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Klein-Beltrami model. Part III

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Summary. Timothy Makarios (with Isabelle/HOL[1]) and John Harrison (with HOL-Light[2]) shown that "the Klein-Beltrami model of the hyperbolic plane satisfy all of Tarski's axioms except his Euclidean axiom" [2],[3],[4],[5].

With the Mizar system [1] we use some ideas taken from Tim Makarios's MSc thesis [10] to formalize some definitions (like the absolute) and lemmas necessary for the verification of the independence of the parallel postulate. In this article we prove that our constructed model (we prefer "Beltrami-Klein" name over "Klein-Beltrami", which can be seen in the naming convention for Mizar functors, and even MML identifiers) satisfies the congruence symmetry, the congruence equivalence relation, and the congruence identity axioms formulated by Tarski (and formalized in Mizar as described briefly in [8]).

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Keywords: Tarski's geometry axioms; foundations of geometry; Klein-Beltrami model

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1. Preliminaries

Now we state the propositions:

(1) Let us consider real numbers x, y. If $x \cdot y < 0$, then $0 < \frac{x}{x-y} < 1$.

(2) Let us consider a non zero real number a, and real numbers b, r. Suppose $r = \sqrt{a^{\mathbf{2}} + b^{\mathbf{2}}}$. Then

(i) r is positive, and

[1] https://www.isa-afp.org/entries/Tarskis_Geometry.html
[2] https://github.com/jrh13/hol-light/blob/master/100/independence.ml

(ii) $(\frac{a}{r})^{\mathbf{2}} + (\frac{b}{r})^{\mathbf{2}} = 1$.

(3) Let us consider a non zero real number a, and real numbers b, c, d, e. Suppose $a \cdot b = c - d \cdot e$. Then $b^{\mathbf{2}} = \frac{c^{\mathbf{2}}}{a^{\mathbf{2}}} - 2 \cdot \frac{c \cdot d}{a \cdot a} \cdot e + \frac{d^{\mathbf{2}}}{a^{\mathbf{2}}} \cdot e^{\mathbf{2}}$.

Let us consider complex numbers a, b, c. Now we state the propositions:

(4) If $a \neq 0$, then $\frac{a^{\mathbf{2}} \cdot b \cdot c}{a^{\mathbf{2}}} = b \cdot c$.

(5) If $a \neq 0$, then $\frac{2 \cdot a^{\mathbf{2}} \cdot b \cdot c}{a^{\mathbf{2}}} = 2 \cdot b \cdot c$. The theorem is a consequence of (4).

(6) Let us consider a real number a. If $1 < a$, then $\frac{1}{a} - 1 < 0$.

(7) Let us consider real numbers a, b. If $0 < a$ and $1 < b$, then $\frac{a}{b} - a < 0$. The theorem is a consequence of (6).

(8) Let us consider a non zero real number a, and real numbers b, c, d. Suppose $a^{\mathbf{2}} + c^{\mathbf{2}} = b^{\mathbf{2}}$ and $1 < b^{\mathbf{2}}$. Then $\frac{(b^{\mathbf{2}})^{\mathbf{2}}}{a^{\mathbf{2}}} - 2 \cdot \frac{b^{\mathbf{2}} \cdot c}{a \cdot a} \cdot d + \frac{c^{\mathbf{2}}}{a^{\mathbf{2}}} \cdot d^{\mathbf{2}} + d^{\mathbf{2}} \neq 1$. The theorem is a consequence of (5) and (7).

(9) Let us consider real numbers a, b, c. If $a \cdot (-b) = c$ and $a \cdot c = b$, then $c = 0$ and $b = 0$.

(10) Let us consider a positive real number a. Then $\frac{\sqrt{a}}{a} = \frac{1}{\sqrt{a}}$.

2. Planar Lemmas

Let a be a non zero real number and b, c be real numbers. Observe that $[a, b, c]$ is non zero as an element of $\mathcal{E}^3_{\mathrm{T}}$ and $[c, a, b]$ is non zero as an element of $\mathcal{E}^3_{\mathrm{T}}$ and $[b, c, a]$ is non zero as an element of $\mathcal{E}^3_{\mathrm{T}}$.

Let P be an element of the real projective plane. Assume $P \in$ (the absolute)$\cup$ (the BK-model). The functor $\# P$ yielding a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ is defined by

(Def. 1) the direction of $it = P$ and $it(3) = 1$.

Now we state the propositions:

(11) Let us consider an element P of the real projective plane. Then there exists an element Q of the BK-model such that $P \neq Q$.

From now on P denotes an element of the BK-model.

(12) There exist elements P_1, P_2 of the absolute such that

(i) $P_1 \neq P_2$, and

(ii) P_1, P and P_2 are collinear.

The theorem is a consequence of (11).

(13) Let us consider an element Q of the absolute. Then there exists an element R of the BK-model such that

(i) $P \neq R$, and

(ii) P, Q and R are collinear.

(14) Let us consider a line L of Inc-ProjSp(the real projective plane). Suppose $P \in L$. Then there exist elements P_1, P_2 of the absolute such that

(i) $P_1 \neq P_2$, and

(ii) P_1, $P_2 \in L$.

Let N be an invertible square matrix over $\mathbb{R}_{\rm F}$ of dimension 3. The functor Line-homography(N) yielding a function from the lines of Inc-ProjSp(the real projective plane) into the lines of Inc-ProjSp(the real projective plane) is defined by

(Def. 2) for every line x of Inc-ProjSp(the real projective plane), $it(x) =$ $\{$(the homography of N)(P), where P is a point of Inc-ProjSp(the real projective plane) : P lies on $x\}$.

In the sequel N, N_1, N_2 denote invertible square matrices over $\mathbb{R}_{\rm F}$ of dimension 3 and l, l_1, l_2 denote elements of the lines of Inc-ProjSp(the real projective plane). Now we state the propositions:

(15) (Line-homography(N_1))((Line-homography(N_2))(l)) = (Line-homography($N_1 \cdot N_2$))(l).

PROOF: Reconsider $l_2 =$ (Line-homography(N_2))(l) as a line of Inc-ProjSp (the real projective plane). $\{$(the homography of N_1)(P), where P is a point of Inc-ProjSp(the real projective plane) : P lies on $l_2\}$ = $\{$(the homography of $N_1 \cdot N_2$)(P), where P is a point of Inc-ProjSp(the real projective plane) : P lies on $l\}$ by [9, (3), (4), (5)], [6, (13)]. $\square$

(16) (Line-homography($I^{3\times 3}_{\mathbb{R}_{\rm F}}$))($l$) = l.

PROOF: Set $X = \{$(the homography of $I^{3\times 3}_{\mathbb{R}_{\rm F}}$)($P$), where P is a point of Inc-ProjSp(the real projective plane) : P lies on $l\}$. $X \subseteq l$. $l \subseteq X$. $\square$

(17) (i) (Line-homography(N))((Line-homography($N^{\smile}$))(l)) = l, and

(ii) (Line-homography($N^{\smile}$))((Line-homography(N))(l)) = l.

The theorem is a consequence of (15) and (16).

(18) If (Line-homography(N))(l_1) = (Line-homography(N))(l_2), then $l_1 = l_2$. The theorem is a consequence of (17).

The functor SetLineHom3 yielding a set is defined by the term

(Def. 3) the set of all Line-homography(N) where N is an invertible square matrix over $\mathbb{R}_{\rm F}$ of dimension 3.

Observe that SetLineHom3 is non empty. Let h_1, h_2 be elements of SetLineHom3. The functor $h_1 \circ h_2$ yielding an element of SetLineHom3 is defined by

(Def. 4) there exist invertible square matrices N_1, N_2 over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that $h_1 = \text{Line-homography}(N_1)$ and $h_2 = \text{Line-homography}(N_2)$ and $it = \text{Line-homography}(N_1 \cdot N_2)$.

Now we state the propositions:

(19) Let us consider elements h_1, h_2 of SetLineHom3. Suppose $h_1 = \text{Line-homography}(N_1)$ and $h_2 = \text{Line-homography}(N_2)$. Then $\text{Line-homography}(N_1 \cdot N_2) = h_1 \circ h_2$. The theorem is a consequence of (15).

(20) Let us consider elements x, y, z of SetLineHom3. Then $(x \circ y) \circ z = x \circ (y \circ z)$. The theorem is a consequence of (19).

The functor BinOpLineHom3 yielding a binary operation on SetLineHom3 is defined by

(Def. 5) for every elements h_1, h_2 of SetLineHom3, $it(h_1, h_2) = h_1 \circ h_2$.

The functor GroupLineHom3 yielding a strict multiplicative magma is defined by the term

(Def. 6) $\langle \text{SetLineHom3}, \text{BinOpLineHom3} \rangle$.

Let us observe that GroupLineHom3 is non empty, associative, and group-like. Now we state the propositions:

(21) $\mathbf{1}_{\text{GroupLineHom3}} = \text{Line-homography}(I^{3 \times 3}_{\mathbb{R}_{\mathrm{F}}})$.

(22) Let us consider elements h, g of GroupLineHom3, and invertible square matrices N, N_1 over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose $h = \text{Line-homography}(N)$ and $g = \text{Line-homography}(N_1)$ and $N_1 = N^{\smile}$. Then $g = h^{-1}$. The theorem is a consequence of (21).

In the sequel P denotes a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ and l denotes a line of Inc-ProjSp(the real projective plane).

(23) If (the homography of $N)(P) \in l$, then $P \in (\text{Line-homography}(N^{\smile}))(l)$.

(24) If $P \in (\text{Line-homography}(N))(l)$, then (the homography of $N^{\smile})(P) \in l$.

(25) $P \in l$ if and only if (the homography of $N)(P) \in (\text{Line-homography}(N))(l)$. The theorem is a consequence of (23) and (17).

(26) Let us consider non zero elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $(u)_{\mathbf{3}} = 1$ and $(v)_{\mathbf{1}} = -(u)_{\mathbf{2}}$ and $(v)_{\mathbf{2}} = (u)_{\mathbf{1}}$ and $(v)_{\mathbf{3}} = 0$ and $(w)_{\mathbf{3}} = 1$ and $\langle |u, v, w| \rangle = 0$. Then $((u)_{\mathbf{1}})^{\mathbf{2}} + ((u)_{\mathbf{2}})^{\mathbf{2}} - (u)_{\mathbf{1}} \cdot (w)_{\mathbf{1}} - (u)_{\mathbf{2}} \cdot (w)_{\mathbf{2}} = 0$.

(27) Let us consider a non zero real number a, and real numbers b, c. Then $a \cdot [\frac{b}{a}, \frac{c}{a}, 1] = [b, c, a]$.

Let us consider non zero elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Now we state the propositions:

(28) Suppose $(u)_{\mathbf{1}} \neq 0$ and $(u)_{\mathbf{3}} = 1$ and $(v)_{\mathbf{1}} = -(u)_{\mathbf{2}}$ and $(v)_{\mathbf{2}} = (u)_{\mathbf{1}}$ and $(v)_{\mathbf{3}} = 0$ and $(w)_{\mathbf{3}} = 1$ and $\langle|u, v, w|\rangle = 0$ and $1 < ((u)_{\mathbf{1}})^{\mathbf{2}} + ((u)_{\mathbf{2}})^{\mathbf{2}}$. Then $((w)_{\mathbf{1}})^{\mathbf{2}} + ((w)_{\mathbf{2}})^{\mathbf{2}} \neq 1$. The theorem is a consequence of (26), (2), (3), and (8).

(29) Suppose $(u)_{\mathbf{2}} \neq 0$ and $(u)_{\mathbf{3}} = 1$ and $(v)_{\mathbf{1}} = -(u)_{\mathbf{2}}$ and $(v)_{\mathbf{2}} = (u)_{\mathbf{1}}$ and $(v)_{\mathbf{3}} = 0$ and $(w)_{\mathbf{3}} = 1$ and $\langle|u, v, w|\rangle = 0$ and $1 < ((u)_{\mathbf{1}})^{\mathbf{2}} + ((u)_{\mathbf{2}})^{\mathbf{2}}$. Then $((w)_{\mathbf{1}})^{\mathbf{2}} + ((w)_{\mathbf{2}})^{\mathbf{2}} \neq 1$. The theorem is a consequence of (26), (2), (3), and (8).

(30) Let us consider an element P of the absolute. Then there exists a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that

   (i) $u(3) = 1$, and

   (ii) $P =$ the direction of u.

(31) Let us consider real numbers a, b, c, d, and non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $u = [a, b, 1]$ and $v = [c, d, 0]$. Then the direction of $u \neq$ the direction of v.

(32) Let us consider a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $u(1)^{\mathbf{2}} + u(2)^{\mathbf{2}} < 1$ and $u(3) = 1$. Then the direction of u is an element of the BK-model.

(33) Let us consider real numbers a, b. Suppose $a^{\mathbf{2}} + b^{\mathbf{2}} \leqslant 1$. Then the direction of $[a, b, 1] \in$ (the BK-model) $\cup$ (the absolute). The theorem is a consequence of (32).

(34) If $P \notin$ (the BK-model) $\cup$ (the absolute), then there exists l such that $P \in l$ and l misses the absolute. The theorem is a consequence of (9), (30), (27), (31), (33), (28), and (29).

(35) Let us consider a point P of the real projective plane, an element h of the subgroup of K-isometries, and an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose $h =$ the homography of N. Then P is an element of the absolute if and only if (the homography of N)(P) is an element of the absolute.

Let us consider an element P of the BK-model, an element h of the subgroup of K-isometries, and an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3.

(36) If $h =$ the homography of N, then (the homography of N)(P) is an element of the BK-model.

   Proof: Set $h_1 =$ (the homography of N)(P). h_1 is not an element of the absolute by (35), [7, (1)]. Consider l such that $h_1 \in l$ and l misses the absolute. Reconsider $L =$ (Line-homography$(N^{\smile})$)(l) as a line of the real projective plane. Reconsider $L' = L$ as a line of Inc-ProjSp(the real projective plane). Consider P_1, P_2 being elements of the absolute such that

$P_1 \neq P_2$ and $P_1 \in L'$ and $P_2 \in L'$. (The homography of $N)(P_1)$ is an element of the absolute. (The homography of $N)(P_1) \in$ (Line-homography(N)) (L). (The homography of $N)(P_1) \in l$. $\square$

(37) Suppose $h =$ the homography of N. Then there exists a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that

(i) (the homography of $N)(P) =$ the direction of u, and

(ii) $u(3) = 1$.

The theorem is a consequence of (36).

3. The Construction of Beltrami-Klein Model

The functor BK-model-Betweenness yielding a relation between (the BK-model) $\times$ (the BK-model) and the BK-model is defined by

(Def. 7) for every elements a, b, c of the BK-model, $\langle\langle a, b\rangle, c\rangle \in it$ iff BK-to-REAL2(b) $\in \mathcal{L}$(BK-to-REAL2(a), BK-to-REAL2(c)).

The functor BK-model-Equidistance yielding a relation between (the BK-model) $\times$ (the BK-model) and (the BK-model) $\times$ (the BK-model) is defined by

(Def. 8) for every elements a, b, c, d of the BK-model, $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in it$ iff there exists an element h of the subgroup of K-isometries and there exists an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that $h =$ the homography of N and (the homography of $N)(a) = c$ and (the homography of $N)(b) = d$.

The functor BK-model-Plane yielding a Tarski plane is defined by the term

(Def. 9) $\langle$the BK-model, BK-model-Betweenness, BK-model-Equidistance$\rangle$.

4. Congruence Symmetry

Now we state the proposition:

(38) BK-model-Plane satisfies the axiom of congruence symmetry.

5. Congruence Equivalence Relation

Now we state the proposition:

(39) BK-model-Plane satisfies the axiom of congruence equivalence relation.

6. Congruence Identity

Now we state the proposition:

(40) BK-model-Plane satisfies the axiom of congruence identity.

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Klein-Beltrami model. Part IV

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Summary. Timothy Makarios (with Isabelle/HOL[1]) and John Harrison (with HOL-Light[2]) shown that "the Klein-Beltrami model of the hyperbolic plane satisfy all of Tarski's axioms except his Euclidean axiom" [2],[3],[4, 5].

With the Mizar system [1] we use some ideas taken from Tim Makarios's MSc thesis [10] to formalize some definitions and lemmas necessary for the verification of the independence of the parallel postulate. In this article, which is the continuation of [8], we prove that our constructed model satisfies the axioms of segment construction, the axiom of betweenness identity, and the axiom of Pasch due to Tarski, as formalized in [11] and related Mizar articles.

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Keywords: Tarski's geometry axioms; foundations of geometry; Klein-Beltrami model

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1. Preliminaries

Let us consider real numbers a, b. Now we state the propositions:

(1) If $a \neq b$, then $1 - \frac{a}{a-b} = -\frac{b}{a-b}$.

(2) If $0 < a \cdot b$, then $0 < \frac{a}{b}$.

Now we state the propositions:

(3) Let us consider real numbers a, b, c. Suppose $0 \leqslant a \leqslant 1$ and $0 < b \cdot c$. Then $0 \leqslant \frac{a \cdot c}{(1-a) \cdot b + a \cdot c} \leqslant 1$.

[1] `https://www.isa-afp.org/entries/Tarskis_Geometry.html`

[2] `https://github.com/jrh13/hol-light/blob/master/100/independence.ml`

(4) Let us consider real numbers a, b, c. Suppose $(1-a)\cdot b + a\cdot c \neq 0$. Then $1 - \frac{a\cdot c}{(1-a)\cdot b+a\cdot c} = \frac{(1-a)\cdot b}{(1-a)\cdot b+a\cdot c}$.

(5) Let us consider real numbers a, b, c, d. If $b \neq 0$, then $\frac{\frac{a\cdot b}{c}\cdot d}{b} = \frac{a\cdot d}{c}$.

(6) Let us consider an element u of $\mathcal{E}^3_{\mathrm{T}}$. Then $u = [u(1), u(2), u(3)]$.

(7) Let us consider an element P of the BK-model. Then BK-to-REAL2$(P) \in$ TarskiEuclid2Space.

Let P be a point of BK-model-Plane. The functor BKtoT2(P) yielding a point of TarskiEuclid2Space is defined by

(Def. 1) there exists an element p of the BK-model such that $P = p$ and $it =$ BK-to-REAL2(p).

Let P be a point of TarskiEuclid2Space. Assume $\hat{P} \in$ the inside of circle(0,0,1). The functor T2toBK(P) yielding a point of BK-model-Plane is defined by

(Def. 2) there exists a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that $it =$ the direction of u and $(u)_{\mathbf{3}} = 1$ and $\hat{P} = [(u)_{\mathbf{1}}, (u)_{\mathbf{2}}]$.

Let us consider a point P of BK-model-Plane. Now we state the propositions:

(8) $\widehat{\mathrm{BKtoT2}(P)} \in$ the inside of circle(0,0,1).

(9) T2toBK(BKtoT2$(P)) = P$.

(10) Let us consider a point P of TarskiEuclid2Space. Suppose $\hat{P}$ is an element of the inside of circle(0,0,1). Then BKtoT2(T2toBK$(P)) = P$.

(11) Let us consider a point P of BK-model-Plane, and an element p of the BK-model. Suppose $P = p$. Then

 (i) BKtoT2$(P) =$ BK-to-REAL2(p), and

 (ii) $\widehat{\mathrm{BKtoT2}(P)} =$ BK-to-REAL2(p).

(12) Let us consider points P, Q, R of BK-model-Plane, and points P_2, Q_2, R_2 of TarskiEuclid2Space. Suppose $P_2 =$ BKtoT2(P) and $Q_2 =$ BKtoT2(Q) and $R_2 =$ BKtoT2(R). Then Q lies between P and R if and only if Q_2 lies between P_2 and R_2. The theorem is a consequence of (11).

(13) Let us consider elements P, Q of $\mathcal{E}^2_{\mathrm{T}}$. If $P \neq Q$, then $P(1) \neq Q(1)$ or $P(2) \neq Q(2)$.

(14) Let us consider real numbers a, b, and elements P, Q of $\mathcal{E}^2_{\mathrm{T}}$. If $P \neq Q$ and $(1-a)\cdot P + a\cdot Q = (1-b)\cdot P + b\cdot Q$, then $a = b$. The theorem is a consequence of (13).

(15) Let us consider points P, Q of BK-model-Plane. If $\widehat{\mathrm{BKtoT2}(P)} = \widehat{\mathrm{BKtoT2}(Q)}$, then $P = Q$. The theorem is a consequence of (11).

Let P, Q, R be points of BK-model-Plane. Assume Q lies between P and R and $P \neq R$. The functor length(P, Q, R) yielding a real number is defined by

(Def. 3) $0 \leqslant it \leqslant 1$ and $\mathrm{BKto\hat{T}2}(Q) = (1 - it) \cdot (\mathrm{BKto\hat{T}2}(P)) + it \cdot (\mathrm{BKto\hat{T}2}(R))$.

Let us consider points P, Q of BK-model-Plane. Now we state the propositions:

(16) (i) P lies between P and Q, and

(ii) Q lies between P and Q.

The theorem is a consequence of (12).

(17) If $P \neq Q$, then $\mathrm{length}(P, P, Q) = 0$ and $\mathrm{length}(P, Q, Q) = 1$. The theorem is a consequence of (16).

(18) Let us consider a square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose $N = \langle\langle 2, 0, -1\rangle, \langle 0, \sqrt{3}, 0\rangle, \langle 1, 0, -2\rangle\rangle$. Then

(i) $\mathrm{Det}\, N = (-3) \cdot \sqrt{3}$, and

(ii) N is invertible.

(19) Let us consider elements a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}, b_{11}, b_{12}, b_{13}, b_{21}, b_{22}, b_{23}, b_{31}, b_{32}, b_{33}, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 of $\mathbb{R}_{\mathrm{F}}$, and square matrices A, B over $\mathbb{R}_{\mathrm{F}}$ of dimension 3.

Suppose $A = \langle\langle a_{11}, a_{12}, a_{13}\rangle, \langle a_{21}, a_{22}, a_{23}\rangle, \langle a_{31}, a_{32}, a_{33}\rangle\rangle$ and $B = \langle\langle b_{11}, b_{12}, b_{13}\rangle, \langle b_{21}, b_{22}, b_{23}\rangle, \langle b_{31}, b_{32}, b_{33}\rangle\rangle$ and $a_1 = a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31}$ and $a_2 = a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32}$ and $a_3 = a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33}$ and $a_4 = a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31}$.

Suppose $a_5 = a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32}$ and $a_6 = a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33}$ and $a_7 = a_{31} \cdot b_{11} + a_{32} \cdot b_{21} + a_{33} \cdot b_{31}$ and $a_8 = a_{31} \cdot b_{12} + a_{32} \cdot b_{22} + a_{33} \cdot b_{32}$ and $a_9 = a_{31} \cdot b_{13} + a_{32} \cdot b_{23} + a_{33} \cdot b_{33}$.

Then $A \cdot B = \langle\langle a_1, a_2, a_3\rangle, \langle a_4, a_5, a_6\rangle, \langle a_7, a_8, a_9\rangle\rangle$.

Let us consider square matrices N_1, N_2 over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Now we state the propositions:

(20) Suppose $N_1 = \langle\langle 2, 0, -1\rangle, \langle 0, \sqrt{3}, 0\rangle, \langle 1, 0, -2\rangle\rangle$ and $N_2 = \langle\langle \frac{2}{3}, 0, -\frac{1}{3}\rangle, \langle 0, \frac{1}{\sqrt{3}}, 0\rangle, \langle \frac{1}{3}, 0, -\frac{2}{3}\rangle\rangle$. Then $N_1 \cdot N_2 = \langle\langle 1, 0, 0\rangle, \langle 0, 1, 0\rangle, \langle 0, 0, 1\rangle\rangle$. The theorem is a consequence of (19).

(21) Suppose $N_2 = \langle\langle 2, 0, -1\rangle, \langle 0, \sqrt{3}, 0\rangle, \langle 1, 0, -2\rangle\rangle$ and $N_1 = \langle\langle \frac{2}{3}, 0, -\frac{1}{3}\rangle, \langle 0, \frac{1}{\sqrt{3}}, 0\rangle, \langle \frac{1}{3}, 0, -\frac{2}{3}\rangle\rangle$. Then $N_1 \cdot N_2 = \langle\langle 1, 0, 0\rangle, \langle 0, 1, 0\rangle, \langle 0, 0, 1\rangle\rangle$. The theorem is a consequence of (19).

(22) Suppose $N_1 = \langle\langle 2, 0, -1\rangle, \langle 0, \sqrt{3}, 0\rangle, \langle 1, 0, -2\rangle\rangle$ and $N_2 = \langle\langle \frac{2}{3}, 0, -\frac{1}{3}\rangle, \langle 0, \frac{1}{\sqrt{3}}, 0\rangle, \langle \frac{1}{3}, 0, -\frac{2}{3}\rangle\rangle$. Then N_1 is inverse of N_2. The theorem is a consequence of (20) and (21).

Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Now we state the propositions:

(23) Suppose $N = \langle\langle \frac{2}{3}, 0, -\frac{1}{3}\rangle, \langle 0, \frac{1}{\sqrt{3}}, 0\rangle, \langle \frac{1}{3}, 0, -\frac{2}{3}\rangle\rangle$. Then (the homography of N)$^\circ$(the absolute) $\subseteq$ the absolute.

PROOF: (The homography of N)$^\circ$(the absolute) $\subseteq$ the absolute by [7, (89)], [9, (7)]. $\square$

(24) Suppose $N = \langle\langle 2, 0, -1\rangle, \langle 0, \sqrt{3}, 0\rangle, \langle 1, 0, -2\rangle\rangle$. Then (the homography of N)$^\circ$(the absolute) = the absolute.

PROOF: (The homography of N)$^\circ$(the absolute) $\subseteq$ the absolute.
The absolute $\subseteq$ (the homography of N)$^\circ$(the absolute) by [6, (19)], (22), (23). $\square$

(25) Let us consider real numbers a, b, r, and elements P, Q, R of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $Q \in \mathcal{L}(P, R)$ and P, $R \in$ the inside of circle(a,b,r). Then $Q \in$ the inside of circle(a,b,r).

(26) Let us consider non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$. Suppose the direction of $u =$ the direction of v and $u(3) \neq 0$ and $u(3) = v(3)$. Then $u = v$.

(27) Let us consider an element R of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, elements P, Q of the BK-model, non zero elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$, and a real number r. Suppose $0 \leqslant r \leqslant 1$ and $P =$ the direction of u and $Q =$ the direction of v and $R =$ the direction of w and $u(3) = 1$ and $v(3) = 1$ and $w = r \cdot u + (1 - r) \cdot v$. Then R is an element of the BK-model.

PROOF: Consider u_2 being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $u_2 = P$ and $u_2(3) = 1$ and BK-to-REAL2(P) $= [u_2(1), u_2(2)]$. $u = u_2$. Reconsider $r_4 = [u_2(1), u_2(2)]$ as an element of $\mathcal{E}^2_{\mathrm{T}}$. Consider v_2 being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $v_2 = Q$ and $v_2(3) = 1$ and BK-to-REAL2(Q) $= [v_2(1), v_2(2)]$. $v = v_2$. Reconsider $r_6 = [v_2(1), v_2(2)]$ as an element of $\mathcal{E}^2_{\mathrm{T}}$. Reconsider $r_8 = [w(1), w(2)]$ as an element of $\mathcal{E}^2_{\mathrm{T}}$. $r_8 = r \cdot r_4 + (1 - r) \cdot r_6$. Consider R_3 being an element of $\mathcal{E}^2_{\mathrm{T}}$ such that $R_3 = r_8$ and REAL2-to-BK(r_8) = the direction of $[(R_3)_{\mathbf{1}}, (R_3)_{\mathbf{2}}, 1]$. $\square$

(28) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, elements n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\mathrm{F}}$, points P, Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$ and $P =$ the direction of u and $Q =$ the direction of v and $Q =$ (the homography of N)(P) and $u(3) = 1$. Then there exists a non zero real number a such that

(i) $v(1) = a \cdot (n_{11} \cdot u(1) + n_{12} \cdot u(2) + n_{13})$, and

(ii) $v(2) = a \cdot (n_{21} \cdot u(1) + n_{22} \cdot u(2) + n_{23})$, and

(iii) $v(3) = a \cdot (n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33})$.

(29) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, elements n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\mathrm{F}}$, an element

P of the BK-model, a point Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$ and P = the direction of u and Q = the direction of v and Q = (the homography of N)(P) and $u(3) = 1$ and $v(3) = 1$. Then

(i) $n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33} \neq 0$, and

(ii) $v(1) = \frac{n_{11} \cdot u(1) + n_{12} \cdot u(2) + n_{13}}{n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}}$, and

(iii) $v(2) = \frac{n_{21} \cdot u(1) + n_{22} \cdot u(2) + n_{23}}{n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}}$.

The theorem is a consequence of (28).

(30) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, an element h of the subgroup of K-isometries, elements n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\mathrm{F}}$, an element P of the BK-model, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose h = the homography of N and $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$ and P = the direction of u and $u(3) = 1$. Then $n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33} \neq 0$. The theorem is a consequence of (29).

(31) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, elements n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\mathrm{F}}$, an element P of the absolute, a point Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$ and P = the direction of u and Q = the direction of v and Q = (the homography of N)(P) and $u(3) = 1$ and $v(3) = 1$. Then

(i) $n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33} \neq 0$, and

(ii) $v(1) = \frac{n_{11} \cdot u(1) + n_{12} \cdot u(2) + n_{13}}{n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}}$, and

(iii) $v(2) = \frac{n_{21} \cdot u(1) + n_{22} \cdot u(2) + n_{23}}{n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}}$.

The theorem is a consequence of (28).

(32) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, an element h of the subgroup of K-isometries, elements n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\mathrm{F}}$, an element P of the absolute, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose h = the homography of N and $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$ and P = the direction of u and $u(3) = 1$. Then $n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33} \neq 0$. The theorem is a consequence of (31).

(33) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, an element h of the subgroup of K-isometries, elements n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\mathrm{F}}$, an element P of the BK-model, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose h = the homography of N and $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$ and P = the direction

of u and $u(3) = 1$. Then (the homography of N)(P) = the direction of $[\frac{n_{11} \cdot u(1) + n_{12} \cdot u(2) + n_{13}}{n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}}, \frac{n_{21} \cdot u(1) + n_{22} \cdot u(2) + n_{23}}{n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}}, 1]$. The theorem is a consequence of (29).

