$ and $(F_{\mathbb{E}})(e) = \tilde{e}$.

We say that F is semi-directed-continuous if and only if

(Def. 17) for every objects e, v, w such that $e \in \mathrm{dom}(F_{\mathbb{E}})$ and $v, w \in \mathrm{dom}(F_{\mathbb{V}})$ holds if $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v)$ to $(F_{\mathbb{V}})(w)$ in G_2, then e joins v to w in G_1.

We say that F is directed-continuous if and only if

(Def. 18) for every objects $\tilde{e}$, v, w such that $v, w \in \mathrm{dom}(F_{\mathbb{V}})$ and $\tilde{e}$ joins $(F_{\mathbb{V}})(v)$ to $(F_{\mathbb{V}})(w)$ in G_2 there exists an object e such that e joins v to w in G_1 and $e \in \mathrm{dom}(F_{\mathbb{E}})$ and $(F_{\mathbb{E}})(e) = \tilde{e}$.

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(13) F is directed if and only if for every object e such that $e \in \mathrm{dom}(F_{\mathbb{E}})$ holds (the source of G_2)$((F_{\mathbb{E}})(e)) = (F_{\mathbb{V}})($(the source of G_1)$(e))$ and (the target of G_2)$((F_{\mathbb{E}})(e)) = (F_{\mathbb{V}})($(the target of G_1)$(e))$. The theorem is a consequence of (5).

(14) F is directed if and only if (the source of G_2) $\cdot (F_{\mathbb{E}}) = (F_{\mathbb{V}}) \cdot$ ((the source of G_1)$\restriction \mathrm{dom}(F_{\mathbb{E}})$) and (the target of G_2) $\cdot (F_{\mathbb{E}}) = (F_{\mathbb{V}}) \cdot$ ((the target of G_1)$\restriction \mathrm{dom}(F_{\mathbb{E}})$). The theorem is a consequence of (13) and (5).

(15) F is semi-continuous if and only if for every objects e, v, w such that $e \in \mathrm{dom}(F_{\mathbb{E}})$ and $v, w \in \mathrm{dom}(F_{\mathbb{V}})$ holds e joins v and w in G_1 iff $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v)$ and $(F_{\mathbb{V}})(w)$ in G_2.

(16) F is semi-directed-continuous if and only if for every objects e, v, w such that $e \in \mathrm{dom}(F_{\mathbb{E}})$ and $v, w \in \mathrm{dom}(F_{\mathbb{V}})$ holds e joins v to w in G_1 iff $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v)$ to $(F_{\mathbb{V}})(w)$ in G_2.

Proof: If F is semi-directed-continuous, then for every objects e, v, w such that $e \in \mathrm{dom}(F_{\mathbb{E}})$ and $v, w \in \mathrm{dom}(F_{\mathbb{V}})$ holds e joins v to w in G_1 iff $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v)$ to $(F_{\mathbb{V}})(w)$ in G_2. $\square$

Let G_1, G_2 be graphs. Note that there exists a partial graph mapping from G_1 to G_2 which is empty, one-to-one, directed-continuous, directed, continuous, semi-directed-continuous, and semi-continuous and there exists a partial graph mapping from G_1 to G_2 which is non empty, one-to-one, directed, semi-directed-continuous, and semi-continuous.

Let F be an empty partial graph mapping from G_1 to G_2. One can verify that $F_{\mathbb{V}}$ is empty as a set and $F_{\mathbb{E}}$ is empty as a set.

Let F be a non empty partial graph mapping from G_1 to G_2. One can verify that $F_{\mathbb{V}}$ is non empty as a set.

Let F be a one-to-one partial graph mapping from G_1 to G_2. One can verify that $F_{\mathbb{V}}$ is one-to-one as a function and $F_{\mathbb{E}}$ is one-to-one as a function.

Now we state the propositions:

(17) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. If $F_{\mathbb{V}}$ is one-to-one, then F is semi-continuous. The theorem is a consequence of (5) and (4).

(18) Let us consider graphs G_1, G_2, and a directed partial graph mapping F from G_1 to G_2. If $F_{\mathbb{V}}$ is one-to-one, then F is semi-directed-continuous. The theorem is a consequence of (5).

(19) Let us consider graphs G_1, G_2, and a semi-continuous partial graph mapping F from G_1 to G_2. Suppose $\mathrm{rng}\, F_{\mathbb{E}} =$ the edges of G_2. Then F is continuous.

(20) Let us consider graphs G_1, G_2, and a semi-directed-continuous partial graph mapping F from G_1 to G_2. Suppose $\operatorname{rng} F_{\mathbb{E}} =$ the edges of G_2. Then F is directed-continuous.

(21) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Suppose $F_{\mathbb{V}}$ is one-to-one and $\operatorname{rng} F_{\mathbb{E}} =$ the edges of G_2. Then F is continuous. The theorem is a consequence of (17) and (19).

(22) Let us consider graphs G_1, G_2, and a directed partial graph mapping F from G_1 to G_2. Suppose $F_{\mathbb{V}}$ is one-to-one and $\operatorname{rng} F_{\mathbb{E}} =$ the edges of G_2. Then F is directed-continuous. The theorem is a consequence of (18) and (20).

(23) Let us consider graphs G_1, G_2, and a continuous partial graph mapping F from G_1 to G_2. If $F_{\mathbb{E}}$ is one-to-one, then F is semi-continuous.

Let us consider graphs G_1, G_2 and a directed-continuous partial graph mapping F from G_1 to G_2. Now we state the propositions:

(24) If $F_{\mathbb{E}}$ is one-to-one, then F is semi-directed-continuous.

(25) If $F_{\mathbb{E}}$ is one-to-one, then F is directed. The theorem is a consequence of (4).

(26) Let us consider graphs G_1, G_2, a semi-continuous partial graph mapping F from G_1 to G_2, and objects v_1, v_2. Suppose $v_1, v_2 \in \operatorname{dom}(F_{\mathbb{V}})$ and $(F_{\mathbb{V}})(v_1) = (F_{\mathbb{V}})(v_2)$ and there exist objects e, w such that $e \in \operatorname{dom}(F_{\mathbb{E}})$ and $w \in \operatorname{dom}(F_{\mathbb{V}})$ and $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v_1)$ and $(F_{\mathbb{V}})(w)$ in G_2. Then $v_1 = v_2$.

(27) Let us consider graphs G_1, G_2, and a semi-continuous partial graph mapping F from G_1 to G_2. Suppose for every object v such that $v \in \operatorname{dom}(F_{\mathbb{V}})$ there exist objects e, w such that $e \in \operatorname{dom}(F_{\mathbb{E}})$ and $w \in \operatorname{dom}(F_{\mathbb{V}})$ and $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v)$ and $(F_{\mathbb{V}})(w)$ in G_2. Then $F_{\mathbb{V}}$ is one-to-one. The theorem is a consequence of (26).

(28) Let us consider graphs G_1, G_2, a semi-directed-continuous partial graph mapping F from G_1 to G_2, and objects v_1, v_2. Suppose $v_1, v_2 \in \operatorname{dom}(F_{\mathbb{V}})$ and $(F_{\mathbb{V}})(v_1) = (F_{\mathbb{V}})(v_2)$ and there exist objects e, w such that $e \in \operatorname{dom}(F_{\mathbb{E}})$ and $w \in \operatorname{dom}(F_{\mathbb{V}})$ and $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v_1)$ to $(F_{\mathbb{V}})(w)$ in G_2. Then $v_1 = v_2$.

(29) Let us consider graphs G_1, G_2, and a semi-directed-continuous partial graph mapping F from G_1 to G_2. Suppose for every object v such that $v \in \operatorname{dom}(F_{\mathbb{V}})$ there exist objects e, w such that $e \in \operatorname{dom}(F_{\mathbb{E}})$ and $w \in \operatorname{dom}(F_{\mathbb{V}})$ and $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v)$ to $(F_{\mathbb{V}})(w)$ in G_2. Then $F_{\mathbb{V}}$ is one-to-one. The theorem is a consequence of (28).

Let G_1, G_2 be graphs. One can verify that every partial graph mapping from

G_1 to G_2 which is one-to-one is also semi-continuous and every partial graph mapping from G_1 to G_2 which is one-to-one and directed is also semi-directed-continuous and every partial graph mapping from G_1 to G_2 which is one-to-one and onto is also continuous and every partial graph mapping from G_1 to G_2 which is directed, one-to-one, and onto is also directed-continuous.

Every partial graph mapping from G_1 to G_2 which is semi-continuous and onto is also continuous and every partial graph mapping from G_1 to G_2 which is semi-directed-continuous is also directed and semi-continuous and every partial graph mapping from G_1 to G_2 which is semi-directed-continuous and onto is also directed-continuous and every partial graph mapping from G_1 to G_2 which is directed-continuous is also continuous.

Every partial graph mapping from G_1 to G_2 which is directed-continuous and one-to-one is also directed and semi-directed-continuous and every partial graph mapping from G_1 to G_2 which is empty is also one-to-one, directed-continuous, directed, and continuous and every partial graph mapping from G_1 to G_2 which is total is also non empty and every partial graph mapping from G_1 to G_2 which is onto is also non empty.

Let G be a graph. One can verify that id_G is total, non empty, onto, one-to-one, and directed-continuous.

Let us consider graphs G_1, G_2, a partial function f from the vertices of G_1 to the vertices of G_2, and a partial function g from the edges of G_1 to the edges of G_2. Now we state the propositions:

(30) Suppose for every object e such that $e \in \mathrm{dom}\, g$ holds (the source of $G_1)(e)$, (the target of $G_1)(e) \in \mathrm{dom}\, f$ and for every objects e, v, w such that $e \in \mathrm{dom}\, g$ and v, $w \in \mathrm{dom}\, f$ holds if e joins v to w in G_1, then $g(e)$ joins $f(v)$ to $f(w)$ in G_2. Then $\langle f, g\rangle$ is a directed partial graph mapping from G_1 to G_2. The theorem is a consequence of (8).

(31) Suppose for every object e such that $e \in \mathrm{dom}\, g$ holds (the source of $G_1)(e)$, (the target of $G_1)(e) \in \mathrm{dom}\, f$ and for every objects e, v, w such that $e \in \mathrm{dom}\, g$ and v, $w \in \mathrm{dom}\, f$ holds e joins v and w in G_1 iff $g(e)$ joins $f(v)$ and $f(w)$ in G_2. Then $\langle f, g\rangle$ is a semi-continuous partial graph mapping from G_1 to G_2. The theorem is a consequence of (8).

(32) Suppose for every object e such that $e \in \mathrm{dom}\, g$ holds (the source of $G_1)(e)$, (the target of $G_1)(e) \in \mathrm{dom}\, f$ and for every objects e, v, w such that $e \in \mathrm{dom}\, g$ and v, $w \in \mathrm{dom}\, f$ holds e joins v to w in G_1 iff $g(e)$ joins $f(v)$ to $f(w)$ in G_2. Then $\langle f, g\rangle$ is a semi-directed-continuous partial graph mapping from G_1 to G_2. The theorem is a consequence of (8).

(33) Let us consider graphs G_1, G_2. Then $\langle \emptyset, \emptyset\rangle$ is an empty, one-to-one, directed-continuous partial graph mapping from G_1 to G_2.

(34) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Suppose F is total. Let us consider a vertex v of G_1. Then $(F_{\mathbb{V}})(v)$ is a vertex of G_2.

(35) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Suppose F is total. Then

(i) if G_2 is loopless, then G_1 is loopless, and

(ii) if G_2 is edgeless, then G_1 is edgeless.

The theorem is a consequence of (4).

(36) Let us consider graphs G_1, G_2, and a continuous partial graph mapping F from G_1 to G_2. Suppose $\operatorname{rng} F_{\mathbb{V}} =$ the vertices of G_2. If G_1 is loopless, then G_2 is loopless.

Proof: For every object v, there exists no object e such that e joins v and v in G_2. □

(37) Let us consider graphs G_1, G_2, and a semi-continuous partial graph mapping F from G_1 to G_2. If F is onto, then if G_1 is loopless, then G_2 is loopless.

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(38) If $\operatorname{rng} F_{\mathbb{E}} =$ the edges of G_2, then if G_1 is edgeless, then G_2 is edgeless.

(39) If F is onto, then if G_1 is edgeless, then G_2 is edgeless.

(40) Let us consider a graph G_1, a non-multi graph G_2, and partial graph mappings F_1, F_2 from G_1 to G_2. Suppose $F_{1\mathbb{V}} = F_{2\mathbb{V}}$ and $\operatorname{dom}(F_{1\mathbb{E}}) = \operatorname{dom}(F_{2\mathbb{E}})$. Then $F_1 = F_2$. The theorem is a consequence of (5) and (4).

(41) Let us consider a graph G_1, a non-directed-multi graph G_2, and directed partial graph mappings F_1, F_2 from G_1 to G_2. Suppose $F_{1\mathbb{V}} = F_{2\mathbb{V}}$ and $\operatorname{dom}(F_{1\mathbb{E}}) = \operatorname{dom}(F_{2\mathbb{E}})$. Then $F_1 = F_2$. The theorem is a consequence of (5).

(42) Let us consider a non-multi graph G_1, a graph G_2, and a semi-continuous partial graph mapping F from G_1 to G_2. Then $F_{\mathbb{E}}$ is one-to-one. The theorem is a consequence of (5) and (4).

(43) Let us consider a non-multi graph G_1, a graph G_2, and a partial graph mapping F from G_1 to G_2. If $F_{\mathbb{V}}$ is one-to-one, then $F_{\mathbb{E}}$ is one-to-one. The theorem is a consequence of (5) and (4).

(44) Let us consider a non-directed-multi graph G_1, a graph G_2, and a directed partial graph mapping F from G_1 to G_2. If $F_{\mathbb{V}}$ is one-to-one, then $F_{\mathbb{E}}$ is one-to-one. The theorem is a consequence of (5).

Let G_1 be a graph and G_2 be a loopless graph. Observe that every partial graph mapping from G_1 to G_2 which is directed and semi-continuous is also semi-directed-continuous and every partial graph mapping from G_1 to G_2 which is directed and continuous is also directed-continuous.

Let G_1 be a trivial graph and G_2 be a graph. Observe that every partial graph mapping from G_1 to G_2 is directed and every partial graph mapping from G_1 to G_2 which is semi-continuous is also semi-directed-continuous and every partial graph mapping from G_1 to G_2 which is continuous is also directed-continuous.

Let G_1 be a trivial, non-directed-multi graph. Note that every partial graph mapping from G_1 to G_2 is one-to-one.

Let G_1 be a trivial, edgeless graph. Observe that every partial graph mapping from G_1 to G_2 which is non empty is also total.

Let G_1 be a graph and G_2 be a trivial, edgeless graph. Note that every partial graph mapping from G_1 to G_2 which is non empty is also onto and every partial graph mapping from G_1 to G_2 is semi-continuous and continuous.

Let G_1, G_2 be graphs and F be a partial graph mapping from G_1 to G_2. We say that F is weak subgraph embedding if and only if

(Def. 19) F is total and one-to-one.

We say that F is strong subgraph embedding if and only if

(Def. 20) F is total, one-to-one, and continuous.

We say that F is isomorphism if and only if

(Def. 21) F is total, one-to-one, and onto.

We say that F is directed-isomorphism if and only if

(Def. 22) F is directed, total, one-to-one, and onto.

One can check that every partial graph mapping from G_1 to G_2 which is weak subgraph embedding is also total, non empty, one-to-one, and semi-continuous and every partial graph mapping from G_1 to G_2 which is total and one-to-one is also weak subgraph embedding and every partial graph mapping from G_1 to G_2 which is strong subgraph embedding is also total, non empty, one-to-one, continuous, and weak subgraph embedding and every partial graph mapping from G_1 to G_2 which is total, one-to-one, and continuous is also strong subgraph embedding.

Every partial graph mapping from G_1 to G_2 which is weak subgraph embedding and continuous is also strong subgraph embedding and every partial graph mapping from G_1 to G_2 which is isomorphism is also onto, semi-continuous, continuous, total, non empty, one-to-one, weak subgraph embedding, and strong subgraph embedding and every partial graph mapping from G_1 to G_2 which is total, one-to-one, onto, and continuous is also isomorphism and every partial graph mapping from G_1 to G_2 which is strong subgraph embedding and onto is also isomorphism.

Every partial graph mapping from G_1 to G_2 which is weak subgraph embedding, continuous, and onto is also isomorphism and every partial graph mapping from G_1 to G_2 which is directed-isomorphism is also directed, isomorphism, continuous, total, non empty, semi-directed-continuous, semi-continuous, one-to-one, weak subgraph embedding, and strong subgraph embedding and every partial graph mapping from G_1 to G_2 which is directed and isomorphism is also directed-continuous and directed-isomorphism.

Let G be a graph. Let us note that id_G is weak subgraph embedding, strong subgraph embedding, isomorphism, and directed-isomorphism and there exists a partial graph mapping from G to G which is weak subgraph embedding, strong subgraph embedding, isomorphism, and directed-isomorphism.

Now we state the propositions:

(45) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Suppose F is weak subgraph embedding. Then

(i) $\overline{\overline{G_1.\mathrm{order}()}} \subseteq \overline{\overline{G_2.\mathrm{order}()}}$, and

(ii) $\overline{\overline{G_1.\mathrm{size}()}} \subseteq \overline{\overline{G_2.\mathrm{size}()}}$.