(34) Let us consider an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3, an element h of the subgroup of K-isometries, elements n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\rm F}$, an element P of the absolute, and a non zero element u of $\mathcal{E}^3_{\rm T}$. Suppose $h =$ the homography of N and $N = \langle \langle n_{11}, n_{12}, n_{13} \rangle, \langle n_{21}, n_{22}, n_{23} \rangle, \langle n_{31}, n_{32}, n_{33} \rangle \rangle$ and $P =$ the direction of u and $u(3) = 1$. Then (the homography of N)(P) = the direction of $[\frac{n_{11} \cdot u(1) + n_{12} \cdot u(2) + n_{13}}{n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}}, \frac{n_{21} \cdot u(1) + n_{22} \cdot u(2) + n_{23}}{n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}}, 1]$. The theorem is a consequence of (31).

(35) Let us consider a subset A of $\mathcal{E}^3_{\rm T}$, a convex, non empty subset B of $\mathcal{E}^2_{\rm T}$, a real number r, and an element x of $\mathcal{E}^3_{\rm T}$. Suppose $A = \{x$, where x is an element of $\mathcal{E}^3_{\rm T}$: $[(x)_{\mathbf{1}}, (x)_{\mathbf{2}}] \in B$ and $(x)_{\mathbf{3}} = r\}$. Then A is non empty and convex.

(36) Let us consider elements n_1, n_2, n_3 of $\mathbb{R}_{\rm F}$, and elements n, u of $\mathcal{E}^3_{\rm T}$. Suppose $n = \langle n_1, n_2, n_3 \rangle$ and $u(3) = 1$. Then $|(n, u)| = n_1 \cdot u(1) + n_2 \cdot u(2) + n_3$.

(37) Let us consider a convex, non empty subset A of $\mathcal{E}^3_{\rm T}$, and elements n, u, v of $\mathcal{E}^3_{\rm T}$. Suppose for every element w of $\mathcal{E}^3_{\rm T}$ such that $w \in A$ holds $|(n, w)| \neq 0$ and u, $v \in A$. Then $0 < |(n, u)| \cdot |(n, v)|$.
Proof: Set $x = |(n, u)|$. Set $y = |(n, v)|$. Reconsider $l = \frac{x}{x-y}$ as a non zero real number. Reconsider $w = l \cdot v + (1 - l) \cdot u$ as an element of $\mathcal{E}^3_{\rm T}$. $x \neq y$. $1 - l = -\frac{y}{x-y}$. $|(n, w)| = 0$. $\square$

Let us consider elements n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\rm F}$ and elements u, v of $\mathcal{E}^2_{\rm T}$. Now we state the propositions:

(38) Suppose u, $v \in$ the inside of circle(0,0,1) and for every element w of $\mathcal{E}^2_{\rm T}$ such that $w \in$ the inside of circle(0,0,1) holds $n_{31} \cdot w(1) + n_{32} \cdot w(2) + n_{33} \neq 0$. Then $0 < (n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}) \cdot (n_{31} \cdot v(1) + n_{32} \cdot v(2) + n_{33})$. The theorem is a consequence of (35), (36), and (37).

(39) Suppose $u \in$ the inside of circle(0,0,1) and $v \in \text{circle}(0, 0, 1)$ and for every element w of $\mathcal{E}^2_{\rm T}$ such that $w \in$ the closed inside of circle(0,0,1) holds $n_{31} \cdot w(1) + n_{32} \cdot w(2) + n_{33} \neq 0$. Then $0 < (n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}) \cdot (n_{31} \cdot v(1) + n_{32} \cdot v(2) + n_{33})$. The theorem is a consequence of (35), (36), and (37).

(40) Let us consider real numbers l, r, elements u, v, w of $\mathcal{E}^3_{\rm T}$, and real numbers n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33}, m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9.

Suppose $m_3 \neq 0$ and $m_6 \neq 0$ and $m_9 \neq 0$ and $r = \frac{l \cdot m_6}{(1-l) \cdot m_3 + l \cdot m_6}$ and $(1-l) \cdot m_3 + l \cdot m_6 \neq 0$ and $w = (1-l) \cdot u + l \cdot v$ and $m_1 = n_{11} \cdot u(1) + n_{12} \cdot u(2) + n_{13}$ and $m_2 = n_{21} \cdot u(1) + n_{22} \cdot u(2) + n_{23}$ and $m_3 = n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}$ and $m_4 = n_{11} \cdot v(1) + n_{12} \cdot v(2) + n_{13}$.

Suppose $m_5 = n_{21} \cdot v(1) + n_{22} \cdot v(2) + n_{23}$ and $m_6 = n_{31} \cdot v(1) + n_{32} \cdot v(2) + n_{33}$ and $m_7 = n_{11} \cdot w(1) + n_{12} \cdot w(2) + n_{13}$ and $m_8 = n_{21} \cdot w(1) + n_{22} \cdot w(2) + n_{23}$ and $m_9 = n_{31} \cdot w(1) + n_{32} \cdot w(2) + n_{33}$.

Then $(1-r) \cdot [\frac{m_1}{m_3}, \frac{m_2}{m_3}, 1] + r \cdot [\frac{m_4}{m_6}, \frac{m_5}{m_6}, 1] = [\frac{m_7}{m_9}, \frac{m_8}{m_9}, 1]$. The theorem is a consequence of (4) and (5).

(41) Let us consider an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3, an element h of the subgroup of K-isometries, elements n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\rm F}$, and an element P of the BK-model. Suppose $h =$ the homography of N and $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$. Then (the homography of N)$(P) =$ the direction of $[\frac{n_{11} \cdot (\text{BK-to-REAL2}(P))_{\mathbf{1}} + n_{12} \cdot (\text{BK-to-REAL2}(P))_{\mathbf{2}} + n_{13}}{n_{31} \cdot (\text{BK-to-REAL2}(P))_{\mathbf{1}} + n_{32} \cdot (\text{BK-to-REAL2}(P))_{\mathbf{2}} + n_{33}}, \frac{n_{21} \cdot (\text{BK-to-REAL2}(P))_{\mathbf{1}} + n_{22} \cdot (\text{BK-to-REAL2}(P))_{\mathbf{2}} + n_{23}}{n_{31} \cdot (\text{BK-to-REAL2}(P))_{\mathbf{1}} + n_{32} \cdot (\text{BK-to-REAL2}(P))_{\mathbf{2}} + n_{33}}, 1]$. The theorem is a consequence of (33).

(42) Let us consider an element h of the subgroup of K-isometries, an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3, elements n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\rm F}$, and an element u_2 of $\mathcal{E}^2_{\rm T}$. Suppose $h =$ the homography of N and $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$ and $u_2 \in$ the inside of circle(0,0,1). Then $n_{31} \cdot u_2(1) + n_{32} \cdot u_2(2) + n_{33} \neq 0$. The theorem is a consequence of (30).

(43) Let us consider a positive real number r, and an element u of $\mathcal{E}^2_{\rm T}$. If $u \in \text{circle}(0, 0, r)$, then u is not zero.

(44) Let us consider an element h of the subgroup of K-isometries, an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3, elements n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\rm F}$, and an element u_2 of $\mathcal{E}^2_{\rm T}$. Suppose $h =$ the homography of N and $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$ and $u_2 \in$ the closed inside of circle(0,0,1). Then $n_{31} \cdot u_2(1) + n_{32} \cdot u_2(2) + n_{33} \neq 0$. The theorem is a consequence of (30), (43), and (32).

(45) Let us consider real numbers a, b, c, d, e, f, r. Suppose $(1-r) \cdot [a, b, 1] + r \cdot [c, d, 1] = [e, f, 1]$. Then $(1-r) \cdot [a, b] + r \cdot [c, d] = [e, f]$.

(46) Let us consider points P, Q, R, P', Q', R' of BK-model-Plane, elements p, q, r, p', q', r' of the BK-model, an element h of the subgroup of K-isometries, and an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3. Suppose $h =$ the homography of N and Q lies between P and R and $P = p$ and $Q = q$ and $R = r$ and $p' =$ (the homography of N)(p) and $q' =$ (the homography of N)(q) and $r' =$ (the homography of N)(r) and

$P' = p'$ and $Q' = q'$ and $R' = r'$. Then Q' lies between P' and R'.

PROOF: Consider n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} being elements of $\mathbb{R}_{\mathrm{F}}$ such that $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$. Consider u being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $u = p$ and $u(3) = 1$ and BK-to-REAL2$(p) = [u(1), u(2)]$. Consider v being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $v = r$ and $v(3) = 1$ and BK-to-REAL2$(r) = [v(1), v(2)]$. Consider w being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $w = q$ and $w(3) = 1$ and BK-to-REAL2$(q) = [w(1), w(2)]$.

Reconsider $m_1 = n_{11} \cdot u(1) + n_{12} \cdot u(2) + n_{13}$, $m_2 = n_{21} \cdot u(1) + n_{22} \cdot u(2)+n_{23}$, $m_3 = n_{31} \cdot u(1)+n_{32} \cdot u(2)+n_{33}$, $m_4 = n_{11} \cdot v(1)+n_{12} \cdot v(2)+n_{13}$, $m_5 = n_{21} \cdot v(1) + n_{22} \cdot v(2) + n_{23}$, $m_6 = n_{31} \cdot v(1) + n_{32} \cdot v(2) + n_{33}$, $m_7 = n_{11} \cdot w(1)+n_{12} \cdot w(2)+n_{13}$, $m_8 = n_{21} \cdot w(1)+n_{22} \cdot w(2)+n_{23}$, $m_9 = n_{31} \cdot w(1)+ n_{32} \cdot w(2) + n_{33}$ as a real number. BKtoT2(P) = BK-to-REAL2(p) and BKto$\hat{\mathrm{T}}$2(P) = BK-to-REAL2(p) and BKtoT2(Q) = BK-to-REAL2(q) and BKto$\hat{\mathrm{T}}$2(Q) = BK-to-REAL2(q) and BKtoT2(R) = BK-to-REAL2(r) and BKto$\hat{\mathrm{T}}$2(R) = BK-to-REAL2(r). Consider l being a real number such that $0 \leqslant l \leqslant 1$ and BKto$\hat{\mathrm{T}}$2$(Q) = (1-l) \cdot ($BKto$\hat{\mathrm{T}}$2$(P)) + l \cdot ($BKto$\hat{\mathrm{T}}$2$(R))$.

Set $r = \frac{l \cdot m_6}{(1-l) \cdot m_3 + l \cdot m_6}$. $(1 - r) \cdot [\frac{m_1}{m_3}, \frac{m_2}{m_3}, 1] + r \cdot [\frac{m_4}{m_6}, \frac{m_5}{m_6}, 1] = [\frac{m_7}{m_9}, \frac{m_8}{m_9}, 1]$. $0 \leqslant r \leqslant 1$. BKtoT2$(P')$ = BK-to-REAL2(p') and BKto$\hat{\mathrm{T}}$2(P') = BK-to-REAL2(p') and BKtoT2(Q') = BK-to-REAL2(q') and BKto$\hat{\mathrm{T}}$2(Q') = BK-to-REAL2(q') and BKtoT2(R') = BK-to-REAL2(r') and BKto$\hat{\mathrm{T}}$2(R') = BK-to-REAL2(r'). □

Let P be a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. We say that P is point at ∞ if and only if

(Def. 4) there exists a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that P = the direction of u and $(u)_{\mathbf{3}} = 0$.

Now we state the proposition:

(47) Let us consider a point P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose there exists a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that P = the direction of u and $(u)_{\mathbf{3}} \neq 0$. Then P is not point at ∞.

Note that there exists a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ which is point at ∞ and there exists a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ which is non point at ∞.

Let P be a non point at ∞ point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. The functor RP3toREAL2$(P)$ yielding an element of $\mathcal{R}^2$ is defined by

(Def. 5) there exists a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that P = the direction of u and $(u)_{\mathbf{3}} = 1$ and *it* $= [(u)_{\mathbf{1}}, (u)_{\mathbf{2}}]$.

The functor RP3toT2(P) yielding a point of TarskiEuclid2Space is defined by the term

(Def. 6) RP3toREAL2(P).

Now we state the propositions:

(48) Let us consider non point at ∞ elements P, Q, R, P', Q', R' of the projective space over $\mathcal{E}^3_{\rm T}$, an element h of the subgroup of K-isometries, and an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3.

Suppose $h =$ the homography of N and $P, Q \in$ the BK-model and $R \in$ the absolute and $P' =$ (the homography of N)(P) and $Q' =$ (the homography of N)(Q) and $R' =$ (the homography of N)(R) and RP3toT2(Q) lies between RP3toT2(P) and RP3toT2(R).

Then RP3toT2(Q') lies between RP3toT2(P') and RP3toT2(R').

PROOF: Consider n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} being elements of $\mathbb{R}_{\rm F}$ such that $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$. Consider u being a non zero element of $\mathcal{E}^3_{\rm T}$ such that $P =$ the direction of u and $(u)_{\mathbf{3}} = 1$ and RP3toREAL2(P) $= [(u)_{\mathbf{1}}, (u)_{\mathbf{2}}]$. Consider v being a non zero element of $\mathcal{E}^3_{\rm T}$ such that $R =$ the direction of v and $(v)_{\mathbf{3}} = 1$ and RP3toREAL2(R) $= [(v)_{\mathbf{1}}, (v)_{\mathbf{2}}]$. Consider w being a non zero element of $\mathcal{E}^3_{\rm T}$ such that $Q =$ the direction of w and $(w)_{\mathbf{3}} = 1$ and RP3toREAL2(Q) $= [(w)_{\mathbf{1}}, (w)_{\mathbf{2}}]$.

Reconsider $m_1 = n_{11} \cdot u(1) + n_{12} \cdot u(2) + n_{13}$, $m_2 = n_{21} \cdot u(1) + n_{22} \cdot u(2) + n_{23}$, $m_3 = n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}$, $m_4 = n_{11} \cdot v(1) + n_{12} \cdot v(2) + n_{13}$, $m_5 = n_{21} \cdot v(1) + n_{22} \cdot v(2) + n_{23}$, $m_6 = n_{31} \cdot v(1) + n_{32} \cdot v(2) + n_{33}$, $m_7 = n_{11} \cdot w(1) + n_{12} \cdot w(2) + n_{13}$, $m_8 = n_{21} \cdot w(1) + n_{22} \cdot w(2) + n_{23}$, $m_9 = n_{31} \cdot w(1) + n_{32} \cdot w(2) + n_{33}$ as a real number.

Consider l being a real number such that $0 \leqslant l \leqslant 1$ and $\widehat{\text{RP3toT2}}(Q) = (1-l) \cdot (\widehat{\text{RP3toT2}}(P)) + l \cdot (\widehat{\text{RP3toT2}}(R))$. Set $r = \frac{l \cdot m_6}{(1-l) \cdot m_3 + l \cdot m_6}$. $(1-r) \cdot [\frac{m_1}{m_3}, \frac{m_2}{m_3}, 1] + r \cdot [\frac{m_4}{m_6}, \frac{m_5}{m_6}, 1] = [\frac{m_7}{m_9}, \frac{m_8}{m_9}, 1]$. $0 \leqslant r \leqslant 1$. $\square$

(49) Let us consider real numbers a, b, c, and elements u, v, w of $\mathcal{E}^3_{\rm T}$. Suppose $a \neq 0$ and $a + b + c = 0$ and $a \cdot u + b \cdot v + c \cdot w = 0_{\mathcal{E}^3_{\rm T}}$. Then $u \in \text{Line}(v, w)$.

(50) Let us consider non point at ∞ points P, Q, R of the projective space over $\mathcal{E}^3_{\rm T}$, and non zero elements u, v, w of $\mathcal{E}^3_{\rm T}$. Suppose $P =$ the direction of u and $Q =$ the direction of v and $R =$ the direction of w and $(u)_{\mathbf{3}} = 1$ and $(v)_{\mathbf{3}} = 1$ and $(w)_{\mathbf{3}} = 1$. Then P, Q and R are collinear if and only if u, v and w are collinear. The theorem is a consequence of (49).

(51) Let us consider elements u, v, w of $\mathcal{E}^3_{\rm T}$. Suppose $u \in \mathcal{L}(v, w)$. Then $[(u)_{\mathbf{1}}, (u)_{\mathbf{2}}] \in \mathcal{L}([(v)_{\mathbf{1}}, (v)_{\mathbf{2}}], [(w)_{\mathbf{1}}, (w)_{\mathbf{2}}])$.

(52) Let us consider elements u, v, w of $\mathcal{E}^2_{\rm T}$. Suppose $u \in \mathcal{L}(v, w)$. Then $[(u)_{\mathbf{1}}, (u)_{\mathbf{2}}, 1] \in \mathcal{L}([(v)_{\mathbf{1}}, (v)_{\mathbf{2}}, 1], [(w)_{\mathbf{1}}, (w)_{\mathbf{2}}, 1])$.

PROOF: Consider r being a real number such that $0 \leqslant r$ and $r \leqslant 1$ and $u = (1-r) \cdot v + r \cdot w$. Reconsider $u' = [(u)_{\mathbf{1}}, (u)_{\mathbf{2}}, 1]$, $v' = [(v)_{\mathbf{1}}, (v)_{\mathbf{2}}, 1]$, $w' = [(w)_{\mathbf{1}}, (w)_{\mathbf{2}}, 1]$ as an element of $\mathcal{E}^3_{\mathrm{T}}$. $u' = (1-r) \cdot v' + r \cdot w'$. □

(53) Let us consider non point at ∞ points P, Q, R of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Then P, Q and R are collinear if and only if RP3toT2(P), RP3toT2(Q) and RP3toT2(R) are collinear. The theorem is a consequence of (50), (51), and (52).

(54) Let us consider elements u, v, w of $\mathcal{E}^2_{\mathrm{T}}$. Suppose u, v and w are collinear. Then $[(u)_{\mathbf{1}}, (u)_{\mathbf{2}}, 1]$, $[(v)_{\mathbf{1}}, (v)_{\mathbf{2}}, 1]$ and $[(w)_{\mathbf{1}}, (w)_{\mathbf{2}}, 1]$ are collinear. The theorem is a consequence of (52).

(55) Let us consider non point at ∞ elements P, Q, P_1 of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose P, $Q \in$ the BK-model and $P_1 \in$ the absolute. Then RP3toT2(P_1) does not lie between RP3toT2(Q) and RP3toT2(P). The theorem is a consequence of (52) and (27).

The functor Dir001 yielding a non point at ∞ element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 7) the direction of $[0, 0, 1]$.

The functor Dir101 yielding a non point at ∞ element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 8) the direction of $[1, 0, 1]$.

Now we state the propositions:

(56) Let us consider non point at ∞ elements P, Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose P, $Q \in$ the absolute. Then $\overline{\text{RP3toT2(Dir001) RP3toT2}(P)} \cong \overline{\text{RP3toT2(Dir001) RP3toT2}(Q)}$.

(57) Let us consider non point at ∞ elements P, Q, R of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and non zero elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Suppose P, $Q \in$ the absolute and $P \neq Q$ and $P =$ the direction of u and $Q =$ the direction of v and $R =$ the direction of w and $(u)_{\mathbf{3}} = 1$ and $(v)_{\mathbf{3}} = 1$ and $w = [\frac{(u)_{\mathbf{1}}+(v)_{\mathbf{1}}}{2}, \frac{(u)_{\mathbf{2}}+(v)_{\mathbf{2}}}{2}, 1]$. Then $R \in$ the BK-model.
PROOF: Reconsider $u' = [u(1), u(2)]$, $v' = [v(1), v(2)]$ as an element of $\mathcal{E}^2_{\mathrm{T}}$. $u' \neq v'$. Reconsider $r_8 = [(w)_{\mathbf{1}}, (w)_{\mathbf{2}}]$ as an element of the inside of circle(0,0,1). Consider R_3 being an element of $\mathcal{E}^2_{\mathrm{T}}$ such that $R_3 = r_8$ and REAL2-to-BK(r_8) = the direction of $[(R_3)_{\mathbf{1}}, (R_3)_{\mathbf{2}}, 1]$. □

(58) Let us consider points R_1, R_2 of TarskiEuclid2Space. Suppose $\hat{R_1}$, $\hat{R_2} \in$ circle$(0, 0, 1)$ and $R_1 \neq R_2$. Then there exists an element P of BK-model-Plane such that BKtoT2(P) lies between R_1 and R_2. The theorem is a consequence of (47), (57), and (26).

(59) Let us consider non point at ∞ elements P, Q of the projective space

over $\mathcal{E}^3_{\rm T}$. If RP3toT2(P) = RP3toT2(Q), then $P = Q$.

(60) Let us consider non point at ∞ elements R_1, R_2 of the projective space over $\mathcal{E}^3_{\rm T}$. Suppose R_1, $R_2 \in$ the absolute and $R_1 \neq R_2$. Then there exists an element P of BK-model-Plane such that BKtoT2(P) lies between RP3toT2(R_1) and RP3toT2(R_2). The theorem is a consequence of (59) and (58).

(61) Let us consider points P, Q, R of TarskiEuclid2Space. Suppose Q lies between P and R and $\hat{P}$, $\hat{R} \in$ the inside of circle(0,0,1). Then $\hat{Q} \in$ the inside of circle(0,0,1).

Let us consider a non point at ∞ element P of the projective space over $\mathcal{E}^3_{\rm T}$.

(62) If $P \in$ the absolute, then RP3toREAL2$(P) \in$ circle$(0, 0, 1)$.

(63) If $P \in$ the BK-model, then RP3toREAL2$(P) \in$ the inside of circle(0,0,1). The theorem is a consequence of (26).

(64) Let us consider a non point at ∞ point P of the projective space over $\mathcal{E}^3_{\rm T}$, and an element Q of the BK-model. If $P = Q$, then RP3toREAL2(P) = BK-to-REAL2(Q). The theorem is a consequence of (26).

(65) Let us consider non point at ∞ elements P, Q, R_1, R_2 of the projective space over $\mathcal{E}^3_{\rm T}$. Suppose $P \neq Q$ and $P \in$ the BK-model and R_1, $R_2 \in$ the absolute and RP3toT2(Q) lies between RP3toT2(P) and RP3toT2(R_1) and RP3toT2(Q) lies between RP3toT2(P) and RP3toT2(R_2). Then $R_1 = R_2$. The theorem is a consequence of (60), (59), (62), (64), (8), and (61).

(66) Let us consider non point at ∞ elements P, Q, P_1, P_2 of the projective space over $\mathcal{E}^3_{\rm T}$. Suppose $P \neq Q$ and P, $Q \in$ the BK-model and P_1, $P_2 \in$ the absolute and $P_1 \neq P_2$ and P, Q and P_1 are collinear and P, Q and P_2 are collinear. Then

(i) RP3toT2(P) lies between RP3toT2(Q) and RP3toT2(P_1), or

(ii) RP3toT2(P) lies between RP3toT2(Q) and RP3toT2(P_2).

The theorem is a consequence of (55), (53), and (65).

Let us consider elements P, Q of the BK-model. Now we state the propositions:

(67) Suppose $P \neq Q$. Then there exists an element R of the absolute such that for every non point at ∞ elements p, q, r of the projective space over $\mathcal{E}^3_{\rm T}$ such that $p = P$ and $q = Q$ and $r = R$ holds RP3toT2(p) lies between RP3toT2(q) and RP3toT2(r). The theorem is a consequence of (47) and (66).

(68) Suppose $P \neq Q$. Then there exists an element R of the absolute such that for every non point at ∞ elements p, q, r of the projective space over

$\mathcal{E}^3_{\mathrm{T}}$ such that $p = P$ and $q = Q$ and $r = R$ holds RP3toT2(q) lies between RP3toT2(p) and RP3toT2(r). The theorem is a consequence of (67).

(69) The direction of $[1, 0, 1]$ is an element of the absolute.

(70) Let us consider points a, b of BK-model-Plane. Then $\overline{aa} \cong \overline{bb}$. The theorem is a consequence of (69).

(71) Every element of the BK-model is a non point at ∞ element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. The theorem is a consequence of (47).

(72) Every element of the absolute is a non point at ∞ element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. The theorem is a consequence of (47).

(73) Let us consider an element P of the BK-model, and a non point at ∞ element P' of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. If $P = P'$, then RP3toREAL2(P') = BK-to-REAL2(P). The theorem is a consequence of (26).

(74) Let us consider points a, q, b, c of BK-model-Plane. Then there exists a point x of BK-model-Plane such that

(i) a lies between q and x, and

(ii) $\overline{ax} \cong \overline{bc}$.

The theorem is a consequence of (71), (68), (72), (12), (70), (48), and (73).

(75) Let us consider points P, Q of BK-model-Plane.
If BKtoT2(P) = BKtoT2(Q), then $P = Q$.

(76) Let us consider real numbers a, b, r, and elements P, Q, R of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P, $R \in$ the inside of circle(a,b,r). Then $\mathcal{L}(P, R) \subseteq$ the inside of circle(a,b,r).

2. The Axiom of Segment Construction

Now we state the proposition:

(77) BK-model-Plane satisfies the axiom of segment construction.

3. The Axiom of Betweenness Identity

Now we state the proposition:

(78) BK-model-Plane satisfies the axiom of betweenness identity. The theorem is a consequence of (12) and (75).

4. The Axiom of Pasch

Now we state the proposition:

(79) BK-model-Plane satisfies the axiom of Pasch. The theorem is a consequence of (12), (8), (25), and (10).

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Miscellaneous Graph Preliminaries

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Summary. This article contains many auxiliary theorems which were missing in the Mizar Mathematical Library [2] to the best of the author's knowledge. Most of them regard graph theory as formalized in the `GLIB` series (cf. [8]) and most of them are preliminaries needed in [7] or other forthcoming articles.

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0. Introduction

A generalized approach to graph theory as it was done in [3, 5] in contrast to [9], [4] is rather uncommon. To avoid duplication of the same theorems in different formalization frameworks in the Mizar Mathematical Library [1], a generalized approach to formalization is preferable (cf. [8], [6]). However, due to the sheer amount of "obvious facts" such an approach brings with it, it is only natural some of them not immediately needed slip the initial formalization process. This article aims to fill some of the gaps that emerged. Thereby, in most cases, preliminaries needed in [7] are provided.

Many theorems in this article regard the change of incident edge sets and degrees of a vertex when going from one graph to a related one (e.g. when reversing edge directions or adding an edge).

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1. Preliminaries not Directly Related to Graphs

Let us consider sets X, Y, Z. Now we state the propositions:

(1) If $Z \subseteq X$, then $X \cup Y \setminus Z = X \cup Y$.

(2) $X \cap Z$ misses $Y \setminus Z$.

(3) Let us consider objects x, y. Then $\{x, y\} \setminus \{\text{the element of } \{x, y\}\} = \emptyset$ if and only if $x = y$.

Let us consider objects a, b, x, y. Now we state the propositions:

(4) Suppose $a \neq b$ and $x =$ the element of $\{a, b\}$ and $y =$ the element of $\{a, b\} \setminus \{\text{the element of } \{a, b\}\}$. Then

(i) $x = a$ and $y = b$, or

(ii) $x = b$ and $y = a$.

(5) $\{a, b\} = \{x, y\}$ if and only if $x = a$ and $y = b$ or $x = b$ and $y = a$.

(6) Let us consider a set X, and a non empty set Y. Then $X \subset Y$ if and only if X is a proper subset of Y.

Let X be a non empty set. One can check that id_X is non irreflexive and $X \times X$ is non irreflexive and non asymmetric and there exists a binary relation on X which is non irreflexive and non asymmetric and there exists a binary relation on X which is symmetric, irreflexive, and non total and there exists a binary relation on X which is symmetric, non irreflexive, and non empty.

Let X be a non trivial set. Observe that id_X is non connected and there exists a binary relation on X which is symmetric and non connected and $X \times X$ is non antisymmetric and there exists a binary relation on X which is non antisymmetric.

Now we state the propositions:

(7) Let us consider binary relations R, S, and a set X. Then $(R \cup S)^{\circ}X = R^{\circ}X \cup S^{\circ}X$.

(8) Let us consider binary relations R, S, and a set Y. Then $(R \cup S)^{-1}(Y) = R^{-1}(Y) \cup S^{-1}(Y)$.

(9) Let us consider a binary relation R, and sets X, Y. Then $(Y \upharpoonleft R) \upharpoonright X = (Y \upharpoonleft R) \cap (R \upharpoonright X)$.

(10) Let us consider a symmetric binary relation R, and an object x. Then $R^{\circ}x = \mathrm{Coim}(R, x)$.

(11) Let us consider a set X, and a binary relation R on X. Then R is irreflexive if and only if id_X misses R.

(12) Let us consider objects x, y. Then $(\{\langle x, y \rangle\}$ **qua** binary relation$)^{\smile} = \{\langle y, x \rangle\}$.

(13) Let us consider a set X, objects x, y, and a symmetric binary relation R on X. If $\langle x, y\rangle \in R$, then $\langle y, x\rangle \in R$.

Let a, b be cardinal numbers. Note that $a \cap b$ is cardinal and $a \cup b$ is cardinal.

Let X be a $\subseteq$-linear set. One can check that ${}^{\subseteq}{}_{X}$ is connected and $\langle X, \subseteq\rangle$ is connected.

Now we state the propositions:

(14) Let us consider a non empty set X. Suppose for every set a such that $a \in X$ holds a is a cardinal number. Then there exists a cardinal number A such that

(i) $A \in X$, and

(ii) $A = \bigcap X$.

PROOF: Define $\mathcal{P}$[ordinal number] $\equiv$ $\$_1 \in X$ and $\$_1$ is a cardinal number. There exists an ordinal number A such that $\mathcal{P}[A]$. Consider A being an ordinal number such that $\mathcal{P}[A]$ and for every ordinal number B such that $\mathcal{P}[B]$ holds $A \subseteq B$. $\square$

(15) Let us consider a set X. Suppose for every set a such that $a \in X$ holds a is a cardinal number. Then $\bigcap X$ is a cardinal number. The theorem is a consequence of (14).

Let f be a cardinal yielding function and x be an object. Note that $f(x)$ is cardinal.

Let X be a functional set. Note that $\bigcap X$ is function-like and relation-like.

Now we state the propositions:

(16) Let us consider a set X. Then $4 \subseteq \overline{\overline{X}}$ if and only if there exist objects w, x, y, z such that $w, x, y, z \in X$ and $w \neq x$ and $w \neq y$ and $w \neq z$ and $x \neq y$ and $x \neq z$ and $y \neq z$.

PROOF: If $4 \subseteq \overline{\overline{X}}$, then there exist objects w, x, y, z such that w, x, y, $z \in X$ and $w \neq x$ and $w \neq y$ and $w \neq z$ and $x \neq y$ and $x \neq z$ and $y \neq z$. $\square$

(17) Let us consider a set X. Suppose $4 \subseteq \overline{\overline{X}}$. Let us consider objects w, x, y. Then there exists an object z such that

(i) $z \in X$, and

(ii) $w \neq z$, and

(iii) $x \neq z$, and

(iv) $y \neq z$.

The theorem is a consequence of (16).

(18) Let us consider a set X. Then S_X misses $2\mathrm{Set}\, X$.

(19) Let us consider sets X, Y. Suppose $\overline{\overline{X}} = \overline{\overline{Y}}$. Then $\overline{\overline{2\text{Set}\,X}} = \overline{\overline{2\text{Set}\,Y}}$.
PROOF: Consider g being a function such that g is one-to-one and $\text{dom}\,g = X$ and $\text{rng}\,g = Y$. Define $\mathcal{K}(\text{set}) =$ the element of $\$_1$. Define $\mathcal{L}(\text{set}) =$ the element of $\$_1 \setminus \{\mathcal{K}(\$_1)\}$. Define $\mathcal{F}(\text{object}) = \{g(\mathcal{K}(\$_1(\in 2^X))), g(\mathcal{L}(\$_1(\in 2^X)))\}$. Consider f being a function such that $\text{dom}\,f = 2\text{Set}\,X$ and for every object x such that $x \in 2\text{Set}\,X$ holds $f(x) = \mathcal{F}(x)$. $\square$

(20) Let us consider a finite set X. Then $\overline{\overline{2\text{Set}\,X}} = \binom{\overline{\overline{X}}}{2}$. The theorem is a consequence of (19).

2. Into GLIB_000

Now we state the propositions:

(21) Let us consider a graph G, a vertex v of G, and objects e, w. If v is isolated, then e does not join v and w in G.

(22) Let us consider a graph G, a vertex v of G, and objects e, w. Suppose v is isolated. Then

(i) e does not join v to w in G, and

(ii) e does not join w to v in G.

The theorem is a consequence of (21).

(23) Let us consider a graph G, and a vertex v of G. Then v is isolated if and only if $v \notin \text{rng}(\text{the source of } G) \cup \text{rng}(\text{the target of } G)$. The theorem is a consequence of (22).

(24) Let us consider a graph G, a vertex v of G, and an object e. If v is endvertex, then e does not join v and v in G.

(25) Let us consider a graph G, and a vertex v of G. Then

(i) $v.\text{edgesIn}() = (\text{the target of } G)^{-1}(\{v\})$, and

(ii) $v.\text{edgesOut}() = (\text{the source of } G)^{-1}(\{v\})$.

Let us consider a trivial graph G and a vertex v of G. Now we state the propositions:

(26) (i) $v.\text{edgesIn}() =$ the edges of G, and

(ii) $v.\text{edgesOut}() =$ the edges of G, and

(iii) $v.\text{edgesInOut}() =$ the edges of G.

(27) (i) $v.\text{inDegree}() = G.\text{size}()$, and

(ii) $v.\text{outDegree}() = G.\text{size}()$, and

(iii) v.degree() $= G$.size() $+ G$.size().

The theorem is a consequence of (26).

(28) Let us consider a graph G, and sets X, Y. Then G.edgesBetween$(X, Y) =$ G.edgesDBetween$(X, Y) \cup G$.edgesDBetween(Y, X).

(29) Let us consider a graph G, and a vertex v of G. Then v.edgesInOut() $=$ G.edgesBetween(the vertices of G, $\{v\}$). The theorem is a consequence of (28).

Let us consider a graph G and sets X, Y. Now we state the propositions:

(30) G.edgesDBetween$(X, Y) = G$.edgesOutOf$(X) \cap G$.edgesInto(Y).

(31) G.edgesDBetween$(X, Y) \subseteq G$.edgesBetween(X, Y).

Let us consider a graph G and a vertex v of G. Now we state the propositions:

(32) If for every object e, e does not join v and v in G, then $\overline{\overline{v.\mathrm{edgesInOut}()}} =$ v.degree().

PROOF: v.edgesIn() $\cap$ v.edgesOut() $= \emptyset$. $\square$

(33) v is isolated if and only if v.edgesIn() $= \emptyset$ and v.edgesOut() $= \emptyset$.

(34) v is isolated if and only if v.inDegree() $= 0$ and v.outDegree() $= 0$. The theorem is a consequence of (33).

(35) v is isolated if and only if v.degree() $= 0$. The theorem is a consequence of (34).

Let us consider a graph G and a set X. Now we state the propositions:

(36) G.edgesInto$(X) = \bigcup\{v$.edgesIn(), where v is a vertex of $G : v \in X\}$.

(37) G.edgesOutOf$(X) = \bigcup\{v$.edgesOut(), where v is a vertex of $G : v \in X\}$.

(38) G.edgesInOut$(X) = \bigcup\{v$.edgesInOut(), where v is a vertex of $G : v \in X\}$.

Let us consider a graph G and sets X, Y. Now we state the propositions:

(39) G.edgesDBetween$(X, Y) = \bigcup\{v$.edgesOut()$\cap w$.edgesIn(), where v, w are vertices of $G : v \in X$ and $w \in Y\}$.

(40) G.edgesBetween$(X, Y) \subseteq \bigcup\{v$.edgesInOut() $\cap$ w.edgesInOut(), where v, w are vertices of $G : v \in X$ and $w \in Y\}$.

(41) Suppose X misses Y. Then G.edgesBetween$(X, Y) = \bigcup\{v$.edgesInOut()$\cap$ w.edgesInOut(), where v, w are vertices of $G : v \in X$ and $w \in Y\}$. The theorem is a consequence of (40).

(42) Let us consider a graph G_1, a set E, a subgraph G_2 of G_1 with edges E removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

(i) v_2.edgesIn() $= v_1$.edgesIn() $\setminus E$, and

(ii) v_2.edgesOut() $= v_1$.edgesOut() $\setminus E$, and

(iii) v_2.edgesInOut() $= v_1$.edgesInOut() $\setminus E$.

(43) Let us consider graphs G_1, G_2, and a set V. Then G_2 is a subgraph of G_1 with vertices V removed if and only if G_2 is a subgraph of G_1 with vertices $V \cap$ (the vertices of G_1) removed.

(44) Let us consider a graph G_1, a set V, a subgraph G_2 of G_1 with vertices V removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $V \subset$ the vertices of G_1 and $v_1 = v_2$. Then

(i) v_2.edgesIn() $= v_1$.edgesIn() $\setminus$ (G_1.edgesOutOf(V)), and

(ii) v_2.edgesOut() $= v_1$.edgesOut() $\setminus$ (G_1.edgesInto(V)), and

(iii) v_2.edgesInOut() $= v_1$.edgesInOut() $\setminus$ (G_1.edgesInOut(V)).

PROOF: v_1.edgesOut()$\cap G_1$.edgesOutOf$(V) = \emptyset$. v_1.edgesIn()$\cap G_1$.edgesInto $(V) = \emptyset$. $\square$

(45) Let us consider a non trivial graph G_1, a vertex v of G_1, a subgraph G_2 of G_1 with vertex v removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

(i) v_2.edgesIn() $= v_1$.edgesIn() $\setminus$ (v.edgesOut()), and

(ii) v_2.edgesOut() $= v_1$.edgesOut() $\setminus$ (v.edgesIn()), and

(iii) v_2.edgesInOut() $= v_1$.edgesInOut() $\setminus$ (v.edgesInOut()).

The theorem is a consequence of (44).

3. Into GLIB_002

Now we state the proposition:

(46) Let us consider a graph G, a component C of G, a vertex v_1 of G, and a vertex v_2 of C. Suppose $v_1 = v_2$. Then

(i) v_1.edgesIn() $= v_2$.edgesIn(), and

(ii) v_1.inDegree() $= v_2$.inDegree(), and

(iii) v_1.edgesOut() $= v_2$.edgesOut(), and

(iv) v_1.outDegree() $= v_2$.outDegree(), and

(v) v_1.edgesInOut() $= v_2$.edgesInOut(), and

(vi) v_1.degree() $= v_2$.degree().

4. Into GLIB_006

Now we state the propositions:

(47) Let us consider a graph G_2, a set V, a supergraph G_1 of G_2 extended by the vertices from V, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

(i) v_1.edgesIn() $= v_2$.edgesIn(), and

(ii) v_1.inDegree() $= v_2$.inDegree(), and

(iii) v_1.edgesOut() $= v_2$.edgesOut(), and

(iv) v_1.outDegree() $= v_2$.outDegree(), and

(v) v_1.edgesInOut() $= v_2$.edgesInOut(), and

(vi) v_1.degree() $= v_2$.degree().

(48) Let us consider a graph G_2, objects v, w, e, a supergraph G_1 of G_2 extended by e between vertices v and w, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$ and $v_2 \neq v$ and $v_2 \neq w$. Then

(i) v_1.edgesIn() $= v_2$.edgesIn(), and

(ii) v_1.inDegree() $= v_2$.inDegree(), and

(iii) v_1.edgesOut() $= v_2$.edgesOut(), and

(iv) v_1.outDegree() $= v_2$.outDegree(), and

(v) v_1.edgesInOut() $= v_2$.edgesInOut(), and

(vi) v_1.degree() $= v_2$.degree().

(49) Let us consider a graph G_2, vertices v, w of G_2, an object e, a supergraph G_1 of G_2 extended by e between vertices v and w, and a vertex v_1 of G_1. Suppose $e \notin$ the edges of G_2 and $v_1 = v$ and $v \neq w$. Then

(i) v_1.edgesIn() $= v$.edgesIn(), and

(ii) v_1.inDegree() $= v$.inDegree(), and

(iii) v_1.edgesOut() $= v$.edgesOut() $\cup \{e\}$, and

(iv) v_1.outDegree() $= v$.outDegree() $+ 1$, and

(v) v_1.edgesInOut() $= v$.edgesInOut() $\cup \{e\}$, and

(vi) v_1.degree() $= v$.degree() $+ 1$.

(50) Let us consider a graph G_2, vertices v, w of G_2, an object e, a supergraph G_1 of G_2 extended by e between vertices v and w, and a vertex w_1 of G_1. Suppose $e \notin$ the edges of G_2 and $w_1 = w$ and $v \neq w$. Then

(i) w_1.edgesIn() $= w$.edgesIn() $\cup \{e\}$, and

(ii) w_1.inDegree() $= w$.inDegree() $+ 1$, and

(iii) w_1.edgesOut() $= w$.edgesOut(), and

(iv) w_1.outDegree() $= w$.outDegree(), and

(v) w_1.edgesInOut() $= w$.edgesInOut() $\cup \{e\}$, and

(vi) w_1.degree() $= w$.degree() $+ 1$.

(51) Let us consider a graph G_2, a vertex v of G_2, an object e, a supergraph G_1 of G_2 extended by e between vertices v and v, and a vertex v_1 of G_1. Suppose $e \notin$ the edges of G_2 and $v_1 = v$. Then

(i) v_1.edgesIn() $= v$.edgesIn() $\cup \{e\}$, and

(ii) v_1.inDegree() $= v$.inDegree() $+ 1$, and

(iii) v_1.edgesOut() $= v$.edgesOut() $\cup \{e\}$, and

(iv) v_1.outDegree() $= v$.outDegree() $+ 1$, and

(v) v_1.edgesInOut() $= v$.edgesInOut() $\cup \{e\}$, and

(vi) v_1.degree() $= v$.degree() $+ 2$.

5. Into GLIB_007

Now we state the propositions:

(52) Let us consider a graph G_3, a set E, a graph G_4 given by reversing directions of the edges E of G_3, a supergraph G_1 of G_3, and a graph G_2 given by reversing directions of the edges E of G_1. Suppose $E \subseteq$ the edges of G_3. Then G_2 is a supergraph of G_4.

(53) Let us consider a graph G_2, and an object v. Then every supergraph of G_2 extended by v is a supergraph of G_2 extended by vertex v and edges between v and $\emptyset$ of G_2.

(54) Let us consider a graph G_1, a set E, a graph G_2 given by reversing directions of the edges E of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$ and $E \subseteq$ the edges of G_1. Then

(i) v_2.edgesIn() $= v_1$.edgesIn() $\setminus E \cup v_1$.edgesOut() $\cap E$, and

(ii) v_2.edgesOut() $= v_1$.edgesOut() $\setminus E \cup v_1$.edgesIn() $\cap E$.

(55) Let us consider a graph G_1, a graph G_2 given by reversing directions of the edges of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

(i) v_2.edgesIn() $= v_1$.edgesOut(), and

(ii) v_2.inDegree() $= v_1$.outDegree(), and

(iii) v_2.edgesOut() $= v_1$.edgesIn(), and

(iv) v_2.outDegree() $= v_1$.inDegree().

(56) Let us consider a graph G_1, a set E, a graph G_2 given by reversing directions of the edges E of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

(i) v_2.edgesInOut() $= v_1$.edgesInOut(), and

(ii) v_2.degree() $= v_1$.degree().

The theorem is a consequence of (54) and (2).

(57) Let us consider a graph G_2, an object v, a set V, a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2, and a vertex w of G_1. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2 and $v = w$. Then

(i) w.allNeighbors() $= V$, and

(ii) w.degree() $= \overline{\overline{V}}$.

The theorem is a consequence of (29), (32), and (35).

(58) Let us consider a graph G_2, an object v, a set V, a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$ and $v_2 \notin V$. Then

(i) v_1.edgesIn() $= v_2$.edgesIn(), and

(ii) v_1.inDegree() $= v_2$.inDegree(), and

(iii) v_1.edgesOut() $= v_2$.edgesOut(), and

(iv) v_1.outDegree() $= v_2$.outDegree(), and

(v) v_1.edgesInOut() $= v_2$.edgesInOut(), and

(vi) v_1.degree() $= v_2$.degree().

(59) Let us consider a graph G_2, an object v, a subset V of the vertices of G_2, a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v \notin$ the vertices of G_2 and $v_1 = v_2$ and $v_2 \in V$. Then

(i) v_1.allNeighbors() $= v_2$.allNeighbors() $\cup \{v\}$, and

(ii) v_1.degree() $= v_2$.degree() $+ 1$.

(60) Let us consider a graph G_2, an object v, a set V, a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

(i) v_1.degree() $\subseteq v_2$.degree() $+ 1$, and

(ii) v_1.inDegree() $\subseteq v_2$.inDegree() $+ 1$, and

(iii) v_1.outDegree() $\subseteq v_2$.outDegree() $+ 1$.

The theorem is a consequence of (58).

6. Into GLIB_008

Now we state the propositions:

(61) Let us consider a graph G. Then G is edgeless if and only if for every vertices v, w of G, v and w are not adjacent.

(62) Let us consider a loopless graph G. Then G is edgeless if and only if for every vertices v, w of G such that $v \neq w$ holds v and w are not adjacent. The theorem is a consequence of (61).

7. Into GLIB_009

Now we state the propositions:

(63) Let us consider a graph G. Then G.loops() $=$ dom((the source of G) $\cap$ (the target of G)).

(64) Let us consider graphs G_1, G_2, and a set E. Then G_2 is a graph given by reversing directions of the edges E of G_1 if and only if G_2 is a graph given by reversing directions of the edges $E \setminus (G_1$.loops()) of G_1.

(65) Let us consider a graph G_1, a subgraph G_2 of G_1 with loops removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

(i) v_2.inNeighbors() $= v_1$.inNeighbors() $\setminus \{v_1\}$, and

(ii) v_2.outNeighbors() $= v_1$.outNeighbors() $\setminus \{v_1\}$, and

(iii) v_2.allNeighbors() $= v_1$.allNeighbors() $\setminus \{v_1\}$.

(66) Let us consider a graph G_1, a subgraph G_2 of G_1 with parallel edges removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then v_2.allNeighbors() $= v_1$.allNeighbors().

(67) Let us consider a graph G_1, a subgraph G_2 of G_1 with directed-parallel edges removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

(i) v_2.inNeighbors() $= v_1$.inNeighbors(), and

(ii) v_2.outNeighbors() $= v_1$.outNeighbors(), and

(iii) v_2.allNeighbors() $= v_1$.allNeighbors().

(68) Let us consider a graph G_1, a simple graph G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then v_2.allNeighbors() $=$ v_1.allNeighbors() $\setminus \{v_1\}$. The theorem is a consequence of (65) and (66).

(69) Let us consider a graph G_1, a directed-simple graph G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

(i) v_2.inNeighbors() $=$ v_1.inNeighbors() $\setminus \{v_1\}$, and

(ii) v_2.outNeighbors() $=$ v_1.outNeighbors() $\setminus \{v_1\}$, and

(iii) v_2.allNeighbors() $=$ v_1.allNeighbors() $\setminus \{v_1\}$.

The theorem is a consequence of (65) and (67).

Let G be a non loopless graph. One can verify that every subgraph of G with parallel edges removed is non loopless and every subgraph of G with directed-parallel edges removed is non loopless.

Let G be a non edgeless graph. Note that every subgraph of G with parallel edges removed is non edgeless and every subgraph of G with directed-parallel edges removed is non edgeless.

Now we state the propositions:

(70) Let us consider a graph G, and a representative selection of the parallel edges E of G. Then $\overline{\overline{E}} = \overline{\overline{\text{Classes}\,\text{EdgeParEqRel}(G)}}$.
PROOF: Define $\mathcal{F}(\text{object}) = [\$_1]_{\text{EdgeParEqRel}(G)}$. Consider f being a function such that $\text{dom}\, f = E$ and for every object x such that $x \in E$ holds $f(x) = \mathcal{F}(x)$. $\square$

(71) Let us consider a graph G, and a representative selection of the directed-parallel edges E of G. Then $\overline{\overline{E}} = \overline{\overline{\text{Classes}\,\text{DEdgeParEqRel}(G)}}$.
PROOF: Define $\mathcal{F}(\text{object}) = [\$_1]_{\text{DEdgeParEqRel}(G)}$. Consider f being a function such that $\text{dom}\, f = E$ and for every object x such that $x \in E$ holds $f(x) = \mathcal{F}(x)$. $\square$

(72) Let us consider a graph G, a set X, a subset E of the edges of G, and a graph H given by reversing directions of the edges X of G. Then E is a representative selection of the parallel edges of G if and only if E is a representative selection of the parallel edges of H.

(73) Let us consider a graph G, a non empty subset V of the vertices of G, a subgraph H of G induced by V, and a representative selection of the parallel edges E of G. Then $E \cap G$.edgesBetween(V) is a representative selection of the parallel edges of H.

(74) Let us consider a graph G, a non empty subset V of the vertices of G, a subgraph H of G induced by V, and a representative selection of the directed-parallel edges E of G. Then $E \cap G$.edgesBetween(V) is a representative selection of the directed-parallel edges of H.

Let us consider a graph G, a set V, a supergraph H of G extended by the vertices from V, and a subset E of the edges of G. Now we state the propositions:

(75) E is a representative selection of the parallel edges of G if and only if E is a representative selection of the parallel edges of H.

(76) E is a representative selection of the directed-parallel edges of G if and only if E is a representative selection of the directed-parallel edges of H.

Note that there exists a graph which is non non-multi and non-directed-multi.

Let G_F be a graph-yielding function. We say that G_F is plain if and only if

(Def. 1) for every object x such that $x \in \operatorname{dom} G_F$ there exists a graph G such that $G_F(x) = G$ and G is plain.

Let G be a plain graph. Note that $\langle G \rangle$ is plain and $\mathbb{N} \longmapsto G$ is plain.

Let G_F be a non empty, graph-yielding function. One can check that G_F is plain if and only if the condition (Def. 2) is satisfied.

(Def. 2) for every element x of $\operatorname{dom} G_F$, $G_F(x)$ is plain.

Let G_{Sq} be a graph sequence. Note that G_{Sq} is plain if and only if the condition (Def. 3) is satisfied.

(Def. 3) for every natural number n, $G_{Sq}(n)$ is plain.

Observe that every graph-yielding function which is empty is also plain and there exists a graph sequence which is plain and there exists a graph-yielding finite sequence which is non empty and plain.

Let G_F be a plain, non empty, graph-yielding function and x be an element of $\operatorname{dom} G_F$. Let us observe that $G_F(x)$ is plain. Let G_{Sq} be a plain graph sequence and x be a natural number. Let us observe that $G_{Sq}(x)$ is plain. Let p be a plain, graph-yielding finite sequence and n be a natural number. One can check that $p{\restriction}n$ is plain and $p_{\lfloor n}$ is plain. Let m be a natural number.

Observe that $\operatorname{smid}(p, m, n)$ is plain and $\langle p(m), \ldots, p(n) \rangle$ is plain. Let p, q be plain, graph-yielding finite sequences. One can check that $p \frown q$ is plain and $p \frown\!\frown q$ is plain. Let G_1, G_2 be plain graphs. Let us observe that $\langle G_1, G_2 \rangle$ is plain. Let G_3 be a plain graph. One can verify that $\langle G_1, G_2, G_3 \rangle$ is plain.

8. Into GLIB_010

Let us consider graphs G_1, G_2. Now we state the propositions:

(77) If $G_1 \approx G_2$, then there exists a partial graph mapping F from G_1 to G_2 such that $F = \mathrm{id}_{G_1}$ and F is directed-isomorphism.

(78) If $G_1 \approx G_2$, then G_2 is G_1-directed-isomorphic. The theorem is a consequence of (77).

(79) Let us consider a graph G_1, a set E, and a graph G_2 given by reversing directions of the edges E of G_1. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \mathrm{id}_{G_1}$, and

(ii) F is isomorphism.

(80) Let us consider a graph G_1, and a set E. Then every graph given by reversing directions of the edges E of G_1 is G_1-isomorphic. The theorem is a consequence of (79).

(81) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Suppose F is directed-continuous and isomorphism. Then

(i) G_1 is non-directed-multi iff G_2 is non-directed-multi, and

(ii) G_1 is directed-simple iff G_2 is directed-simple.

Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and a vertex v of G_1. Now we state the propositions:

(82) If $v \in \mathrm{dom}(F_{\mathbb{V}})$, then $(F_{\mathbb{E}})^{\circ}(v.\mathrm{edgesInOut}()) \subseteq (F_{\mathbb{V}})_{/v}.\mathrm{edgesInOut}()$.

(83) Suppose F is directed and $v \in \mathrm{dom}(F_{\mathbb{V}})$. Then

(i) $(F_{\mathbb{E}})^{\circ}(v.\mathrm{edgesIn}()) \subseteq (F_{\mathbb{V}})_{/v}.\mathrm{edgesIn}()$, and

(ii) $(F_{\mathbb{E}})^{\circ}(v.\mathrm{edgesOut}()) \subseteq (F_{\mathbb{V}})_{/v}.\mathrm{edgesOut}()$.

(84) Suppose F is onto and semi-continuous and $v \in \mathrm{dom}(F_{\mathbb{V}})$. Then $(F_{\mathbb{E}})^{\circ}(v.\mathrm{edgesInOut}()) = (F_{\mathbb{V}})_{/v}.\mathrm{edgesInOut}()$. The theorem is a consequence of (82).

(85) Suppose F is onto and semi-directed-continuous and $v \in \mathrm{dom}(F_{\mathbb{V}})$. Then

(i) $(F_{\mathbb{E}})^{\circ}(v.\mathrm{edgesIn}()) = (F_{\mathbb{V}})_{/v}.\mathrm{edgesIn}()$, and

(ii) $(F_{\mathbb{E}})^{\circ}(v.\mathrm{edgesOut}()) = (F_{\mathbb{V}})_{/v}.\mathrm{edgesOut}()$.

The theorem is a consequence of (83).

(86) If F is isomorphism, then $(F_{\mathbb{E}})^{\circ}(v.\mathrm{edgesInOut}()) = (F_{\mathbb{V}})_{/v}.\mathrm{edgesInOut}()$. The theorem is a consequence of (84).

(87) Suppose F is directed-isomorphism. Then

(i) $(F_{\mathbb{E}})^{\circ}(v.\mathrm{edgesIn}()) = (F_{\mathbb{V}})_{/v}.\mathrm{edgesIn}()$, and

(ii) $(F_{\mathbb{E}})^{\circ}(v.\mathrm{edgesOut}()) = (F_{\mathbb{V}})_{/v}.\mathrm{edgesOut}()$.

The theorem is a consequence of (85).

Let G_1 be a graph and G_2 be an edgeless graph. Note that every partial graph mapping from G_1 to G_2 is directed.

Let us consider graphs G_1, G_2 and a partial graph mapping F_0 from G_1 to G_2. Now we state the propositions:

(88) Suppose $F_{0\mathbb{E}}$ is one-to-one. Then there exists a subset E of the edges of G_2 such that for every graph G_3 given by reversing directions of the edges E of G_2.

There exists a partial graph mapping F from G_1 to G_3 such that $F = F_0$ and F is directed and if F_0 is not empty, then F is not empty and if F_0 is total, then F is total and if F_0 is one-to-one, then F is one-to-one and if F_0 is onto, then F is onto and if F_0 is semi-continuous, then F is semi-continuous and if F_0 is continuous, then F is continuous. The theorem is a consequence of (79).

(89) Suppose $F_{0\mathbb{E}}$ is one-to-one. Then there exists a subset E of the edges of G_2 such that for every graph G_3 given by reversing directions of the edges E of G_2.

There exists a partial graph mapping F from G_1 to G_3 such that $F = F_0$ and F is directed and if F_0 is weak subgraph embedding, then F is weak subgraph embedding and if F_0 is strong subgraph embedding, then F is strong subgraph embedding and if F_0 is isomorphism, then F is isomorphism. The theorem is a consequence of (88).

Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and a vertex v of G_1. Now we state the propositions:

(90) Suppose F is directed and weak subgraph embedding. Then

(i) $v.\mathrm{inDegree}() \subseteq (F_{\mathbb{V}})_{/v}.\mathrm{inDegree}()$, and

(ii) $v.\mathrm{outDegree}() \subseteq (F_{\mathbb{V}})_{/v}.\mathrm{outDegree}()$.

The theorem is a consequence of (83).

(91) If F is weak subgraph embedding, then $v.\mathrm{degree}() \subseteq (F_{\mathbb{V}})_{/v}.\mathrm{degree}()$. The theorem is a consequence of (89) and (56).

(92) Suppose F is onto and semi-directed-continuous and $v \in \mathrm{dom}(F_{\mathbb{V}})$. Then

(i) $(F_{\mathbb{V}})_{/v}.\mathrm{inDegree}() \subseteq v.\mathrm{inDegree}()$, and

(ii) $(F_{\mathbb{V}})_{/v}.\mathrm{outDegree}() \subseteq v.\mathrm{outDegree}()$.

The theorem is a consequence of (85).

(93) If F is onto and semi-directed-continuous and $v \in \mathrm{dom}(F_{\mathbb{V}})$, then $(F_{\mathbb{V}})_{/v}.\mathrm{degree}() \subseteq v.\mathrm{degree}()$. The theorem is a consequence of (92).

(94) If F is directed-isomorphism, then $v.\mathrm{inDegree}() = (F_{\mathbb{V}})_{/v}.\mathrm{inDegree}()$ and $v.\mathrm{outDegree}() = (F_{\mathbb{V}})_{/v}.\mathrm{outDegree}()$. The theorem is a consequence of (92) and (90).

(95) If F is isomorphism, then $v.\mathrm{degree}() = (F_{\mathbb{V}})_{/v}.\mathrm{degree}()$. The theorem is a consequence of (89), (94), and (56).

9. Into CHORD

Now we state the proposition:

(96) Let us consider a graph G, and vertices u, v, w of G. Suppose v is endvertex and $u \neq w$. Then

(i) u and v are not adjacent, or

(ii) v and w are not adjacent.

Proof: Consider e being an object such that v.edgesInOut() $= \{e\}$ and e does not join v and v in G. Consider v' being a vertex of G such that e joins v and v' in G. Consider e_8 being an object such that e_8 joins v and u in G. There exists no object e' such that e' joins v and w in G. □

Let us consider a graph G and a vertex v of G. Now we state the propositions:

(97) Suppose $3 \subseteq G$.order() and v is endvertex. Then there exist vertices u, w of G such that

(i) $u \neq v$, and

(ii) $w \neq v$, and

(iii) $u \neq w$, and

(iv) u and v are adjacent, and

(v) v and w are not adjacent.

The theorem is a consequence of (96).

(98) Suppose $4 \subseteq G$.order() and v is endvertex. Then there exist vertices x, y, z of G such that

(i) $v \neq x$, and

(ii) $v \neq y$, and

(iii) $v \neq z$, and

(iv) $x \neq y$, and

(v) $x \neq z$, and

(vi) $y \neq z$, and

(vii) v and x are adjacent, and

(viii) v and y are not adjacent, and

(ix) v and z are not adjacent.

The theorem is a consequence of (97), (17), and (96).

Let G_F be a graph-yielding function. We say that G_F is chordal if and only if

(Def. 4) for every object x such that $x \in \operatorname{dom} G_F$ there exists a graph G such that $G_F(x) = G$ and G is chordal.

Let G be a chordal graph. Let us note that $\langle G \rangle$ is chordal and $\mathbb{N} \longmapsto G$ is chordal.

Let G_F be a non empty, graph-yielding function. Note that G_F is chordal if and only if the condition (Def. 5) is satisfied.

(Def. 5) for every element x of $\operatorname{dom} G_F$, $G_F(x)$ is chordal.

Let G_{Sq} be a graph sequence. Let us note that G_{Sq} is chordal if and only if the condition (Def. 6) is satisfied.

(Def. 6) for every natural number n, $G_{Sq}(n)$ is chordal.

Let us observe that every graph-yielding function which is empty is also chordal and there exists a graph sequence which is chordal and there exists a graph-yielding finite sequence which is non empty and chordal.

Let G_F be a chordal, non empty, graph-yielding function and x be an element of $\operatorname{dom} G_F$. One can verify that $G_F(x)$ is chordal. Let G_{Sq} be a chordal graph sequence and x be a natural number. One can verify that $G_{Sq}(x)$ is chordal.

Let p be a chordal, graph-yielding finite sequence and n be a natural number. Note that $p \restriction n$ is chordal and $p_{\lfloor n}$ is chordal. Let m be a natural number. Let us observe that $\operatorname{smid}(p, m, n)$ is chordal and $\langle p(m), \ldots, p(n) \rangle$ is chordal.

Let p, q be chordal, graph-yielding finite sequences. Note that $p \frown q$ is chordal and $p \frown\!\frown q$ is chordal.

Let G_1, G_2 be chordal graphs. One can verify that $\langle G_1, G_2 \rangle$ is chordal. Let G_3 be a chordal graph. One can check that $\langle G_1, G_2, G_3 \rangle$ is chordal.