(46) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and subsets X, Y of the vertices of G_1. Suppose F is weak subgraph embedding. Then $\overline{\overline{G_1.\mathrm{edgesBetween}(X,Y)}} \subseteq \overline{\overline{G_2.\mathrm{edgesBetween}((F_{\mathbb{V}})^\circ X, (F_{\mathbb{V}})^\circ Y)}}$.
PROOF: Set $f = F_{\mathbb{E}}{\restriction}G_1.\mathrm{edgesBetween}(X,Y)$. For every object y such that $y \in \mathrm{rng}\, f$ holds $y \in G_2.\mathrm{edgesBetween}((F_{\mathbb{V}})^\circ X, (F_{\mathbb{V}})^\circ Y)$. $\square$

(47) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and a subset X of the vertices of G_1. Suppose F is weak subgraph embedding. Then $\overline{\overline{G_1.\mathrm{edgesBetween}(X)}} \subseteq \overline{\overline{G_2.\mathrm{edgesBetween}((F_{\mathbb{V}})^\circ X)}}$.
PROOF: Set $f = F_{\mathbb{E}}{\restriction}G_1.\mathrm{edgesBetween}(X)$. For every object y such that $y \in \mathrm{rng}\, f$ holds $y \in G_2.\mathrm{edgesBetween}((F_{\mathbb{V}})^\circ X)$. $\square$

(48) Let us consider graphs G_1, G_2, a directed partial graph mapping F from G_1 to G_2, and subsets X, Y of the vertices of G_1. Suppose F is weak subgraph embedding. Then $\overline{\overline{G_1.\mathrm{edgesDBetween}(X,Y)}} \subseteq$ $\overline{\overline{G_2.\mathrm{edgesDBetween}((F_{\mathbb{V}})^\circ X, (F_{\mathbb{V}})^\circ Y)}}$.
PROOF: Set $f = F_{\mathbb{E}}{\restriction}G_1.\mathrm{edgesDBetween}(X,Y)$. For every object y such that $y \in \mathrm{rng}\, f$ holds $y \in G_2.\mathrm{edgesDBetween}((F_{\mathbb{V}})^\circ X, (F_{\mathbb{V}})^\circ Y)$. $\square$

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(49) Suppose F is weak subgraph embedding. Then

(i) if G_2 is trivial, then G_1 is trivial, and

(ii) if G_2 is non-multi, then G_1 is non-multi, and

(iii) if G_2 is simple, then G_1 is simple, and

(iv) if G_2 is finite, then G_1 is finite.

PROOF: If G_2 is non-multi, then G_1 is non-multi. G_1.order() $\subseteq G_2$.order() and G_1.size() $\subseteq G_2$.size(). □

(50) Suppose F is directed and weak subgraph embedding. Then

(i) if G_2 is non-directed-multi, then G_1 is non-directed-multi, and

(ii) if G_2 is directed-simple, then G_1 is directed-simple.

PROOF: If G_2 is non-directed-multi, then G_1 is non-directed-multi. G_1 is loopless and non-directed-multi. □

(51) Let us consider finite graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Suppose F is strong subgraph embedding and G_1.order() $=$ G_2.order() and G_1.size() $= G_2$.size(). Then F is isomorphism.

(52) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Suppose F is strong subgraph embedding. If G_2 is complete, then G_1 is complete.

Let G_1, G_2 be graphs. We say that G_2 is G_1-isomorphic if and only if

(Def. 23) there exists a partial graph mapping F from G_1 to G_2 such that F is isomorphism.

We say that G_2 is G_1-directed-isomorphic if and only if

(Def. 24) there exists a partial graph mapping F from G_1 to G_2 such that F is directed-isomorphism.

Let G be a graph. Note that every graph which is G-directed-isomorphic is also G-isomorphic and there exists a graph which is G-directed-isomorphic and G-isomorphic.

Now we state the proposition:

(53) Every graph is directed-isomorphic and isomorphic to itself.

Let G_1 be a graph and G_2 be a G_1-isomorphic graph. Let us observe that there exists a partial graph mapping from G_1 to G_2 which is isomorphism, strong subgraph embedding, weak subgraph embedding, total, non empty, one-to-one, onto, semi-continuous, and continuous.

An isomorphism between G_1 and G_2 is an isomorphism partial graph mapping from G_1 to G_2. Let G_2 be a G_1-directed-isomorphic graph. One can verify that there exists a partial graph mapping from G_1 to G_2 which is isomorphism, strong subgraph embedding, weak subgraph embedding, total, non empty, one-to-one, onto, directed, semi-directed-continuous, and directed-continuous.

A directed isomorphism of G_1 and G_2 is a directed-isomorphism partial graph mapping from G_1 to G_2. Let G_1, G_2 be w-graphs and F be a partial graph mapping from G_1 to G_2. We say that F preserves weight if and only if

(Def. 25) (the weight of G_2) $\cdot$ $(F_{\mathbb{E}})$ = (the weight of G_1)$\restriction$dom$(F_{\mathbb{E}})$.

Let G_1, G_2 be e-graphs. We say that F preserves elabel if and only if

(Def. 26) (the elabel of G_2) $\cdot$ $(F_{\mathbb{E}})$ = (the elabel of G_1)$\restriction$dom$(F_{\mathbb{E}})$.

Let G_1, G_2 be v-graphs. We say that F preserves vlabel if and only if

(Def. 27) (the vlabel of G_2) $\cdot$ $(F_{\mathbb{V}})$ = (the vlabel of G_1)$\restriction$dom$(F_{\mathbb{V}})$.

Let G_1, G_2 be ordered graphs. We say that F preserves ordering if and only if

(Def. 28) (the ordering of G_2) $\cdot$ $(F_{\mathbb{V}})$ = the ordering of $G_1$$\restriction$dom$(F_{\mathbb{V}})$.

Let G be a w-graph. Note that id_G preserves weight.

Let G be an e-graph. Let us note that id_G preserves elabel.

Let G be a v-graph. Observe that id_G preserves vlabel.

Let G be an ordered graph. Let us observe that id_G preserves ordering.

Let G_1, G_2 be graphs and F be a partial graph mapping from G_1 to G_2. The functor dom F yielding a subgraph of G_1 induced by dom$(F_{\mathbb{V}})$ and dom$(F_{\mathbb{E}})$ is defined by the term

(Def. 29) the plain subgraph of G_1 induced by dom$(F_{\mathbb{V}})$ and dom$(F_{\mathbb{E}})$.

The functor rng F yielding a subgraph of G_2 induced by rng $F_{\mathbb{V}}$ and rng $F_{\mathbb{E}}$ is defined by the term

(Def. 30) the plain subgraph of G_2 induced by rng $F_{\mathbb{V}}$ and rng $F_{\mathbb{E}}$.

One can verify that dom F is plain and rng F is plain.

Let us consider graphs G_1, G_2 and a non empty partial graph mapping F from G_1 to G_2. Now we state the propositions:

(54) (i) the vertices of dom F = dom$(F_{\mathbb{V}})$, and

(ii) the edges of dom F = dom$(F_{\mathbb{E}})$, and

(iii) the vertices of rng F = rng $F_{\mathbb{V}}$, and

(iv) the edges of rng F = rng $F_{\mathbb{E}}$.

The theorem is a consequence of (7).

(55) F is total if and only if dom $F \approx G_1$. The theorem is a consequence of (54).

(56) F is onto if and only if rng $F \approx G_2$. The theorem is a consequence of (54).

Let G_1, G_2 be graphs, H be a subgraph of G_1, and F be a partial graph mapping from G_1 to G_2. The functor $F\restriction H$ yielding a partial graph mapping from H to G_2 is defined by the term

(Def. 31) $\langle F_{\mathbb{V}}\restriction$(the vertices of H), $F_{\mathbb{E}}\restriction$(the edges of H)$\rangle$.

Now we state the propositions:

(57) Let us consider graphs G_1, G_2, a subgraph H of G_1, and a partial graph mapping F from G_1 to G_2. Then

(i) if F is empty, then $F{\restriction}H$ is empty, and

(ii) if F is total, then $F{\restriction}H$ is total, and

(iii) if F is one-to-one, then $F{\restriction}H$ is one-to-one, and

(iv) if F is weak subgraph embedding, then $F{\restriction}H$ is weak subgraph embedding, and

(v) if F is semi-continuous, then $F{\restriction}H$ is semi-continuous, and

(vi) if F is not onto, then $F{\restriction}H$ is not onto, and

(vii) if F is directed, then $F{\restriction}H$ is directed, and

(viii) if F is semi-directed-continuous, then $F{\restriction}H$ is semi-directed-continuous.

Proof: If F is total, then $F{\restriction}H$ is total. If F is semi-continuous, then $F{\restriction}H$ is semi-continuous. If $F{\restriction}H$ is onto, then F is onto. If F is directed, then $F{\restriction}H$ is directed. If F is semi-directed-continuous, then $F{\restriction}H$ is semi-directed-continuous. □

(58) Let us consider graphs G_1, G_2, a set V, a subgraph H of G_1 induced by V, and a partial graph mapping F from G_1 to G_2. Then

(i) if F is continuous, then $F{\restriction}H$ is continuous, and

(ii) if F is strong subgraph embedding, then $F{\restriction}H$ is strong subgraph embedding, and

(iii) if F is directed-continuous, then $F{\restriction}H$ is directed-continuous.

The theorem is a consequence of (57).

Let G_1, G_2 be graphs, H be a subgraph of G_1, and F be an empty partial graph mapping from G_1 to G_2. Let us observe that $F{\restriction}H$ is empty.

Let F be a one-to-one partial graph mapping from G_1 to G_2. Let us observe that $F{\restriction}H$ is one-to-one.

Let F be a semi-continuous partial graph mapping from G_1 to G_2. Observe that $F{\restriction}H$ is semi-continuous.

Let V be a set, H be a subgraph of G_1 induced by V, and F be a continuous partial graph mapping from G_1 to G_2. Let us observe that $F{\restriction}H$ is continuous.

Let H be a subgraph of G_1 and F be a directed partial graph mapping from G_1 to G_2. Note that $F{\restriction}H$ is directed.

Let F be a semi-directed-continuous partial graph mapping from G_1 to G_2. One can check that $F{\restriction}H$ is semi-directed-continuous.

Let V be a set, H be a subgraph of G_1 induced by V, and F be a directed-continuous partial graph mapping from G_1 to G_2. Note that $F{\restriction}H$ is directed-continuous.

Let F be a non empty partial graph mapping from G_1 to G_2. One can verify that $F \restriction \operatorname{dom} F$ is total.

Now we state the propositions:

(59) Let us consider graphs G_1, G_2, a subgraph H of G_1, and a partial graph mapping F from G_1 to G_2. Then

(i) $\operatorname{dom}((F \restriction H)_{\mathbb{V}}) = \operatorname{dom}(F_{\mathbb{V}}) \cap$ (the vertices of H), and

(ii) $\operatorname{dom}((F \restriction H)_{\mathbb{E}}) = \operatorname{dom}(F_{\mathbb{E}}) \cap$ (the edges of H).

(60) Let us consider w-graphs G_1, G_2, a w-subgraph H of G_1, and a partial graph mapping F from G_1 to G_2. If F preserves weight, then $F \restriction H$ preserves weight. The theorem is a consequence of (59).

(61) Let us consider e-graphs G_1, G_2, an e-subgraph H of G_1, and a partial graph mapping F from G_1 to G_2. If F preserves elabel, then $F \restriction H$ preserves elabel. The theorem is a consequence of (59).

(62) Let us consider v-graphs G_1, G_2, a v-subgraph H of G_1, and a partial graph mapping F from G_1 to G_2. If F preserves vlabel, then $F \restriction H$ preserves vlabel. The theorem is a consequence of (59).

Let G_1, G_2 be graphs, H be a subgraph of G_2, and F be a partial graph mapping from G_1 to G_2. The functor $H \upharpoonright F$ yielding a partial graph mapping from G_1 to H is defined by the term

(Def. 32) $\langle$(the vertices of $H) \upharpoonright F_{\mathbb{V}}$, (the edges of $H) \upharpoonright F_{\mathbb{E}}\rangle$.

Now we state the proposition:

(63) Let us consider graphs G_1, G_2, a subgraph H of G_2, and a partial graph mapping F from G_1 to G_2. Then

(i) if F is empty, then $H \upharpoonright F$ is empty, and

(ii) if F is one-to-one, then $H \upharpoonright F$ is one-to-one, and

(iii) if F is onto, then $H \upharpoonright F$ is onto, and

(iv) if F is not total, then $H \upharpoonright F$ is not total, and

(v) if F is directed, then $H \upharpoonright F$ is directed, and

(vi) if F is semi-continuous, then $H \upharpoonright F$ is semi-continuous, and

(vii) if F is continuous, then $H \upharpoonright F$ is continuous, and

(viii) if F is semi-directed-continuous, then $H \upharpoonright F$ is semi-directed-continuous, and

(ix) if F is directed-continuous, then $H \upharpoonright F$ is directed-continuous.

Proof: If F is onto, then $H \upharpoonright F$ is onto. If F is directed, then $H \upharpoonright F$ is directed. If F is semi-continuous, then $H \upharpoonright F$ is semi-continuous. If F is continuous, then $H \upharpoonright F$ is continuous. If F is semi-directed-continuous, then

$H|F$ is semi-directed-continuous. If F is directed-continuous, then $H|F$ is directed-continuous. □

Let G_1, G_2 be graphs, H be a subgraph of G_2, and F be an empty partial graph mapping from G_1 to G_2. One can verify that $H|F$ is empty.

Let F be a one-to-one partial graph mapping from G_1 to G_2. Let us observe that $H|F$ is one-to-one.

Let F be a semi-continuous partial graph mapping from G_1 to G_2. Observe that $H|F$ is semi-continuous.

Let F be a continuous partial graph mapping from G_1 to G_2. Let us note that $H|F$ is continuous.

Let F be a directed partial graph mapping from G_1 to G_2. Note that $H|F$ is directed.

Let F be a semi-directed-continuous partial graph mapping from G_1 to G_2. One can check that $H|F$ is semi-directed-continuous.

Let F be a directed-continuous partial graph mapping from G_1 to G_2. One can verify that $H|F$ is directed-continuous.

Let F be a non empty partial graph mapping from G_1 to G_2. Observe that $\operatorname{rng} F|F$ is onto.

Now we state the propositions:

(64) Let us consider graphs G_1, G_2, a subgraph H of G_2, and a partial graph mapping F from G_1 to G_2. Then

(i) $\operatorname{rng}(H|F)_{\mathbb{V}} = \operatorname{rng} F_{\mathbb{V}} \cap (\text{the vertices of } H)$, and

(ii) $\operatorname{rng}(H|F)_{\mathbb{E}} = \operatorname{rng} F_{\mathbb{E}} \cap (\text{the edges of } H)$.

(65) Let us consider w-graphs G_1, G_2, a w-subgraph H of G_2, and a partial graph mapping F from G_1 to G_2. If F preserves weight, then $H|F$ preserves weight.

(66) Let us consider e-graphs G_1, G_2, an e-subgraph H of G_2, and a partial graph mapping F from G_1 to G_2. If F preserves elabel, then $H|F$ preserves elabel.

(67) Let us consider v-graphs G_1, G_2, a v-subgraph H of G_2, and a partial graph mapping F from G_1 to G_2. If F preserves vlabel, then $H|F$ preserves vlabel.

(68) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, a subgraph H_1 of G_1, and a subgraph H_2 of G_2. Then $(H_2|F)\upharpoonright H_1 = H_2|(F\upharpoonright H_1)$.

Let G_1, G_2 be graphs and F be a one-to-one partial graph mapping from G_1 to G_2. The functor F^{-1} yielding a partial graph mapping from G_2 to G_1 is defined by the term

(Def. 33) $\langle (F_{\mathbb{V}})^{-1}, (F_{\mathbb{E}})^{-1} \rangle$.

One can verify that F^{-1} is one-to-one and semi-continuous.

Let F be an empty, one-to-one partial graph mapping from G_1 to G_2. One can verify that F^{-1} is empty.

Let F be a non empty, one-to-one partial graph mapping from G_1 to G_2. Let us note that F^{-1} is non empty.

Let F be a one-to-one, semi-directed-continuous partial graph mapping from G_1 to G_2. One can verify that F^{-1} is semi-directed-continuous.

Let us consider graphs G_1, G_2 and a one-to-one partial graph mapping F from G_1 to G_2. Now we state the propositions:

(69) (i) ${F^{-1}}_{\mathbb{V}} = (F_{\mathbb{V}})^{-1}$, and

(ii) ${F^{-1}}_{\mathbb{E}} = (F_{\mathbb{E}})^{-1}$.

(70) $(F^{-1})^{-1} = F$.

(71) F is total if and only if F^{-1} is onto.

(72) F is onto if and only if F^{-1} is total.

(73) If F is total and continuous, then F^{-1} is continuous.

(74) If F is total and directed-continuous, then F^{-1} is directed-continuous.

(75) F is isomorphism if and only if F^{-1} is isomorphism.

(76) Let us consider w-graphs G_1, G_2, and a one-to-one partial graph mapping F from G_1 to G_2. Then F preserves weight if and only if F^{-1} preserves weight. The theorem is a consequence of (2) and (70).

(77) Let us consider e-graphs G_1, G_2, and a one-to-one partial graph mapping F from G_1 to G_2. Then F preserves elabel if and only if F^{-1} preserves elabel. The theorem is a consequence of (2) and (70).

(78) Let us consider v-graphs G_1, G_2, and a one-to-one partial graph mapping F from G_1 to G_2. Then F preserves vlabel if and only if F^{-1} preserves vlabel. The theorem is a consequence of (2) and (70).

(79) Let us consider graphs G_1, G_2, and a one-to-one partial graph mapping F from G_1 to G_2. Suppose F is onto. Let us consider a vertex v of G_2. Then $({F^{-1}}_{\mathbb{V}})(v)$ is a vertex of G_1.