10. Into GLIB_011

Now we state the propositions:

(99) Let us consider non-directed-multi graphs G_1, G_2, a directed partial vertex mapping f from G_1 to G_2, and a vertex v of G_1. Suppose f is directed-isomorphism. Then

(i) v.inDegree() $= f_{/v}$.inDegree(), and

(ii) v.outDegree() $= f_{/v}$.outDegree().

The theorem is a consequence of (94).

(100) Let us consider non-multi graphs G_1, G_2, a partial vertex mapping f from G_1 to G_2, and a vertex v of G_1. If f is isomorphism, then v.degree() $=$ $f_{/v}$.degree(). The theorem is a consequence of (95).

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About Graph Complements

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Summary. This article formalizes different variants of the complement graph in the Mizar system [3], based on the formalization of graphs in [6].

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0. Introduction

In the first section of this article, the property of a graph to be reflexive is rigorously introduced. But since the irreflexive attribute was called `loopless` in [6], `loopfull` was chosen this time.

The following section introduces a mode to add loops to a subset of the vertices of a graph. It is shown that for a finite subset this operation can be done by adding a loop at a time (cf. [5]). It is also shown that adding loops can preserve isomorphism between graphs, if the subset of vertices of the second graph the loops are added to is the image under an isomorphism of the subset of vertices of the first graphs the loops are added to.

The next four sections formalize the directed complement with loops, the undirected complement with loops, the directed complement without loops and the undirected complement without loops, respectively. Given a simple undirected graph, its complement is usually defined on the same vertex set; two different vertices being adjacent iff they weren't adjacent in the original graph

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[8], [2], [1]. A similar definition can be given for simple digraphs [1]. The loop variants are introduced on the base of similarity between graphs and relations.

In contrast to the literature the definitions formalized allow to take the complement of any graphs, with parallel edges simply being ignored. So any complement of a graph is also a complement of that graph with its parallel edges removed. Furthermore on a technical note, the vertex sets of the graph and its complement are required to be the same, while the edge sets have to be disjoint. This choice was made to ensure the union of a graph and its complement would be complete and its intersection edgeless. Since the edge set of the complement graph is otherwise unspecified, for each complement type all possible complements of a graph are only isomorphic to each other. Other theorems include:

- Involutiveness of the graph complement: If a graph is of the right type (e.g. simple for undirected complement without loops), then it is the complement of its complement.
- The complement of an edgeless graph is complete.
- The undirected complement without loops of a complete graph is edgeless.
- The complement of an unconnected graph is connected.
- The neighbors of a vertex in a complement without loops of a graph is the complement of the neighbors in the original graph.
- If a graph has order at least 3, no vertex can be an endvertex in both that graph and its directed complement without loops (the directed K_2 is a counterexample for order equal to 2.)
- If a graph has order at least 4, no vertex can be an endvertex in both that graph and its undirected complement without loops (P_3 with its complement $K_2 + K_1$ is a counterexample for order equal to 3.)

The last section briefly introduces the property of a graph to be self-complementary for all four variants, but without going into depth. However, it is shown that these four variants are mutually exclusive, except for K_1 which is self-complementary with respect to the directed or undirected complement, without loops in both cases.

1. Loopfull Graphs

Let G be a graph. We say that G is loopfull if and only if

(Def. 1) for every vertex v of G, there exists an object e such that e joins v and v in G.

Let us consider a graph G. Now we state the propositions:

(1) G is loopfull if and only if for every vertex v of G, there exists an object e such that e joins v to v in G.

(2) G is loopfull if and only if for every vertex v of G, v and v are adjacent.

One can verify that every graph which is loopfull is also non loopless and every graph which is trivial and non loopless is also loopfull and there exists a graph which is loopfull and complete and there exists a graph which is non loopfull.

Now we state the proposition:

(3) Let us consider a graph G_1, a set E, and a graph G_2 given by reversing directions of the edges E of G_1. Then G_1 is loopfull if and only if G_2 is loopfull.

Let G be a loopfull graph and E be a set. One can check that every graph given by reversing directions of the edges E of G is loopfull.

Let G be a non loopfull graph. Let us observe that every graph given by reversing directions of the edges E of G is non loopfull.

Now we state the propositions:

(4) Let us consider graphs G_1, G_2. If $G_1 \approx G_2$, then if G_1 is loopfull, then G_2 is loopfull.

(5) Let us consider a loopfull graph G_2, and a supergraph G_1 of G_2. Suppose the vertices of $G_1 =$ the vertices of G_2. Then G_1 is loopfull.

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(6) Suppose $\operatorname{rng} F_{\mathbb{V}} =$ the vertices of G_2 and $G_1.\mathrm{loops}() \subseteq \mathrm{dom}(F_{\mathbb{E}})$. Then if G_1 is loopfull, then G_2 is loopfull.

(7) If F is total and onto, then if G_1 is loopfull, then G_2 is loopfull. The theorem is a consequence of (6).

(8) Suppose F is semi-continuous and $\mathrm{dom}(F_{\mathbb{V}}) =$ the vertices of G_1 and $G_2.\mathrm{loops}() \subseteq \operatorname{rng} F_{\mathbb{E}}$. Then if G_2 is loopfull, then G_1 is loopfull.

(9) If F is total, onto, and semi-continuous, then if G_2 is loopfull, then G_1 is loopfull. The theorem is a consequence of (8).

(10) If F is isomorphism, then G_1 is loopfull iff G_2 is loopfull.

Let G be a loopfull graph and V be a set. Let us observe that every subgraph of G induced by V is loopfull and every subgraph of G with vertices V removed is loopfull and every subgraph of G with vertex V removed is loopfull.

Let G be a non loopfull graph. Let us observe that every spanning subgraph of G is non loopfull.

Let E be a set. Let us note that every subgraph of G induced by the vertices of G and E is non loopfull and every subgraph of G with edges E removed is non loopfull and every subgraph of G with edge E removed is non loopfull.

Now we state the proposition:

(11) Let us consider a graph G_2, a set V, and a supergraph G_1 of G_2 extended by the vertices from V. Suppose $V \setminus$ (the vertices of G_2) $\neq \emptyset$. Then G_1 is not loopfull.

Let G be a non loopfull graph and V be a set. Observe that every supergraph of G extended by the vertices from V is non loopfull.

Let G be a loopfull graph and v, e, w be objects. One can verify that every supergraph of G extended by e between vertices v and w is loopfull.

Now we state the propositions:

(12) Let us consider a graph G_2, a vertex v of G_2, objects e, w, and a supergraph G_1 of G_2 extended by v, w and e between them. Suppose $e \notin$ the edges of G_2 and $w \notin$ the vertices of G_2. Then G_1 is not loopfull.

(13) Let us consider a graph G_2, objects v, e, a vertex w of G_2, and a supergraph G_1 of G_2 extended by v, w and e between them. Suppose $e \notin$ the edges of G_2 and $v \notin$ the vertices of G_2. Then G_1 is not loopfull.

Let G be a non loopfull graph and v, e, w be objects. Let us observe that every supergraph of G extended by v, w and e between them is non loopfull.

Now we state the proposition:

(14) Let us consider a graph G_2, an object v, a subset V of the vertices of G_2, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Suppose $v \notin$ the vertices of G_2. Then G_1 is not loopfull.

Let G be a non loopfull graph, v be an object, and V be a set. One can check that every supergraph of G extended by vertex v and edges between v and V of G is non loopfull.

Let G be a loopfull graph. Let us note that every subgraph of G with parallel edges removed is loopfull and every subgraph of G with directed-parallel edges removed is loopfull.

Let G be a non loopfull graph. Note that every subgraph of G with parallel edges removed is non loopfull and every subgraph of G with directed-parallel edges removed is non loopfull.

Let G_F be a graph-yielding function. We say that G_F is loopfull if and only if

(Def. 2) for every object x such that $x \in \operatorname{dom} G_F$ there exists a graph G such that $G_F(x) = G$ and G is loopfull.

Let G be a loopfull graph. Let us note that $\langle G \rangle$ is loopfull and $\mathbb{N} \longmapsto G$ is loopfull.

Let G_F be a non empty, graph-yielding function. Note that G_F is loopfull if and only if the condition (Def. 3) is satisfied.

(Def. 3) for every element x of $\operatorname{dom} G_F$, $G_F(x)$ is loopfull.

Let G_{Sq} be a graph sequence. Let us note that G_{Sq} is loopfull if and only if the condition (Def. 4) is satisfied.

(Def. 4) for every natural number n, $G_{Sq}(n)$ is loopfull.

Let us observe that every graph-yielding function which is empty is also loopfull and every graph-yielding function which is non empty and loopfull is also non loopless and there exists a graph sequence which is loopfull and there exists a graph-yielding finite sequence which is non empty and loopfull.

Let G_F be a loopfull, non empty, graph-yielding function and x be an element of $\operatorname{dom} G_F$. Note that $G_F(x)$ is loopfull.

Let G_{Sq} be a loopfull graph sequence and x be a natural number. Note that $G_{Sq}(x)$ is loopfull.

Let p be a loopfull, graph-yielding finite sequence and n be a natural number. Observe that $p{\restriction}n$ is loopfull and $p_{\downharpoonright n}$ is loopfull.

Let m be a natural number. One can check that $\operatorname{smid}(p, m, n)$ is loopfull and $\langle p(m), \ldots, p(n) \rangle$ is loopfull.

Let p, q be loopfull, graph-yielding finite sequences. Observe that $p \mathbin{^\frown} q$ is loopfull and $p \mathbin{^\frown\!\!\frown} q$ is loopfull.

Let G_1, G_2 be loopfull graphs. Note that $\langle G_1, G_2 \rangle$ is loopfull.

Let G_3 be a loopfull graph. Let us note that $\langle G_1, G_2, G_3 \rangle$ is loopfull.

2. Adding Loops to a Graph

Let G be a graph and V be a set.

A graph by adding a loop to each vertex of G in V is a supergraph of G defined by

(Def. 5) (i) the vertices of *it* = the vertices of G and there exists a set E and there exists a one-to-one function f such that E misses the edges of G and the edges of *it* = (the edges of G) $\cup\, E$ and $\operatorname{dom} f = E$ and $\operatorname{rng} f = V$ and the source of *it* = (the source of G)$+\cdot f$ and the target of *it* = (the target of G)$+\cdot f$, **if** $V \subseteq$ the vertices of G,

(ii) *it* $\approx G$, **otherwise**.

A graph by adding a loop to each vertex of G is a graph by adding a loop to each vertex of G in the vertices of G. Now we state the proposition:

(15) Let us consider a graph G_2, a set V, and a graph G_1 by adding a loop to each vertex of G_2 in V. Then the vertices of G_1 = the vertices of G_2.

Let us consider a graph G_2, a set V, a graph G_1 by adding a loop to each vertex of G_2 in V, and objects e, v, w. Now we state the propositions:

(16) If $v \neq w$, then e joins v to w in G_1 iff e joins v to w in G_2.

(17) If $v \neq w$, then e joins v and w in G_1 iff e joins v and w in G_2. The theorem is a consequence of (16).

(18) Let us consider a graph G_2, a subset V of the vertices of G_2, a graph G_1 by adding a loop to each vertex of G_2 in V, and a vertex v of G_1. If $v \in V$, then v and v are adjacent.

(19) Let us consider a graph G_2, a set V, and a graph G_1 by adding a loop to each vertex of G_2 in V. Then G_1.order() = G_2.order().

(20) Let us consider a graph G_2, a subset V of the vertices of G_2, and a graph G_1 by adding a loop to each vertex of G_2 in V. Then G_1.size() = G_2.size()+ $\overline{\overline{V}}$.

(21) Let us consider graphs G_1, G_2. Then G_1 is a graph by adding a loop to each vertex of G_2 in $\emptyset$ if and only if $G_1 \approx G_2$. The theorem is a consequence of (15).

(22) Every graph is a graph by adding a loop to each vertex of G in $\emptyset$.

(23) Let us consider a graph G, subsets V_1, V_2 of the vertices of G, a graph G_1 by adding a loop to each vertex of G in V_1, and a graph G_2 by adding a loop to each vertex of G_1 in V_2. Suppose V_1 misses V_2. Then G_2 is a graph by adding a loop to each vertex of G in $V_1 \cup V_2$. The theorem is a consequence of (15).

(24) Let us consider a graph G_3, subsets V_1, V_2 of the vertices of G_3, and a graph G_1 by adding a loop to each vertex of G_3 in $V_1 \cup V_2$. Suppose V_1 misses V_2. Then there exists a graph G_2 by adding a loop to each vertex of G_3 in V_1 such that G_1 is a graph by adding a loop to each vertex of G_2 in V_2.

(25) Let us consider a loopless graph G_2, a subset V of the vertices of G_2, and a graph G_1 by adding a loop to each vertex of G_2 in V. Then

(i) the edges of G_2 misses G_1.loops(), and

(ii) the edges of G_1 = (the edges of G_2) $\cup$ G_1.loops().

(26) Let us consider a loopless graph G_1, a set V, a graph G_2 by adding a loop to each vertex of G_1 in V, and a subgraph G_3 of G_2 with loops removed. Then $G_1 \approx G_3$. The theorem is a consequence of (25).

(27) Let us consider graphs G_1, G_2, and a vertex v of G_2. Then G_1 is a graph by adding a loop to each vertex of G_2 in $\{v\}$ if and only if there exists an object e such that $e \notin$ the edges of G_2 and G_1 is a supergraph of G_2 extended by e between vertices v and v.

(28) Let us consider a graph G_2, a finite subset V of the vertices of G_2, and a graph G_1 by adding a loop to each vertex of G_2 in V. Then there exists a non empty, graph-yielding finite sequence p such that

(i) $p(1) \approx G_2$, and

(ii) $p(\operatorname{len} p) = G_1$, and

(iii) $\operatorname{len} p = \overline{\overline{V}} + 1$, and

(iv) for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exists a vertex v of G_2 and there exists an object e such that $p(n+1)$ is a supergraph of $p(n)$ extended by e between vertices v and v and $v \in V$ and $e \notin$ the edges of $p(n)$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every graph G_2 for every finite subset V of the vertices of G_2 for every graph G_1 by adding a loop to each vertex of G_2 in V such that $\overline{\overline{V}} = \$_1$ there exists a non empty, graph-yielding finite sequence p such that $p(1) \approx G_2$ and $p(\operatorname{len} p) = G_1$ and $\operatorname{len} p = \overline{\overline{V}} + 1$.

For every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exists a vertex v of G_2 and there exists an object e such that $p(n+1)$ is a supergraph of $p(n)$ extended by e between vertices v and v and $v \in V$ and $e \notin$ the edges of $p(n)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. $\square$

(29) Let us consider graphs G_3, G_4, sets V_1, V_2, a graph G_1 by adding a loop to each vertex of G_3 in V_1, a graph G_2 by adding a loop to each vertex of G_4 in V_2, and a partial graph mapping F_0 from G_3 to G_4. Suppose $V_1 \subseteq$ the vertices of G_3 and $V_2 \subseteq$ the vertices of G_4 and $F_{0\mathbb{V}} \restriction V_1$ is one-to-one and $\operatorname{dom}(F_{0\mathbb{V}} \restriction V_1) = V_1$ and $\operatorname{rng}(F_{0\mathbb{V}} \restriction V_1) = V_2$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F_{\mathbb{V}} = F_{0\mathbb{V}}$, and

(ii) $F_{\mathbb{E}} \restriction \operatorname{dom}(F_{0\mathbb{E}}) = F_{0\mathbb{E}}$, and

(iii) if F_0 is not empty, then F is not empty, and

(iv) if F_0 is total, then F is total, and

(v) if F_0 is onto, then F is onto, and

(vi) if F_0 is one-to-one, then F is one-to-one, and

(vii) if F_0 is directed, then F is directed, and

(viii) if F_0 is weak subgraph embedding, then F is weak subgraph embedding, and

(ix) if F_0 is isomorphism, then F is isomorphism, and

(x) if F_0 is directed-isomorphism, then F is directed-isomorphism.

PROOF: Reconsider $f = F_{0\mathbb{V}}$ as a partial function from the vertices of G_1 to the vertices of G_2. Consider E_1 being a set, f_1 being a one-to-one function such that E_1 misses the edges of G_3 and the edges of $G_1 =$ (the edges of G_3) $\cup\, E_1$ and $\operatorname{dom} f_1 = E_1$ and $\operatorname{rng} f_1 = V_1$ and the source of $G_1 =$ (the source of G_3)$+\cdot f_1$ and the target of $G_1 =$ (the target of G_3)$+\cdot f_1$.

Consider E_2 being a set, f_2 being a one-to-one function such that E_2 misses the edges of G_4 and the edges of $G_2 =$ (the edges of G_4) $\cup\, E_2$ and $\operatorname{dom} f_2 = E_2$ and $\operatorname{rng} f_2 = V_2$ and the source of $G_2 =$ (the source of G_4)$+\cdot f_2$ and the target of $G_2 =$ (the target of G_4)$+\cdot f_2$. Set $h = {f_2}^{-1} \cdot (F_{0\mathbb{V}} {\restriction} V_1) \cdot f_1$. Set $g = F_{0\mathbb{E}} {+\cdot} h$. Reconsider $F = \langle f, g\rangle$ as a partial graph mapping from G_1 to G_2. If F_0 is total, then F is total. If F_0 is onto, then F is onto by [7, (6)]. If F_0 is one-to-one, then F is one-to-one. If F_0 is directed, then F is directed by [4, (70), (71)]. □

(30) Let us consider a graph G_3, a G_3-isomorphic graph G_4, and a graph G_1 by adding a loop to each vertex of G_3. Then every graph by adding a loop to each vertex of G_4 is G_1-isomorphic. The theorem is a consequence of (29).

(31) Let us consider a graph G_3, a G_3-directed-isomorphic graph G_4, and a graph G_1 by adding a loop to each vertex of G_3. Then every graph by adding a loop to each vertex of G_4 is G_1-directed-isomorphic. The theorem is a consequence of (29).

(32) Let us consider graphs G_3, G_4, a set V, a graph G_1 by adding a loop to each vertex of G_3 in V, and a graph G_2 by adding a loop to each vertex of G_4 in V. If $G_3 \approx G_4$, then G_2 is G_1-directed-isomorphic. The theorem is a consequence of (29).

(33) Let us consider a graph G_3, sets V, E, a graph G_4 given by reversing directions of the edges E of G_3, and a graph G_1 by adding a loop to each vertex of G_3 in V. Then every graph by adding a loop to each vertex of G_4 in V is G_1-isomorphic. The theorem is a consequence of (29).

(34) Let us consider a graph G_3, sets E, V, a graph G_4 given by reversing directions of the edges E of G_3, a graph G_1 by adding a loop to each vertex

of G_3 in V, and a graph G_2 given by reversing directions of the edges E of G_1. Suppose $E \subseteq$ the edges of G_3. Then G_2 is a graph by adding a loop to each vertex of G_4 in V. The theorem is a consequence of (15).

(35) Let us consider a graph G_3, a subset V_1 of the vertices of G_3, a non empty subset V_2 of the vertices of G_3, a subgraph G_4 of G_3 induced by V_2, and a graph G_1 by adding a loop to each vertex of G_3 in V_1. Then every subgraph of G_1 induced by V_2 is a graph by adding a loop to each vertex of G_4 in $V_1 \cap V_2$.

(36) Let us consider a graph G_2, a set V, a graph G_1 by adding a loop to each vertex of G_2 in V, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 \notin V$ and $v_1 = v_2$. Then

(i) v_1 is isolated iff v_2 is isolated, and

(ii) v_1 is endvertex iff v_2 is endvertex.

The theorem is a consequence of (17).

(37) Let us consider a graph G_2, a set V, a graph G_1 by adding a loop to each vertex of G_2 in V, and a path P of G_1. Then

(i) P is a path of G_2, or

(ii) there exist objects v, e such that e joins v and v in G_1 and $P = G_1.\text{walkOf}(v, e, v)$.

The theorem is a consequence of (15).

(38) Let us consider a graph G_2, a set V, a graph G_1 by adding a loop to each vertex of G_2 in V, and a walk W of G_1. Suppose W.edges() misses (G_1.loops()) $\setminus$ (G_2.loops()). Then W is a walk of G_2. The theorem is a consequence of (15).

Let G be a graph. Observe that every graph by adding a loop to each vertex of G is loopfull.

Let V be a non empty subset of the vertices of G. Observe that every graph by adding a loop to each vertex of G in V is non loopless.

Now we state the proposition:

(39) Let us consider a graph G_2, a set V, and a graph G_1 by adding a loop to each vertex of G_2 in V. Then G_1 is finite if and only if G_2 is finite. The theorem is a consequence of (15).

Let G be a finite graph and V be a set. Observe that every graph by adding a loop to each vertex of G in V is finite.

Let G be a non finite graph. Note that every graph by adding a loop to each vertex of G in V is non finite.

Now we state the proposition:

(40) Let us consider a graph G_2, a set V, and a graph G_1 by adding a loop to each vertex of G_2 in V. Then G_1 is connected if and only if G_2 is connected. The theorem is a consequence of (15) and (37).

Let G be a connected graph and V be a set. Let us observe that every graph by adding a loop to each vertex of G in V is connected.

Let G be a non connected graph. Let us note that every graph by adding a loop to each vertex of G in V is non connected.

Now we state the proposition:

(41) Let us consider a graph G_2, a set V, and a graph G_1 by adding a loop to each vertex of G_2 in V. Then G_1 is chordal if and only if G_2 is chordal. The theorem is a consequence of (17) and (37).

Let G be a chordal graph and V be a set. Let us observe that every graph by adding a loop to each vertex of G in V is chordal.

Let G be a non edgeless graph. Let us note that every graph by adding a loop to each vertex of G in V is non edgeless.

Let G be a loopfull graph. Note that every graph by adding a loop to each vertex of G in V is loopfull.

Let G be a simple graph. Let us note that every graph by adding a loop to each vertex of G in V is non-multi.

Let G be a directed-simple graph. Note that every graph by adding a loop to each vertex of G in V is non-directed-multi.

Let us consider a graph G_2, a subset V of the vertices of G_2, a graph G_1 by adding a loop to each vertex of G_2 in V, a vertex v_1 of G_1, and a vertex v_2 of G_2. Now we state the propositions:

(42) Suppose $v_1 = v_2$ and $v_1 \in V$. Then there exists an object e such that

(i) e joins v_1 to v_1 in G_1, and

(ii) $e \notin$ the edges of G_2, and

(iii) v_1.edgesIn() $= v_2$.edgesIn() $\cup \{e\}$, and

(iv) v_1.edgesOut() $= v_2$.edgesOut() $\cup \{e\}$, and

(v) v_1.edgesInOut() $= v_2$.edgesInOut() $\cup \{e\}$.

(43) If $v_1 = v_2$ and $v_1 \in V$, then v_1.inDegree() $= v_2$.inDegree() $+ 1$ and v_1.outDegree() $= v_2$.outDegree() $+ 1$ and v_1.degree() $= v_2$.degree() $+ 2$. The theorem is a consequence of (42).

(44) Suppose $v_1 = v_2$ and $v_1 \notin V$. Then

(i) v_1.edgesIn() $= v_2$.edgesIn(), and

(ii) v_1.inDegree() $= v_2$.inDegree(), and

(iii) v_1.edgesOut() $= v_2$.edgesOut(), and

(iv) v_1.outDegree() $= v_2$.outDegree(), and

(v) v_1.edgesInOut() $= v_2$.edgesInOut(), and

(vi) v_1.degree() $= v_2$.degree().

3. Directed Graph Complement with Loops

Let G be a graph.

A directed graph complement of G with loops is a non-directed-multi graph defined by

(Def. 6) the vertices of *it* $=$ the vertices of G and the edges of *it* misses the edges of G and for every vertices v, w of G, there exists an object e_1 such that e_1 joins v to w in G iff there exists no object e_2 such that e_2 joins v to w in *it*.

Now we state the proposition:

(45) Let us consider graphs G_1, G_2, G_3, and a directed graph complement G_4 of G_1 with loops. Suppose $G_1 \approx G_2$ and $G_3 \approx G_4$. Then G_3 is a directed graph complement of G_2 with loops.

Let G be a graph. Observe that there exists a directed graph complement of G with loops which is plain.

Now we state the propositions:

(46) Let us consider a graph G_1, a directed graph complement G_2 of G_1 with loops, and objects e_1, e_2, v, w. If e_1 joins v to w in G_1, then e_2 does not join v to w in G_2.

(47) Let us consider a graph G_1, and a subgraph G_2 of G_1 with directed-parallel edges removed. Then every directed graph complement of G_1 with loops is a directed graph complement of G_2 with loops. The theorem is a consequence of (46).

(48) Let us consider graphs G_1, G_2, a subgraph G_3 of G_1 with directed-parallel edges removed, a subgraph G_4 of G_2 with directed-parallel edges removed, a directed graph complement G_5 of G_1 with loops, and a directed graph complement G_6 of G_2 with loops. Suppose G_4 is G_3-directed-isomorphic. Then G_6 is G_5-directed-isomorphic. The theorem is a consequence of (47).

(49) Let us consider a graph G_1, a G_1-directed-isomorphic graph G_2, and a directed graph complement G_3 of G_1 with loops. Then every directed graph complement of G_2 with loops is G_3-directed-isomorphic. The theorem is a consequence of (48).

(50) Let us consider a graph G_1, and directed graph complements G_2, G_3 of G_1 with loops. Then G_3 is G_2-directed-isomorphic. The theorem is a consequence of (49).

(51) Let us consider a graph G_1, a graph G_2 given by reversing directions of the edges of G_1, and a directed graph complement G_3 of G_1 with loops. Then every graph given by reversing directions of the edges of G_3 is a directed graph complement of G_2 with loops. The theorem is a consequence of (46).

(52) Let us consider a graph G_1, a non empty subset V of the vertices of G_1, a subgraph G_2 of G_1 induced by V, and a directed graph complement G_3 of G_1 with loops. Then every subgraph of G_3 induced by V is a directed graph complement of G_2 with loops. The theorem is a consequence of (46).

(53) Let us consider a graph G_1, a proper subset V of the vertices of G_1, a subgraph G_2 of G_1 with vertices V removed, and a directed graph complement G_3 of G_1 with loops. Then every subgraph of G_3 with vertices V removed is a directed graph complement of G_2 with loops. The theorem is a consequence of (52).

(54) Let us consider a non-directed-multi graph G_1, and a directed graph complement G_2 of G_1 with loops. Then G_1 is a directed graph complement of G_2 with loops.

Let us consider a graph G_1 and a directed graph complement G_2 of G_1 with loops. Now we state the propositions:

(55) G_1.order() = G_2.order().

(56) (i) G_1 is trivial iff G_2 is trivial, and

(ii) G_1 is loopfull iff G_2 is loopless, and

(iii) G_1 is loopless iff G_2 is loopfull.

The theorem is a consequence of (55), (1), and (46).

Let G be a trivial graph. One can verify that every directed graph complement of G with loops is trivial. Let G be a non trivial graph. One can check that every directed graph complement of G with loops is non trivial. Let G be a loopfull graph. Note that every directed graph complement of G with loops is loopless.

Let G be a non loopfull graph. Let us note that every directed graph complement of G with loops is non loopless. Let G be a loopless graph. Observe that every directed graph complement of G with loops is loopfull. Let G be a non loopless graph. Let us observe that every directed graph complement of G with loops is non loopfull.

Now we state the proposition:

(57) Let us consider a graph G_1, and a directed graph complement G_2 of G_1 with loops. Suppose the edges of $G_1 = G_1$.loops(). Then G_2 is complete.

Let G be an edgeless graph. One can verify that every directed graph complement of G with loops is complete. Let G be a non connected graph. One can check that every directed graph complement of G with loops is connected.

Now we state the propositions:

(58) Let us consider a graph G_1, a directed graph complement G_2 of G_1 with loops, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

(i) if v_1 is isolated, then v_2 is not isolated, and

(ii) if v_1 is endvertex, then v_2 is not endvertex.

(59) Let us consider a graph G_1, a directed graph complement G_2 of G_1 with loops, and vertices v, w of G_1. Suppose there exists no object e such that e joins v and w in G_1. Then there exists an object e such that e joins v and w in G_2.

Proof: There exists no object e such that e joins v to w in G_1. Consider e being an object such that e joins v to w in G_2. $\square$

Let us consider a graph G_1, a directed graph complement G_2 of G_1 with loops, a vertex v_1 of G_1, and a vertex v_2 of G_2. Now we state the propositions:

(60) Suppose $v_1 = v_2$. Then

(i) v_2.inNeighbors() = (the vertices of G_2) $\setminus$ (v_1.inNeighbors()), and

(ii) v_2.outNeighbors() = (the vertices of G_2) $\setminus$ (v_1.outNeighbors()).

(61) Suppose $v_1 = v_2$ and v_1 is isolated. Then

(i) v_2.inNeighbors() = the vertices of G_2, and

(ii) v_2.outNeighbors() = the vertices of G_2, and

(iii) v_2.allNeighbors() = the vertices of G_2.

The theorem is a consequence of (60).

4. Undirected Graph Complement with Loops

Let G be a graph.

An undirected graph complement of G with loops is a non-multi graph defined by

(Def. 7) the vertices of *it* = the vertices of G and the edges of *it* misses the edges of G and for every vertices v, w of G, there exists an object e_1 such that e_1 joins v and w in G iff there exists no object e_2 such that e_2 joins v and w in *it*.

Now we state the proposition:

(62) Let us consider graphs G_1, G_2, G_3, and an undirected graph complement G_4 of G_1 with loops. Suppose $G_1 \approx G_2$ and $G_3 \approx G_4$. Then G_3 is an undirected graph complement of G_2 with loops.

Let G be a graph. Note that there exists an undirected graph complement of G with loops which is plain.

Now we state the propositions:

(63) Let us consider a graph G_1, and a non-multi graph G_2. Then G_2 is an undirected graph complement of G_1 with loops if and only if the vertices of $G_2 =$ the vertices of G_1 and the edges of G_2 misses the edges of G_1 and for every vertices v_1, w_1 of G_1 and for every vertices v_2, w_2 of G_2 such that $v_1 = v_2$ and $w_1 = w_2$ holds v_1 and w_1 are adjacent iff v_2 and w_2 are not adjacent.

(64) Let us consider a graph G_1, an undirected graph complement G_2 of G_1 with loops, and objects e_1, e_2, v, w. If e_1 joins v and w in G_1, then e_2 does not join v and w in G_2.

(65) Let us consider a graph G_1, and a subgraph G_2 of G_1 with parallel edges removed. Then every undirected graph complement of G_1 with loops is an undirected graph complement of G_2 with loops. The theorem is a consequence of (64).

(66) Let us consider graphs G_1, G_2, a subgraph G_3 of G_1 with parallel edges removed, a subgraph G_4 of G_2 with parallel edges removed, an undirected graph complement G_5 of G_1 with loops, and an undirected graph complement G_6 of G_2 with loops. If G_4 is G_3-isomorphic, then G_6 is G_5-isomorphic. The theorem is a consequence of (65).

(67) Let us consider a graph G_1, a G_1-isomorphic graph G_2, and an undirected graph complement G_3 of G_1 with loops. Then every undirected graph complement of G_2 with loops is G_3-isomorphic. The theorem is a consequence of (66).

(68) Let us consider a graph G_1, and undirected graph complements G_2, G_3 of G_1 with loops. Then G_3 is G_2-isomorphic. The theorem is a consequence of (67).

(69) Let us consider a graph G_1, a non empty subset V of the vertices of G_1, a subgraph G_2 of G_1 induced by V, and an undirected graph complement G_3 of G_1 with loops. Then every subgraph of G_3 induced by V is an undirected graph complement of G_2 with loops. The theorem is a consequence of (64).

(70) Let us consider a graph G_1, a proper subset V of the vertices of G_1, a subgraph G_2 of G_1 with vertices V removed, and an undirected graph

complement G_3 of G_1 with loops. Then every subgraph of G_3 with vertices V removed is an undirected graph complement of G_2 with loops. The theorem is a consequence of (69).

(71) Let us consider a non-multi graph G_1, and an undirected graph complement G_2 of G_1 with loops. Then G_1 is an undirected graph complement of G_2 with loops.

Let us consider a graph G_1 and an undirected graph complement G_2 of G_1 with loops. Now we state the propositions:

(72) G_1.order() = G_2.order().

(73) (i) G_1 is trivial iff G_2 is trivial, and

(ii) G_1 is loopfull iff G_2 is loopless, and

(iii) G_1 is loopless iff G_2 is loopfull.

The theorem is a consequence of (72) and (64).

Let G be a trivial graph. Observe that every undirected graph complement of G with loops is trivial.

Let G be a non trivial graph. Let us observe that every undirected graph complement of G with loops is non trivial.

Let G be a loopfull graph. One can verify that every undirected graph complement of G with loops is loopless.

Let G be a non loopfull graph. One can check that every undirected graph complement of G with loops is non loopless.

Let G be a loopless graph. Note that every undirected graph complement of G with loops is loopfull.

Let G be a non loopless graph. Let us note that every undirected graph complement of G with loops is non loopfull.

Now we state the proposition:

(74) Let us consider a graph G_1, and an undirected graph complement G_2 of G_1 with loops. Suppose the edges of $G_1 = G_1$.loops(). Then G_2 is complete.

Let G be an edgeless graph. Observe that every undirected graph complement of G with loops is complete.

Now we state the proposition:

(75) Let us consider a complete graph G_1, and an undirected graph complement G_2 of G_1 with loops. Then the edges of $G_2 = G_2$.loops(). The theorem is a consequence of (64).

Let G be a complete, loopfull graph. Observe that every undirected graph complement of G with loops is edgeless.

Let G be a non connected graph. Note that every undirected graph complement of G with loops is connected.

Let us consider a graph G_1, an undirected graph complement G_2 of G_1 with loops, a vertex v_1 of G_1, and a vertex v_2 of G_2. Now we state the propositions:

(76) If $v_1 = v_2$, then if v_1 is isolated, then v_2 is not isolated and if v_1 is endvertex, then v_2 is not endvertex.

(77) If $v_1 = v_2$, then v_2.allNeighbors() = (the vertices of G_2)$\setminus$(v_1.allNeighbors ()).

(78) If $v_1 = v_2$ and v_1 is isolated, then v_2.allNeighbors() = the vertices of G_2. The theorem is a consequence of (77).

5. Directed Graph Complement without Loops

Let G be a graph.

A directed graph complement of G is a directed-simple graph defined by

(Def. 8) there exists a directed graph complement G' of G with loops such that *it* is a subgraph of G' with loops removed.

Now we state the proposition:

(79) Let us consider graphs G_1, G_2, G_3, and a directed graph complement G_4 of G_1. Suppose $G_1 \approx G_2$ and $G_3 \approx G_4$. Then G_3 is a directed graph complement of G_2. The theorem is a consequence of (45).

Let G be a graph. One can check that there exists a directed graph complement of G which is plain. Now we state the propositions:

(80) Let us consider a graph G_1, and a directed-simple graph G_2. Then G_2 is a directed graph complement of G_1 if and only if the vertices of G_2 = the vertices of G_1 and the edges of G_2 misses the edges of G_1 and for every vertices v, w of G_1 such that $v \neq w$ holds there exists an object e_1 such that e_1 joins v to w in G_1 iff there exists no object e_2 such that e_2 joins v to w in G_2. The theorem is a consequence of (46), (26), and (1).

(81) Let us consider a graph G_1, a directed graph complement G_2 of G_1, and objects e_1, e_2, v, w. If e_1 joins v to w in G_1, then e_2 does not join v to w in G_2. The theorem is a consequence of (80).

(82) Let us consider a graph G_1, and a directed-simple graph G_2 of G_1. Then every directed graph complement of G_1 is a directed graph complement of G_2. The theorem is a consequence of (80) and (81).

(83) Let us consider graphs G_1, G_2, a directed-simple graph G_3 of G_1, a directed-simple graph G_4 of G_2, a directed graph complement G_5 of G_1,

and a directed graph complement G_6 of G_2. Suppose G_4 is G_3-directed-isomorphic. Then G_6 is G_5-directed-isomorphic. The theorem is a consequence of (82) and (80).

(84) Let us consider a graph G_1, a G_1-directed-isomorphic graph G_2, and a directed graph complement G_3 of G_1. Then every directed graph complement of G_2 is G_3-directed-isomorphic. The theorem is a consequence of (83).

(85) Let us consider a graph G_1, and directed graph complements G_2, G_3 of G_1. Then G_3 is G_2-directed-isomorphic. The theorem is a consequence of (84).

(86) Let us consider a graph G_1, a graph G_2 given by reversing directions of the edges of G_1, and a directed graph complement G_3 of G_1. Then every graph given by reversing directions of the edges of G_3 is a directed graph complement of G_2. The theorem is a consequence of (80) and (81).

(87) Let us consider a graph G_1, a non empty subset V of the vertices of G_1, a subgraph G_2 of G_1 induced by V, and a directed graph complement G_3 of G_1. Then every subgraph of G_3 induced by V is a directed graph complement of G_2. The theorem is a consequence of (80) and (81).

(88) Let us consider a graph G_1, a proper subset V of the vertices of G_1, a subgraph G_2 of G_1 with vertices V removed, and a directed graph complement G_3 of G_1. Then every subgraph of G_3 with vertices V removed is a directed graph complement of G_2. The theorem is a consequence of (80) and (87).

(89) Let us consider a directed-simple graph G_1, and a directed graph complement G_2 of G_1. Then G_1 is a directed graph complement of G_2. The theorem is a consequence of (80).

Let us consider a graph G_1 and a directed graph complement G_2 of G_1. Now we state the propositions:

(90) G_1.order() $= G_2$.order().

(91) G_1 is trivial if and only if G_2 is trivial. The theorem is a consequence of (90).

Let G be a trivial graph. One can verify that every directed graph complement of G is trivial. Let G be a non trivial graph. One can check that every directed graph complement of G is non trivial. Now we state the proposition:

(92) Let us consider a graph G_1, and a directed graph complement G_2 of G_1. Suppose the edges of $G_1 = G_1$.loops(). Then G_2 is complete. The theorem is a consequence of (80).

Let G be an edgeless graph. One can check that every directed graph complement of G is complete. Let G be a trivial, edgeless graph. Let us observe that every directed graph complement of G is edgeless. Let G be a non connected graph. One can check that every directed graph complement of G is connected. Now we state the proposition:

(93) Let us consider a non trivial graph G_1, a directed graph complement G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then if v_1 is isolated, then v_2 is not isolated. The theorem is a consequence of (80).

Let us consider a graph G_1, a directed graph complement G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. Now we state the propositions:

(94) If $v_1 = v_2$ and $3 \subseteq G_1$.order(), then if v_1 is endvertex, then v_2 is not endvertex.

PROOF: Consider u, w being vertices of G_1 such that $u \neq v_1$ and $w \neq v_1$ and $u \neq w$ and u and v_1 are adjacent and v_1 and w are not adjacent. There exists no object e such that e joins v_1 to w in G_1. Consider e_1 being an object such that e_1 joins v_1 to w in G_2. There exists no object e such that e joins w to v_1 in G_1. Consider e_2 being an object such that e_2 joins w to v_1 in G_2. Consider e' being an object such that v_2.edgesInOut() $= \{e'\}$ and e' does not join v_2 and v_2 in G_2. $\square$

(95) Suppose $v_1 = v_2$. Then

(i) v_2.inNeighbors() $=$ (the vertices of G_2) $\setminus$ (v_1.inNeighbors() $\cup \{v_2\}$), and

(ii) v_2.outNeighbors() $=$ (the vertices of G_2) $\setminus$ (v_1.outNeighbors() $\cup \{v_2\}$).

The theorem is a consequence of (60).

(96) Suppose $v_1 = v_2$ and v_1 is isolated. Then

(i) v_2.inNeighbors() $=$ (the vertices of G_2) $\setminus \{v_2\}$, and

(ii) v_2.outNeighbors() $=$ (the vertices of G_2) $\setminus \{v_2\}$, and

(iii) v_2.allNeighbors() $=$ (the vertices of G_2) $\setminus \{v_2\}$.

The theorem is a consequence of (95).

6. Undirected Graph Complement without Loops

Let G be a graph.

A graph complement of G is a simple graph defined by

(Def. 9) there exists an undirected graph complement G' of G with loops such that *it* is a subgraph of G' with loops removed.

Now we state the proposition:

(97) Let us consider graphs G_1, G_2, G_3, and a graph complement G_4 of G_1. Suppose $G_1 \approx G_2$ and $G_3 \approx G_4$. Then G_3 is a graph complement of G_2. The theorem is a consequence of (62).

Let G be a graph. Observe that there exists a graph complement of G which is plain. Let us consider a graph G_1 and a simple graph G_2. Now we state the propositions:

(98) G_2 is a graph complement of G_1 if and only if the vertices of G_2 = the vertices of G_1 and the edges of G_2 misses the edges of G_1 and for every vertices v, w of G_1 such that $v \neq w$ holds there exists an object e_1 such that e_1 joins v and w in G_1 iff there exists no object e_2 such that e_2 joins v and w in G_2. The theorem is a consequence of (64) and (26).

(99) G_2 is a graph complement of G_1 if and only if the vertices of G_2 = the vertices of G_1 and the edges of G_2 misses the edges of G_1 and for every vertices v_1, w_1 of G_1 and for every vertices v_2, w_2 of G_2 such that $v_1 = v_2$ and $w_1 = w_2$ and $v_1 \neq w_1$ holds v_1 and w_1 are adjacent iff v_2 and w_2 are not adjacent. The theorem is a consequence of (98).

(100) Let us consider a graph G_1, a graph complement G_2 of G_1, and objects e_1, e_2, v, w. If e_1 joins v and w in G_1, then e_2 does not join v and w in G_2. The theorem is a consequence of (98).

(101) Let us consider a graph G_1, and a simple graph G_2 of G_1. Then every graph complement of G_1 is a graph complement of G_2. The theorem is a consequence of (98) and (100).

(102) Let us consider graphs G_1, G_2, a simple graph G_3 of G_1, a simple graph G_4 of G_2, a graph complement G_5 of G_1, and a graph complement G_6 of G_2. If G_4 is G_3-isomorphic, then G_6 is G_5-isomorphic. The theorem is a consequence of (101) and (98).

(103) Let us consider a graph G_1, a G_1-isomorphic graph G_2, and a graph complement G_3 of G_1. Then every graph complement of G_2 is G_3-isomorphic. The theorem is a consequence of (102).

(104) Let us consider a graph G_1, and graph complements G_2, G_3 of G_1. Then G_3 is G_2-isomorphic. The theorem is a consequence of (103).

(105) Let us consider a graph G_1, an object v, a subset V of the vertices of G_1, a supergraph G_2 of G_1 extended by vertex v and edges between v and V of G_1, and a graph complement G_3 of G_1. Suppose $v \notin$ the vertices of G_1 and the edges of G_2 misses the edges of G_3. Then there exists a supergraph G_4 of G_3 extended by vertex v and edges between v and (the vertices of G_1) $\setminus V$ of G_3 such that G_4 is a graph complement of G_2. The theorem is a consequence of (98).

(106) Let us consider a graph G_1, an object v, a supergraph G_2 of G_1 extended

by v, and a graph complement G_3 of G_1. Suppose $v \notin$ the vertices of G_1. Then there exists a supergraph G_4 of G_3 extended by vertex v and edges between v and the vertices of G_3 such that G_4 is a graph complement of G_2. The theorem is a consequence of (98) and (105).

(107) Let us consider a graph G_1, an object v, a supergraph G_2 of G_1 extended by vertex v and edges between v and the vertices of G_1, a graph complement G_3 of G_1, and a supergraph G_4 of G_3 extended by v. Suppose $v \notin$ the vertices of G_1 and the edges of G_2 misses the edges of G_3. Then G_4 is a graph complement of G_2. The theorem is a consequence of (105) and (97).

(108) Let us consider a graph G_1, a non empty subset V of the vertices of G_1, a subgraph G_2 of G_1 induced by V, and a graph complement G_3 of G_1. Then every subgraph of G_3 induced by V is a graph complement of G_2. The theorem is a consequence of (98) and (100).

(109) Let us consider a graph G_1, a proper subset V of the vertices of G_1, a subgraph G_2 of G_1 with vertices V removed, and a graph complement G_3 of G_1. Then every subgraph of G_3 with vertices V removed is a graph complement of G_2. The theorem is a consequence of (98) and (108).

(110) Let us consider a simple graph G_1, and a graph complement G_2 of G_1. Then G_1 is a graph complement of G_2. The theorem is a consequence of (98).

Let us consider a graph G_1 and a graph complement G_2 of G_1. Now we state the propositions:

(111) G_1.order() $= G_2$.order().

(112) G_1 is trivial if and only if G_2 is trivial. The theorem is a consequence of (111).

Let G be a trivial graph. Observe that every graph complement of G is trivial. Let G be a non trivial graph. Let us observe that every graph complement of G is non trivial. Now we state the proposition:

(113) Let us consider a graph G_1, and a graph complement G_2 of G_1. Then

  (i) G_1 is complete iff G_2 is edgeless, and

  (ii) the edges of $G_1 = G_1$.loops() iff G_2 is complete.

  The theorem is a consequence of (99) and (98).

Let G be a complete graph. Observe that every graph complement of G is edgeless.

Let G be a non complete graph. Let us observe that every graph complement of G is non edgeless.

Let G be an edgeless graph. One can verify that every graph complement of G is complete.

Let G be a non connected graph. One can check that every graph complement of G is connected.

Now we state the propositions:

(114) Let us consider a non trivial graph G_1, a graph complement G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then if v_1 is isolated, then v_2 is not isolated. The theorem is a consequence of (98).

(115) Let us consider a graph G_1, a graph complement G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$ and G_1.order() $= 2$. Then

(i) if v_1 is endvertex, then v_2 is isolated, and

(ii) if v_1 is isolated, then v_2 is endvertex.

The theorem is a consequence of (111), (98), and (100).

(116) Let us consider a simple graph G_1, a graph complement G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$ and G_1.order() $= 2$. Then

(i) v_1 is endvertex iff v_2 is isolated, and

(ii) v_1 is isolated iff v_2 is endvertex.

The theorem is a consequence of (110), (111), and (115).

Let us consider a graph G_1, a graph complement G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. Now we state the propositions:

(117) If $v_1 = v_2$ and $4 \subseteq G_1$.order(), then if v_1 is endvertex, then v_2 is not endvertex. The theorem is a consequence of (99).

(118) If $v_1 = v_2$, then v_2.allNeighbors() = (the vertices of G_2)$\setminus$(v_1.allNeighbors() $\cup \{v_2\}$). The theorem is a consequence of (77).

(119) If $v_1 = v_2$ and v_1 is isolated, then v_2.allNeighbors() = (the vertices of G_2) $\setminus \{v_2\}$. The theorem is a consequence of (118).

7. Self-complementary Graphs

Let G be a graph. We say that G is self-DLcomplementary if and only if

(Def. 10) every directed graph complement of G with loops is G-directed-isomorphic.

We say that G is self-Lcomplementary if and only if

(Def. 11) every undirected graph complement of G with loops is G-isomorphic.

We say that G is self-Dcomplementary if and only if

(Def. 12) every directed graph complement of G is G-directed-isomorphic.

We say that G is self-complementary if and only if

(Def. 13) every graph complement of G is G-isomorphic.

Let us consider a graph G. Now we state the propositions:

(120) G is self-DLcomplementary if and only if there exists a directed graph complement H of G with loops such that H is G-directed-isomorphic. The theorem is a consequence of (50).

(121) G is self-Lcomplementary if and only if there exists an undirected graph complement H of G with loops such that H is G-isomorphic. The theorem is a consequence of (68).

(122) G is self-Dcomplementary if and only if there exists a directed graph complement H of G such that H is G-directed-isomorphic. The theorem is a consequence of (85).

(123) G is self-complementary if and only if there exists a graph complement H of G such that H is G-isomorphic. The theorem is a consequence of (104).

Let us observe that every graph which is self-DLcomplementary is also non loopless, non loopfull, non-directed-multi, and connected and every graph which is self-Lcomplementary is also non loopless, non loopfull, non-multi, and connected and every graph which is self-Dcomplementary is also directed-simple and connected and every graph which is self-complementary is also simple and connected.

Every graph which is trivial and edgeless is also self-Dcomplementary and self-complementary and every graph which is self-Dcomplementary and self-complementary is also trivial and edgeless and every graph which is self-DLcomplementary is also non trivial, non self-Lcomplementary, non self-Dcomplementary, and non self-complementary and every graph which is self-Lcomplementary is also non trivial, non self-DLcomplementary, non self-Dcomplementary, and non self-complementary and there exists a graph which is self-Dcomplementary and self-complementary.

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Stability of the 7-3 Compressor Circuit for Wallace Tree. Part I[1]

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Summary. To evaluate our formal verification method on a real-size calculation circuit, in this article, we continue to formalize the concept of the 7-3 Compressor (STC) Circuit [6] for Wallace Tree [11], to define the structures of calculation units for a very fast multiplication algorithm for VLSI implementation [10]. We define the circuit structure of the tree constructions of the Generalized Full Adder Circuits (GFAs). We then successfully prove its circuit stability of the calculation outputs after four and six steps. The motivation for this research is to establish a technique based on formalized mathematics and its applications for calculation circuits with high reliability, and to implement the applications of the reliable logic synthesizer and hardware compiler [5].

MSC: 68M07 68W35 68V20

Keywords: arithmetic processor; high order compressor; high-speed multiplier; Wallace tree; logic circuit stability

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0. Introduction

Since calculation models of the arithmetic logic unit based on many sorted algebra have been proposed, we continue to verify the structure and design of these circuits using the Mizar [2], [3], [4] proof checking system. Actually, the stability of circuit primitives is proved based on the definitions and theorems on logic operations, hardware gates, and signal lines [8], [9], [7].

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The various concepts for the Boolean operations, the logic gate elements needed to define the digital circuit, and the connections are defined and have been proved [1]. For logic gate elements that compose a calculation circuit using many Boolean operations, we have prepared a practical collection of logic gates [13]. To construct the adder circuit structure for the RSD numeric representation, we then formalized the definitions and properties of the Generalized Full Adder Circuits (GFAs) to have three inputs and two outputs [14]. Since we have to scale the size of evaluation up to this formal verification method on a real-size calculation circuit, we have already completed formalize the concept of the 4-2 Binary Addition Cell primitives (FTAs) [12] to construct the structures of calculation units for a very fast multiplication algorithm for VLSI implementation [10].

There is the Wallace tree multiplication method [11] as achieved high-speed multiplier, which is connected like the tree using the usual full adder (FA) circuit cell. Since it transforms the Wallace tree multiplication method to improve the high-speed computation and circuit regularity, there is also a refinement multiplication method using the 7-3 Compressor Circuit [6].

We show the component symbol and the block diagram of a 7-3 Compressor Circuit implementation in Figure 1 and Figure 2 using four GFAs. First two GFAs take six of the seven inputs (x1,x2,x3,x5,x6,x7) and generate two sum (A1,A2) and two carry outputs (C1,C2) in Layer-I (after 2-steps). The sum outputs are combined with the seventh input (x4) in another GFA to generate the s0 output of the 7-3 Compressor in Layer-II (after 4-steps). The carry output of this GFA is combined with the carry outputs from the two first level GFAs using fourth GFA to yield s1 and s2, with weights of two and four respectively in Layer-III (after 6-steps).

```
   x1 x2 x3 x4 x5 x6 x7     Inputs  : x1,x2,...,x7 (without pair)
   |  |  |  |  |  |  |
 +-*--*--*--*--*--*--*-+
 |                     |
 |      STC TYPE-0     |
 |                     |
 +------*---*---*------+
        |   |   |
       s2  s1  s0           Outputs : s2,s1,s0 (pair)

Composition : Cascading tree together with four GFA TYPE-0
Function    : [s2:s1:s0] = bit_count_of_<x1,x2,...,x7>
```

Fig.1 7-3 Compressor Circuit (Seven-to-Three Compressor:STC): TYPE-0, Component Symbol.

```
        7-inputs      x1   x2  x3  x4   x5   x6  x7   ..............
                      |    /   /   /    |    /   /
                  +---*---*   /   / +---*---*   /
                  |  GFA  *__/   /  |  GFA  *__/
                  | TYPE0 |     /   | TYPE0 |         LAYER-I
                  *---*---+    /    *---*---+         (2-steps)
                 /    |       /    /   /                      |
          C1 ___/   A1|      /  C2/ A2/     ..................V......
            /    _____|_____/____/   /
           /    /     |    /        /
          /    /  +---*---*        /
         /    /   |  GFA  *_______/                   LAYER-II
        /    /    | TYPE0 |                           (4-steps)
       /    /     *---*---+                                   |
    C1|  C2/   C3/ A3/                ........................V......
  +---*---*     /   /
  |  GFA  *____/   /
  | TYPE0 |   ____/                                   LAYER-III
  *---*---+  /                                        (6-steps)
 /    |     /                                                 |
s2   s1    s0     3-outputs     ..............................V......
```

```
Intermediate Outputs (2-steps):
  C1 := GFA0CarryOutput(x1,x2,x3)
  C2 := GFA0CarryOutput(x5,x6,x7)
  A1 := GFA0AdderOutput(x1,x2,x3)
  A2 := GFA0AdderOutput(x5,x6,x7)
Intermediate Output (4-steps):
  C3 := GFA0CarryOutput(A1,A2,x4)
External Outputs    (4,6-steps):
  s0 := GFA0AdderOutput(A1,A2,x4) (=A3)
  s1 := GFA0AdderOutput(C1,C2,C3)
  s2 := GFA0CarryOutput(C1,C2,C3)

Composite Circuit Structure:
  ( ( BitGFA0Str(x1,x2,x3) ++ BitGFA0Str(x5,x6,x7) )  # STC0IIStr
   ++ BitGFA0Str(A1,A2,x4) )                          # STC0IStr
   ++ BitGFA0Str(C1,C2,C3)                            # STC0Str
   --->
      STC0Str(x1,x2,x3,x4,x5,x6,x7)
```

Fig.2 7-3 Compressor Circuit, Block Diagram and Calculation Stability: Following(s,6) is stable.

1. Properties of 'Intermediate' STC Circuit Structure (LAYER-I)

Let x_1, x_2, x_3, x_4 be non pair objects. Let us note that $\{x_1, x_2, x_3, x_4\}$ has no pairs.

Let x_5 be a non pair object. Observe that $\{x_1, x_2, x_3, x_4, x_5\}$ has no pairs.

Let x_6 be a non pair object. Let us note that $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ has no pairs.

Let x_7 be a non pair object. One can verify that $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ has no pairs.

Let x_1, x_2, x_3, x_5, x_6, x_7 be sets. The functor STC0IIStr$(x_1, x_2, x_3, x_5, x_6, x_7)$ yielding an unsplit, non void, strict, non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by the term

(Def. 1) BitGFA0Str$(x_1, x_2, x_3) +\cdot$ BitGFA0Str(x_5, x_6, x_7).

The functor STC0IICirc$(x_1, x_2, x_3, x_5, x_6, x_7)$ yielding a strict, Boolean circuit of STC0IIStr$(x_1, x_2, x_3, x_5, x_6, x_7)$ with denotation held in gates is defined by the term

(Def. 2) BitGFA0Circ$(x_1, x_2, x_3) +\cdot$ BitGFA0Circ(x_5, x_6, x_7).

Let us consider sets x_1, x_2, x_3, x_5, x_6, x_7. Now we state the propositions:

(1) InnerVertices(STC0IIStr$(x_1, x_2, x_3, x_5, x_6, x_7)$) $= ((\{\langle\langle x_1, x_2\rangle, \mathrm{xor}_2\rangle$, GFA0AdderOutput$(x_1, x_2, x_3)\} \cup \{\langle\langle x_1, x_2\rangle, \mathrm{and}_2\rangle, \langle\langle x_2, x_3\rangle, \mathrm{and}_2\rangle, \langle\langle x_3, x_1\rangle, \mathrm{and}_2\rangle$, GFA0CarryOutput$(x_1, x_2, x_3)\}) \cup \{\langle\langle x_5, x_6\rangle, \mathrm{xor}_2\rangle$, GFA0AdderOutput$(x_5, x_6, x_7)\}) \cup \{\langle\langle x_5, x_6\rangle, \mathrm{and}_2\rangle, \langle\langle x_6, x_7\rangle, \mathrm{and}_2\rangle, \langle\langle x_7, x_5\rangle, \mathrm{and}_2\rangle$, GFA0CarryOutput$(x_5, x_6, x_7)\}$.

(2) InnerVertices(STC0IIStr$(x_1, x_2, x_3, x_5, x_6, x_7)$) is a binary relation.

Let us consider non pair sets x_1, x_2, x_3, x_5, x_6, x_7. Now we state the propositions:

(3) InputVertices(STC0IIStr$(x_1, x_2, x_3, x_5, x_6, x_7)$) $= \{x_1, x_2, x_3, x_5, x_6, x_7\}$.

(4) InputVertices(STC0IIStr$(x_1, x_2, x_3, x_5, x_6, x_7)$) has no pairs.

Let us consider sets x_1, x_2, x_3, x_5, x_6, x_7. Now we state the propositions:

(5) x_1, x_2, x_3, x_5, x_6, x_7, $\langle\langle x_1, x_2\rangle, \mathrm{xor}_2\rangle$, GFA0AdderOutput$(x_1, x_2, x_3)$, $\langle\langle x_1, x_2\rangle, \mathrm{and}_2\rangle$, $\langle\langle x_2, x_3\rangle, \mathrm{and}_2\rangle$, $\langle\langle x_3, x_1\rangle, \mathrm{and}_2\rangle$, GFA0CarryOutput$(x_1, x_2, x_3)$, $\langle\langle x_5, x_6\rangle, \mathrm{xor}_2\rangle$, GFA0AdderOutput$(x_5, x_6, x_7)$, $\langle\langle x_5, x_6\rangle, \mathrm{and}_2\rangle$, $\langle\langle x_6, x_7\rangle, \mathrm{and}_2\rangle$, $\langle\langle x_7, x_5\rangle, \mathrm{and}_2\rangle$, GFA0CarryOutput$(x_5, x_6, x_7) \in$ the carrier of STC0IIStr$(x_1, x_2, x_3, x_5, x_6, x_7)$.

(6) $\langle\langle x_1, x_2\rangle, \mathrm{xor}_2\rangle$, GFA0AdderOutput$(x_1, x_2, x_3)$, $\langle\langle x_1, x_2\rangle, \mathrm{and}_2\rangle$, $\langle\langle x_2, x_3\rangle, \mathrm{and}_2\rangle$, $\langle\langle x_3, x_1\rangle, \mathrm{and}_2\rangle$, GFA0CarryOutput$(x_1, x_2, x_3)$, $\langle\langle x_5, x_6\rangle, \mathrm{xor}_2\rangle$,

GFA0AdderOutput(x_5, x_6, x_7), $\langle\langle x_5, x_6\rangle$, and$_2\,\rangle$, $\langle\langle x_6, x_7\rangle$, and$_2\,\rangle$, $\langle\langle x_7, x_5\rangle$, and$_2\,\rangle$, GFA0CarryOutput$(x_5, x_6, x_7) \in$ InnerVertices(STC0IIStr$(x_1, x_2, x_3, x_5, x_6, x_7)$). The theorem is a consequence of (1).

(7) Let us consider non pair sets x_1, x_2, x_3, x_5, x_6, x_7. Then x_1, x_2, x_3, x_5, x_6, $x_7 \in$ InputVertices(STC0IIStr$(x_1, x_2, x_3, x_5, x_6, x_7)$). The theorem is a consequence of (3).