(80) Let us consider a graph G. Then $(\mathrm{id}_G)^{-1} = \mathrm{id}_G$.

(81) Let us consider graphs G_1, G_2, and a non empty, one-to-one partial graph mapping F from G_1 to G_2. Then

(i) $\operatorname{dom} F = \operatorname{rng} F^{-1}$, and

(ii) $\operatorname{rng} F = \operatorname{dom}(F^{-1})$.

The theorem is a consequence of (54).

(82) Let us consider graphs G_1, G_2, a one-to-one partial graph mapping F from G_1 to G_2, and a subgraph H of G_1. Then $(F{\restriction}H)^{-1} = H{\upharpoonleft}F^{-1}$.

(83) Let us consider graphs G_1, G_2, a one-to-one partial graph mapping F from G_1 to G_2, and a subgraph H of G_2. Then $(H{\upharpoonleft}F)^{-1} = F^{-1}{\restriction}H$. The theorem is a consequence of (82) and (70).

(84) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Suppose F is isomorphism. Then

(i) G_1.order() $= G_2$.order(), and

(ii) G_1.size() $= G_2$.size().

The theorem is a consequence of (45) and (75).

(85) Let us consider finite graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Suppose F is strong subgraph embedding. If there exists a partial graph mapping F_0 from G_1 to G_2 such that F_0 is isomorphism, then F is isomorphism. The theorem is a consequence of (84) and (51).

(86) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and subsets X, Y of the vertices of G_1. Suppose F is isomorphism. Then $\overline{\overline{G_1.\text{edgesBetween}(X,Y)}} = \overline{\overline{G_2.\text{edgesBetween}((F_{\mathbb{V}})^{\circ}X, (F_{\mathbb{V}})^{\circ}Y)}}$. The theorem is a consequence of (46) and (75).

(87) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and a subset X of the vertices of G_1. Suppose F is isomorphism. Then $\overline{\overline{G_1.\text{edgesBetween}(X)}} = \overline{\overline{G_2.\text{edgesBetween}((F_{\mathbb{V}})^{\circ}X)}}$. The theorem is a consequence of (47) and (75).

(88) Let us consider graphs G_1, G_2, a directed partial graph mapping F from G_1 to G_2, and subsets X, Y of the vertices of G_1. Suppose F is isomorphism. Then $\overline{\overline{G_1.\text{edgesDBetween}(X,Y)}} = \overline{\overline{G_2.\text{edgesDBetween}((F_{\mathbb{V}})^{\circ}X, (F_{\mathbb{V}})^{\circ}Y)}}$. The theorem is a consequence of (48) and (75).

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(89) Suppose F is isomorphism. Then

(i) G_1 is trivial iff G_2 is trivial, and

(ii) G_1 is loopless iff G_2 is loopless, and

(iii) G_1 is edgeless iff G_2 is edgeless, and

(iv) G_1 is non-multi iff G_2 is non-multi, and

(v) G_1 is simple iff G_2 is simple, and

(vi) G_1 is finite iff G_2 is finite, and

(vii) G_1 is complete iff G_2 is complete.

The theorem is a consequence of (75), (35), (49), and (52).

(90) Suppose F is directed-continuous and isomorphism. Then

(i) G_1 is non-directed-multi iff G_2 is non-directed-multi, and

(ii) G_1 is directed-simple iff G_2 is directed-simple.

The theorem is a consequence of (74), (75), and (50).

(91) Let us consider graphs G_1, G_2, and a non empty, one-to-one partial graph mapping F from G_1 to G_2. Then $\overline{\overline{\mathrm{dom}\, F.\mathrm{loops}()}} = \overline{\overline{\mathrm{rng}\, F.\mathrm{loops}()}}$. The theorem is a consequence of (81).

Let us consider graphs G_1, G_2 and a one-to-one partial graph mapping F from G_1 to G_2. Now we state the propositions:

(92) If F is total, then $\overline{\overline{G_1.\mathrm{loops}()}} \subseteq \overline{\overline{G_2.\mathrm{loops}()}}$. The theorem is a consequence of (55).

(93) If F is onto, then $\overline{\overline{G_2.\mathrm{loops}()}} \subseteq \overline{\overline{G_1.\mathrm{loops}()}}$. The theorem is a consequence of (72) and (92).

(94) If F is isomorphism, then $\overline{\overline{G_1.\mathrm{loops}()}} = \overline{\overline{G_2.\mathrm{loops}()}}$. The theorem is a consequence of (92) and (93).

(95) Let us consider a graph G_1, and a G_1-isomorphic graph G_2. Then G_1 is G_2-isomorphic. The theorem is a consequence of (75).

(96) Let us consider a graph G_1, and a G_1-directed-isomorphic graph G_2. Then G_1 is G_2-directed-isomorphic. The theorem is a consequence of (71) and (72).

Let us consider a graph G_1, a G_1-isomorphic graph G_2, a G_2-isomorphic graph G_3, and an isomorphism F between G_1 and G_2. Now we state the propositions:

(97) Suppose there exists a set E such that G_3 is a graph given by reversing directions of the edges E of G_1. Then F^{-1} is an isomorphism between G_2 and G_3.

Proof: Reconsider $F_2 = F^{-1}$ as a partial graph mapping from G_2 to G_3. F_2 is total. F_2 is onto. □

(98) If $G_1 \approx G_3$, then F^{-1} is an isomorphism between G_2 and G_3. The theorem is a consequence of (97).

(99) Let us consider a graph G_1, a G_1-directed-isomorphic graph G_2, a G_2-directed-isomorphic graph G_3, and a directed isomorphism F of G_1 and G_2. Suppose $G_1 \approx G_3$. Then F^{-1} is a directed isomorphism of G_2 and G_3.

Proof: Reconsider $F_2 = F^{-1}$ as a partial graph mapping from G_2 to G_3. F_2 is total. F_2 is onto. □

Let G_1, G_2, G_3 be graphs, F_1 be a partial graph mapping from G_1 to G_2, and F_2 be a partial graph mapping from G_2 to G_3. The functor $F_2 \cdot F_1$ yielding a partial graph mapping from G_1 to G_3 is defined by the term

(Def. 34) $\langle (F_{2\mathbb{V}}) \cdot (F_{1\mathbb{V}}), (F_{2\mathbb{E}}) \cdot (F_{1\mathbb{E}}) \rangle$.

Let us consider graphs G_1, G_2, G_3, a partial graph mapping F_1 from G_1 to G_2, and a partial graph mapping F_2 from G_2 to G_3. Now we state the propositions:

(100) (i) $F_2 \cdot F_{1\mathbb{V}} = (F_{2\mathbb{V}}) \cdot (F_{1\mathbb{V}})$, and

(ii) $F_2 \cdot F_{1\mathbb{E}} = (F_{2\mathbb{E}}) \cdot (F_{1\mathbb{E}})$.

(101) If $F_2 \cdot F_1$ is onto, then F_2 is onto.

(102) If $F_2 \cdot F_1$ is total, then F_1 is total.

Let G_1, G_2, G_3 be graphs, F_1 be a one-to-one partial graph mapping from G_1 to G_2, and F_2 be a one-to-one partial graph mapping from G_2 to G_3. Observe that $F_2 \cdot F_1$ is one-to-one.

Let F_1 be a semi-continuous partial graph mapping from G_1 to G_2 and F_2 be a semi-continuous partial graph mapping from G_2 to G_3. Let us observe that $F_2 \cdot F_1$ is semi-continuous.

Let F_1 be a continuous partial graph mapping from G_1 to G_2 and F_2 be a continuous partial graph mapping from G_2 to G_3. One can check that $F_2 \cdot F_1$ is continuous.

Let F_1 be a directed partial graph mapping from G_1 to G_2 and F_2 be a directed partial graph mapping from G_2 to G_3. One can check that $F_2 \cdot F_1$ is directed.

Let F_1 be a semi-directed-continuous partial graph mapping from G_1 to G_2 and F_2 be a semi-directed-continuous partial graph mapping from G_2 to G_3. Note that $F_2 \cdot F_1$ is semi-directed-continuous.

Let F_1 be a directed-continuous partial graph mapping from G_1 to G_2 and F_2 be a directed-continuous partial graph mapping from G_2 to G_3. Observe that $F_2 \cdot F_1$ is directed-continuous.

Let F_1 be an empty partial graph mapping from G_1 to G_2 and F_2 be a partial graph mapping from G_2 to G_3. Observe that $F_2 \cdot F_1$ is empty.

Let F_1 be a partial graph mapping from G_1 to G_2 and F_2 be an empty partial graph mapping from G_2 to G_3. Let us observe that $F_2 \cdot F_1$ is empty.

Let us consider graphs G_1, G_2, G_3, a partial graph mapping F_1 from G_1 to G_2, and a partial graph mapping F_2 from G_2 to G_3. Now we state the propositions:

(103) Suppose F_1 is total and $\operatorname{rng} F_{1\mathbb{V}} \subseteq \operatorname{dom}(F_{2\mathbb{V}})$ and $\operatorname{rng} F_{1\mathbb{E}} \subseteq \operatorname{dom}(F_{2\mathbb{E}})$. Then $F_2 \cdot F_1$ is total.

(104) If F_1 is total and F_2 is total, then $F_2 \cdot F_1$ is total. The theorem is a consequence of (103).

(105) Suppose F_2 is onto and $\mathrm{dom}(F_{2\mathbb{V}}) \subseteq \mathrm{rng}\, F_{1\mathbb{V}}$ and $\mathrm{dom}(F_{2\mathbb{E}}) \subseteq \mathrm{rng}\, F_{1\mathbb{E}}$. Then $F_2 \cdot F_1$ is onto.

(106) If F_1 is onto and F_2 is onto, then $F_2 \cdot F_1$ is onto. The theorem is a consequence of (105).

(107) If F_1 is weak subgraph embedding and F_2 is weak subgraph embedding, then $F_2 \cdot F_1$ is weak subgraph embedding.

(108) If F_1 is strong subgraph embedding and F_2 is strong subgraph embedding, then $F_2 \cdot F_1$ is strong subgraph embedding.

(109) If F_1 is isomorphism and F_2 is isomorphism, then $F_2 \cdot F_1$ is isomorphism.

(110) If F_1 is directed-isomorphism and F_2 is directed-isomorphism, then $F_2 \cdot F_1$ is directed-isomorphism. The theorem is a consequence of (109).

(111) Let us consider w-graphs G_1, G_2, G_3, a partial graph mapping F_1 from G_1 to G_2, and a partial graph mapping F_2 from G_2 to G_3. Suppose F_1 preserves weight and F_2 preserves weight. Then $F_2 \cdot F_1$ preserves weight. The theorem is a consequence of (1).

(112) Let us consider e-graphs G_1, G_2, G_3, a partial graph mapping F_1 from G_1 to G_2, and a partial graph mapping F_2 from G_2 to G_3. Suppose F_1 preserves elabel and F_2 preserves elabel. Then $F_2 \cdot F_1$ preserves elabel. The theorem is a consequence of (1).

(113) Let us consider v-graphs G_1, G_2, G_3, a partial graph mapping F_1 from G_1 to G_2, and a partial graph mapping F_2 from G_2 to G_3. Suppose F_1 preserves vlabel and F_2 preserves vlabel. Then $F_2 \cdot F_1$ preserves vlabel. The theorem is a consequence of (1).

(114) Let us consider graphs G_1, G_2, G_3, G_4, a partial graph mapping F_1 from G_1 to G_2, a partial graph mapping F_2 from G_2 to G_3, and a partial graph mapping F_3 from G_3 to G_4. Then $F_3 \cdot (F_2 \cdot F_1) = (F_3 \cdot F_2) \cdot F_1$.

(115) Let us consider graphs G_1, G_2, and a one-to-one partial graph mapping F from G_1 to G_2. Suppose F is isomorphism. Then

(i) $F \cdot (F^{-1}) = \mathrm{id}_{G_2}$, and

(ii) $F^{-1} \cdot F = \mathrm{id}_{G_1}$.

(116) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Then

(i) $F \cdot (\mathrm{id}_{G_1}) = F$, and

(ii) $\mathrm{id}_{G_2} \cdot F = F$.

(117) Let us consider graphs G_1, G_2, G_3, a partial graph mapping F_1 from G_1 to G_2, a partial graph mapping F_2 from G_2 to G_3, and a subgraph H of G_1. Then $F_2 \cdot (F_1 \restriction H) = (F_2 \cdot F_1) \restriction H$.

(118) Let us consider graphs G_1, G_2, G_3, a partial graph mapping F_1 from G_1 to G_2, a partial graph mapping F_2 from G_2 to G_3, and a subgraph H of G_3. Then $(H \upharpoonleft F_2) \cdot F_1 = H \upharpoonleft (F_2 \cdot F_1)$.

Let G_1 be a graph and G_2 be a G_1-isomorphic graph. Let us note that every graph which is G_2-isomorphic is also G_1-isomorphic.

Let G_2 be a G_1-directed-isomorphic graph. Note that every graph which is G_2-directed-isomorphic is also G_1-directed-isomorphic.

4. Walks Induced by Graph Mappings

Let G_1, G_2 be graphs, F be a partial graph mapping from G_1 to G_2, and W_1 be a walk of G_1. We say that W_1 is F-defined if and only if

(Def. 35) W_1.vertices() $\subseteq \mathrm{dom}(F_{\mathbb{V}})$ and W_1.edges() $\subseteq \mathrm{dom}(F_{\mathbb{E}})$.

Let W_2 be a walk of G_2. We say that W_2 is F-valued if and only if

(Def. 36) W_2.vertices() $\subseteq \mathrm{rng}\, F_{\mathbb{V}}$ and W_2.edges() $\subseteq \mathrm{rng}\, F_{\mathbb{E}}$.

Let F be a non empty partial graph mapping from G_1 to G_2. Observe that there exists a walk of G_1 which is F-defined and trivial and there exists a walk of G_2 which is F-valued and trivial.

Let us consider graphs G_1, G_2 and an empty partial graph mapping F from G_1 to G_2. Now we state the propositions:

(119) Every walk of G_1 is not F-defined.

(120) Every walk of G_2 is not F-valued.

(121) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and a walk W_1 of G_1. If F is total, then W_1 is F-defined.

(122) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and a walk W_2 of G_2. If F is onto, then W_2 is F-valued.

Let G_1, G_2 be graphs and F be a one-to-one partial graph mapping from G_1 to G_2. Observe that every walk of G_1 which is F-defined is also (F^{-1})-valued and every walk of G_2 which is F-valued is also (F^{-1})-defined.

Let F be a non empty partial graph mapping from G_1 to G_2 and W_1 be an F-defined walk of G_1. The functor $F^{\circ}W_1$ yielding a walk of G_2 is defined by

(Def. 37) $(F_{\mathbb{V}}) \cdot (W_1.\mathrm{vertexSeq}()) = \mathit{it}.\mathrm{vertexSeq}()$ and $(F_{\mathbb{E}}) \cdot (W_1.\mathrm{edgeSeq}()) = \mathit{it}.\mathrm{edgeSeq}()$.

Note that $F^{\circ}W_1$ is F-valued.

Let us observe that the functor $F^{\circ}W_1$ yields an F-valued walk of G_2. Let F be a non empty, one-to-one partial graph mapping from G_1 to G_2 and W_2 be an F-valued walk of G_2. The functor $F^{-1}(W_2)$ yielding an F-defined walk of G_1 is defined by the term

(Def. 38) $(F^{-1})^{\circ}W_2$.

Let us observe that the functor $F^{-1}(W_2)$ is defined by

(Def. 39) $(F_{\mathbb{V}}) \cdot (it.\mathrm{vertexSeq}()) = W_2.\mathrm{vertexSeq}()$ and $(F_{\mathbb{E}}) \cdot (it.\mathrm{edgeSeq}()) = W_2.\mathrm{edgeSeq}()$.

Now we state the propositions:

(123) Let us consider graphs G_1, G_2, a non empty, one-to-one partial graph mapping F from G_1 to G_2, and an F-defined walk W_1 of G_1. Then $F^{-1}(F^{\circ}W_1) = W_1$.

(124) Let us consider graphs G_1, G_2, a non empty, one-to-one partial graph mapping F from G_1 to G_2, and an F-valued walk W_2 of G_2. Then $F^{\circ}(F^{-1}(W_2)) = W_2$.

(125) Let us consider graphs G_1, G_2, a non empty partial graph mapping F from G_1 to G_2, and an F-defined walk W_1 of G_1. Then

(i) $W_1.\mathrm{length}() = (F^{\circ}W_1).\mathrm{length}()$, and

(ii) $\mathrm{len}\, W_1 = \mathrm{len}(F^{\circ}W_1)$.

(126) Let us consider graphs G_1, G_2, a non empty, one-to-one partial graph mapping F from G_1 to G_2, and an F-valued walk W_2 of G_2. Then

(i) $W_2.\mathrm{length}() = (F^{-1}(W_2)).\mathrm{length}()$, and

(ii) $\mathrm{len}\, W_2 = \mathrm{len}(F^{-1}(W_2))$.

(127) Let us consider graphs G_1, G_2, a non empty partial graph mapping F from G_1 to G_2, and an F-defined walk W_1 of G_1. Then

(i) $(F_{\mathbb{V}})(W_1.\mathrm{first}()) = (F^{\circ}W_1).\mathrm{first}()$, and

(ii) $(F_{\mathbb{V}})(W_1.\mathrm{last}()) = (F^{\circ}W_1).\mathrm{last}()$.

(128) Let us consider graphs G_1, G_2, a non empty, one-to-one partial graph mapping F from G_1 to G_2, and an F-valued walk W_2 of G_2. Then

(i) $((F_{\mathbb{V}})^{-1})(W_2.\mathrm{first}()) = (F^{-1}(W_2)).\mathrm{first}()$, and

(ii) $((F_{\mathbb{V}})^{-1})(W_2.\mathrm{last}()) = (F^{-1}(W_2)).\mathrm{last}()$.

(129) Let us consider graphs G_1, G_2, a non empty partial graph mapping F from G_1 to G_2, an F-defined walk W_1 of G_1, and an odd element n of $\mathbb{N}$. If $n \leqslant \mathrm{len}\, W_1$, then $(F_{\mathbb{V}})(W_1(n)) = (F^{\circ}W_1)(n)$. The theorem is a consequence of (125).

(130) Let us consider graphs G_1, G_2, a non empty partial graph mapping F from G_1 to G_2, an F-defined walk W_1 of G_1, and an even element n of $\mathbb{N}$. Suppose $1 \leqslant n \leqslant \operatorname{len} W_1$. Then $(F_{\mathbb{E}})(W_1(n)) = (F^{\circ}W_1)(n)$. The theorem is a consequence of (125).

Let us consider graphs G_1, G_2, a non empty partial graph mapping F from G_1 to G_2, an F-defined walk W_1 of G_1, and objects v, w. Now we state the propositions:

(131) If W_1 is walk from v to w, then $v, w \in \operatorname{dom}(F_{\mathbb{V}})$.

(132) If W_1 is walk from v to w, then $F^{\circ}W_1$ is walk from $(F_{\mathbb{V}})(v)$ to $(F_{\mathbb{V}})(w)$. The theorem is a consequence of (129) and (125).

(133) Let us consider graphs G_1, G_2, a non empty, one-to-one partial graph mapping F from G_1 to G_2, an F-defined walk W_1 of G_1, and objects v, w. Then W_1 is walk from v to w if and only if $v, w \in \operatorname{dom}(F_{\mathbb{V}})$ and $F^{\circ}W_1$ is walk from $(F_{\mathbb{V}})(v)$ to $(F_{\mathbb{V}})(w)$. The theorem is a consequence of (131), (132), and (123).

(134) Let us consider graphs G_1, G_2, a non empty, one-to-one partial graph mapping F from G_1 to G_2, and an F-defined walk W_1 of G_1. Suppose $(F_{\mathbb{V}})(W_1.\text{first}()) = (F_{\mathbb{V}})(W_1.\text{last}())$. Then $W_1.\text{first}() = W_1.\text{last}()$. The theorem is a consequence of (4).

Let us consider graphs G_1, G_2, a non empty partial graph mapping F from G_1 to G_2, and an F-defined walk W_1 of G_1. Now we state the propositions:

(135) $(F^{\circ}W_1).\text{vertices}() = (F_{\mathbb{V}})^{\circ}(W_1.\text{vertices}())$.
PROOF: For every object y, $y \in \operatorname{rng}(F_{\mathbb{V}}) \cdot (W_1.\text{vertexSeq}())$ iff $y \in (F_{\mathbb{V}})^{\circ}(W_1.\text{vertices}())$. $\square$

(136) $(F^{\circ}W_1).\text{edges}() = (F_{\mathbb{E}})^{\circ}(W_1.\text{edges}())$.
PROOF: For every object y, $y \in \operatorname{rng}(F_{\mathbb{E}}) \cdot (W_1.\text{edgeSeq}())$ iff $y \in (F_{\mathbb{E}})^{\circ}(W_1.\text{edges}())$. $\square$

(137) (i) if W_1 is trivial, then $F^{\circ}W_1$ is trivial, and

(ii) if W_1 is closed, then $F^{\circ}W_1$ is closed, and

(iii) if $F^{\circ}W_1$ is trail-like, then W_1 is trail-like, and

(iv) if $F^{\circ}W_1$ is path-like, then W_1 is path-like.

PROOF: If $F^{\circ}W_1$ is trail-like, then W_1 is trail-like. For every odd elements m, n of $\mathbb{N}$ such that $m < n \leqslant \operatorname{len} W_1$ holds if $W_1(m) = W_1(n)$, then $m = 1$ and $n = \operatorname{len} W_1$. $\square$

(138) Let us consider graphs G_1, G_2, a non empty, one-to-one partial graph mapping F from G_1 to G_2, and an F-defined walk W_1 of G_1. Then

(i) W_1 is trivial iff $F^{\circ}W_1$ is trivial, and

(ii) W_1 is closed iff $F^\circ W_1$ is closed, and

(iii) W_1 is trail-like iff $F^\circ W_1$ is trail-like, and

(iv) W_1 is path-like iff $F^\circ W_1$ is path-like, and

(v) W_1 is circuit-like iff $F^\circ W_1$ is circuit-like, and

(vi) W_1 is cycle-like iff $F^\circ W_1$ is cycle-like.

The theorem is a consequence of (123) and (137).

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(139) If F is strong subgraph embedding, then if G_2 is acyclic, then G_1 is acyclic. The theorem is a consequence of (121) and (138).

(140) Suppose F is isomorphism. Then

(i) G_1 is acyclic iff G_2 is acyclic, and

(ii) G_1 is chordal iff G_2 is chordal, and

(iii) G_1 is connected iff G_2 is connected.

Proof: F^{-1} is isomorphism and semi-continuous. For every vertices u, v of G_1, there exists a walk W_1 of G_1 such that W_1 is walk from u to v. □

5. Graph Mappings and Graph Modes

Let us consider graphs G_1, G_2, sets E_1, E_2, a graph G_3 given by reversing directions of the edges E_1 of G_1, a graph G_4 given by reversing directions of the edges E_2 of G_2, and a partial graph mapping F_0 from G_1 to G_2. Now we state the propositions:

(141) There exists a partial graph mapping F from G_3 to G_4 such that

(i) $F = F_0$, and

(ii) if F_0 is not empty, then F is not empty, and

(iii) if F_0 is total, then F is total, and

(iv) if F_0 is onto, then F is onto, and

(v) if F_0 is one-to-one, then F is one-to-one, and

(vi) if F_0 is semi-continuous, then F is semi-continuous, and

(vii) if F_0 is continuous, then F is continuous.

Proof: Reconsider $F = F_0$ as a partial graph mapping from G_3 to G_4. If F_0 is semi-continuous, then F is semi-continuous. If F_0 is continuous, then F is continuous by [13, (9)]. □

(142) There exists a partial graph mapping F from G_3 to G_4 such that

(i) $F = F_0$, and

(ii) if F_0 is weak subgraph embedding, then F is weak subgraph embedding, and

(iii) if F_0 is strong subgraph embedding, then F is strong subgraph embedding, and

(iv) if F_0 is isomorphism, then F is isomorphism.

The theorem is a consequence of (141).

(143) Let us consider a graph G_1, a G_1-isomorphic graph G_2, sets E_1, E_2, and a graph G_3 given by reversing directions of the edges E_1 of G_1. Then every graph given by reversing directions of the edges E_2 of G_2 is G_3-isomorphic. The theorem is a consequence of (142).

Let us consider graphs G_3, G_4, sets V_1, V_2, a supergraph G_1 of G_3 extended by the vertices from V_1, a supergraph G_2 of G_4 extended by the vertices from V_2, a partial graph mapping F_0 from G_3 to G_4, and a one-to-one function f. Now we state the propositions:

(144) Suppose $\mathrm{dom}\, f = V_1 \setminus$ (the vertices of G_3) and $\mathrm{rng}\, f = V_2 \setminus$ (the vertices of G_4). Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}} {+\cdot} f, F_{0\mathbb{E}} \rangle$, and

(ii) if F_0 is not empty, then F is not empty, and

(iii) if F_0 is total, then F is total, and

(iv) if F_0 is onto, then F is onto, and

(v) if F_0 is one-to-one, then F is one-to-one, and

(vi) if F_0 is directed, then F is directed, and

(vii) if F_0 is semi-continuous, then F is semi-continuous, and

(viii) if F_0 is continuous, then F is continuous, and

(ix) if F_0 is semi-directed-continuous, then F is semi-directed-continuous, and

(x) if F_0 is directed-continuous, then F is directed-continuous.

Proof: Set $h = F_{0\mathbb{V}} {+\cdot} f$. Reconsider $g = F_{0\mathbb{E}}$ as a partial function from the edges of G_1 to the edges of G_2. Reconsider $F = \langle h, g \rangle$ as a partial graph mapping from G_1 to G_2. If F_0 is total, then F is total. If F_0 is onto, then F is onto. If F_0 is directed, then F is directed. If F_0 is semi-continuous, then F is semi-continuous. If F_0 is continuous, then F is continuous. If F_0 is semi-directed-continuous, then F is semi-directed-continuous. If F_0 is directed-continuous, then F is directed-continuous. $\square$

(145) Suppose $\operatorname{dom} f = V_1 \setminus$ (the vertices of G_3) and $\operatorname{rng} f = V_2 \setminus$ (the vertices of G_4). Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}} {+\cdot} f, F_{0\mathbb{E}} \rangle$, and

(ii) if F_0 is weak subgraph embedding, then F is weak subgraph embedding, and

(iii) if F_0 is strong subgraph embedding, then F is strong subgraph embedding, and

(iv) if F_0 is isomorphism, then F is isomorphism, and

(v) if F_0 is directed-isomorphism, then F is directed-isomorphism.

The theorem is a consequence of (144).

(146) Let us consider a graph G_3, a G_3-isomorphic graph G_4, sets V_1, V_2, a supergraph G_1 of G_3 extended by the vertices from V_1, and a supergraph G_2 of G_4 extended by the vertices from V_2. Suppose $\overline{\overline{V_1 \setminus \alpha}} = \overline{\overline{V_2 \setminus \beta}}$. Then G_2 is G_1-isomorphic, where α is the vertices of G_3 and β is the vertices of G_4. The theorem is a consequence of (145).

(147) Let us consider a graph G_3, a G_3-directed-isomorphic graph G_4, sets V_1, V_2, a supergraph G_1 of G_3 extended by the vertices from V_1, and a supergraph G_2 of G_4 extended by the vertices from V_2. Suppose $\overline{\overline{V_1 \setminus \alpha}} = \overline{\overline{V_2 \setminus \beta}}$. Then G_2 is G_1-directed-isomorphic, where α is the vertices of G_3 and β is the vertices of G_4. The theorem is a consequence of (145).

Let us consider graphs G_3, G_4, objects v_1, v_2, a supergraph G_1 of G_3 extended by v_1, a supergraph G_2 of G_4 extended by v_2, and a partial graph mapping F_0 from G_3 to G_4. Now we state the propositions:

(148) Suppose $v_1 \notin$ the vertices of G_3 and $v_2 \notin$ the vertices of G_4. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}} {+\cdot} (v_1 \mathrel{\dot{\longmapsto}} v_2), F_{0\mathbb{E}} \rangle$, and

(ii) if F_0 is total, then F is total, and

(iii) if F_0 is onto, then F is onto, and

(iv) if F_0 is one-to-one, then F is one-to-one, and

(v) if F_0 is directed, then F is directed, and

(vi) if F_0 is semi-continuous, then F is semi-continuous, and

(vii) if F_0 is continuous, then F is continuous, and

(viii) if F_0 is semi-directed-continuous, then F is semi-directed-continuous, and

(ix) if F_0 is directed-continuous, then F is directed-continuous.

The theorem is a consequence of (144).

(149) Suppose $v_1 \notin$ the vertices of G_3 and $v_2 \notin$ the vertices of G_4. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}} {+\cdot} (v_1 \dot{\longmapsto} v_2), F_{0\mathbb{E}} \rangle$, and

(ii) if F_0 is weak subgraph embedding, then F is weak subgraph embedding, and

(iii) if F_0 is strong subgraph embedding, then F is strong subgraph embedding, and

(iv) if F_0 is isomorphism, then F is isomorphism, and

(v) if F_0 is directed-isomorphism, then F is directed-isomorphism.

The theorem is a consequence of (148).

(150) Let us consider a graph G_3, a G_3-isomorphic graph G_4, objects v_1, v_2, a supergraph G_1 of G_3 extended by v_1, and a supergraph G_2 of G_4 extended by v_2. Suppose $v_1 \in$ the vertices of G_3 iff $v_2 \in$ the vertices of G_4. Then G_2 is G_1-isomorphic. The theorem is a consequence of (146).

(151) Let us consider a graph G_3, a G_3-directed-isomorphic graph G_4, objects v_1, v_2, a supergraph G_1 of G_3 extended by v_1, and a supergraph G_2 of G_4 extended by v_2. Suppose $v_1 \in$ the vertices of G_3 iff $v_2 \in$ the vertices of G_4. Then G_2 is G_1-directed-isomorphic. The theorem is a consequence of (147).

Let us consider graphs G_3, G_4, vertices v_1, v_3 of G_3, vertices v_2, v_4 of G_4, objects e_1, e_2, a supergraph G_1 of G_3 extended by e_1 between vertices v_1 and v_3, a supergraph G_2 of G_4 extended by e_2 between vertices v_2 and v_4, and a partial graph mapping F_0 from G_3 to G_4. Now we state the propositions:

(152) Suppose $e_1 \notin$ the edges of G_3 and $e_2 \notin$ the edges of G_4 and $v_1, v_3 \in \mathrm{dom}(F_{0\mathbb{V}})$ and $((F_{0\mathbb{V}})(v_1) = v_2$ and $(F_{0\mathbb{V}})(v_3) = v_4$ or $(F_{0\mathbb{V}})(v_1) = v_4$ and $(F_{0\mathbb{V}})(v_3) = v_2)$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}}, F_{0\mathbb{E}} {+\cdot} (e_1 \dot{\longmapsto} e_2) \rangle$, and

(ii) if F_0 is total, then F is total, and

(iii) if F_0 is onto, then F is onto, and

(iv) if F_0 is one-to-one, then F is one-to-one.

The theorem is a consequence of (5), (4), and (8).

(153) Suppose $e_1 \notin$ the edges of G_3 and $e_2 \notin$ the edges of G_4 and $v_1, v_3 \in \mathrm{dom}(F_{0\mathbb{V}})$ and $((F_{0\mathbb{V}})(v_1) = v_2$ and $(F_{0\mathbb{V}})(v_3) = v_4$ or $(F_{0\mathbb{V}})(v_1) = v_4$ and

$(F_{0\mathbb{V}})(v_3) = v_2)$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}}, F_{0\mathbb{E}} +\cdot (e_1 \dot{\longmapsto} e_2) \rangle$, and

(ii) if F_0 is weak subgraph embedding, then F is weak subgraph embedding, and

(iii) if F_0 is isomorphism, then F is isomorphism.

The theorem is a consequence of (152).

(154) Suppose $e_1 \notin$ the edges of G_3 and $e_2 \notin$ the edges of G_4 and v_1, $v_3 \in \operatorname{dom}(F_{0\mathbb{V}})$ and $(F_{0\mathbb{V}})(v_1) = v_2$ and $(F_{0\mathbb{V}})(v_3) = v_4$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}}, F_{0\mathbb{E}} +\cdot (e_1 \dot{\longmapsto} e_2) \rangle$, and

(ii) if F_0 is directed, then F is directed, and

(iii) if F_0 is directed-isomorphism, then F is directed-isomorphism.

Proof: Consider F being a partial graph mapping from G_1 to G_2 such that $F = \langle F_{0\mathbb{V}}, F_{0\mathbb{E}} +\cdot (e_1 \dot{\longmapsto} e_2) \rangle$ and if F_0 is total, then F is total and if F_0 is onto, then F is onto and if F_0 is one-to-one, then F is one-to-one. If F_0 is directed, then F is directed by [15, (16)], [12, (71),(70),(106)]. $\square$

Let us consider graphs G_3, G_4, a vertex v_3 of G_3, a vertex v_4 of G_4, objects e_1, e_2, v_1, v_2, a supergraph G_1 of G_3 extended by v_1, v_3 and e_1 between them, a supergraph G_2 of G_4 extended by v_2, v_4 and e_2 between them, and a partial graph mapping F_0 from G_3 to G_4. Now we state the propositions:

(155) Suppose $e_1 \notin$ the edges of G_3 and $e_2 \notin$ the edges of G_4 and $v_1 \notin$ the vertices of G_3 and $v_2 \notin$ the vertices of G_4 and $v_3 \in \operatorname{dom}(F_{0\mathbb{V}})$ and $(F_{0\mathbb{V}})(v_3) = v_4$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}} +\cdot (v_1 \dot{\longmapsto} v_2), F_{0\mathbb{E}} +\cdot (e_1 \dot{\longmapsto} e_2) \rangle$, and

(ii) if F_0 is total, then F is total, and

(iii) if F_0 is onto, then F is onto, and

(iv) if F_0 is one-to-one, then F is one-to-one, and

(v) if F_0 is directed, then F is directed.

Proof: Consider G_5 being a supergraph of G_3 extended by v_1 such that G_1 is a supergraph of G_5 extended by e_1 between vertices v_1 and v_3. Consider G_6 being a supergraph of G_4 extended by v_2 such that G_2 is a supergraph of G_6 extended by e_2 between vertices v_2 and v_4.

Consider F_1 being a partial graph mapping from G_5 to G_6 such that $F_1 = \langle F_{0\mathbb{V}} +\cdot (v_1 \dot{\longmapsto} v_2), F_{0\mathbb{E}} \rangle$ and if F_0 is total, then F_1 is total and if F_0

is onto, then F_1 is onto and if F_0 is one-to-one, then F_1 is one-to-one and if F_0 is directed, then F_1 is directed and if F_0 is semi-continuous, then F_1 is semi-continuous and if F_0 is continuous, then F_1 is continuous and if F_0 is semi-directed-continuous, then F_1 is semi-directed-continuous and if F_0 is directed-continuous, then F_1 is directed-continuous. v_1, $v_3 \in \mathrm{dom}(F_{1\mathbb{V}})$ and $(F_{1\mathbb{V}})(v_1) = v_2$ and $(F_{1\mathbb{V}})(v_3) = v_4$.