Let $x_1, x_2, x_3, x_5, x_6, x_7$ be sets. The functors: STC0IICarryOutC1$(x_1, x_2, x_3, x_5, x_6, x_7)$, STC0IIAdderOutA1$(x_1, x_2, x_3, x_5, x_6, x_7)$, STC0IICarryOutC2$(x_1, x_2, x_3, x_5, x_6, x_7)$, and STC0IIAdderOutA2$(x_1, x_2, x_3, x_5, x_6, x_7)$ yielding elements of InnerVertices(STC0IIStr$(x_1, x_2, x_3, x_5, x_6, x_7)$) are defined by terms

(Def. 3) GFA0CarryOutput(x_1, x_2, x_3),

(Def. 4) GFA0AdderOutput(x_1, x_2, x_3),

(Def. 5) GFA0CarryOutput(x_5, x_6, x_7),

(Def. 6) GFA0AdderOutput(x_5, x_6, x_7),

respectively. Now we state the propositions:

(8) Let us consider non pair sets x_1, x_2, x_3, x_5, x_6, x_7, a state s of STC0IICirc $(x_1, x_2, x_3, x_5, x_6, x_7)$, and elements a_1, a_2, a_3, a_5, a_6, a_7 of *Boolean*. Suppose $a_1 = s(x_1)$ and $a_2 = s(x_2)$ and $a_3 = s(x_3)$ and $a_5 = s(x_5)$ and $a_6 = s(x_6)$ and $a_7 = s(x_7)$. Then

(i) (Following$(s, 2)$)(STC0IICarryOutC1$(x_1, x_2, x_3, x_5, x_6, x_7)$) $= (a_1 \wedge a_2 \vee a_2 \wedge a_3) \vee a_3 \wedge a_1$, and

(ii) (Following$(s, 2)$)(STC0IIAdderOutA1$(x_1, x_2, x_3, x_5, x_6, x_7)$) $= (a_1 \oplus a_2) \oplus a_3$, and

(iii) (Following$(s, 2)$)(STC0IICarryOutC2$(x_1, x_2, x_3, x_5, x_6, x_7)$) $= (a_5 \wedge a_6 \vee a_6 \wedge a_7) \vee a_7 \wedge a_5$, and

(iv) (Following$(s, 2)$)(STC0IIAdderOutA2$(x_1, x_2, x_3, x_5, x_6, x_7)$) $= (a_5 \oplus a_6) \oplus a_7$, and

(v) (Following$(s, 2)$)$(x_1) = a_1$, and

(vi) (Following$(s, 2)$)$(x_2) = a_2$, and

(vii) (Following$(s, 2)$)$(x_3) = a_3$, and

(viii) (Following$(s, 2)$)$(x_5) = a_5$, and

(ix) (Following$(s, 2)$)$(x_6) = a_6$, and

(x) (Following$(s, 2)$)$(x_7) = a_7$.

The theorem is a consequence of (7).

(9) Let us consider non pair sets x_1, x_2, x_3, x_5, x_6, x_7, and a state s of STC0IICirc$(x_1, x_2, x_3, x_5, x_6, x_7)$. Then Following$(s, 2)$ is stable.

2. Properties of 'Intermediate' STC Circuit Structure (LAYER-II)

Let x_1, x_2, x_3, x_4, x_5, x_6, x_7 be sets. The functor STC0IStr$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ yielding an unsplit, non void, strict, non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by the term

(Def. 7) STC0IIStr$(x_1, x_2, x_3, x_5, x_6, x_7)$+· BitGFA0Str(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput$(x_5, x_6, x_7), x_4)$.

The functor STC0ICirc$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ yielding a strict, Boolean circuit of STC0IStr$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ with denotation held in gates is defined by the term

(Def. 8) STC0IICirc$(x_1, x_2, x_3, x_5, x_6, x_7)$+· BitGFA0Circ(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput$(x_5, x_6, x_7), x_4)$.

Let us consider sets x_1, x_2, x_3, x_4, x_5, x_6, x_7.

Now we state the propositions:

(10) InnerVertices(STC0IStr$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$) = $\{\langle\langle x_1, x_2\rangle, \mathrm{xor}_2\rangle$, GFA0AdderOutput$(x_1, x_2, x_3)\}\cup$ $\{\langle\langle x_1, x_2\rangle, \mathrm{and}_2\rangle, \langle\langle x_2, x_3\rangle, \mathrm{and}_2\rangle, \langle\langle x_3, x_1\rangle, \mathrm{and}_2\rangle$, GFA0CarryOutput$(x_1, x_2, x_3)\}\cup$ $\{\langle\langle x_5, x_6\rangle, \mathrm{xor}_2\rangle$, GFA0AdderOutput$(x_5, x_6, x_7)\}\cup$ $\{\langle\langle x_5, x_6\rangle, \mathrm{and}_2\rangle, \langle\langle x_6, x_7\rangle, \mathrm{and}_2\rangle, \langle\langle x_7, x_5\rangle, \mathrm{and}_2\rangle$, GFA0CarryOutput$(x_5, x_6, x_7)\}\cup$ $\{\langle\langle$GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput$(x_5, x_6, x_7)\rangle, \mathrm{xor}_2\rangle$, GFA0AdderOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput $(x_5, x_6, x_7), x_4)\}\cup$ $\{\langle\langle$GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput$(x_5, x_6, x_7)\rangle, \mathrm{and}_2\rangle$, $\langle\langle$GFA0AdderOutput$(x_5, x_6, x_7), x_4\rangle, \mathrm{and}_2\rangle, \langle\langle x_4$, GFA0AdderOutput $(x_1, x_2, x_3)\rangle, \mathrm{and}_2\rangle$, GFA0CarryOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput$(x_5, x_6, x_7), x_4)\}$.

The theorem is a consequence of (1).

(11) InnerVertices(STC0IStr$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$) is a binary relation.

(12) Let us consider non pair sets x_1, x_2, x_3, x_5, x_6, x_7, and a set x_4. Suppose $x_4 \neq \langle\langle$GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput$(x_5, x_6, x_7)\rangle$, $\mathrm{xor}_2\rangle$ and $x_4 \neq \langle\langle$GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput$(x_5, x_6, x_7)\rangle, \mathrm{and}_2\rangle$ and $x_4 \notin$ InnerVertices(STC0IIStr$(x_1, x_2, x_3, x_5, x_6, x_7)$). Then InputVertices(STC0IStr$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$) $= \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. The theorem is a consequence of (1) and (3).

Let us consider non pair sets x_1, x_2, x_3, x_4, x_5, x_6, x_7.

(13) InputVertices(STC0IStr($x_1, x_2, x_3, x_4, x_5, x_6, x_7$)) $= \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. The theorem is a consequence of (12).

(14) InputVertices(STC0IStr($x_1, x_2, x_3, x_4, x_5, x_6, x_7$)) has no pairs. The theorem is a consequence of (13).

Let us consider sets x_1, x_2, x_3, x_4, x_5, x_6, x_7.

(15) $x_1, x_2, x_3, x_4, x_5, x_6, x_7$, $\langle\langle x_1, x_2\rangle$, xor$_2\rangle$, GFA0AdderOutput($x_1, x_2, x_3$), $\langle\langle x_1, x_2\rangle$, and$_2\rangle$, $\langle\langle x_2, x_3\rangle$, and$_2\rangle$, $\langle\langle x_3, x_1\rangle$, and$_2\rangle$, GFA0CarryOutput($x_1$, x_2, x_3), $\langle\langle$GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput(x_5, x_6, x_7)$\rangle$, xor$_2\rangle$, GFA0AdderOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0Adder Output(x_5, x_6, x_7), x_4), $\langle\langle$GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOut− put(x_5, x_6, x_7)$\rangle$, and$_2\rangle$, $\langle\langle$GFA0AdderOutput(x_5, x_6, x_7), $x_4\rangle$, and$_2\rangle$, $\langle\langle x_4$, GFA0AdderOutput(x_1, x_2, x_3)$\rangle$, and$_2\rangle$ $\in$ the carrier of STC0IStr(x_1, x_2, x_3, x_4, x_5, x_6, x_7).

And also GFA0CarryOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0 AdderOutput(x_5, x_6, x_7), x_4), $\langle\langle x_5, x_6\rangle$, xor$_2\rangle$, GFA0AdderOutput($x_5, x_6$, x_7), $\langle\langle x_5, x_6\rangle$, and$_2\rangle$, $\langle\langle x_6, x_7\rangle$, and$_2\rangle$, $\langle\langle x_7, x_5\rangle$, and$_2\rangle$, GFA0CarryOut− put($x_5, x_6, x_7$) $\in$ the carrier of STC0IStr($x_1, x_2, x_3, x_4, x_5, x_6, x_7$). The theorem is a consequence of (5).

(16) $\langle\langle x_1, x_2\rangle$, xor$_2\rangle$, GFA0AdderOutput($x_1, x_2, x_3$), $\langle\langle x_1, x_2\rangle$, and$_2\rangle$, $\langle\langle x_2$, $x_3\rangle$, and$_2\rangle$, $\langle\langle x_3, x_1\rangle$, and$_2\rangle$, GFA0CarryOutput($x_1, x_2, x_3$), $\langle\langle$GFA0Adder Output(x_1, x_2, x_3), GFA0AdderOutput(x_5, x_6, x_7)$\rangle$, xor$_2\rangle$, GFA0Adder Output(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput(x_5, x_6, x_7), x_4), $\langle\langle$GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput(x_5, x_6, x_7)$\rangle$, and$_2\rangle$, $\langle\langle$GFA0AdderOutput(x_5, x_6, x_7), $x_4\rangle$, and$_2\rangle$, $\langle\langle x_4$, GFA0AdderOutput (x_1, x_2, x_3)$\rangle$, and$_2\rangle$, GFA0CarryOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput(x_5, x_6, x_7), x_4), $\langle\langle x_5, x_6\rangle$, xor$_2\rangle$, GFA0AdderOutput ($x_5, x_6, x_7$), $\langle\langle x_5, x_6\rangle$, and$_2\rangle$, $\langle\langle x_6, x_7\rangle$, and$_2\rangle$, $\langle\langle x_7, x_5\rangle$, and$_2\rangle$, GFA0Carry Output($x_5, x_6, x_7$) $\in$ InnerVertices(STC0IStr($x_1, x_2, x_3, x_4, x_5, x_6, x_7$)). The theorem is a consequence of (10).

(17) Let us consider non pair sets x_1, x_2, x_3, x_5, x_6, x_7, and a set x_4. Suppose $x_4 \neq \langle\langle$GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput(x_5, x_6, x_7)$\rangle$, xor$_2\rangle$ and $x_4 \neq \langle\langle$GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput(x_5, x_6, x_7)$\rangle$, and$_2\rangle$ and $x_4 \notin$ InnerVertices(STC0IIStr($x_1, x_2, x_3, x_5, x_6, x_7$)). Then x_1, x_2, x_3, x_4, x_5, x_6, $x_7 \in$ InputVertices(STC0IStr(x_1, x_2, x_3, x_4, x_5, x_6, x_7)). The theorem is a consequence of (12).

(18) Let us consider non pair sets x_1, x_2, x_3, x_4, x_5, x_6, x_7. Then x_1, x_2, x_3, x_4, x_5, x_6, $x_7 \in$ InputVertices(STC0IStr($x_1, x_2, x_3, x_4, x_5, x_6, x_7$)). The theorem is a consequence of (13).

Let $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ be sets. The functors: STC0ICarryOutC1($x_1, x_2, x_3, x_4, x_5, x_6, x_7$), STC0ICarryOutC2($x_1, x_2, x_3, x_4, x_5, x_6, x_7$), STC0ICarry OutC3($x_1, x_2, x_3, x_4, x_5, x_6, x_7$), and STC0IAdderOutA3($x_1, x_2, x_3, x_4, x_5, x_6, x_7$) yielding elements of InnerVertices(STC0IStr($x_1, x_2, x_3, x_4, x_5, x_6, x_7$)) are defined by terms

(Def. 9) GFA0CarryOutput(x_1, x_2, x_3),

(Def. 10) GFA0CarryOutput(x_5, x_6, x_7),

(Def. 11) GFA0CarryOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput (x_5, x_6, x_7), x_4),

(Def. 12) GFA0AdderOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput (x_5, x_6, x_7), x_4),

respectively.

Now we state the propositions:

(19) Let us consider non pair sets x_1, x_2, x_3, x_4, x_5, x_6, x_7, a state s of STC0ICirc($x_1, x_2, x_3, x_4, x_5, x_6, x_7$), and elements a_1, a_2, a_3, a_4, a_5, a_6, a_7 of *Boolean*. Suppose $a_1 = s(x_1)$ and $a_2 = s(x_2)$ and $a_3 = s(x_3)$ and $a_4 = s(x_4)$ and $a_5 = s(x_5)$ and $a_6 = s(x_6)$ and $a_7 = s(x_7)$. Then

(i) (Following$(s, 2)$)(STC0ICarryOutC1($x_1, x_2, x_3, x_4, x_5, x_6, x_7$)) $= (a_1 \wedge a_2 \vee a_2 \wedge a_3) \vee a_3 \wedge a_1$, and

(ii) (Following$(s, 2)$)(STC0ICarryOutC2($x_1, x_2, x_3, x_4, x_5, x_6, x_7$)) $= (a_5 \wedge a_6 \vee a_6 \wedge a_7) \vee a_7 \wedge a_5$, and

(iii) (Following$(s, 4)$)(STC0ICarryOutC3($x_1, x_2, x_3, x_4, x_5, x_6, x_7$)) $= (((a_1 \oplus a_2) \oplus a_3) \wedge ((a_5 \oplus a_6) \oplus a_7) \vee ((a_5 \oplus a_6) \oplus a_7) \wedge a_4) \vee a_4 \wedge ((a_1 \oplus a_2) \oplus a_3)$, and

(iv) (Following$(s, 4)$)(STC0IAdderOutA3($x_1, x_2, x_3, x_4, x_5, x_6, x_7$)) $= (((((a_1 \oplus a_2) \oplus a_3) \oplus a_4) \oplus a_5) \oplus a_6) \oplus a_7$, and

(v) (Following$(s, 4)$)$(x_1) = a_1$, and

(vi) (Following$(s, 4)$)$(x_2) = a_2$, and

(vii) (Following$(s, 4)$)$(x_3) = a_3$, and

(viii) (Following$(s, 4)$)$(x_4) = a_4$, and

(ix) (Following$(s, 4)$)$(x_5) = a_5$, and

(x) (Following$(s, 4)$)$(x_6) = a_6$, and

(xi) (Following$(s, 4)$)$(x_7) = a_7$.

(20) Let us consider non pair sets x_1, x_2, x_3, x_4, x_5, x_6, x_7, and a state s of STC0ICirc($x_1, x_2, x_3, x_4, x_5, x_6, x_7$). Then Following$(s, 4)$ is stable. The theorem is a consequence of (9).

3. Properties of STC Circuit Structure (LAYER-III)

Let x_1, x_2, x_3, x_4, x_5, x_6, x_7 be sets. The functor STC0Str($x_1, x_2, x_3, x_4, x_5, x_6, x_7$) yielding an unsplit, non void, strict, non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by the term

(Def. 13) STC0IStr($x_1, x_2, x_3, x_4, x_5, x_6, x_7$)$+\cdot$ BitGFA0Str(STC0ICarryOutC1($x_1, x_2, x_3, x_4, x_5, x_6, x_7$), STC0ICarryOutC2($x_1, x_2, x_3, x_4, x_5, x_6, x_7$), STC0ICarryOutC3($x_1, x_2, x_3, x_4, x_5, x_6, x_7$)).

The functor STC0Circ($x_1, x_2, x_3, x_4, x_5, x_6, x_7$) yielding a strict, Boolean circuit of STC0Str($x_1, x_2, x_3, x_4, x_5, x_6, x_7$) with denotation held in gates is defined by the term

(Def. 14) STC0ICirc($x_1, x_2, x_3, x_4, x_5, x_6, x_7$)$+\cdot$ BitGFA0Circ(STC0ICarryOutC1($x_1, x_2, x_3, x_4, x_5, x_6, x_7$), STC0ICarryOutC2($x_1, x_2, x_3, x_4, x_5, x_6, x_7$), STC0ICarryOutC3($x_1, x_2, x_3, x_4, x_5, x_6, x_7$)).

Let us consider sets x_1, x_2, x_3, x_4, x_5, x_6, x_7. Now we state the propositions:

(21) InnerVertices(STC0Str($x_1, x_2, x_3, x_4, x_5, x_6, x_7$)) = InnerVertices(STC0IStr($x_1, x_2, x_3, x_4, x_5, x_6, x_7$))$\cup$ $\{\langle\langle$STC0ICarryOutC1($x_1, x_2, x_3, x_4, x_5, x_6, x_7$), STC0ICarryOutC2($x_1, x_2, x_3, x_4, x_5, x_6, x_7$)$\rangle$, $\mathrm{xor}_2\rangle$, GFA0AdderOutput(STC0ICarryOutC1($x_1, x_2, x_3, x_4, x_5, x_6, x_7$), STC0ICarryOutC2($x_1, x_2, x_3, x_4, x_5, x_6, x_7$), STC0ICarryOutC3($x_1, x_2, x_3, x_4, x_5, x_6, x_7$))$\}\cup$ $\{\langle\langle$STC0ICarryOutC1($x_1, x_2, x_3, x_4, x_5, x_6, x_7$), STC0ICarryOutC2($x_1, x_2, x_3, x_4, x_5, x_6, x_7$)$\rangle$, $\mathrm{and}_2\rangle$, $\langle\langle$STC0ICarryOutC2($x_1, x_2, x_3, x_4, x_5, x_6, x_7$), STC0ICarryOutC3($x_1, x_2, x_3, x_4, x_5, x_6, x_7$)$\rangle$, $\mathrm{and}_2\rangle$, $\langle\langle$STC0ICarryOutC3($x_1, x_2, x_3, x_4, x_5, x_6, x_7$), STC0ICarryOutC1($x_1, x_2, x_3, x_4, x_5, x_6, x_7$)$\rangle$, $\mathrm{and}_2\rangle$, GFA0CarryOutput(STC0ICarryOutC1($x_1, x_2, x_3, x_4, x_5, x_6, x_7$), STC0ICarryOutC2($x_1, x_2, x_3, x_4, x_5, x_6, x_7$), STC0ICarryOutC3($x_1, x_2, x_3, x_4, x_5, x_6, x_7$))$\}$.

(22) InnerVertices(STC0Str($x_1, x_2, x_3, x_4, x_5, x_6, x_7$)) = $\{\langle\langle x_1, x_2\rangle, \mathrm{xor}_2\rangle$, GFA0AdderOutput($x_1, x_2, x_3$)$\}\cup\{\langle\langle x_1, x_2\rangle, \mathrm{and}_2\rangle, \langle\langle x_2, x_3\rangle, \mathrm{and}_2\rangle, \langle\langle x_3, x_1\rangle, \mathrm{and}_2\rangle$, GFA0CarryOutput($x_1, x_2, x_3$)$\}\cup\{\langle\langle x_5, x_6\rangle, \mathrm{xor}_2\rangle$, GFA0AdderOutput($x_5, x_6, x_7$)$\}\cup\{\langle\langle x_5, x_6\rangle, \mathrm{and}_2\rangle, \langle\langle x_6, x_7\rangle, \mathrm{and}_2\rangle, \langle\langle x_7, x_5\rangle, \mathrm{and}_2\rangle$, GFA0CarryOutput($x_5, x_6, x_7$)$\}\cup\{\langle\langle$GFA0AdderOutput($x_1, x_2, x_3$), GFA0AdderOutput($x_5, x_6, x_7$)$\rangle, \mathrm{xor}_2\rangle$, GFA0AdderOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput(x_5, x_6, x_7), x_4)$\}\cup\{\langle\langle$GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput(x_5, x_6, x_7)$\rangle, \mathrm{and}_2\rangle, \langle\langle$GFA0AdderOutput($x_5, x_6, x_7$), $x_4\rangle, \mathrm{and}_2\rangle, \langle\langle x_4$, GFA0AdderOutput($x_1, x_2, x_3$)$\rangle, \mathrm{and}_2\rangle$, GFA0CarryOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput(x_5, x_6, x_7),

$x_4)\} \cup \{\langle\langle$GFA0CarryOutput$(x_1, x_2, x_3)$, GFA0CarryOutput$(x_5, x_6, x_7)\rangle$, $\mathrm{xor}_2\rangle$, GFA0AdderOutput(GFA0CarryOutput(x_1, x_2, x_3), GFA0Carry Output(x_5, x_6, x_7), GFA0CarryOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput$(x_5, x_6, x_7), x_4))\} \cup \{\langle\langle$GFA0CarryOutput$(x_1, x_2, x_3)$, GFA0CarryOutput$(x_5, x_6, x_7)\rangle$, $\mathrm{and}_2\rangle$, $\langle\langle$GFA0CarryOutput(x_5, x_6, x_7), GFA0CarryOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput $(x_5, x_6, x_7), x_4)\rangle$, $\mathrm{and}_2\rangle$, $\langle\langle$GFA0CarryOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput$(x_5, x_6, x_7), x_4)$, GFA0CarryOutput$(x_1, x_2, x_3)\rangle$, $\mathrm{and}_2\rangle$, GFA0CarryOutput(GFA0CarryOutput(x_1, x_2, x_3), GFA0Carry Output(x_5, x_6, x_7), GFA0CarryOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput$(x_5, x_6, x_7), x_4))\}$. The theorem is a consequence of (21) and (10).

(23) InnerVertices(STC0Str$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$) is a binary relation.

Let us consider non pair sets x_1, x_2, x_3, x_4, x_5, x_6, x_7.

(24) InputVertices(STC0Str$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$) $= \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. The theorem is a consequence of (10), (14), and (13).

(25) InputVertices(STC0Str$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$) has no pairs. The theorem is a consequence of (24).

Let us consider sets x_1, x_2, x_3, x_4, x_5, x_6, x_7.

(26) x_1, x_2, x_3, x_4, x_5, x_6, x_7, $\langle\langle x_1, x_2\rangle, \mathrm{xor}_2\rangle$, $\langle\langle x_1, x_2\rangle, \mathrm{and}_2\rangle$, $\langle\langle x_2, x_3\rangle, \mathrm{and}_2\rangle$, $\langle\langle x_3, x_1\rangle, \mathrm{and}_2\rangle$, $\langle\langle x_5, x_6\rangle, \mathrm{xor}_2\rangle$, $\langle\langle x_5, x_6\rangle, \mathrm{and}_2\rangle$, $\langle\langle x_6, x_7\rangle, \mathrm{and}_2\rangle$, $\langle\langle x_7, x_5\rangle, \mathrm{and}_2\rangle$, GFA0AdderOutput$(x_1, x_2, x_3)$, GFA0CarryOutput$(x_1, x_2, x_3)$, GFA0AdderOutput$(x_5, x_6, x_7)$, GFA0CarryOutput$(x_5, x_6, x_7)$, $\langle\langle$GFA0 AdderOutput(x_1, x_2, x_3), GFA0AdderOutput$(x_5, x_6, x_7)\rangle$, $\mathrm{xor}_2\rangle$, $\langle\langle$GFA0Ad derOutput(x_1, x_2, x_3), GFA0AdderOutput$(x_5, x_6, x_7)\rangle$, $\mathrm{and}_2\rangle$, $\langle\langle$GFA0Add erOutput$(x_5, x_6, x_7), x_4\rangle$, $\mathrm{and}_2\rangle$, $\langle\langle x_4$, GFA0AdderOutput$(x_1, x_2, x_3)\rangle$, $\mathrm{and}_2\rangle$ $\in$ the carrier of STC0Str$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$.

And also GFA0AdderOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0Add erOutput$(x_5, x_6, x_7), x_4)$, GFA0CarryOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput$(x_5, x_6, x_7), x_4)$, $\langle\langle$GFA0CarryOutput(x_1, x_2, x_3), GFA0 CarryOutput$(x_5, x_6, x_7)\rangle$, $\mathrm{xor}_2\rangle$, GFA0AdderOutput(GFA0CarryOutput(x_1, x_2, x_3), GFA0CarryOutput(x_5, x_6, x_7), GFA0CarryOutput(GFA0AdderOut− put(x_1, x_2, x_3), GFA0AdderOuput$(x_5, x_6, x_7), x_4))$, $\langle\langle$GFA0CarryOutput(x_1, x_2, x_3), GFA0CarryOutput$(x_5, x_6, x_7)\rangle$, $\mathrm{and}_2\rangle$, $\langle\langle$GFA0CarryOutput(x_5, x_6, x_7), GFA0CarryOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput $(x_5,x_6,x_7), x_4)\rangle$, $\mathrm{and}_2\rangle$, $\langle\langle$GFA0CarryOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput$(x_5, x_6, x_7), x_4)$, GFA0CarryOutput$(x_1, x_2, x_3)\rangle$, $\mathrm{and}_2\rangle$, GFA0CarryOutput(GFA0CarryOutput(x_1, x_2, x_3), GFA0CarryOutput(x_5, x_6, x_7), GFA0CarryOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOut put$(x_5, x_6, x_7), x_4)) \in$ the carrier of STC0Str$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$. The theorem is a consequence of (15).

(27) $\langle\langle x_1, x_2\rangle$, xor$_2\rangle$, $\langle\langle x_1, x_2\rangle$, and$_2\rangle$, $\langle\langle x_2, x_3\rangle$, and$_2\rangle$, $\langle\langle x_3, x_1\rangle$, and$_2\rangle$, $\langle\langle x_5, x_6\rangle$, xor$_2\rangle$, $\langle\langle x_5, x_6\rangle$, and$_2\rangle$, $\langle\langle x_6, x_7\rangle$, and$_2\rangle$, $\langle\langle x_7, x_5\rangle$, and$_2\rangle$, GFA0AdderOutput$(x_1, x_2, x_3)$, GFA0CarryOutput$(x_1, x_2, x_3)$, GFA0AdderOutput (x_5, x_6, x_7), GFA0CarryOutput(x_5, x_6, x_7), $\langle\langle$GFA0AdderOutput(x_1, x_2, x_3),GFA0AdderOutput$(x_5,x_6, x_7)\rangle$,xor$_2\rangle$, $\langle\langle$GFA0AdderOutput(x_1,x_2, x_3), GFA0AdderOutput$(x_5, x_6, x_7)\rangle$, and$_2\rangle$, $\langle\langle$GFA0AdderOutput(x_5, x_6, x_7), $x_4\rangle$, and$_2\rangle$, $\langle\langle x_4$, GFA0AdderOutput$(x_1, x_2, x_3)\rangle$, and$_2\rangle$, GFA0AdderOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput$(x_5, x_6, x_7), x_4)$ GFA0CarryOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput$(x_5, x_6,x_7),x_4)$, $\langle\langle$GFA0CarryOutput(x_1,x_2, x_3), GFA0CarryOutput$(x_5,x_6, x_7)\rangle$, xor$_2\rangle$, GFA0AdderOutput(GFA0CarryOutput(x_1, x_2, x_3), GFA0CarryOutput(x_5, x_6, x_7), GFA0CarryOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0 Adder Output$(x_5, x_6, x_7), x_4)) \in$ InnerVertices(STC0Str$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$).

And also $\langle\langle$GFA0CarryOutput(x_1, x_2, x_3), GFA0CarryOutput $(x_5, x_6, x_7)\rangle$, and$_2\rangle$, $\langle\langle$GFA0CarryOutput(x_5, x_6, x_7), GFA0CarryOutput (GFA0AdderOutput(x_1,x_2, x_3),GFA0AdderOutput$(x_5, x_6, x_7), x_4)\rangle$, and$_2\rangle$, $\langle\langle$GFA0CarryOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput $(x_5, x_6, x_7), x_4)$, GFA0CarryOutput$(x_1, x_2, x_3)\rangle$, and$_2\rangle$, GFA0CarryOutput (GFA0CarryOutput(x_1, x_2, x_3), GFA0CarryOutput(x_5, x_6, x_7), GFA0CarryOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput$(x_5, x_6, x_7), x_4)) \in$ Inner Vertices(STC0Str$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$). The theorem is a consequence of (22).

(28) Let us consider non pair sets x_1, x_2, x_3, x_4, x_5, x_6, x_7. Then x_1, x_2, x_3, x_4, x_5, x_6, $x_7 \in$ InputVertices(STC0Str$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$). The theorem is a consequence of (24).

Let x_1, x_2, x_3, x_4, x_5, x_6, x_7 be sets. The functors: STC0OutS0$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$, STC0OutS1$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$, and STC0OutS2$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ yielding elements of InnerVertices(STC0Str$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$) are defined by terms

(Def. 15) GFA0AdderOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0AdderOutput $(x_5, x_6, x_7), x_4)$,

(Def. 16) GFA0AdderOutput(GFA0CarryOutput(x_1, x_2, x_3), GFA0CarryOutput (x_5, x_6, x_7), GFA0CarryOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0Ad−derOutput$(x_5, x_6, x_7), x_4)$),

(Def. 17) GFA0CarryOutput(GFA0CarryOutput(x_1, x_2, x_3), GFA0CarryOutput (x_5, x_6, x_7), GFA0CarryOutput(GFA0AdderOutput(x_1, x_2, x_3), GFA0Ad−

derOutput$(x_5, x_6, x_7), x_4)$),

respectively. Now we state the propositions:

(29) Let us consider non pair sets x_1, x_2, x_3, x_4, x_5, x_6, x_7, a state s of STC0Circ$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$, and elements a_1, a_2, a_3, a_4, a_5, a_6, a_7 of *Boolean*. Suppose $a_1 = s(x_1)$ and $a_2 = s(x_2)$ and $a_3 = s(x_3)$ and $a_4 = s(x_4)$ and $a_5 = s(x_5)$ and $a_6 = s(x_6)$ and $a_7 = s(x_7)$. Then

(i) (Following$(s, 4)$)(STC0OutS0$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$) = $(((((a_1 \oplus a_2) \oplus a_3) \oplus a_4) \oplus a_5) \oplus a_6) \oplus a_7$, and

(ii) (Following$(s, 6)$)(STC0OutS1$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$) = $(((a_1 \wedge a_2 \vee a_2 \wedge a_3) \vee a_3 \wedge a_1) \oplus ((a_5 \wedge a_6 \vee a_6 \wedge a_7) \vee a_7 \wedge a_5)) \oplus ((((a_1 \oplus a_2) \oplus a_3) \wedge ((a_5 \oplus a_6) \oplus a_7) \vee ((a_5 \oplus a_6) \oplus a_7) \wedge a_4) \vee a_4 \wedge ((a_1 \oplus a_2) \oplus a_3))$, and

(iii) (Following$(s, 6)$)(STC0OutS2$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$) = $(((a_1 \wedge a_2 \vee a_2 \wedge a_3) \vee a_3 \wedge a_1) \wedge ((a_5 \wedge a_6 \vee a_6 \wedge a_7) \vee a_7 \wedge a_5) \vee ((a_5 \wedge a_6 \vee a_6 \wedge a_7) \vee a_7 \wedge a_5) \wedge ((((a_1 \oplus a_2) \oplus a_3) \wedge ((a_5 \oplus a_6) \oplus a_7) \vee ((a_5 \oplus a_6) \oplus a_7) \wedge a_4) \vee a_4 \wedge ((a_1 \oplus a_2) \oplus a_3))) \vee ((((a_1 \oplus a_2) \oplus a_3) \wedge ((a_5 \oplus a_6) \oplus a_7) \vee ((a_5 \oplus a_6) \oplus a_7) \wedge a_4) \vee a_4 \wedge ((a_1 \oplus a_2) \oplus a_3)) \wedge ((a_1 \wedge a_2 \vee a_2 \wedge a_3) \vee a_3 \wedge a_1)$, and

(iv) (Following$(s, 6)$)$(x_1) = a_1$, and

(v) (Following$(s, 6)$)$(x_2) = a_2$, and

(vi) (Following$(s, 6)$)$(x_3) = a_3$, and

(vii) (Following$(s, 6)$)$(x_4) = a_4$, and

(viii) (Following$(s, 6)$)$(x_5) = a_5$, and

(ix) (Following$(s, 6)$)$(x_6) = a_6$, and

(x) (Following$(s, 6)$)$(x_7) = a_7$.