Consider F_2 being a partial graph mapping from G_1 to G_2 such that $F_2 = \langle F_{1\mathbb{V}}, F_{1\mathbb{E}} {+\cdot} (e_1 \dot{\longmapsto} e_2) \rangle$ and if F_1 is total, then F_2 is total and if F_1 is onto, then F_2 is onto and if F_1 is one-to-one, then F_2 is one-to-one. Consider F_3 being a partial graph mapping from G_1 to G_2 such that $F_3 = \langle F_{1\mathbb{V}}, F_{1\mathbb{E}} {+\cdot} (e_1 \dot{\longmapsto} e_2) \rangle$ and if F_1 is directed, then F_3 is directed and if F_1 is directed-isomorphism, then F_3 is directed-isomorphism. □

(156) Suppose $e_1 \notin$ the edges of G_3 and $e_2 \notin$ the edges of G_4 and $v_1 \notin$ the vertices of G_3 and $v_2 \notin$ the vertices of G_4 and $v_3 \in \mathrm{dom}(F_{0\mathbb{V}})$ and $(F_{0\mathbb{V}})(v_3) = v_4$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}} {+\cdot} (v_1 \dot{\longmapsto} v_2), F_{0\mathbb{E}} {+\cdot} (e_1 \dot{\longmapsto} e_2) \rangle$, and

(ii) if F_0 is weak subgraph embedding, then F is weak subgraph embedding, and

(iii) if F_0 is isomorphism, then F is isomorphism, and

(iv) if F_0 is directed-isomorphism, then F is directed-isomorphism.

The theorem is a consequence of (155).

Let us consider graphs G_3, G_4, a vertex v_3 of G_3, a vertex v_4 of G_4, objects e_1, e_2, v_1, v_2, a supergraph G_1 of G_3 extended by v_3, v_1 and e_1 between them, a supergraph G_2 of G_4 extended by v_4, v_2 and e_2 between them, and a partial graph mapping F_0 from G_3 to G_4. Now we state the propositions:

(157) Suppose $e_1 \notin$ the edges of G_3 and $e_2 \notin$ the edges of G_4 and $v_1 \notin$ the vertices of G_3 and $v_2 \notin$ the vertices of G_4 and $v_3 \in \mathrm{dom}(F_{0\mathbb{V}})$ and $(F_{0\mathbb{V}})(v_3) = v_4$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}} {+\cdot} (v_1 \dot{\longmapsto} v_2), F_{0\mathbb{E}} {+\cdot} (e_1 \dot{\longmapsto} e_2) \rangle$, and

(ii) if F_0 is total, then F is total, and

(iii) if F_0 is onto, then F is onto, and

(iv) if F_0 is one-to-one, then F is one-to-one, and

(v) if F_0 is directed, then F is directed.

Proof: Consider G_5 being a supergraph of G_3 extended by v_1 such that G_1 is a supergraph of G_5 extended by e_1 between vertices v_3 and v_1.

Consider G_6 being a supergraph of G_4 extended by v_2 such that G_2 is a supergraph of G_6 extended by e_2 between vertices v_4 and v_2.

Consider F_1 being a partial graph mapping from G_5 to G_6 such that $F_1 = \langle F_{0\mathbb{V}} +\cdot (v_1 \dot{\longmapsto} v_2), F_{0\mathbb{E}} \rangle$ and if F_0 is total, then F_1 is total and if F_0 is onto, then F_1 is onto and if F_0 is one-to-one, then F_1 is one-to-one and if F_0 is directed, then F_1 is directed and if F_0 is semi-continuous, then F_1 is semi-continuous and if F_0 is continuous, then F_1 is continuous and if F_0 is semi-directed-continuous, then F_1 is semi-directed-continuous and if F_0 is directed-continuous, then F_1 is directed-continuous. $v_1, v_3 \in \mathrm{dom}(F_{1\mathbb{V}})$ and $(F_{1\mathbb{V}})(v_1) = v_2$ and $(F_{1\mathbb{V}})(v_3) = v_4$.

Consider F_2 being a partial graph mapping from G_1 to G_2 such that $F_2 = \langle F_{1\mathbb{V}}, F_{1\mathbb{E}} +\cdot (e_1 \dot{\longmapsto} e_2) \rangle$ and if F_1 is total, then F_2 is total and if F_1 is onto, then F_2 is onto and if F_1 is one-to-one, then F_2 is one-to-one. Consider F_3 being a partial graph mapping from G_1 to G_2 such that $F_3 = \langle F_{1\mathbb{V}}, F_{1\mathbb{E}} +\cdot (e_1 \dot{\longmapsto} e_2) \rangle$ and if F_1 is directed, then F_3 is directed and if F_1 is directed-isomorphism, then F_3 is directed-isomorphism. $\square$

(158) Suppose $e_1 \notin$ the edges of G_3 and $e_2 \notin$ the edges of G_4 and $v_1 \notin$ the vertices of G_3 and $v_2 \notin$ the vertices of G_4 and $v_3 \in \mathrm{dom}(F_{0\mathbb{V}})$ and $(F_{0\mathbb{V}})(v_3) = v_4$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}} +\cdot (v_1 \dot{\longmapsto} v_2), F_{0\mathbb{E}} +\cdot (e_1 \dot{\longmapsto} e_2) \rangle$, and

(ii) if F_0 is weak subgraph embedding, then F is weak subgraph embedding, and

(iii) if F_0 is isomorphism, then F is isomorphism, and

(iv) if F_0 is directed-isomorphism, then F is directed-isomorphism.

The theorem is a consequence of (157).

(159) Let us consider graphs G_3, G_4, a vertex v_3 of G_3, a vertex v_4 of G_4, objects e_1, e_2, v_1, v_2, a supergraph G_1 of G_3 extended by v_1, v_3 and e_1 between them, a supergraph G_2 of G_4 extended by v_4, v_2 and e_2 between them, and a partial graph mapping F_0 from G_3 to G_4. Suppose $e_1 \notin$ the edges of G_3 and $e_2 \notin$ the edges of G_4 and $v_1 \notin$ the vertices of G_3 and $v_2 \notin$ the vertices of G_4 and $v_3 \in \mathrm{dom}(F_{0\mathbb{V}})$ and $(F_{0\mathbb{V}})(v_3) = v_4$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}} +\cdot (v_1 \dot{\longmapsto} v_2), F_{0\mathbb{E}} +\cdot (e_1 \dot{\longmapsto} e_2) \rangle$, and

(ii) if F_0 is total, then F is total, and

(iii) if F_0 is onto, then F is onto, and

(iv) if F_0 is one-to-one, then F is one-to-one, and

(v) if F_0 is weak subgraph embedding, then F is weak subgraph embedding, and

(vi) if F_0 is isomorphism, then F is isomorphism.

PROOF: Consider G_5 being a supergraph of G_3 extended by v_1 such that G_1 is a supergraph of G_5 extended by e_1 between vertices v_1 and v_3. Consider G_6 being a supergraph of G_4 extended by v_2 such that G_2 is a supergraph of G_6 extended by e_2 between vertices v_4 and v_2.

Consider F_1 being a partial graph mapping from G_5 to G_6 such that $F_1 = \langle F_{0\mathbb{V}} +\cdot (v_1 \dot{\longmapsto} v_2), F_{0\mathbb{E}} \rangle$ and if F_0 is total, then F_1 is total and if F_0 is onto, then F_1 is onto and if F_0 is one-to-one, then F_1 is one-to-one and if F_0 is directed, then F_1 is directed and if F_0 is semi-continuous, then F_1 is semi-continuous and if F_0 is continuous, then F_1 is continuous and if F_0 is semi-directed-continuous, then F_1 is semi-directed-continuous and if F_0 is directed-continuous, then F_1 is directed-continuous. $v_1, v_3 \in \mathrm{dom}(F_{1\mathbb{V}})$ and $(F_{1\mathbb{V}})(v_1) = v_2$ and $(F_{1\mathbb{V}})(v_3) = v_4$.

Consider F_2 being a partial graph mapping from G_1 to G_2 such that $F_2 = \langle F_{1\mathbb{V}}, F_{1\mathbb{E}} +\cdot (e_1 \dot{\longmapsto} e_2) \rangle$ and if F_1 is total, then F_2 is total and if F_1 is onto, then F_2 is onto and if F_1 is one-to-one, then F_2 is one-to-one. □

(160) Let us consider graphs G_3, G_4, a vertex v_3 of G_3, a vertex v_4 of G_4, objects e_1, e_2, v_1, v_2, a supergraph G_1 of G_3 extended by v_3, v_1 and e_1 between them, a supergraph G_2 of G_4 extended by v_2, v_4 and e_2 between them, and a partial graph mapping F_0 from G_3 to G_4. Suppose $e_1 \notin$ the edges of G_3 and $e_2 \notin$ the edges of G_4 and $v_1 \notin$ the vertices of G_3 and $v_2 \notin$ the vertices of G_4 and $v_3 \in \mathrm{dom}(F_{0\mathbb{V}})$ and $(F_{0\mathbb{V}})(v_3) = v_4$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}} +\cdot (v_1 \dot{\longmapsto} v_2), F_{0\mathbb{E}} +\cdot (e_1 \dot{\longmapsto} e_2) \rangle$, and

(ii) if F_0 is total, then F is total, and

(iii) if F_0 is onto, then F is onto, and

(iv) if F_0 is one-to-one, then F is one-to-one, and

(v) if F_0 is weak subgraph embedding, then F is weak subgraph embedding, and

(vi) if F_0 is isomorphism, then F is isomorphism.

PROOF: Consider G_5 being a supergraph of G_3 extended by v_1 such that G_1 is a supergraph of G_5 extended by e_1 between vertices v_3 and v_1. Consider G_6 being a supergraph of G_4 extended by v_2 such that G_2 is a supergraph of G_6 extended by e_2 between vertices v_2 and v_4.

Consider F_1 being a partial graph mapping from G_5 to G_6 such that $F_1 = \langle F_{0\mathbb{V}} +\cdot (v_1 \dot{\longmapsto} v_2), F_{0\mathbb{E}} \rangle$ and if F_0 is total, then F_1 is total and if F_0

is onto, then F_1 is onto and if F_0 is one-to-one, then F_1 is one-to-one and if F_0 is directed, then F_1 is directed and if F_0 is semi-continuous, then F_1 is semi-continuous and if F_0 is continuous, then F_1 is continuous and if F_0 is semi-directed-continuous, then F_1 is semi-directed-continuous and if F_0 is directed-continuous, then F_1 is directed-continuous. v_1, $v_3 \in \mathrm{dom}(F_{1\mathbb{V}})$ and $(F_{1\mathbb{V}})(v_1) = v_2$ and $(F_{1\mathbb{V}})(v_3) = v_4$.

Consider F_2 being a partial graph mapping from G_1 to G_2 such that $F_2 = \langle F_{1\mathbb{V}}, F_{1\mathbb{E}} {+\cdot} (e_1 \dot{\longmapsto} e_2)\rangle$ and if F_1 is total, then F_2 is total and if F_1 is onto, then F_2 is onto and if F_1 is one-to-one, then F_2 is one-to-one. □

(161) Let us consider a graph G, an object v, a set V, and supergraphs G_1, G_2 of G extended by vertex v and edges between v and V of G. Then G_2 is G_1-isomorphic. The theorem is a consequence of (8), (53), and (143).

(162) Let us consider graphs G_3, G_4, objects v_1, v_2, sets V_1, V_2, a supergraph G_1 of G_3 extended by vertex v_1 and edges between v_1 and V_1 of G_3, a supergraph G_2 of G_4 extended by vertex v_2 and edges between v_2 and V_2 of G_4, and a partial graph mapping F_0 from G_3 to G_4. Suppose $V_1 \subseteq$ the vertices of G_3 and $V_2 \subseteq$ the vertices of G_4 and $v_1 \notin$ the vertices of G_3 and $v_2 \notin$ the vertices of G_4 and $F_{0\mathbb{V}} \restriction V_1$ is one-to-one and $\mathrm{dom}(F_{0\mathbb{V}} \restriction V_1) = V_1$ and $\mathrm{rng}(F_{0\mathbb{V}} \restriction V_1) = V_2$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F_{\mathbb{V}} = F_{0\mathbb{V}} {+\cdot} (v_1 \dot{\longmapsto} v_2)$, and

(ii) $F_{\mathbb{E}} \restriction \mathrm{dom}(F_{0\mathbb{E}}) = F_{0\mathbb{E}}$, and

(iii) if F_0 is total, then F is total, and

(iv) if F_0 is onto, then F is onto, and

(v) if F_0 is one-to-one, then F is one-to-one, and

(vi) if F_0 is weak subgraph embedding, then F is weak subgraph embedding, and

(vii) if F_0 is isomorphism, then F is isomorphism.

Proof: $V_1 \subseteq \mathrm{dom}(F_{0\mathbb{V}})$. Set $f = F_{0\mathbb{V}} {+\cdot} (v_1 \dot{\longmapsto} v_2)$. Consider h_1 being a function from V_1 into G_1.edgesBetween(V_1, $\{v_1\}$) such that h_1 is one-to-one and onto and for every object w such that $w \in V_1$ holds $h_1(w)$ joins w and v_1 in G_1. Consider h_2 being a function from V_2 into G_2.edgesBetween(V_2, $\{v_2\}$) such that h_2 is one-to-one and onto and for every object w such that $w \in V_2$ holds $h_2(w)$ joins w and v_2 in G_2. Set $g = F_{0\mathbb{E}} {+\cdot} h_2 \cdot (F_{0\mathbb{V}}) \cdot ({h_1}^{-1})$.

$\mathrm{dom}(F_{0\mathbb{E}})$ misses $\mathrm{dom}(h_2 \cdot (F_{0\mathbb{V}}) \cdot ({h_1}^{-1}))$. $\mathrm{rng}\, F_{0\mathbb{E}}$ misses $\mathrm{rng}\, h_2 \cdot (F_{0\mathbb{V}}) \cdot ({h_1}^{-1})$. Consider E_1 being a set such that $\overline{\overline{V_1}} = \overline{\overline{E_1}}$ and E_1 misses the edges of G_3 and the edges of $G_1 =$ (the edges of G_3) $\cup E_1$ and for every object w_1 such that $w_1 \in V_1$ there exists an object e_1 such that $e_1 \in E_1$ and e_1

joins w_1 and v_1 in G_1 and for every object $\tilde{e}$ such that $\tilde{e}$ joins w_1 and v_1 in G_1 holds $e_1 = \tilde{e}$.

Consider E_2 being a set such that $\overline{\overline{V_2}} = \overline{\overline{E_2}}$ and E_2 misses the edges of G_4 and the edges of $G_2 =$ (the edges of $G_4) \cup E_2$ and for every object w_2 such that $w_2 \in V_2$ there exists an object e_2 such that $e_2 \in E_2$ and e_2 joins w_2 and v_2 in G_2 and for every object $\tilde{e}$ such that $\tilde{e}$ joins w_2 and v_2 in G_2 holds $e_2 = \tilde{e}$. Reconsider $F = \langle f, g \rangle$ as a partial graph mapping from G_1 to G_2. If F_0 is total, then F is total. If F_0 is onto, then F is onto. □

(163) Let us consider a graph G_3, a G_3-isomorphic graph G_4, objects v_1, v_2, a supergraph G_1 of G_3 extended by vertex v_1 and edges between v_1 and the vertices of G_3, and a supergraph G_2 of G_4 extended by vertex v_2 and edges between v_2 and the vertices of G_4. Suppose $v_1 \in$ the vertices of G_3 iff $v_2 \in$ the vertices of G_4. Then G_2 is G_1-isomorphic. The theorem is a consequence of (162) and (143).

Let us consider graphs G_1, G_2, a subgraph G_3 of G_1 with loops removed, a subgraph G_4 of G_2 with loops removed, and a one-to-one partial graph mapping F_0 from G_1 to G_2. Now we state the propositions:

(164) There exists a one-to-one partial graph mapping F from G_3 to G_4 such that

(i) $F = F_0 \restriction G_3$, and

(ii) if F_0 is total, then F is total, and

(iii) if F_0 is onto, then F is onto, and

(iv) if F_0 is directed, then F is directed, and

(v) if F_0 is semi-directed-continuous, then F is semi-directed-continuous.

Proof: Reconsider $F = G_4 \upharpoonleft (F_0 \restriction G_3)$ as a one-to-one partial graph mapping from G_3 to G_4. If F_0 is total, then F is total. If F_0 is onto, then F is onto. □

(165) There exists a one-to-one partial graph mapping F from G_3 to G_4 such that

(i) $F = F_0 \restriction G_3$, and

(ii) if F_0 is weak subgraph embedding, then F is weak subgraph embedding, and

(iii) if F_0 is isomorphism, then F is isomorphism, and

(iv) if F_0 is directed-isomorphism, then F is directed-isomorphism.

The theorem is a consequence of (164).

(166) Let us consider a graph G_1, a G_1-isomorphic graph G_2, and a subgraph G_3 of G_1 with loops removed. Then every subgraph of G_2 with loops removed is G_3-isomorphic. The theorem is a consequence of (165).

(167) Let us consider a graph G_1, a G_1-directed-isomorphic graph G_2, and a subgraph G_3 of G_1 with loops removed. Then every subgraph of G_2 with loops removed is G_3-directed-isomorphic. The theorem is a consequence of (165).

(168) Let us consider a graph G_1, a G_1-isomorphic graph G_2, and a subgraph G_3 of G_1 with parallel edges removed. Then every subgraph of G_2 with parallel edges removed is G_3-isomorphic.