(30) Let us consider non pair sets x_1, x_2, x_3, x_4, x_5, x_6, x_7, and a state s of STC0Circ$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$. Then Following$(s, 6)$ is stable. The theorem is a consequence of (20).

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Rings of Fractions and Localization

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Summary. This article formalized rings of fractions in the Mizar system [3], [4]. A construction of the ring of fractions from an integral domain, namely a quotient field was formalized in [7].

This article generalizes a construction of fractions to a ring which is commutative and has zero divisor by means of a multiplicatively closed set, say S, by known manner. Constructed ring of fraction is denoted by $S\tilde{\ }R$ instead of $S^{-1}R$ appeared in [1], [6]. As an important example we formalize a ring of fractions by a particular multiplicatively closed set, namely $R \setminus \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of R. The resulted local ring is denoted by $R_{\mathfrak{p}}$. In our Mizar article it is coded by $R\tilde{\ }\mathfrak{p}$ as a synonym.

This article contains also the formal proof of a universal property of a ring of fractions, the total-quotient ring, a proof of the equivalence between the total-quotient ring and the quotient field of an integral domain.

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1. Preliminaries: Units, Zero Divisors and Multiplicatively-closed Set

From now on R, R_1 denote commutative rings, A, B denote non degenerated, commutative rings, o, o_1, o_2 denote objects, r, r_1, r_2 denote elements of R, a, a_1, a_2, b, b_1 denote elements of A, f denotes a function from R into R_1, and $\mathfrak{p}$ denotes an element of the spectrum of A.

Let R be a commutative ring and r be an element of R. We say that r is zero-divisible if and only if

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(Def. 1) there exists an element r_1 of R such that $r_1 \neq 0_R$ and $r \cdot r_1 = 0_R$.

Let A be a non degenerated, commutative ring. Let us observe that there exists an element of A which is zero-divisible.

Let us consider A.

A zero-divisor of A is a zero-divisible element of A. Now we state the propositions:

(1) 0_A is a zero-divisor of A.

(2) 1_A is not a zero-divisor of A.

Let us consider A. The functor ZeroDivSet(A) yielding a subset of A is defined by the term

(Def. 2) $\{a$, where a is an element of $A : a$ is a zero-divisor of $A\}$.

The functor NonZeroDivSet(A) yielding a subset of A is defined by the term

(Def. 3) $\Omega_A \setminus (\mathrm{ZeroDivSet}(A))$.

Let us note that ZeroDivSet(A) is non empty and NonZeroDivSet(A) is non empty.

Now we state the propositions:

(3) $0_A \notin \mathrm{NonZeroDivSet}(A)$. The theorem is a consequence of (1).

(4) If A is an integral domain, then $\{0_A\} = \mathrm{ZeroDivSet}(A)$. The theorem is a consequence of (1).

(5) $\{1_R\}$ is multiplicatively closed.

Let us consider R. One can check that there exists a non empty subset of R which is multiplicatively closed.

Let us consider A. Let V be a subset of A. We say that V is without zero if and only if

(Def. 4) $0_A \notin V$.

Let us observe that there exists a non empty, multiplicatively closed subset of A which is without zero.

Now we state the propositions:

(6) $\Omega_A \setminus \mathfrak{p}$ is multiplicatively closed.

(7) Let us consider a proper ideal J of A. Then multClSet(J, a) is multiplicatively closed.

Let us consider A. One can check that NonZeroDivSet(A) is multiplicatively closed.

Let us consider R. The functor UnitSet(R) yielding a subset of R is defined by the term

(Def. 5) $\{a$, where a is an element of $R : a$ is a unit of $R\}$.

Let us observe that UnitSet(R) is non empty.

Now we state the proposition:

(8) If $r_1 \in \mathrm{UnitSet}(R)$, then r_1 is right mult-cancelable.

PROOF: Consider r_2 such that $r_2 \cdot r_1 = 1_R$. For every elements u, v of R such that $u \cdot r_1 = v \cdot r_1$ holds $u = v$. □

Let us consider R. Let r be an element of R. Assume $r \in \mathrm{UnitSet}(R)$. The functor recip($r$) yielding an element of R is defined by

(Def. 6) $it \cdot r = 1_R$.

We introduce the notation r^{-1} as a synonym of recip(r).

Let u, v be elements of R. The functor u/v yielding an element of R is defined by the term

(Def. 7) $u \cdot \mathrm{recip}(u)$.

Let us consider a unit u of R and an element v of R. Now we state the propositions:

(9) If f inherits ring homomorphism, then $f(u)$ is a unit of R_1 and $f(u)^{-1} = f(u^{-1})$.

(10) If f inherits ring homomorphism, then $f(v \cdot (u^{-1})) = f(v) \cdot (f(u)^{-1})$. The theorem is a consequence of (9).

2. Equivalence Relation of Fractions

In the sequel S denotes a non empty, multiplicatively closed subset of R.

Let us consider R and S. The functor Frac(S) yielding a subset of (the carrier of R) × (the carrier of R) is defined by

(Def. 8) for every set x, $x \in it$ iff there exist elements a, b of R such that $x = \langle a, b\rangle$ and $b \in S$.

Now we state the proposition:

(11) $\mathrm{Frac}(S) = \Omega_R \times S$.

Let us consider R and S. Let us observe that Frac(S) is non empty.

The functor frac1(S) yielding a function from R into Frac(S) is defined by

(Def. 9) for every object o such that $o \in$ the carrier of R holds $it(o) = \langle o, 1_R\rangle$.

From now on u, v, w, x, y, z denote elements of Frac(S).

Let us consider R and S. Let u, v be elements of Frac(S). The functor FracAdd(u, v) yielding an element of Frac(S) is defined by the term

(Def. 10) $\langle (u)_{\mathbf{1}} \cdot (v)_{\mathbf{2}} + (v)_{\mathbf{1}} \cdot (u)_{\mathbf{2}}, (u)_{\mathbf{2}} \cdot (v)_{\mathbf{2}}\rangle$.

One can verify that the functor is commutative.

The functor FracMult(u, v) yielding an element of Frac(S) is defined by the term

(Def. 11) $\langle (u)_{\mathbf{1}} \cdot (v)_{\mathbf{1}}, (u)_{\mathbf{2}} \cdot (v)_{\mathbf{2}} \rangle$.

One can check that the functor is commutative.

Let us consider x and y. The functors: $x + y$ and $x \cdot y$ yielding elements of $\mathrm{Frac}(S)$ are defined by terms

(Def. 12) $\mathrm{FracAdd}(x, y)$,

(Def. 13) $\mathrm{FracMult}(x, y)$,

respectively. Now we state the propositions:

(12) $\mathrm{FracAdd}(x, \mathrm{FracAdd}(y, z)) = \mathrm{FracAdd}(\mathrm{FracAdd}(x, y), z)$.

(13) $\mathrm{FracMult}(x, \mathrm{FracMult}(y, z)) = \mathrm{FracMult}(\mathrm{FracMult}(x, y), z)$.

Let us consider R and S. Let x, y be elements of $\mathrm{Frac}(S)$. We say that $x =_{Fr_S} y$ if and only if

(Def. 14) there exists an element s_1 of R such that $s_1 \in S$ and $((x)_{\mathbf{1}} \cdot ((y)_{\mathbf{2}}) - (y)_{\mathbf{1}} \cdot ((x)_{\mathbf{2}})) \cdot s_1 = 0_R$.

Now we state the propositions:

(14) If $0_R \in S$, then $x =_{Fr_S} y$.

(15) $x =_{Fr_S} x$.

(16) If $x =_{Fr_S} y$, then $y =_{Fr_S} x$.

(17) If $x =_{Fr_S} y$ and $y =_{Fr_S} z$, then $x =_{Fr_S} z$.

Let us consider R and S. The functor $\mathrm{EqRel}(S)$ yielding an equivalence relation of $\mathrm{Frac}(S)$ is defined by

(Def. 15) $\langle u, v \rangle \in it$ iff $u =_{Fr_S} v$.

Now we state the propositions:

(18) $x \in [y]_{\mathrm{EqRel}(S)}$ if and only if $x =_{Fr_S} y$.

(19) $[x]_{\mathrm{EqRel}(S)} = [y]_{\mathrm{EqRel}(S)}$ if and only if $x =_{Fr_S} y$.
PROOF: Set $E = \mathrm{EqRel}(S)$. If $[x]_E = [y]_E$, then $x =_{Fr_S} y$. $x \in [y]_E$. $\square$

(20) If $x =_{Fr_S} u$ and $y =_{Fr_S} v$, then $\mathrm{FracMult}(x, y) =_{Fr_S} \mathrm{FracMult}(u, v)$.

(21) If $x =_{Fr_S} u$ and $y =_{Fr_S} v$, then $\mathrm{FracAdd}(x, y) =_{Fr_S} \mathrm{FracAdd}(u, v)$.

(22) $(x + y) \cdot z =_{Fr_S} x \cdot z + y \cdot z$.

Let us consider R and S. The functors: $0_R^{S \times S}$ and $I_R^{S \times S}$ yielding elements of $\mathrm{Frac}(S)$ are defined by terms

(Def. 16) $\langle 0_R, 1_R \rangle$,

(Def. 17) $\langle 1_R, 1_R \rangle$,

respectively. Now we state the proposition:

(23) Let us consider an element s of S. If $x = \langle s, s \rangle$, then $x =_{Fr_S} I_R^{S \times S}$.

3. CONSTRUCTION OF RING OF FRACTIONS

Let us consider R and S. The functor FracRing(S) yielding a strict double loop structure is defined by

(Def. 18) the carrier of it = Classes EqRel(S) and $1_{it} = [I_R^{S \times S}]_{\mathrm{EqRel}(S)}$ and $0_{it} = [0_R^{S \times S}]_{\mathrm{EqRel}(S)}$ and for every elements x, y of it, there exist elements a, b of Frac(S) such that $x = [a]_{\mathrm{EqRel}(S)}$ and $y = [b]_{\mathrm{EqRel}(S)}$ and (the addition of it)$(x, y) = [a + b]_{\mathrm{EqRel}(S)}$ and for every elements x, y of it, there exist elements a, b of Frac(S) such that $x = [a]_{\mathrm{EqRel}(S)}$ and $y = [b]_{\mathrm{EqRel}(S)}$ and (the multiplication of it)$(x, y) = [a \cdot b]_{\mathrm{EqRel}(S)}$.

We introduce the notation $S{\sim}R$ as a synonym of FracRing(S).

One can verify that $S{\sim}R$ is non empty.

Now we state the proposition:

(24) $0_R \in S$ if and only if $S{\sim}R$ is degenerated. The theorem is a consequence of (19).

In the sequel a, b, c denote elements of Frac(S) and x, y, z denote elements of $S{\sim}R$.

Now we state the propositions:

(25) There exists an element a of Frac(S) such that $x = [a]_{\mathrm{EqRel}(S)}$.

(26) If $x = [a]_{\mathrm{EqRel}(S)}$ and $y = [b]_{\mathrm{EqRel}(S)}$, then $x \cdot y = [a \cdot b]_{\mathrm{EqRel}(S)}$. The theorem is a consequence of (19) and (20).

(27) $x \cdot y = y \cdot x$. The theorem is a consequence of (25) and (26).

(28) If $x = [a]_{\mathrm{EqRel}(S)}$ and $y = [b]_{\mathrm{EqRel}(S)}$, then $x + y = [a + b]_{\mathrm{EqRel}(S)}$. The theorem is a consequence of (19) and (21).

(29) $S{\sim}R$ is a ring.

PROOF: $x + y = y + x$. $(x + y) + z = x + (y + z)$. $x + 0_{S \sim R} = x$. x is right complementable. $(x + y) \cdot z = x \cdot z + y \cdot z$. $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. $x \cdot (1_{S \sim R}) = x$ and $1_{S \sim R} \cdot x = x$. □

Let us consider R and S. One can verify that $S{\sim}R$ is commutative, Abelian, add-associative, right zeroed, right complementable, associative, well unital, and distributive.

Now we state the proposition:

(30) There exist elements r_1, r_2 of R such that

(i) $r_2 \in S$, and

(ii) $z = [\langle r_1, r_2 \rangle]_{\mathrm{EqRel}(S)}$.

The theorem is a consequence of (25).

In the sequel S denotes a without zero, non empty, multiplicatively closed subset of A.

Let us consider A and S. The canonical homomorphism of S into quotient field yielding a function from A into $S{\sim}A$ is defined by

(Def. 19) for every object o such that $o \in$ the carrier of A holds $it(o) = [(\mathrm{frac1}(S))(o)]_{\mathrm{EqRel}(S)}$.

Let us observe that the canonical homomorphism of S into quotient field is additive, multiplicative, and unity-preserving.

Now we state the propositions:

(31) Let us consider elements a, b of A. Then (the canonical homomorphism of S into quotient field)$(a - b)$ = (the canonical homomorphism of S into quotient field)(a) − (the canonical homomorphism of S into quotient field)(b).

(32) Suppose $0_A \notin S$. Then ker the canonical homomorphism of S into quotient field $\subseteq$ ZeroDivSet(A).

PROOF: For every o such that $o \in$ ker the canonical homomorphism of S into quotient field holds $o \in$ ZeroDivSet(A). □

(33) Suppose $0_A \notin S$ and A is an integral domain. Then

(i) ker the canonical homomorphism of S into quotient field $= \{0_A\}$, and

(ii) the canonical homomorphism of S into quotient field is one-to-one.

PROOF: ker the canonical homomorphism of S into quotient field $\subseteq$ ZeroDiv Set(A). ZeroDivSet$(A) = \{0_A\}$. For every objects x, y such that x, $y \in$ dom(the canonical homomorphism of S into quotient field) and (the canonical homomorphism of S into quotient field)(x) = (the canonical homomorphism of S into quotient field)(y) holds $x = y$. □

4. Localization in Terms of Prime Ideals

From now on $\mathfrak{p}$ denotes an element of the spectrum of A.

Let us consider A and $\mathfrak{p}$. The functor $\mathrm{Loc}(A, \mathfrak{p})$ yielding a subset of A is defined by the term

(Def. 20) $\Omega_A \setminus \mathfrak{p}$.

One can check that $\mathrm{Loc}(A, \mathfrak{p})$ is non empty and $\mathrm{Loc}(A, \mathfrak{p})$ is multiplicatively closed and $\mathrm{Loc}(A, \mathfrak{p})$ is without zero.

The functor $A{\sim}\mathfrak{p}$ yielding a ring is defined by the term

(Def. 21) $\mathrm{Loc}(A, \mathfrak{p}){\sim}A$.

One can verify that $A{\sim}\mathfrak{p}$ is non degenerated and $A{\sim}\mathfrak{p}$ is commutative.

The functor LocIdeal$(\mathfrak{p})$ yielding a subset of $\Omega_{A\sim\mathfrak{p}}$ is defined by the term

(Def. 22) $\{y$, where y is an element of $A{\sim}\mathfrak{p}$: there exists an element a of $\mathrm{Frac}(\mathrm{Loc}(A,\mathfrak{p}))$ such that $a \in \mathfrak{p} \times \mathrm{Loc}(A,\mathfrak{p})$ and $y = [a]_{\mathrm{EqRel}(\mathrm{Loc}(A,\mathfrak{p}))}\}$.

Observe that LocIdeal($\mathfrak{p}$) is non empty.

In the sequel a, m, n denote elements of $A{\sim}\mathfrak{p}$.

Now we state the propositions:

(34) LocIdeal($\mathfrak{p}$) is a proper ideal of $A{\sim}\mathfrak{p}$.
PROOF: Reconsider $M = \mathrm{LocIdeal}(\mathfrak{p})$ as a subset of $A{\sim}\mathfrak{p}$. For every elements m, n of $A{\sim}\mathfrak{p}$ such that m, $n \in M$ holds $m + n \in M$. For every elements x, m of $A{\sim}\mathfrak{p}$ such that $m \in M$ holds $x \cdot m \in M$. M is proper by [2, (19)], (19). □

(35) Let us consider an object x. Suppose $x \in \Omega_{A\sim\mathfrak{p}} \setminus (\mathrm{LocIdeal}(\mathfrak{p}))$. Then x is a unit of $A{\sim}\mathfrak{p}$. The theorem is a consequence of (25) and (11).

(36) (i) $A{\sim}\mathfrak{p}$ is local, and

(ii) LocIdeal($\mathfrak{p}$) is a maximal ideal of $A{\sim}\mathfrak{p}$.
PROOF: Reconsider $J = \mathrm{LocIdeal}(\mathfrak{p})$ as a proper ideal of $A{\sim}\mathfrak{p}$. $A{\sim}\mathfrak{p}$ is local. J is a maximal ideal of $A{\sim}\mathfrak{p}$ by [8, (8), (11)], (35). □

5. Universal Property of Ring of Fractions

From now on f denotes a function from A into B.

Now we state the proposition:

(37) Let us consider an element s of S. Suppose f inherits ring homomorphism and $f^{\circ}S \subseteq \mathrm{UnitSet}(B)$. Then $f(s)$ is a unit of B.

Let us consider A, B, S, and f. Assume f inherits ring homomorphism and $f^{\circ}S \subseteq \mathrm{UnitSet}(B)$. The functor UnivMap($S, f$) yielding a function from $S{\sim}A$ into B is defined by

(Def. 23) for every object x such that $x \in$ the carrier of $S{\sim}A$ there exist elements a, s of A such that $s \in S$ and $x = [\langle a, s\rangle]_{\mathrm{EqRel}(S)}$ and $it(x) = f(a) \cdot (f(s)^{-1})$.

Now we state the propositions:

(38) If f inherits ring homomorphism and $f^{\circ}S \subseteq \mathrm{UnitSet}(B)$, then UnivMap($S, f$) is additive.
PROOF: For every elements x, y of $S{\sim}A$, $(\mathrm{UnivMap}(S,f))(x+y) = (\mathrm{UnivMap}(S,f))(x) + (\mathrm{UnivMap}(S,f))(y)$. □

(39) If f inherits ring homomorphism and $f^{\circ}S \subseteq \mathrm{UnitSet}(B)$, then UnivMap($S, f$) is multiplicative.
PROOF: For every elements x, y of $S{\sim}A$, $(\mathrm{UnivMap}(S,f))(x \cdot y) = (\mathrm{UnivMap}(S,f))(x) \cdot (\mathrm{UnivMap}(S,f))(y)$. □

(40) If f inherits ring homomorphism and $f^\circ S \subseteq \mathrm{UnitSet}(B)$, then $\mathrm{UnivMap}(S, f)$ is unity-preserving.
PROOF: $(\mathrm{UnivMap}(S, f))(1_{S\sim A}) = 1_B$. □

(41) If f inherits ring homomorphism and $f^\circ S \subseteq \mathrm{UnitSet}(B)$, then $\mathrm{UnivMap}(S, f)$ inherits ring homomorphism.

(42) Suppose f inherits ring homomorphism and $f^\circ S \subseteq \mathrm{UnitSet}(B)$. Then $f = (\mathrm{UnivMap}(S, f)) \cdot$ (the canonical homomorphism of S into quotient field).
PROOF: Set $g_1 = (\mathrm{UnivMap}(S, f)) \cdot$ (the canonical homomorphism of S into quotient field). For every object x such that $x \in \mathrm{dom}\, f$ holds $f(x) = g_1(x)$ by (19), (37), [5, (8)]. □

6. The Total-Quotient Ring and the Quotient Field of Integral Domain

Let us consider A. The functor TotalQuotRing(A) yielding a ring is defined by the term

(Def. 24) NonZeroDivSet(A)$\sim A$.

Observe that TotalQuotRing(A) is non degenerated.

In the sequel x denotes an object.

Now we state the proposition:

(43) If A is a field, then $\mathrm{Ideals}\, A = \{\{0_A\}, \text{the carrier of } A\}$.
PROOF: If $x \in \mathrm{Ideals}\, A$, then $x \in \{\{0_A\}, \text{the carrier of } A\}$. If $x \in \{\{0_A\}, \text{the carrier of } A\}$, then $x \in \mathrm{Ideals}\, A$. □

From now on A denotes an integral domain.

(44) (i) $\mathrm{NonZeroDivSet}(A) = \Omega_A \setminus \{0_A\}$, and

(ii) NonZeroDivSet(A) is a without zero, non empty, multiplicatively closed subset of A.

The theorem is a consequence of (4).

(45) Let us consider an element a of A. Then $a \in \mathrm{NonZeroDivSet}(A)$ if and only if $a \neq 0_A$. The theorem is a consequence of (44).

(46) TotalQuotRing(A) is a field. The theorem is a consequence of (4), (30), and (19).

(47) Let us consider an integral domain A. Then the field of quotients of A is ring isomorphic to TotalQuotRing(A).
PROOF: Set $S = \mathrm{NonZeroDivSet}(A)$. Set $B =$ the field of quotients of A. Set $f =$ the canonical homomorphism of A into quotient field. $f^\circ S \subseteq \mathrm{UnitSet}(B)$. Reconsider $S = \mathrm{NonZeroDivSet}(A)$ as a without zero, non

empty, multiplicatively closed subset of A. UnivMap(S, f) inherits ring homomorphism. TotalQuotRing(A) is a field. Set $g = \mathrm{UnivMap}(S, f)$. For every object y such that $y \in \Omega_B$ holds $y \in \operatorname{rng} g$. $\square$

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Dynamic Programming for the Subset Sum Problem[1]

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Summary. The subset sum problem is a basic problem in the field of theoretical computer science, especially in the complexity theory [8]. The input is a sequence of positive integers and a target positive integer. The task is to determine if there exists a subsequence of the input sequence with sum equal to the target integer. It is known that the problem is NP-hard [2] and can be solved by dynamic programming in pseudo-polynomial time [1]. In this article we formalize the recurrence relation of the dynamic programming.

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Keywords: dynamic programming; subset sum problem; complexity theory

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1. Preliminaries

Let x be a finite sequence and I be a set. The functor $\mathrm{Seq}(x, I)$ yielding a finite sequence is defined by the term

(Def. 1) $\mathrm{Seq}(x \restriction I)$.

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Let D be a set and x be a D-valued finite sequence. One can check that $\operatorname{Seq}(x, I)$ is D-valued.

Let x be a real-valued finite sequence. Let us observe that $\operatorname{Seq}(x, I)$ is real-valued.

Let D be a set, x be a D-valued finite sequence, and i be a natural number. Let us observe that $x \restriction i$ is D-valued as a finite sequence-like function.

Let x be a real-valued finite sequence. One can verify that $x \restriction i$ is real-valued as a finite sequence-like function.

2. Summing Up Finite Sequences

Let x be an $\mathbb{R}$-valued finite sequence and a be a real number. We say that the sum of x is equal to a if and only if

(Def. 2) there exists a set I such that $I \subseteq \operatorname{dom} x$ and $\sum \operatorname{Seq}(x, I) = a$.

The functor Q_x yielding a function from $\operatorname{Seg} \operatorname{len} x \times \mathbb{R}$ into *Boolean* is defined by

(Def. 3) for every natural number i and for every real number s such that $1 \leqslant i \leqslant \operatorname{len} x$ holds if the sum of $x \restriction i$ is equal to s, then $it(i, s) = \mathit{true}$ and if the sum of $x \restriction i$ is not equal to s, then $it(i, s) = \mathit{false}$.

Let A be a subset of $\mathbb{N}$, i be a natural number, s be a real number, and f be a function from $A \times \mathbb{R}$ into *Boolean*. Let us note that $f(i, s)$ is Boolean.

Let a, b be objects. The functor $a =_{\Sigma} b$ yielding an object is defined by the term

(Def. 4) $(a = b \rightarrow \mathit{true}, \mathit{false})$.

Note that $a =_{\Sigma} b$ is Boolean.

Let a, b be extended reals. The functor $a \leqslant_{\Sigma} b$ yielding an object is defined by the term

(Def. 5) $(a > b \rightarrow \mathit{false}, \mathit{true})$.

Let us note that $a \leqslant_{\Sigma} b$ is Boolean.

Now we state the propositions:

(1) Let us consider a real number s, and an $\mathbb{R}$-valued finite sequence x. Suppose $1 \leqslant \operatorname{len} x$. Then $\mathrm{Q}_x(1, s) = (x(1) =_{\Sigma} s) \vee (s =_{\Sigma} 0)$.

(2) Let us consider functions f, g, and sets X, Y. Suppose $\operatorname{rng} g \subseteq X$. Then $(f \restriction (X \cup Y)) \cdot g = (f \restriction X) \cdot g$.
Proof: For every object i, $i \in \operatorname{dom}((f \restriction (X \cup Y)) \cdot g)$ iff $i \in \operatorname{dom} g$ and $g(i) \in \operatorname{dom}(f \restriction X)$. For every object i such that $i \in \operatorname{dom}((f \restriction (X \cup Y)) \cdot g)$ holds $((f \restriction (X \cup Y)) \cdot g)(i) = (f \restriction X)(g(i))$. $\square$

(3) Let us consider an $\mathbb{R}$-valued finite sequence x, a natural number i, and a set I_0. Suppose $I_0 \subseteq \operatorname{Seg} i$ and $\operatorname{Seg}(i+1) \subseteq \operatorname{dom} x$. Then $\operatorname{Seq}(x \restriction (i+1), I_0 \cup \{i+1\}) = \operatorname{Seq}(x \restriction i, I_0) \frown \langle x(i+1) \rangle$. The theorem is a consequence of (2).

(4) Let us consider a real-valued finite sequence x. If $x \neq \emptyset$ and x is positive, then $0 < \sum x$.

(5) Let us consider a real-valued finite sequence x, and a natural number i. Suppose x is positive and $1 \leqslant i \leqslant \operatorname{len} x$. Then

(i) $x \restriction i$ is positive, and

(ii) $x \restriction i \neq \emptyset$.

PROOF: For every natural number j such that $j \in \operatorname{dom}(x \restriction i)$ holds $0 < (x \restriction i)(j)$ by [4, (112)]. $\square$

(6) Let us consider a real-valued finite sequence x, and a set I. Suppose x is positive and $I \subseteq \operatorname{dom} x$ and $I \neq \emptyset$. Then

(i) $\operatorname{Seq}(x, I)$ is positive, and

(ii) $\operatorname{Seq}(x, I) \neq \emptyset$.

PROOF: For every natural number j such that $j \in \operatorname{dom}(\operatorname{Seq}(x, I))$ holds $0 < (\operatorname{Seq}(x, I))(j)$. $\square$

3. Recurrence Relation of Dynamic Programming for the Subset Sum Problem

Now we state the proposition:

(7) Let us consider an $\mathbb{R}$-valued finite sequence x. Suppose x is positive. Let us consider a natural number i, and a real number s. Suppose $1 \leqslant i < \operatorname{len} x$. Then $\mathrm{Q}_x(i+1, s) = \mathrm{Q}_x(i, s) \vee (x(i+1) \leqslant_{\Sigma} s) \wedge \mathrm{Q}_x(i, s - x(i+1))$.
PROOF: $\mathrm{Q}_x(i+1, s) = \mathit{true}$ iff $\mathrm{Q}_x(i, s) \vee (x(i+1) \leqslant_{\Sigma} s) \wedge \mathrm{Q}_x(i, s - x(i+1)) = \mathit{true}$. $\square$

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Reconstruction of the One-Dimensional Lebesgue Measure

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Summary. In the Mizar system ([1], [2]), Józef Białas has already given the one-dimensional Lebesgue measure [4]. However, the measure introduced by Białas limited the outer measure to a field with finite additivity. So, although it satisfies the nature of the measure, it cannot specify the length of measurable sets and also it cannot determine what kind of set is a measurable set. From the above, the authors first determined the length of the interval by the outer measure. Specifically, we used the compactness of the real space. Next, we constructed the pre-measure by limiting the outer measure to a semialgebra of intervals. Furthermore, by repeating the extension of the previous measure, we reconstructed the one-dimensional Lebesgue measure [7], [3].

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1. Properties of Intervals

Now we state the propositions:

(1) Let us consider non empty intervals A, B. Suppose A is open interval and B is open interval and $A \cup B$ is an interval. Then

(i) $A \cup B$ is open interval, and

(ii) A meets B, and

(iii) $\inf A < \sup B$ or $\inf B < \sup A$.

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(2) Let us consider open interval subsets A, B of $\mathbb{R}$. If A meets B, then $A \cup B$ is an open interval subset of $\mathbb{R}$. The theorem is a consequence of (1).

(3) Let us consider an interval A, and open interval subsets B, C of $\mathbb{R}$. If $A \subseteq B \cup C$ and A meets B and A meets C, then B meets C.

Let us consider non empty sets A, B and extended real numbers p, q, r, s. Now we state the propositions:

(4) If $A = [p,q]$ and $B = [r,s]$ and A misses B, then $q < r$ or $s < p$.

(5) If $A = [p,q]$ and $B = [r,s[$ and A misses B, then $q < r$ or $s \leqslant p$.

(6) If $A = [p,q]$ and $B =]r,s]$ and A misses B, then $q \leqslant r$ or $s < p$.

(7) If $A = [p,q]$ and $B =]r,s[$ and A misses B, then $q \leqslant r$ or $s \leqslant p$.

(8) If $A = [p,q[$ and $B = [r,s[$ and A misses B, then $q \leqslant r$ or $s \leqslant p$.

(9) If $A = [p,q[$ and $B =]r,s]$ and A misses B, then $q \leqslant r$ or $s < p$.

(10) If $A = [p,q[$ and $B =]r,s[$ and A misses B, then $q \leqslant r$ or $s \leqslant p$.

(11) If $A =]p,q]$ and $B =]r,s]$ and A misses B, then $q \leqslant r$ or $s \leqslant p$.

(12) If $A =]p,q]$ and $B =]r,s[$ and A misses B, then $q \leqslant r$ or $s \leqslant p$.

(13) If $A =]p,q[$ and $B =]r,s[$ and A misses B, then $q \leqslant r$ or $s \leqslant p$.