PROOF: Consider G being a partial graph mapping from G_1 to G_2 such that G is isomorphism. Consider E_1 being a representative selection of the parallel edges of G_1 such that G_3 is a subgraph of G_1 induced by the vertices of G_1 and E_1.

Consider E_2 being a representative selection of the parallel edges of G_2 such that G_4 is a subgraph of G_2 induced by the vertices of G_2 and E_2. Define $\mathcal{P}[\text{object}, \text{object}] \equiv \$_2 \in E_2$ and $\langle \$_1, \$_2 \rangle \in \text{EdgeParEqRel}(G_2)$. For every objects x, y_1, y_2 such that $x \in$ the edges of G_2 and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. For every object x such that $x \in$ the edges of G_2 there exists an object y such that $\mathcal{P}[x, y]$.

Consider h being a function such that $\operatorname{dom} h =$ the edges of G_2 and for every object x such that $x \in$ the edges of G_2 holds $\mathcal{P}[x, h(x)]$. □

(169) Let us consider a graph G_1, and subgraphs G_2, G_3 of G_1 with parallel edges removed. Then G_3 is G_2-isomorphic. The theorem is a consequence of (53) and (168).

(170) Let us consider a graph G_1, a G_1-directed-isomorphic graph G_2, and a subgraph G_3 of G_1 with directed-parallel edges removed. Then every subgraph of G_2 with directed-parallel edges removed is G_3-directed-isomorphic.

PROOF: Consider G being a partial graph mapping from G_1 to G_2 such that G is directed-isomorphism. Consider E_1 being a representative selection of the directed-parallel edges of G_1 such that G_3 is a subgraph of G_1 induced by the vertices of G_1 and E_1.

Consider E_2 being a representative selection of the directed-parallel edges of G_2 such that G_4 is a subgraph of G_2 induced by the vertices of G_2 and E_2. Define $\mathcal{P}[\text{object}, \text{object}] \equiv \$_2 \in E_2$ and $\langle \$_1, \$_2 \rangle \in \text{DEdgeParEqRel}(G_2)$. For every objects x, y_1, y_2 such that $x \in$ the edges of G_2 and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. For every object x such that $x \in$ the edges of G_2 there exists an object y such that $\mathcal{P}[x, y]$.

Consider h being a function such that $\operatorname{dom} h =$ the edges of G_2 and for every object x such that $x \in$ the edges of G_2 holds $\mathcal{P}[x, h(x)]$. $\square$

(171) Let us consider a graph G_1, and subgraphs G_2, G_3 of G_1 with directed-parallel edges removed. Then G_3 is G_2-directed-isomorphic. The theorem is a consequence of (53) and (170).

(172) Let us consider a graph G_1, a G_1-isomorphic graph G_2, and a simple graph G_3 of G_1. Then every simple graph of G_2 is G_3-isomorphic. The theorem is a consequence of (166) and (168).

(173) Let us consider a graph G_1, and simple graphs G_2, G_3 of G_1. Then G_3 is G_2-isomorphic. The theorem is a consequence of (53) and (172).

(174) Let us consider a graph G_1, a G_1-directed-isomorphic graph G_2, and a directed-simple graph G_3 of G_1. Then every directed-simple graph of G_2 is G_3-directed-isomorphic. The theorem is a consequence of (167) and (170).

(175) Let us consider a graph G_1, and directed-simple graphs G_2, G_3 of G_1. Then G_3 is G_2-directed-isomorphic. The theorem is a consequence of (53) and (174).

(176) Let us consider trivial, loopless graphs G_1, G_2, and a non empty partial graph mapping F from G_1 to G_2. Then

(i) F is directed-isomorphism, and

(ii) $F = \langle$the vertex of $G_1 \dot{\longmapsto}$the vertex of G_2, $\emptyset\rangle$.

(177) Let us consider trivial graphs G_1, G_2. Suppose G_1.size() $=$ G_2.size(). Then there exists a partial graph mapping F from G_1 to G_2 such that F is directed-isomorphism. The theorem is a consequence of (31).

(178) Let us consider trivial, loopless graphs G_1, G_2. Then G_2 is G_1-directed-isomorphic and G_1-isomorphic.

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Accepted August 29, 2019

FORMALIZED MATHEMATICS
Vol. 27, No. 3, Pages 303–313, 2019
DOI: 10.2478/forma-2019-0025

About Vertex Mappings

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Summary. In [6] partial graph mappings were formalized in the Mizar system [3]. Such mappings map some vertices and edges of a graph to another while preserving adjacency. While this general approach is appropriate for the general form of (multidi)graphs as introduced in [7], a more specialized version for graphs without parallel edges seems convenient. As such, partial vertex mappings preserving adjacency between the mapped verticed are formalized here.

MSC: 05C60 68T99 03B35

Keywords: graph homomorphism; graph isomorphism

MML identifier: `GLIB_011`, version: `8.1.09 5.59.1363`

0. Introduction

This article is a brief introduction to partial vertex mappings in Mizar [2]. As discussed in the introduction of [6] almost no graph theory book discusses graph homomorphisms in a scope as general as it was done in [5] and [6]. Most of the time, graph homomorphisms are only discussed in the form of vertex mappings, often only in the context of simple graphs. But of course that choice is not without reason and in many cases considering vertex mappings is enough, which is especially useful since one does not need to think about an edge mapping then. Given that the graph definitions change slightly between different authors, a quick overview of the formalized notation seems in order.

A *partial vertex mapping* f between two graphs G_1, G_2 is a partial function of their vertex sets $V(G_1)$, $V(G_2)$ with the additional property that if vertices

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$v, w \in \operatorname{dom} f$ are adjacent in G_1, then their images $f(v)$, $f(w)$ are adjacent in G_2. The properties of f to be *total* (or a homomorphism), *one-to-one* (or injective) and *onto* (or surjective) have the usual meaning for f as a partial function. f is *continuous* if for any $v, w \in \operatorname{dom} f$ such that $f(v)$ and $f(w)$ are adjacent, v and w are adjacent as well. f is an *isomorphism* if it is total, one-to-one, onto and the cardinality of edges between to vertices v and w of G_1 is the same as the cardinality of the edges between $f(v)$ and $f(w)$. Corresponding attributes for directed vertex mappings are given as well in this article.

The attribute *continuous* is the generalization for not necessarily simple graphs of the *continuous* of [5]. The *isomorphism* attribute was inspired by [1]. It is shown that for graphs G_1, G_2 without multiple edges that a total bijective and continuous vertex mapping f between them is already an isomorphism, just like a graph isomorphism is usually described (cf. [4], [8], [5]). This article does not go into depth like [6], but the inverse and composition of partial vertex mappings are covered.

A partial graph mapping does not always induce a partial vertex mapping (since any subset of the set of edges of G_1 can be mapped) and a partial vertex mapping can give rise to several partial graph mappings. In the second part of this article it is shown when the induced partial vertex mapping exists and when the induced partial graph mapping is unique. Furthermore it is formally stated that for two graphs without parallel edges there exists a graph mapping that is an isomorphism iff there exists a vertex mapping that is an isomorphism.

1. Vertex Mappings

Let G_1, G_2 be graphs.

A partial vertex mapping from G_1 to G_2 is a partial function from the vertices of G_1 to the vertices of G_2 defined by

(Def. 1) for every vertices v, w of G_1 such that v, $w \in \operatorname{dom} \mathit{it}$ and v and w are adjacent holds $\mathit{it}_{/v}$ and $\mathit{it}_{/w}$ are adjacent.

Now we state the proposition:

(1) Let us consider graphs G_1, G_2, and a partial function f from the vertices of G_1 to the vertices of G_2. Then f is a partial vertex mapping from G_1 to G_2 if and only if for every objects v, w, e such that v, $w \in \operatorname{dom} f$ and e joins v and w in G_1 there exists an object $\tilde{e}$ such that $\tilde{e}$ joins $f(v)$ and $f(w)$ in G_2.

Let G_1, G_2 be graphs and f be a partial vertex mapping from G_1 to G_2. We say that f is directed if and only if

(Def. 2) for every objects v, w, e such that $v, w \in \operatorname{dom} f$ and e joins v to w in G_1 there exists an object $\tilde{e}$ such that $\tilde{e}$ joins $f(v)$ to $f(w)$ in G_2.

We say that f is continuous if and only if

(Def. 3) for every vertices v, w of G_1 such that $v, w \in \operatorname{dom} f$ and $f_{/v}$ and $f_{/w}$ are adjacent holds v and w are adjacent.

We say that f is directed-continuous if and only if

(Def. 4) for every objects v, w, $\tilde{e}$ such that $v, w \in \operatorname{dom} f$ and $\tilde{e}$ joins $f(v)$ to $f(w)$ in G_2 there exists an object e such that e joins v to w in G_1.

Let us consider graphs G_1, G_2 and a partial vertex mapping f from G_1 to G_2. Now we state the propositions:

(2) f is continuous if and only if for every objects v, w, $\tilde{e}$ such that $v, w \in \operatorname{dom} f$ and $\tilde{e}$ joins $f(v)$ and $f(w)$ in G_2 there exists an object e such that e joins v and w in G_1.

(3) f is continuous if and only if for every vertices v, w of G_1 such that $v, w \in \operatorname{dom} f$ holds v and w are adjacent iff $f_{/v}$ and $f_{/w}$ are adjacent.

Let G_1, G_2 be graphs. One can check that every partial vertex mapping from G_1 to G_2 which is directed-continuous is also continuous and every partial vertex mapping from G_1 to G_2 which is empty is also one-to-one, directed-continuous, directed, and continuous and every partial vertex mapping from G_1 to G_2 which is total is also non empty and every partial vertex mapping from G_1 to G_2 which is onto is also non empty.

Let G_1 be a simple graph and G_2 be a graph. Observe that every partial vertex mapping from G_1 to G_2 which is directed-continuous is also directed.

Let G_1 be a graph and G_2 be a simple graph. Observe that every partial vertex mapping from G_1 to G_2 which is directed and continuous is also directed-continuous.

Let G_1 be a trivial graph and G_2 be a graph. Let us observe that every partial vertex mapping from G_1 to G_2 is directed and every partial vertex mapping from G_1 to G_2 which is continuous is also directed-continuous and every partial vertex mapping from G_1 to G_2 which is non empty is also total.

Let G_1 be a graph and G_2 be a trivial graph. One can verify that every partial vertex mapping from G_1 to G_2 which is non empty is also onto.

Let G_2 be a trivial, loopless graph. Let us note that every partial vertex mapping from G_1 to G_2 is directed-continuous and continuous.

Let G_1, G_2 be graphs. Observe that there exists a partial vertex mapping from G_1 to G_2 which is empty, one-to-one, directed, continuous, and directed-continuous.

Now we state the proposition:

(4) Let us consider graphs G_1, G_2, and a partial function f from the vertices of G_1 to the vertices of G_2. Then f is a directed partial vertex mapping from G_1 to G_2 if and only if for every objects v, w, e such that $v, w \in \operatorname{dom} f$ and e joins v to w in G_1 there exists an object $\tilde{e}$ such that $\tilde{e}$ joins $f(v)$ to $f(w)$ in G_2. The theorem is a consequence of (1).

Let G_1 be a loopless graph and G_2 be a graph. One can verify that there exists a partial vertex mapping from G_1 to G_2 which is non empty, one-to-one, and directed.

Let G_1, G_2 be loopless graphs. Let us observe that there exists a partial vertex mapping from G_1 to G_2 which is non empty, one-to-one, directed, continuous, and directed-continuous.

Let G_1, G_2 be non loopless graphs. One can verify that there exists a partial vertex mapping from G_1 to G_2 which is non empty, one-to-one, directed, continuous, and directed-continuous.

Now we state the propositions:

(5) Let us consider a graph G. Then id_α is a directed, continuous, directed-continuous partial vertex mapping from G to G, where α is the vertices of G. The theorem is a consequence of (1) and (2).

(6) Let us consider graphs G_1, G_2, and a partial vertex mapping f from G_1 to G_2. Suppose f is total. Then

(i) if G_2 is loopless, then G_1 is loopless, and

(ii) if G_2 is edgeless, then G_1 is edgeless.

The theorem is a consequence of (1).

(7) Let us consider graphs G_1, G_2, and a continuous partial vertex mapping f from G_1 to G_2. Suppose f is onto. Then

(i) if G_1 is loopless, then G_2 is loopless, and

(ii) if G_1 is edgeless, then G_2 is edgeless.

The theorem is a consequence of (2).

Let G_1, G_2 be graphs and f be a partial vertex mapping from G_1 to G_2. We say that f is isomorphism if and only if

(Def. 5) f is total, one-to-one, and onto and for every vertices v, w of G_1, $\overline{\overline{G_1.\mathrm{edgesBetween}(\{v\},\{w\})}} = \overline{\overline{G_2.\mathrm{edgesBetween}(\{f(v)\},\{f(w)\})}}$.

We say that f is directed-isomorphism if and only if

(Def. 6) f is total, one-to-one, and onto and for every vertices v, w of G_1, $\overline{\overline{G_1.\mathrm{edgesDBetween}(\{v\},\{w\})}} = \overline{\overline{G_2.\mathrm{edgesDBetween}(\{f(v)\},\{f(w)\})}}$ and $\overline{\overline{G_1.\mathrm{edgesDBetween}(\{w\},\{v\})}} = \overline{\overline{G_2.\mathrm{edgesDBetween}(\{f(w)\},\{f(v)\})}}$.

Let us note that every partial vertex mapping from G_1 to G_2 which is isomorphism is also total, one-to-one, onto, and continuous and every partial vertex mapping from G_1 to G_2 which is directed-isomorphism is also total, one-to-one, onto, isomorphism, continuous, directed, and directed-continuous.

Now we state the proposition:

(8) Let us consider non-multi graphs G_1, G_2, and a partial vertex mapping f from G_1 to G_2. Suppose f is total, one-to-one, and continuous. Let us consider vertices v, w of G_1. Then $\overline{\overline{G_1.\text{edgesBetween}(\{v\},\{w\})}} = \overline{\overline{G_2.\text{edgesBetween}(\{f(v)\},\{f(w)\})}}$. The theorem is a consequence of (2) and (1).

Let G_1, G_2 be non-multi graphs and f be a partial vertex mapping from G_1 to G_2. Note that f is isomorphism if and only if the condition (Def. 7) is satisfied.

(Def. 7) f is total, one-to-one, onto, and continuous.

Observe that every partial vertex mapping from G_1 to G_2 which is total, one-to-one, onto, and continuous is also isomorphism.

Now we state the proposition:

(9) Let us consider non-directed-multi graphs G_1, G_2, and a partial vertex mapping f from G_1 to G_2. Suppose f is total, one-to-one, directed, and directed-continuous. Let us consider vertices v, w of G_1. Then

  (i) $\overline{\overline{G_1.\text{edgesDBetween}(\{v\},\{w\})}} = \overline{\overline{G_2.\text{edgesDBetween}(\{f(v)\},\{f(w)\})}}$, and

  (ii) $\overline{\overline{G_1.\text{edgesDBetween}(\{w\},\{v\})}} = \overline{\overline{G_2.\text{edgesDBetween}(\{f(w)\},\{f(v)\})}}$.

Let G_1, G_2 be non-directed-multi graphs and f be a partial vertex mapping from G_1 to G_2. Observe that f is directed-isomorphism if and only if the condition (Def. 8) is satisfied.

(Def. 8) f is total, one-to-one, onto, directed, and directed-continuous.

One can check that every partial vertex mapping from G_1 to G_2 which is total, one-to-one, onto, directed, and directed-continuous is also directed-isomorphism.

Let G be a graph. Let us observe that there exists a partial vertex mapping from G to G which is directed-isomorphism and isomorphism.

Now we state the proposition:

(10) Let us consider a graph G. Then id_α is a directed-isomorphism, isomorphism partial vertex mapping from G to G, where α is the vertices of G. The theorem is a consequence of (5).

Let G_1, G_2 be graphs and f be a partial vertex mapping from G_1 to G_2. We say that f is invertible if and only if

(Def. 9) f is one-to-one and continuous.

Note that every partial vertex mapping from G_1 to G_2 which is invertible is also one-to-one and continuous and every partial vertex mapping from G_1 to G_2 which is one-to-one and continuous is also invertible and every partial vertex mapping from G_1 to G_2 which is isomorphism is also invertible and every partial vertex mapping from G_1 to G_2 which is directed-isomorphism is also invertible and there exists a partial vertex mapping from G_1 to G_2 which is empty and invertible.

Let G_1, G_2 be loopless graphs. Note that there exists a partial vertex mapping from G_1 to G_2 which is non empty, directed, and invertible.

Let G_1, G_2 be non loopless graphs. Observe that there exists a partial vertex mapping from G_1 to G_2 which is non empty, directed, and invertible.

Let G_1, G_2 be graphs and f be an invertible partial vertex mapping from G_1 to G_2. Note that the functor f^{-1} yields a partial vertex mapping from G_2 to G_1. Observe that f^{-1} is one-to-one, continuous, and invertible as a partial vertex mapping from G_2 to G_1.

Let G_1, G_2, G_3 be graphs, f be a partial vertex mapping from G_1 to G_2, and g be a partial vertex mapping from G_2 to G_3. One can check that the functor $g \cdot f$ yields a partial vertex mapping from G_1 to G_3.

Let us consider graphs G_1, G_2, G_3, a partial vertex mapping f from G_1 to G_2, and a partial vertex mapping g from G_2 to G_3. Now we state the propositions:

(11) If f is continuous and g is continuous, then $g \cdot f$ is continuous. The theorem is a consequence of (2).

(12) If f is directed and g is directed, then $g \cdot f$ is directed.

(13) If f is directed-continuous and g is directed-continuous, then $g \cdot f$ is directed-continuous.

(14) If f is isomorphism and g is isomorphism, then $g \cdot f$ is isomorphism.

(15) If f is directed-isomorphism and g is directed-isomorphism, then $g \cdot f$ is directed-isomorphism.

2. The Relation Between Graph Mappings and Vertex Mappings

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(16) Suppose for every vertices v, w of G_1 such that $v, w \in \mathrm{dom}(F_{\mathbb{V}})$ and v and w are adjacent there exists an object e such that $e \in \mathrm{dom}(F_{\mathbb{E}})$ and e joins v and w in G_1. Then $F_{\mathbb{V}}$ is a partial vertex mapping from G_1 to G_2.

(17) If $\mathrm{dom}(F_{\mathbb{E}}) =$ the edges of G_1, then $F_{\mathbb{V}}$ is a partial vertex mapping from G_1 to G_2. The theorem is a consequence of (16).

(18) If F is total, then $F_{\mathbb{V}}$ is a partial vertex mapping from G_1 to G_2. The theorem is a consequence of (17).

Let us consider graphs G_1, G_2 and a directed partial graph mapping F from G_1 to G_2. Now we state the propositions:

(19) Suppose for every objects v, w such that $v, w \in \operatorname{dom}(F_{\mathbb{V}})$ and there exists an object e such that e joins v to w in G_1 there exists an object e such that $e \in \operatorname{dom}(F_{\mathbb{E}})$ and e joins v to w in G_1. Then $F_{\mathbb{V}}$ is a directed partial vertex mapping from G_1 to G_2. The theorem is a consequence of (1).

(20) Suppose $\operatorname{dom}(F_{\mathbb{E}}) =$ the edges of G_1. Then $F_{\mathbb{V}}$ is a directed partial vertex mapping from G_1 to G_2. The theorem is a consequence of (19).

(21) If F is total, then $F_{\mathbb{V}}$ is a directed partial vertex mapping from G_1 to G_2. The theorem is a consequence of (20).

Let us consider graphs G_1, G_2 and a semi-continuous partial graph mapping F from G_1 to G_2. Now we state the propositions:

(22) Suppose $F_{\mathbb{V}}$ is a partial vertex mapping from G_1 to G_2 and for every vertices v, w of G_1 such that $v, w \in \operatorname{dom}(F_{\mathbb{V}})$ and $(F_{\mathbb{V}})_{/v}$ and $(F_{\mathbb{V}})_{/w}$ are adjacent there exists an object $\tilde{e}$ such that $\tilde{e} \in \operatorname{rng} F_{\mathbb{E}}$ and $\tilde{e}$ joins $(F_{\mathbb{V}})(v)$ and $(F_{\mathbb{V}})(w)$ in G_2. Then $F_{\mathbb{V}}$ is a continuous partial vertex mapping from G_1 to G_2. The theorem is a consequence of (2).

(23) Suppose $\operatorname{dom}(F_{\mathbb{E}}) =$ the edges of G_1 and $\operatorname{rng} F_{\mathbb{E}} =$ the edges of G_2. Then $F_{\mathbb{V}}$ is a continuous partial vertex mapping from G_1 to G_2. The theorem is a consequence of (17) and (22).

(24) If F is total and onto, then $F_{\mathbb{V}}$ is a continuous partial vertex mapping from G_1 to G_2. The theorem is a consequence of (23).

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(25) If F is isomorphism, then there exists a partial vertex mapping f from G_1 to G_2 such that $F_{\mathbb{V}} = f$ and f is isomorphism. The theorem is a consequence of (18).

(26) If F is directed-isomorphism, then there exists a directed partial vertex mapping f from G_1 to G_2 such that $F_{\mathbb{V}} = f$ and f is directed-isomorphism. The theorem is a consequence of (21).

(27) Let us consider graphs G_1, G_2, a partial vertex mapping f from G_1 to G_2, a representative selection of the parallel edges E_1 of G_1, and a representative selection of the parallel edges E_2 of G_2. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F_{\mathbb{V}} = f$, and

(ii) $\mathrm{dom}(F_{\mathbb{E}}) = E_1 \cap G_1.\mathrm{edgesBetween}(\mathrm{dom}\, f)$, and

(iii) $\mathrm{rng}\, F_{\mathbb{E}} \subseteq E_2 \cap G_2.\mathrm{edgesBetween}(\mathrm{rng}\, f)$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exist objects v, w such that v, $w \in \mathrm{dom}\, f$ and $\$_1 \in E_1$ and $\$_2 \in E_2$ and $\$_1$ joins v and w in G_1 and $\$_2$ joins $f(v)$ and $f(w)$ in G_2. For every objects e_1, e_2, e_3 such that $e_1 \in E_1 \cap G_1.\mathrm{edgesBetween}(\mathrm{dom}\, f)$ and $\mathcal{P}[e_1, e_2]$ and $\mathcal{P}[e_1, e_3]$ holds $e_2 = e_3$.

For every object e_1 such that $e_1 \in E_1 \cap G_1.\mathrm{edgesBetween}(\mathrm{dom}\, f)$ there exists an object e_2 such that $\mathcal{P}[e_1, e_2]$. Consider g being a function such that $\mathrm{dom}\, g = E_1 \cap G_1.\mathrm{edgesBetween}(\mathrm{dom}\, f)$ and for every object e_1 such that $e_1 \in E_1 \cap G_1.\mathrm{edgesBetween}(\mathrm{dom}\, f)$ holds $\mathcal{P}[e_1, g(e_1)]$. For every object y such that $y \in \mathrm{rng}\, g$ holds $y \in E_2 \cap G_2.\mathrm{edgesBetween}(\mathrm{rng}\, f)$. □

Let G_1, G_2 be non-multi graphs and f be a partial vertex mapping from G_1 to G_2. The functor PVM2PGM(f) yielding a partial graph mapping from G_1 to G_2 is defined by

(Def. 10) $it_{\mathbb{V}} = f$ and $\mathrm{dom}(it_{\mathbb{E}}) = G_1.\mathrm{edgesBetween}(\mathrm{dom}\, f)$ and $\mathrm{rng}\, it_{\mathbb{E}} \subseteq G_2.\mathrm{edgesBetween}(\mathrm{rng}\, f)$.

Now we state the proposition:

(28) Let us consider non-multi graphs G_1, G_2, and a partial vertex mapping f from G_1 to G_2. Then $\mathrm{PVM2PGM}(f)_{\mathbb{V}} = f$.

Let G_1, G_2 be non-multi graphs and f be a partial vertex mapping from G_1 to G_2. Observe that $\mathrm{PVM2PGM}(f)_{\mathbb{V}}$ reduces to f.

Now we state the proposition:

(29) Let us consider graphs G_1, G_2, a directed partial vertex mapping f from G_1 to G_2, a representative selection of the directed-parallel edges E_1 of G_1, and a representative selection of the directed-parallel edges E_2 of G_2. Then there exists a directed partial graph mapping F from G_1 to G_2 such that

(i) $F_{\mathbb{V}} = f$, and

(ii) $\mathrm{dom}(F_{\mathbb{E}}) = E_1 \cap G_1.\mathrm{edgesBetween}(\mathrm{dom}\, f)$, and

(iii) $\mathrm{rng}\, F_{\mathbb{E}} \subseteq E_2 \cap G_2.\mathrm{edgesBetween}(\mathrm{rng}\, f)$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exist objects v, w such that v, $w \in \mathrm{dom}\, f$ and $\$_1 \in E_1$ and $\$_2 \in E_2$ and $\$_1$ joins v to w in G_1 and $\$_2$ joins $f(v)$ to $f(w)$ in G_2. For every objects e_1, e_2, e_3 such that $e_1 \in E_1 \cap G_1.\mathrm{edgesBetween}(\mathrm{dom}\, f)$ and $\mathcal{P}[e_1, e_2]$ and $\mathcal{P}[e_1, e_3]$ holds $e_2 = e_3$.

For every object e_1 such that $e_1 \in E_1 \cap G_1.\mathrm{edgesBetween}(\mathrm{dom}\, f)$ there exists an object e_2 such that $\mathcal{P}[e_1, e_2]$. Consider g being a function such that $\mathrm{dom}\, g = E_1 \cap G_1.\mathrm{edgesBetween}(\mathrm{dom}\, f)$ and for every object e_1 such that $e_1 \in E_1 \cap G_1$.edgesBetween(dom f) holds $\mathcal{P}[e_1, g(e_1)]$. For every object y such that $y \in \operatorname{rng} g$ holds $y \in E_2 \cap G_2$.edgesBetween(rng f). $\square$

Let G_1, G_2 be non-directed-multi graphs and f be a directed partial vertex mapping from G_1 to G_2. The functor DPVM2PGM(f) yielding a directed partial graph mapping from G_1 to G_2 is defined by

(Def. 11) $it_{\mathbb{V}} = f$ and $\operatorname{dom}(it_{\mathbb{E}}) = G_1$.edgesBetween(dom f) and $\operatorname{rng} it_{\mathbb{E}} \subseteq G_2$.edgesBetween(rng f).

Now we state the proposition:

(30) Let us consider non-directed-multi graphs G_1, G_2, and a directed partial vertex mapping f from G_1 to G_2. Then $\mathrm{DPVM2PGM}(f)_{\mathbb{V}} = f$.

Let G_1, G_2 be non-directed-multi graphs and f be a directed partial vertex mapping from G_1 to G_2. One can check that $\mathrm{DPVM2PGM}(f)_{\mathbb{V}}$ reduces to f.

Now we state the propositions:

(31) Let us consider non-multi graphs G_1, G_2, and a directed partial vertex mapping f from G_1 to G_2. Then $\mathrm{PVM2PGM}(f) = \mathrm{DPVM2PGM}(f)$.

(32) Let us consider non-multi graphs G_1, G_2, and a partial vertex mapping f from G_1 to G_2. If f is total, then PVM2PGM(f) is total.

(33) Let us consider non-directed-multi graphs G_1, G_2, and a directed partial vertex mapping f from G_1 to G_2. If f is total, then DPVM2PGM(f) is total.

(34) Let us consider non-multi graphs G_1, G_2, and a partial vertex mapping f from G_1 to G_2. If f is one-to-one, then PVM2PGM(f) is one-to-one.
Proof: Set $g = \mathrm{PVM2PGM}(f)_{\mathbb{E}}$. For every objects x_1, x_2 such that x_1, $x_2 \in \operatorname{dom} g$ and $g(x_1) = g(x_2)$ holds $x_1 = x_2$. $\square$

(35) Let us consider non-directed-multi graphs G_1, G_2, and a directed partial vertex mapping f from G_1 to G_2. If f is one-to-one, then DPVM2PGM(f) is one-to-one.
Proof: Set $g = \mathrm{DPVM2PGM}(f)_{\mathbb{E}}$. For every objects x_1, x_2 such that x_1, $x_2 \in \operatorname{dom} g$ and $g(x_1) = g(x_2)$ holds $x_1 = x_2$. $\square$

(36) Let us consider non-multi graphs G_1, G_2, and a partial vertex mapping f from G_1 to G_2. If f is onto and continuous, then PVM2PGM(f) is onto.
Proof: Set $g = \mathrm{PVM2PGM}(f)_{\mathbb{E}}$. For every object e such that $e \in$ the edges of G_2 holds $e \in \operatorname{rng} g$. $\square$

(37) Let us consider non-directed-multi graphs G_1, G_2, and a directed partial vertex mapping f from G_1 to G_2. If f is onto and directed-continuous, then DPVM2PGM(f) is onto.
Proof: Set $g = \mathrm{DPVM2PGM}(f)_{\mathbb{E}}$. For every object e such that $e \in$ the edges of G_2 holds $e \in \operatorname{rng} g$. $\square$

Let us consider non-multi graphs G_1, G_2 and a partial vertex mapping f from G_1 to G_2. Now we state the propositions:

(38) If f is continuous and one-to-one, then PVM2PGM(f) is semi-continuous. The theorem is a consequence of (2) and (34).

(39) If f is continuous, then PVM2PGM(f) is continuous. The theorem is a consequence of (2).

Let us consider non-directed-multi graphs G_1, G_2 and a directed partial vertex mapping f from G_1 to G_2. Now we state the propositions:

(40) If f is one-to-one, then DPVM2PGM(f) is semi-directed-continuous and semi-continuous. The theorem is a consequence of (35).

(41) If f is directed-continuous, then DPVM2PGM(f) is directed-continuous.

(42) Let us consider non-multi graphs G_1, G_2, and a partial vertex mapping f from G_1 to G_2. If f is one-to-one, then PVM2PGM(f) is one-to-one.

(43) Let us consider non-directed-multi graphs G_1, G_2, and a directed partial vertex mapping f from G_1 to G_2. If f is one-to-one, then DPVM2PGM(f) is one-to-one.

(44) Let us consider non-multi graphs G_1, G_2, and a partial vertex mapping f from G_1 to G_2. Suppose f is total and one-to-one. Then PVM2PGM(f) is weak subgraph embedding. The theorem is a consequence of (32) and (34).

(45) Let us consider non-directed-multi graphs G_1, G_2, and a directed partial vertex mapping f from G_1 to G_2. Suppose f is total and one-to-one. Then DPVM2PGM(f) is weak subgraph embedding. The theorem is a consequence of (33) and (35).

Let us consider non-multi graphs G_1, G_2 and a partial vertex mapping f from G_1 to G_2. Now we state the propositions:

(46) If f is total, one-to-one, and continuous, then PVM2PGM(f) is strong subgraph embedding. The theorem is a consequence of (32), (34), and (39).

(47) If f is isomorphism, then PVM2PGM(f) is isomorphism. The theorem is a consequence of (32), (34), and (36).

(48) Let us consider non-directed-multi graphs G_1, G_2, and a directed partial vertex mapping f from G_1 to G_2. Suppose f is directed-isomorphism. Then DPVM2PGM(f) is directed-isomorphism. The theorem is a consequence of (33), (35), (37), and (41).

(49) Let us consider non-multi graphs G_1, G_2. Then G_2 is G_1-isomorphic if and only if there exists a partial vertex mapping f from G_1 to G_2 such that f is isomorphism. The theorem is a consequence of (25) and (47).

(50) Let us consider non-directed-multi graphs G_1, G_2. Then G_2 is G_1-directed-isomorphic if and only if there exists a directed partial vertex mapping f from G_1 to G_2 such that f is directed-isomorphism. The theorem is a consequence of (26) and (48).

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Accepted August 29, 2019

FORMALIZED MATHEMATICS
Vol. 27, No. 3, Pages 315–320, 2019
DOI: 10.2478/forma-2019-0026

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Operations of Points on Elliptic Curve in Affine Coordinates[1]

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Summary. In this article, we formalize in Mizar [1], [2] a binary operation of points on an elliptic curve over $\mathbf{GF}(\mathbf{p})$ in affine coordinates. We show that the operation is unital, complementable and commutative. Elliptic curve cryptography [3], whose security is based on a difficulty of discrete logarithm problem of elliptic curves, is important for information security.

MSC: 14H52 14K05 68T99 03B35

Keywords: elliptic curve; commutative operation

MML identifier: EC_PF_3, version: 8.1.09 5.59.1363

1. Set of Points on Elliptic Curve in Affine Coordinates

From now on p denotes a 5 or greater prime number and z denotes an element of the parameters of elliptic curve p.

Now we state the propositions:

(1) Let us consider a prime number p, elements a, b of $\mathrm{GF}(p)$, and an element P of $\mathrm{ProjCo}(\mathrm{GF}(p))$. Suppose $P = \langle 0, 1, 0\rangle$ or $(P)_{\mathbf{3},3} = 1$. Then the represent point of $P = P$.

[1]This work was supported by JSPS KAKENHI Grant Numbers JP15K00183 and JP17K00182.

ISSN 1426–2630(Print), 1898-9934(Online)

PROOF: If $P = \langle 0, 1, 0 \rangle$, then the represent point of $P = P$. If $(P)_{\mathbf{3},3} = 1$, then the represent point of $P = P$ by [5, (2)], [6, (3)]. $\square$

(2) Let us consider a 5 or greater prime number p, an element z of the parameters of elliptic curve p, and elements P, O of $\mathrm{EC}_{\mathrm{SetProjCo}}((z)_{\mathbf{1}})$. Suppose $O = \langle 0, 1, 0 \rangle$. Then $(P)_{\mathbf{3},3} = 0$ if and only if $P \equiv O$. The theorem is a consequence of (1).

(3) Let us consider a 5 or greater prime number p, an element z of the parameters of elliptic curve p, and an element P of $\mathrm{EC}_{\mathrm{SetProjCo}}((z)_{\mathbf{1}})$. If $(P)_{\mathbf{3},3} = 0$, then $P \equiv (\mathrm{compell}_{\mathrm{ProjCo}}(z, p))(P)$. The theorem is a consequence of (2).

(4) Let us consider elements P, O of $\mathrm{EC}_{\mathrm{SetProjCo}}((z)_{\mathbf{1}})$. Suppose $O = \langle 0, 1, 0 \rangle$. Then $(\mathrm{addell}_{\mathrm{ProjCo}}(z, p))(P, (\mathrm{compell}_{\mathrm{ProjCo}}(z, p))(P)) \equiv O$. The theorem is a consequence of (2) and (3).

Let p be a 5 or greater prime number and z be an element of the parameters of elliptic curve p. The functor EC-SetAffCo(z, p) yielding a non empty subset of $\mathrm{EC}_{\mathrm{SetProjCo}}((z)_{\mathbf{1}})$ is defined by the term

(Def. 1) $\{P$, where P is an element of $\mathrm{EC}_{\mathrm{SetProjCo}}((z)_{\mathbf{1}}) : (P)_{\mathbf{3},3} = 1$ or $P = \langle 0, 1, 0 \rangle\}$.

Now we state the proposition:

(5) $\langle 0, 1, 0 \rangle$ is an element of EC-SetAffCo(z, p).

Let us consider a 5 or greater prime number p, an element z of the parameters of elliptic curve p, and an element P of $\mathrm{EC}_{\mathrm{SetProjCo}}((z)_{\mathbf{1}})$. Now we state the propositions:

(6) The represent point of P is an element of EC-SetAffCo(z, p).

(7) If $P \in$ EC-SetAffCo(z, p), then the represent point of $P = P$. The theorem is a consequence of (1).

Let us consider elements P, O of $\mathrm{EC}_{\mathrm{SetProjCo}}((z)_{\mathbf{1}})$. Now we state the propositions:

(8) If $O = \langle 0, 1, 0 \rangle$ and $P \not\equiv O$, then (the represent point of $P)_{\mathbf{3},3} = 1$. The theorem is a consequence of (2).

(9) Suppose $O = \langle 0, 1, 0 \rangle$ and the represent point of $P \equiv O$. Then

(i) the represent point of $P = O$, and

(ii) $P \equiv O$.

The theorem is a consequence of (2) and (1).

(10) Let us consider an element P of ProjCo(GF(p)). Then the represent point of the represent point of $P =$ the represent point of P. The theorem is a consequence of (1).

(11) Let us consider elements P, Q of $\mathrm{EC}_{\mathrm{SetProjCo}}((z)_{\mathbf{1}})$. Suppose the represent point of $P \equiv$ the represent point of Q. Then the represent point of $P =$ the represent point of Q. The theorem is a consequence of (10).

Let p be a 5 or greater prime number and z be an element of the parameters of elliptic curve p. The functor compell-AffCo(z, p) yielding a unary operation on EC-SetAffCo(z, p) is defined by

(Def. 2) for every element P of EC-SetAffCo(z, p), $it(P) =$ the represent point of $(\mathrm{compell}_{\mathrm{ProjCo}}(z, p))(P)$.

Let F be a function from EC-SetAffCo(z, p) into EC-SetAffCo(z, p) and P be an element of EC-SetAffCo(z, p). Let us observe that the functor $F(P)$ yields an element of EC-SetAffCo(z, p). The functor addell-AffCo(z, p) yielding a binary operation on EC-SetAffCo(z, p) is defined by

(Def. 3) for every elements P, Q of EC-SetAffCo(z, p), $it(P, Q) =$ the represent point of $(\mathrm{addell}_{\mathrm{ProjCo}}(z, p))(P, Q)$.

Let F be a function from EC-SetAffCo$(z, p) \times$ EC-SetAffCo(z, p) into

EC-SetAffCo(z, p) and Q, R be elements of EC-SetAffCo(z, p). Let us observe that the functor $F(Q, R)$ yields an element of EC-SetAffCo(z, p). Now we state the proposition:

(12) Let us consider elements P, O of $\mathrm{EC}_{\mathrm{SetProjCo}}((z)_{\mathbf{1}})$. Suppose $O = \langle 0, 1, 0 \rangle$. Then

   (i) $(\mathrm{addell}_{\mathrm{ProjCo}}(z, p))(P, O) \equiv P$, and

   (ii) $(\mathrm{addell}_{\mathrm{ProjCo}}(z, p))(O, P) \equiv P$.

Let us consider elements P, O of EC-SetAffCo(z, p). Now we state the propositions:

(13) If $O = \langle 0, 1, 0 \rangle$, then $(\text{addell-AffCo}(z, p))(O, P) = P$. The theorem is a consequence of (12) and (7).

(14) If $O = \langle 0, 1, 0 \rangle$, then $(\text{addell-AffCo}(z, p))(P, O) = P$. The theorem is a consequence of (12) and (7).

(15) Let us consider an element O of EC-SetAffCo(z, p). Suppose $O = \langle 0, 1, 0 \rangle$. Then O is a unity w.r.t. addell-AffCo(z, p). The theorem is a consequence of (13) and (14).

(16) Let us consider elements P, O of EC-SetAffCo(z, p). Suppose $O = \langle 0, 1, 0 \rangle$. Then $(\text{addell-AffCo}(z, p))(P, (\text{compell-AffCo}(z, p))(P)) = O$. The theorem is a consequence of (7), (4), and (2).

2. Commutative Property of Operations of Points on Elliptic Curve

Now we state the propositions:

(17) Let us consider a 5 or greater prime number p, an element z of the parameters of elliptic curve p, and elements P, Q, O, P_3, Q_3 of $\mathrm{EC}_{\mathrm{SetProjCo}}((z)_{\mathbf{1}})$. Suppose $O = \langle 0, 1, 0 \rangle$ and $P \not\equiv O$ and $Q \not\equiv O$ and $P \not\equiv Q$. Suppose $P_3 = (\mathrm{addell}_{\mathrm{ProjCo}}(z, p))(P, Q)$ and $Q_3 = (\mathrm{addell}_{\mathrm{ProjCo}}(z, p))(Q, P)$. Then

(i) $(Q_3)_{\mathbf{1},3} = -(P_3)_{\mathbf{1},3}$, and

(ii) $(Q_3)_{\mathbf{2},3} = -(P_3)_{\mathbf{2},3}$, and

(iii) $(Q_3)_{\mathbf{3},3} = -(P_3)_{\mathbf{3},3}$.

Proof: Reconsider $g_2 = 2 \bmod p$ as an element of $\mathrm{GF}(p)$. Set $gf_{1PQ} = (Q)_{\mathbf{2},3} \cdot ((P)_{\mathbf{3},3}) - (P)_{\mathbf{2},3} \cdot ((Q)_{\mathbf{3},3})$. Set $gf_{2PQ} = (Q)_{\mathbf{1},3} \cdot ((P)_{\mathbf{3},3}) - (P)_{\mathbf{1},3} \cdot ((Q)_{\mathbf{3},3})$. Set $gf_{3PQ} = {gf_{1PQ}}^2 \cdot ((P)_{\mathbf{3},3}) \cdot ((Q)_{\mathbf{3},3}) - {gf_{2PQ}}^3 - g_2 \cdot ({gf_{2PQ}}^2) \cdot ((P)_{\mathbf{1},3}) \cdot ((Q)_{\mathbf{3},3})$. Set $gf_{1QP} = (P)_{\mathbf{2},3} \cdot ((Q)_{\mathbf{3},3}) - (Q)_{\mathbf{2},3} \cdot ((P)_{\mathbf{3},3})$. Set $gf_{2QP} = (P)_{\mathbf{1},3} \cdot ((Q)_{\mathbf{3},3}) - (Q)_{\mathbf{1},3} \cdot ((P)_{\mathbf{3},3})$. Set $gf_{3QP} = {gf_{1QP}}^2 \cdot ((Q)_{\mathbf{3},3}) \cdot ((P)_{\mathbf{3},3}) - {gf_{2QP}}^3 - g_2 \cdot ({gf_{2QP}}^2) \cdot ((Q)_{\mathbf{1},3}) \cdot ((P)_{\mathbf{3},3})$. $gf_{3QP} = gf_{3PQ}$. $(Q_3)_{\mathbf{1},3} = -(P_3)_{\mathbf{1},3}$. $(Q_3)_{\mathbf{2},3} = -(P_3)_{\mathbf{2},3}$. $(Q_3)_{\mathbf{3},3} = -(P_3)_{\mathbf{3},3}$. $\square$

(18) Let us consider elements P, Q, O, P_3, Q_3 of $\mathrm{EC}_{\mathrm{SetProjCo}}((z)_{\mathbf{1}})$, and an element d of $\mathrm{GF}(p)$. Suppose $O = \langle 0, 1, 0 \rangle$ and $d \neq 0_{\mathrm{GF}(p)}$ and $(Q)_{\mathbf{1},3} = d \cdot ((P)_{\mathbf{1},3})$ and $(Q)_{\mathbf{2},3} = d \cdot ((P)_{\mathbf{2},3})$ and $(Q)_{\mathbf{3},3} = d \cdot ((P)_{\mathbf{3},3})$ and $P \not\equiv O$ and $Q \not\equiv O$ and $P \equiv Q$ and $P_3 = (\mathrm{addell}_{\mathrm{ProjCo}}(z, p))(P, Q)$ and $Q_3 = (\mathrm{addell}_{\mathrm{ProjCo}}(z, p))(Q, P)$. Then

(i) $(Q_3)_{\mathbf{1},3} = d^6 \cdot ((P_3)_{\mathbf{1},3})$, and

(ii) $(Q_3)_{\mathbf{2},3} = d^6 \cdot ((P_3)_{\mathbf{2},3})$, and

(iii) $(Q_3)_{\mathbf{3},3} = d^6 \cdot ((P_3)_{\mathbf{3},3})$.

(19) Let us consider elements P, Q of $\mathrm{EC}_{\mathrm{SetProjCo}}((z)_{\mathbf{1}})$. Then $(\mathrm{addell}_{\mathrm{ProjCo}}(z, p))(P, Q) \equiv (\mathrm{addell}_{\mathrm{ProjCo}}(z, p))(Q, P)$. The theorem is a consequence of (17) and (18).

(20) Let us consider elements P, Q of EC-SetAffCo(z, p). Then $(\text{addell-AffCo}(z, p))(P, Q) = (\text{addell-AffCo}(z, p))(Q, P)$. The theorem is a consequence of (19).

Let p be a 5 or greater prime number and z be an element of the parameters of elliptic curve p. One can verify that addell-AffCo(z, p) is non empty, commutative, and unital.

The functor 0-EC(z, p) yielding an element of EC-SetAffCo(z, p) is defined by the term

(Def. 4) $\langle 0, 1, 0\rangle$.

Let us consider p and z. Let us observe that $\langle$EC-SetAffCo(z,p), addell-AffCo $(z,p)\rangle$ is Abelian and $\langle$EC-SetAffCo(z,p), addell-AffCo(z,p), 0-EC$(z,p)\rangle$ is left zeroed and right zeroed and $\langle$EC-SetAffCo(z,p), addell-AffCo(z,p), 0-EC$(z,p)\rangle$ is complementable.

Let p be a 5 or greater prime number and z be an element of the parameters of elliptic curve p. One can verify that $\langle$EC-SetAffCo(z,p), addell-AffCo$(z,p)\rangle$ is unital.

Now we state the proposition:

(21) Let us consider a 5 or greater prime number p, and an element z of the parameters of elliptic curve p. Then $\mathbf{1}_{\langle \text{EC-SetAffCo}(z,p), \text{addell-AffCo}(z,p)\rangle} =$ 0-EC(z,p). The theorem is a consequence of (15).

Let p be a 5 or greater prime number and z be an element of the parameters of elliptic curve p. One can check that $\langle$EC-SetAffCo(z,p), addell-AffCo$(z,p)\rangle$ is commutative, group-like, and non empty.

Now we state the propositions:

(22) Let us consider elements P_1, P_2, Q of $\text{EC}_{\text{SetProjCo}}((z)_{\mathbf{1}})$. Suppose $P_1 \equiv P_2$. Then $(\text{addell}_{\text{ProjCo}}(z,p))(P_1,Q) \equiv (\text{addell}_{\text{ProjCo}}(z,p))(P_2,Q)$. The theorem is a consequence of (19).

(23) Let us consider elements P, Q_1, Q_2 of $\text{EC}_{\text{SetProjCo}}((z)_{\mathbf{1}})$. Suppose $Q_1 \equiv Q_2$. Then $(\text{addell}_{\text{ProjCo}}(z,p))(P,Q_1) \equiv (\text{addell}_{\text{ProjCo}}(z,p))(P,Q_2)$. The theorem is a consequence of (19) and (22).

(24) Let us consider elements P_1, P_2, Q_1, Q_2 of $\text{EC}_{\text{SetProjCo}}((z)_{\mathbf{1}})$. Suppose $P_1 \equiv P_2$ and $Q_1 \equiv Q_2$. Then $(\text{addell}_{\text{ProjCo}}(z,p))(P_1,Q_1) \equiv (\text{addell}_{\text{ProjCo}}(z,p))(P_2,Q_2)$. The theorem is a consequence of (22) and (23).

(25) Let us consider elements P, O of $\text{EC}_{\text{SetProjCo}}((z)_{\mathbf{1}})$. Suppose $O = \langle 0, 1, 0\rangle$. Then $P \equiv O$ if and only if $(\text{compell}_{\text{ProjCo}}(z,p))(P) \equiv O$.

(26) Let us consider elements P, Q of $\text{EC}_{\text{SetProjCo}}((z)_{\mathbf{1}})$, and an element a of GF(p). Suppose $a \neq 0_{\text{GF}(p)}$ and $(P)_{\mathbf{1},3} = a \cdot ((Q)_{\mathbf{1},3})$ and $(P)_{\mathbf{2},3} = a \cdot ((Q)_{\mathbf{2},3})$ and $(P)_{\mathbf{3},3} = a \cdot ((Q)_{\mathbf{3},3})$. Then $P \equiv Q$.

(27) Let us consider elements P, Q of $\text{EC}_{\text{SetProjCo}}((z)_{\mathbf{1}})$, and elements g_2, gf_1, gf_2, gf_3 of GF(p). Suppose $P \not\equiv Q$ and $(P)_{\mathbf{3},3} = 1$ and $(Q)_{\mathbf{3},3} = 1$ and $g_2 = 2 \bmod p$ and $gf_1 = (Q)_{\mathbf{2},3} - (P)_{\mathbf{2},3}$ and $gf_2 = (Q)_{\mathbf{1},3} - (P)_{\mathbf{1},3}$ and $gf_3 = {gf_1}^2 - {gf_2}^3 - g_2 \cdot ({gf_2}^2) \cdot ((P)_{\mathbf{1},3})$. Then $(\text{addell}_{\text{ProjCo}}(z,p))(P,Q) = \langle gf_2 \cdot gf_3, gf_1 \cdot ({gf_2}^2 \cdot ((P)_{\mathbf{1},3}) - gf_3) - {gf_2}^3 \cdot ((P)_{\mathbf{2},3}), {gf_2}^3\rangle$. The theorem is a consequence of (2).

(28) Let us consider elements P, Q of $\text{EC}_{\text{SetProjCo}}((z)_{\mathbf{1}})$, and elements g_2, g_3, g_4, g_8, gf_1, gf_2, gf_3, gf_4 of GF(p). Suppose $P \equiv Q$ and $(P)_{\mathbf{3},3} = 1$ and $(Q)_{\mathbf{3},3} = 1$ and $g_2 = 2 \bmod p$ and $g_3 = 3 \bmod p$ and $g_4 = 4 \bmod p$ and $g_8 =$

8 mod p and $gf_1 = (z)_{\mathbf{1}} + g_3 \cdot (((P)_{\mathbf{1},3})^2)$ and $gf_2 = (P)_{\mathbf{2},3}$ and $gf_3 = (P)_{\mathbf{1},3} \cdot ((P)_{\mathbf{2},3}) \cdot gf_2$ and $gf_4 = {gf_1}^2 - g_8 \cdot gf_3$. Then $(\mathrm{addell}_{\mathrm{ProjCo}}(z,p))(P,Q) = \langle g_2 \cdot gf_4 \cdot gf_2,\ gf_1 \cdot (g_4 \cdot gf_3 - gf_4) - g_8 \cdot (((P)_{\mathbf{2},3})^2) \cdot ({gf_2}^2),\ g_8 \cdot ({gf_2}^3)\rangle$. The theorem is a consequence of (2).

Let us consider elements P, Q of $\mathrm{EC}_{\mathrm{SetProjCo}}((z)_{\mathbf{1}})$. Now we state the propositions:

(29) Suppose $(P)_{\mathbf{3},3} = 1$ and $(Q)_{\mathbf{3},3} = 1$.
Then $(\mathrm{compell}_{\mathrm{ProjCo}}(z,p))((\mathrm{addell}_{\mathrm{ProjCo}}(z,p))(P,Q)) \equiv (\mathrm{addell}_{\mathrm{ProjCo}}(z,p))((\mathrm{compell}_{\mathrm{ProjCo}}(z,p))(P), (\mathrm{compell}_{\mathrm{ProjCo}}(z,p))(Q))$. The theorem is a consequence of (27), (28), and (26).

(30) $(\mathrm{compell}_{\mathrm{ProjCo}}(z,p))((\mathrm{addell}_{\mathrm{ProjCo}}(z,p))(P,Q)) \equiv (\mathrm{addell}_{\mathrm{ProjCo}}(z,p))((\mathrm{compell}_{\mathrm{ProjCo}}(z,p))(P), (\mathrm{compell}_{\mathrm{ProjCo}}(z,p))(Q))$. The theorem is a consequence of (25), (8), (29), (24), and (2).

(31) Let us consider elements P, O of $\mathrm{EC}_{\mathrm{SetProjCo}}((z)_{\mathbf{1}})$. Suppose $O = \langle 0, 1, 0\rangle$ and $P \not\equiv O$. Then $(P)_{\mathbf{2},3} = 0_{\mathrm{GF}(p)}$ if and only if $(\mathrm{addell}_{\mathrm{ProjCo}}(z,p))(P,P) \equiv O$.
Proof: Reconsider $g_8 = 8 \bmod p$ as an element of $\mathrm{GF}(p)$. $((\mathrm{addell}_{\mathrm{ProjCo}}(z,p))(P,P))_{\mathbf{3},3} = 0$. $g_8 \neq 0_{\mathrm{GF}(p)}$. $(P)_{\mathbf{3},3} \neq 0$ by [4, (23)], [5, (28)]. □

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Accepted August 29, 2019