(14) Let us consider non empty intervals A, B, and extended real numbers p, q, r, s. Suppose $A = [p,q]$ and $B = [r,s]$ and A misses B. Then $A \cup B$ is not an interval. The theorem is a consequence of (4).

Let us consider non empty intervals A, B and extended real numbers p, q, r, s. Now we state the propositions:

(15) If $A = [p,q]$ and $B = [r,s[$ and A misses B and $A \cup B$ is an interval, then $p = s$ and $A \cup B = [r,q]$. The theorem is a consequence of (5).

(16) If $A = [p,q]$ and $B =]r,s]$ and A misses B and $A \cup B$ is an interval, then $q = r$ and $A \cup B = [p,s]$. The theorem is a consequence of (6).

(17) Suppose $A = [p,q]$ and $B =]r,s[$ and A misses B and $A \cup B$ is an interval. Then

(i) $p = s$ and $A \cup B =]r,q]$, or

(ii) $q = r$ and $A \cup B = [p,s[$.

The theorem is a consequence of (7).

(18) Suppose $A = [p,q[$ and $B = [r,s[$ and A misses B and $A \cup B$ is an interval. Then

(i) $p = s$ and $A \cup B = [r,q[$, or

(ii) $q = r$ and $A \cup B = [p,s[$.

The theorem is a consequence of (8).

(19) Let us consider non empty intervals A, B, and extended real numbers p, q, r, s. Suppose $A = [p, q[$ and $B =]r, s]$ and A misses B. Then $A \cup B$ is not an interval. The theorem is a consequence of (9).

Let us consider non empty intervals A, B and extended real numbers p, q, r, s. Now we state the propositions:

(20) Suppose $A = [p, q[$ and $B =]r, s[$ and A misses B and $A \cup B$ is an interval. Then

(i) $p = s$, and

(ii) $A \cup B =]r, q[$.

The theorem is a consequence of (10).

(21) Suppose $A =]p, q]$ and $B =]r, s]$ and A misses B and $A \cup B$ is an interval. Then

(i) $p = s$ and $A \cup B =]r, q]$, or

(ii) $q = r$ and $A \cup B =]p, s]$.

The theorem is a consequence of (11).

(22) Suppose $A =]p, q]$ and $B =]r, s[$ and A misses B and $A \cup B$ is an interval. Then

(i) $q = r$, and

(ii) $A \cup B =]p, s[$.

The theorem is a consequence of (12).

(23) Let us consider non empty intervals A, B, and extended real numbers p, q, r, s. Suppose $A =]p, q[$ and $B =]r, s[$ and A misses B. Then $A \cup B$ is not an interval. The theorem is a consequence of (13).

(24) Let us consider real numbers a, b, and a subset I of $\mathbb{R}^{\mathbf{1}}$. If $I = [a, b]$, then I is compact.

2. Tools for Extended Real Sequences

Let f be a finite sequence of elements of $\overline{\mathbb{R}}$. The functor $\max_{\rm p} f$ yielding a natural number is defined by

(Def. 1) if $\operatorname{len} f = 0$, then $it = 0$ and if $\operatorname{len} f > 0$, then $it \in \operatorname{dom} f$ and for every natural number i and for every extended reals r_1, r_2 such that $i \in \operatorname{dom} f$ and $r_1 = f(i)$ and $r_2 = f(it)$ holds $r_1 \leqslant r_2$ and for every natural number j such that $j \in \operatorname{dom} f$ and $f(j) = f(it)$ holds $it \leqslant j$.

The functor $\min_{\rm p} f$ yielding a natural number is defined by

(Def. 2) if $\operatorname{len} f = 0$, then $it = 0$ and if $\operatorname{len} f > 0$, then $it \in \operatorname{dom} f$ and for every natural number i and for every extended reals r_1, r_2 such that $i \in \operatorname{dom} f$ and $r_1 = f(i)$ and $r_2 = f(it)$ holds $r_1 \geqslant r_2$ and for every natural number j such that $j \in \operatorname{dom} f$ and $f(j) = f(it)$ holds $it \leqslant j$.

The functors: $\max f$ and $\min f$ yielding extended reals are defined by terms

(Def. 3) $f(\max_{\mathrm{p}} f)$,

(Def. 4) $f(\min_{\mathrm{p}} f)$,

respectively.

Let us consider a finite sequence f of elements of $\overline{\mathbb{R}}$ and a natural number i. Now we state the propositions:

(25) If $1 \leqslant i \leqslant \operatorname{len} f$, then $f(i) \leqslant f(\max_{\mathrm{p}} f)$ and $f(i) \leqslant \max f$.

(26) If $1 \leqslant i \leqslant \operatorname{len} f$, then $f(i) \geqslant f(\min_{\mathrm{p}} f)$ and $f(i) \geqslant \min f$.

Let us consider a function F and objects x, y. Now we state the propositions:

(27) If x, $y \in \operatorname{dom} F$, then $\operatorname{Swap}(F, x, y) = F \cdot (\operatorname{Swap}(\operatorname{id}_{\operatorname{dom} F}, x, y))$.

(28) If x, $y \in \operatorname{dom} F$, then F and $\operatorname{Swap}(F, x, y)$ are fiberwise equipotent. The theorem is a consequence of (27).

Now we state the proposition:

(29) Let us consider a set X, a function F, and objects x, y. Suppose $x \notin X$ and $y \notin X$. Then $F \restriction X = \operatorname{Swap}(F, x, y) \restriction X$.

3. Open Covering of Intervals

Let A be a subset of $\mathbb{R}$.

An open interval covering of A is an interval covering of A defined by

(Def. 5) for every element n of $\mathbb{N}$, $it(n)$ is open interval.

Let F be an open interval covering of A and n be an element of $\mathbb{N}$. One can verify that the functor $F(n)$ yields an open interval subset of $\mathbb{R}$. Let F be a sequence of $2^{\mathbb{R}}$.

An open interval covering of F is an interval covering of F defined by

(Def. 6) for every element n of $\mathbb{N}$, $it(n)$ is an open interval covering of $F(n)$.

Let H be an open interval covering of F and n be an element of $\mathbb{N}$. Let us note that the functor $H(n)$ yields an open interval covering of $F(n)$. Let A be a subset of $\mathbb{R}$. The functor $\operatorname{Svc2}(A)$ yielding a subset of $\overline{\mathbb{R}}$ is defined by

(Def. 7) for every extended real number x, $x \in it$ iff there exists an open interval covering F of A such that $x = \operatorname{vol}(F)$.

Let us note that $\operatorname{Svc2}(A)$ is non empty. Now we state the propositions:

(30) Let us consider a subset A of $\mathbb{R}$. Then

(i) $\mathrm{Svc2}(A) \subseteq \mathrm{Svc}(A)$, and

(ii) $\inf \mathrm{Svc}(A) \leqslant \inf \mathrm{Svc2}(A)$.

(31) Let us consider a sequence F of $2^{\mathbb{R}}$, an open interval covering G of F, and a sequence H of $\mathbb{N} \times \mathbb{N}$. Suppose $\mathrm{rng}\, H = \mathbb{N} \times \mathbb{N}$. Then $\mathrm{On}(G, H)$ is an open interval covering of $\bigcup \mathrm{rng}\, F$.

(32) Let us consider a subset A of $\mathbb{R}$, and a sequence G of $2^{\mathbb{R}}$. Suppose $A \subseteq \bigcup \mathrm{rng}\, G$ and for every element n of $\mathbb{N}$, $G(n)$ is open interval. Then G is an open interval covering of A.

(33) Let us consider a sequence F of $2^{\mathbb{R}}$, and a sequence G of $(2^{\mathbb{R}})^{\mathbb{N}}$. Suppose for every element n of $\mathbb{N}$, $G(n)$ is an open interval covering of $F(n)$. Then G is an open interval covering of F.

(34) Let us consider a sequence H of $\mathbb{N} \times \mathbb{N}$. Suppose H is one-to-one and $\mathrm{rng}\, H = \mathbb{N} \times \mathbb{N}$. Let us consider a natural number k. Then there exists an element m of $\mathbb{N}$ such that for every sequence F of $2^{\mathbb{R}}$ for every open interval covering G of F, $(\mathrm{Ser}((\mathrm{On}(G, H))\,\mathrm{vol}))(k) \leqslant (\mathrm{Ser}\,\mathrm{vol}(G))(m)$.

(35) Let us consider a sequence F of $2^{\mathbb{R}}$, and an open interval covering G of F. Then $\inf \mathrm{Svc2}(\bigcup \mathrm{rng}\, F) \leqslant \overline{\sum} \mathrm{vol}(G)$. The theorem is a consequence of (34) and (31).

Let F be a non empty family of subsets of $\mathbb{R}$. One can verify that an element of F is a subset of $\mathbb{R}$. Now we state the propositions:

(36) Let us consider an element A of $\mathrm{Intervals}_{\mathbb{R}}$. Suppose A is open interval. Then there exists an open interval covering F of A such that

(i) $F(0) = A$, and

(ii) for every natural number n such that $n \neq 0$ holds $F(n) = \emptyset$, and

(iii) $\bigcup \mathrm{rng}\, F = A$, and

(iv) $\overline{\sum}((F)\,\mathrm{vol}) = \varnothing A$.

Proof: Define $\mathcal{P}[\text{natural number}, \text{set}] \equiv$ if $\$_1 = 0$, then $\$_2 = A$ and if $\$_1 \neq 0$, then $\$_2 = \emptyset_{\mathbb{R}}$. For every element n of $\mathbb{N}$, there exists an element E of $2^{\mathbb{R}}$ such that $\mathcal{P}[n, E]$. Consider F being a function from $\mathbb{N}$ into $2^{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $\mathcal{P}[n, F(n)]$. For every natural number n such that $n \neq 0$ holds $F(n) = \emptyset$. For every object n, $0 \leqslant ((F)\,\mathrm{vol})(n)$. Define $\mathcal{P}[\text{natural number}] \equiv ((\sum_{\alpha=0}^{\kappa} (F)\,\mathrm{vol}(\alpha))_{\kappa \in \mathbb{N}})(\$_1) = \varnothing A$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. $\overline{\sum}((F)\,\mathrm{vol}) = \varnothing A$ by [6, (2)], [9, (32)], [5, (52)]. $\square$

(37) Let us consider subsets A, B of $\mathbb{R}$, and an interval covering F of A. If $B \subseteq A$, then F is an interval covering of B.

(38) Let us consider subsets A, B of $\mathbb{R}$, and an open interval covering F of A. If $B \subseteq A$, then F is an open interval covering of B. The theorem is a consequence of (37).

(39) Let us consider subsets A, B of $\mathbb{R}$, an interval covering F of A, and an interval covering G of B. If $F = G$, then $(F)\,\mathrm{vol} = (G)\,\mathrm{vol}$.

(40) Let us consider a finite sequence F of elements of $2^{\mathbb{R}}$, and a natural number k. Suppose for every natural number n such that $n \in \mathrm{dom}\, F$ holds $F(n)$ is an open interval subset of $\mathbb{R}$ and for every natural number n such that $1 \leqslant n < \mathrm{len}\, F$ holds $\bigcup \mathrm{rng}(F{\restriction}n)$ meets $F(n+1)$. Then $\bigcup \mathrm{rng}(F{\restriction}k)$ is an open interval subset of $\mathbb{R}$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv \bigcup \mathrm{rng}(F{\restriction}\$_1)$ is an open interval subset of $\mathbb{R}$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. $\square$

(41) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, and a finite sequence F of elements of $2^{\mathbb{R}}$. Suppose $A \subseteq \bigcup \mathrm{rng}\, F$ and for every natural number n such that $n \in \mathrm{dom}\, F$ holds A meets $F(n)$ and for every natural number n such that $n \in \mathrm{dom}\, F$ holds $F(n)$ is an open interval subset of $\mathbb{R}$. Then there exists a finite sequence G of elements of $2^{\mathbb{R}}$ such that

(i) F and G are fiberwise equipotent, and

(ii) for every natural number n such that $1 \leqslant n < \mathrm{len}\, G$ holds $\bigcup \mathrm{rng}(G{\restriction}n)$ meets $G(n+1)$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant \mathrm{len}\, F$, then there exists a finite sequence G of elements of $2^{\mathbb{R}}$ such that F and G are fiberwise equipotent and for every natural number n such that $1 \leqslant n < \$_1$ holds $\bigcup \mathrm{rng}(G{\restriction}n)$ meets $G(n+1)$. For every non zero natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every non zero natural number k, $\mathcal{P}[k]$. Consider G being a finite sequence of elements of $2^{\mathbb{R}}$ such that F and G are fiberwise equipotent and for every natural number n such that $1 \leqslant n < \mathrm{len}\, F$ holds $\bigcup \mathrm{rng}(G{\restriction}n)$ meets $G(n+1)$. $\square$

4. Measure of Intervals by OS-Meas

Let us consider an element I of $\mathrm{Intervals}_{\mathbb{R}}$. Now we state the propositions:

(42) If I is open interval, then $(\text{OS-Meas})(I) \leqslant \varnothing I$. The theorem is a consequence of (36) and (30).

(43) If $I \neq \emptyset$ and I is right open interval, then $(\text{OS-Meas})(I) \leqslant \varnothing I$. The theorem is a consequence of (36), (38), (39), and (30).

(44) If I is an interval, then (OS-Meas)(I) $\leqslant \varnothing I$. The theorem is a consequence of (42) and (43).

(45) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a finite sequence F of elements of $2^{\mathbb{R}}$, and a finite sequence G of elements of $\overline{\mathbb{R}}$. Suppose $A \subseteq \bigcup \operatorname{rng} F$ and $\operatorname{len} F = \operatorname{len} G$ and for every natural number n such that $n \in \operatorname{dom} F$ holds $F(n)$ is an open interval subset of $\mathbb{R}$ and for every natural number n such that $n \in \operatorname{dom} F$ holds $G(n) = \varnothing F(n)$ and for every natural number n such that $n \in \operatorname{dom} F$ holds A meets $F(n)$. Then $\varnothing A \leqslant \sum G$.

PROOF: Consider F_1 being a finite sequence of elements of $2^{\mathbb{R}}$ such that F and F_1 are fiberwise equipotent and for every natural number n such that $1 \leqslant n < \operatorname{len} F_1$ holds $\bigcup \operatorname{rng}(F_1 \restriction n)$ meets $F_1(n+1)$. Consider P being a permutation of $\operatorname{dom} F$ such that $F = F_1 \cdot P$. Reconsider $G_1 = G \cdot (P^{-1})$ as a finite sequence of elements of $\overline{\mathbb{R}}$. For every natural number n such that $n \in \operatorname{dom} F_1$ holds $G_1(n) = \varnothing F_1(n)$. Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \in \operatorname{dom} F_1$, then $\varnothing \bigcup \operatorname{rng}(F_1 \restriction \$_1) \leqslant \sum(G_1 \restriction \$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. $\bigcup \operatorname{rng}(F_1 \restriction \operatorname{len} F_1)$ is an open interval subset of $\mathbb{R}$. □

(46) Let us consider a non empty set X, a sequence f of X, and natural numbers i, j. Then there exists a sequence g of X such that

(i) for every natural number n such that $n \neq i$ and $n \neq j$ holds $f(n) = g(n)$, and

(ii) $f(i) = g(j)$, and

(iii) $f(j) = g(i)$.

PROOF: Define $\mathcal{P}$[object, object] $\equiv$ if $\$_1 \neq i$ and $\$_1 \neq j$, then $\$_2 = f(\$_1)$ and if $\$_1 = i$, then $\$_2 = f(j)$ and if $\$_1 = j$, then $\$_2 = f(i)$. For every element n of $\mathbb{N}$, there exists an element x of X such that $\mathcal{P}[n, x]$. Consider g being a function from $\mathbb{N}$ into X such that for every element n of $\mathbb{N}$, $\mathcal{P}[n, g(n)]$. □

(47) Let us consider sequences f, g of $\overline{\mathbb{R}}$. Suppose f is non-negative and there exists a natural number N such that $(\operatorname{Ser} f)(N) \leqslant (\operatorname{Ser} g)(N)$ and for every natural number n such that $n > N$ holds $f(n) \leqslant g(n)$. Then $\overline{\sum} f \leqslant \overline{\sum} g$.

PROOF: Consider N being a natural number such that $(\operatorname{Ser} f)(N) \leqslant (\operatorname{Ser} g)(N)$ and for every natural number n such that $n > N$ holds $f(n) \leqslant g(n)$. Define $\mathcal{P}$[natural number] $\equiv (\operatorname{Ser} f)(N + \$_1) \leqslant (\operatorname{Ser} g)(N + \$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number m, $\mathcal{P}[m]$. For every extended real x such that $x \in \operatorname{rng} \operatorname{Ser} f$ there exists an extended real y such that $y \in \operatorname{rng} \operatorname{Ser} g$ and $x \leqslant y$. □

(48) Let us consider sequences f, g of $\overline{\mathbb{R}}$, and natural numbers j, k. Suppose $k < j$ and for every natural number n such that $n < j$ holds $f(n) = g(n)$. Then $(\mathrm{Ser}\, f)(k) = (\mathrm{Ser}\, g)(k)$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant k$, then $(\mathrm{Ser}\, f)(\$_1) = (\mathrm{Ser}\, g)(\$_1)$. For every natural number m such that $\mathcal{P}[m]$ holds $\mathcal{P}[m+1]$. For every natural number m, $\mathcal{P}[m]$. □

(49) Let us consider sequences f, g of $\overline{\mathbb{R}}$, and natural numbers i, j. Suppose f is non-negative and $i \geqslant j$ and for every natural number n such that $n \neq i$ and $n \neq j$ holds $f(n) = g(n)$ and $f(i) = g(j)$ and $f(j) = g(i)$. Then $(\mathrm{Ser}\, f)(i) = (\mathrm{Ser}\, g)(i)$.
PROOF: For every element k of $\mathbb{N}$, $0 \leqslant g(k)$. □

(50) Let us consider sequences f, g of $\overline{\mathbb{R}}$, and natural numbers i, j. Suppose f is non-negative and $f(i) = g(j)$ and $f(j) = g(i)$ and for every natural number n such that $n \neq i$ and $n \neq j$ holds $f(n) = g(n)$. Let us consider a natural number n. If $n \geqslant i$ and $n \geqslant j$, then $(\mathrm{Ser}\, f)(n) = (\mathrm{Ser}\, g)(n)$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \geqslant i$ and $\$_1 \geqslant j$, then $(\mathrm{Ser}\, f)(\$_1) = (\mathrm{Ser}\, g)(\$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

(51) Let us consider sequences f, g of $\overline{\mathbb{R}}$, and natural numbers i, j. Suppose f is non-negative and $i \geqslant j$ and for every natural number n such that $n \neq i$ and $n \neq j$ holds $f(n) = g(n)$ and $f(i) = g(j)$ and $f(j) = g(i)$. Then $\overline{\sum} f = \overline{\sum} g$.
PROOF: For every element k of $\mathbb{N}$, $0 \leqslant g(k)$. □

(52) Let us consider a subset A of $\mathbb{R}$, interval coverings F_1, F_2 of A, and natural numbers n, m. Suppose for every natural number k such that $k \neq n$ and $k \neq m$ holds $F_1(k) = F_2(k)$ and $F_1(n) = F_2(m)$ and $F_1(m) = F_2(n)$. Then $\mathrm{vol}(F_1) = \mathrm{vol}(F_2)$. The theorem is a consequence of (51).

(53) Let us consider a subset A of $\mathbb{R}$, interval coverings F_1, F_2 of A, and natural numbers n, m. Suppose for every natural number k such that $k \neq n$ and $k \neq m$ holds $F_1(k) = F_2(k)$ and $F_1(n) = F_2(m)$ and $F_1(m) = F_2(n)$. Let us consider a natural number k. Suppose $k \geqslant n$ and $k \geqslant m$. Then $(\mathrm{Ser}((F_1)\,\mathrm{vol}))(k) = (\mathrm{Ser}((F_2)\,\mathrm{vol}))(k)$. The theorem is a consequence of (50).

(54) Let us consider a non empty set X, a sequence s_2 of X, and a finite sequence f of elements of X. Suppose $\mathrm{rng}\, f \subseteq \mathrm{rng}\, s_2$. Then there exists a natural number N such that $\mathrm{rng}\, f \subseteq \mathrm{rng}(s_2 \restriction \mathbb{Z}_N)$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence F of elements of X such that $\mathrm{len}\, F = \$_1$ and $\mathrm{rng}\, F \subseteq \mathrm{rng}\, s_2$ there exists a natural number N such that $\mathrm{rng}\, F \subseteq \mathrm{rng}(s_2 \restriction \mathbb{Z}_N)$. For every natural number k

such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

(55) Let us consider a non empty subset A of $\mathbb{R}$, an interval covering F of A, and a one-to-one finite sequence G of elements of $2^{\mathbb{R}}$. Suppose $\operatorname{rng} G \subseteq \operatorname{rng} F$. Then there exists an interval covering F_1 of A such that

(i) for every natural number n such that $n \in \operatorname{dom} G$ holds $G(n) = F_1(n)$, and

(ii) $\operatorname{vol}(F_1) = \operatorname{vol}(F)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ there exists an interval covering F_0 of A such that for every natural number n such that $n \in \operatorname{dom}(G \restriction \$_1)$ holds $(G \restriction \$_1)(n) = F_0(n)$ and F_0 and F are fiberwise equipotent and $\operatorname{vol}(F_0) = \operatorname{vol}(F)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

(56) Let us consider a non empty subset A of $\mathbb{R}$, an interval covering F of A, a one-to-one finite sequence G of elements of $2^{\mathbb{R}}$, and a finite sequence H of elements of $\overline{\mathbb{R}}$. Suppose $\operatorname{rng} G \subseteq \operatorname{rng} F$ and $\operatorname{dom} G = \operatorname{dom} H$ and for every natural number n, $H(n) = \varnothing G(n)$. Then $\sum H \leqslant \operatorname{vol}(F)$.

PROOF: Consider F_1 being an interval covering of A such that for every natural number n such that $n \in \operatorname{dom} G$ holds $G(n) = F_1(n)$ and $\operatorname{vol}(F_1) = \operatorname{vol}(F)$. Consider S being a sequence of $\overline{\mathbb{R}}$ such that $\sum H = S(\operatorname{len} H)$ and $S(0) = 0$ and for every natural number n such that $n < \operatorname{len} H$ holds $S(n+1) = S(n) + H(n+1)$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant \operatorname{len} H$, then $S(\$_1) \leqslant (\operatorname{Ser}((F_1)\operatorname{vol}))(\$_1)$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. □

(57) Let us consider an interval I. Then $\varnothing I = (\text{OS-Meas})(I)$. The theorem is a consequence of (44).

5. CONSTRUCTION OF THE ONE-DIMENSIONAL LEBESGUE MEASURE

Let F be a finite sequence of elements of $\text{Intervals}_{\mathbb{R}}$ and n be a natural number. Let us note that the functor $F(n)$ yields an interval subset of $\mathbb{R}$. The functor pre-Meas yielding a non-negative, zeroed function from $\text{Intervals}_{\mathbb{R}}$ into $\overline{\mathbb{R}}$ is defined by the term

(Def. 8) $\text{OS-Meas} \restriction \text{Intervals}_{\mathbb{R}}$.

Now we state the propositions:

(58) Let us consider an element I of $\text{Intervals}_{\mathbb{R}}$. Then $(\text{pre-Meas})(I) = \varnothing I$. The theorem is a consequence of (57).

(59) Let us consider an interval I. Then $(\text{pre-Meas})(I) = \varnothing I$. The theorem is a consequence of (58).

(60) Let us consider elements A, B of $\text{Intervals}_{\mathbb{R}}$. Suppose A misses B and $A \cup B$ is an interval. Then $(\text{pre-Meas})(A \cup B) = (\text{pre-Meas})(A) + (\text{pre-Meas})(B)$. The theorem is a consequence of (58), (14), (15), (59), (16), (17), (19), (18), (20), (21), (22), and (23).

(61) Let us consider a non empty, disjoint valued finite sequence F of elements of $\text{Intervals}_{\mathbb{R}}$. Suppose $\bigcup F$ is an interval. Then there exists a natural number n such that

(i) $n \in \operatorname{dom} F$, and

(ii) $(\bigcup F) \setminus F(n)$ is an interval.

The theorem is a consequence of (26).

(62) Let us consider an interval A. Then $(\text{pre-Meas}) \cdot \langle A \rangle = \langle (\text{pre-Meas})(A) \rangle$.
PROOF: Reconsider $F = \langle A \rangle$ as a finite sequence of elements of $\text{Intervals}_{\mathbb{R}}$. For every natural number n such that $n \in \operatorname{dom}((\text{pre-Meas}) \cdot F)$ holds $((\text{pre-Meas}) \cdot F)(n) = \langle (\text{pre-Meas})(A) \rangle (n)$. $\square$

(63) Let us consider a disjoint valued finite sequence F of elements of $\text{Intervals}_{\mathbb{R}}$. Suppose $\bigcup F \in \text{Intervals}_{\mathbb{R}}$. Then there exists a disjoint valued finite sequence G of elements of $\text{Intervals}_{\mathbb{R}}$ such that

(i) F and G are fiberwise equipotent, and

(ii) for every natural number n such that $n \in \operatorname{dom} G$ holds $\bigcup (G \restriction n) \in \text{Intervals}_{\mathbb{R}}$ and $(\text{pre-Meas})(\bigcup (G \restriction n)) = \sum (\text{pre-Meas}) \cdot (G \restriction n)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every disjoint valued finite sequence H of elements of $\text{Intervals}_{\mathbb{R}}$ such that $\operatorname{len} H = \$_1$ and $\bigcup H \in \text{Intervals}_{\mathbb{R}}$ there exists a disjoint valued finite sequence G of elements of $\text{Intervals}_{\mathbb{R}}$ such that H and G are fiberwise equipotent and for every natural number n such that $n \in \operatorname{dom} G$ holds $\bigcup (G \restriction n) \in \text{Intervals}_{\mathbb{R}}$ and $(\text{pre-Meas})(\bigcup (G \restriction n)) = \sum (\text{pre-Meas}) \cdot (G \restriction n)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. $\square$

(64) Let us consider finite sequences F, G of elements of $\overline{\mathbb{R}}$. Then

(i) if F is without $-\infty$ and G is without $-\infty$, then $F \frown G$ is without $-\infty$, and

(ii) if F is without $+\infty$ and G is without $+\infty$, then $F \frown G$ is without $+\infty$.

(65) Let us consider a finite sequence F of elements of $\overline{\mathbb{R}}$, and a natural number k. Then

(i) if F is without $-\infty$, then $F_{\lfloor k}$ is without $-\infty$, and

(ii) if F is without $+\infty$, then $F_{\lfloor k}$ is without $+\infty$.

(66) Let us consider a finite sequence F of elements of $\overline{\mathbb{R}}$. Then

(i) if F is without $-\infty$, then $\sum F \neq -\infty$, and

(ii) if F is without $+\infty$, then $\sum F \neq +\infty$.

PROOF: Consider S being a sequence of $\overline{\mathbb{R}}$ such that $\sum F = S(\operatorname{len} F)$ and $S(0) = 0$ and for every natural number n such that $n < \operatorname{len} F$ holds $S(n+1) = S(n)+F(n+1)$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant \operatorname{len} F$, then $S(\$_1) \neq +\infty$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. $\square$

(67) Let us consider without $-\infty$ finite sequences R_1, R_2 of elements of $\overline{\mathbb{R}}$. If R_1 and R_2 are fiberwise equipotent, then $\sum R_1 = \sum R_2$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every without $-\infty$ finite sequences f, g of elements of $\overline{\mathbb{R}}$ such that f and g are fiberwise equipotent and $\operatorname{len} f = \$_1$ holds $\sum f = \sum g$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. $\mathcal{P}[0]$. For every natural number n, $\mathcal{P}[n]$. $\square$

(68) Let us consider a disjoint valued finite sequence F of elements of $\text{Intervals}_{\mathbb{R}}$. Suppose $\bigcup F \in \text{Intervals}_{\mathbb{R}}$. Then $(\text{pre-Meas})(\bigcup F) = \sum(\text{pre-Meas}) \cdot F$. The theorem is a consequence of (63), (59), and (67).

(69) Let us consider a disjoint valued function K from $\mathbb{N}$ into $\text{Intervals}_{\mathbb{R}}$. Suppose $\bigcup K \in \text{Intervals}_{\mathbb{R}}$. Then $(\text{pre-Meas})(\bigcup K) \leqslant \overline{\sum}(\text{pre-Meas}) \cdot K$.
PROOF: Reconsider $F = K$ as a sequence of $2^{\mathbb{R}}$. For every element n of $\mathbb{N}$, $((\text{OS-Meas}) \cdot F)(n) = ((\text{pre-Meas}) \cdot K)(n)$. $\square$

One can verify that the functor pre-Meas yields a pre-measure of $\text{Intervals}_{\mathbb{R}}$. The functor J-Meas yielding a measure on the field generated by $\text{Intervals}_{\mathbb{R}}$ is defined by

(Def. 9) for every set A such that $A \in$ the field generated by $\text{Intervals}_{\mathbb{R}}$ for every disjoint valued finite sequence F of elements of $\text{Intervals}_{\mathbb{R}}$ such that $A = \bigcup F$ holds $\mathit{it}(A) = \sum(\text{pre-Meas}) \cdot F$.

Note that the functor J-Meas yields an induced measure of $\text{Intervals}_{\mathbb{R}}$ and pre-Meas. Now we state the proposition:

(70) J-Meas is completely-additive.

The functor B-Meas yielding a σ-measure on the Borel sets is defined by the term

(Def. 10) σ-Meas(the Caratheodory measure determined by J-Meas)$\restriction$(the Borel sets).

Let us consider an interval A. Now we state the propositions:

(71) $(\text{J-Meas})(A) = \varnothing A$. The theorem is a consequence of (62) and (59).

(72) $(\text{B-Meas})(A) = \varnothing A$. The theorem is a consequence of (71).

(73) $A \in$ the Borel sets.

The functor L-Field yielding a σ-field of subsets of $\mathbb{R}$ is defined by the term

(Def. 11) COM(the Borel sets, B-Meas).

The functor L-Meas yielding a σ-measure on L-Field is defined by the term

(Def. 12) COM(B-Meas).

Observe that L-Meas is complete. Now we state the propositions:

(74) $\emptyset$ is a set with measure zero w.r.t. B-Meas. The theorem is a consequence of (72).

(75) Let us consider a real number a. Then $\{a\}$ is a set with measure zero w.r.t. B-Meas. The theorem is a consequence of (72).

(76) The Borel sets $\subseteq$ L-Field. The theorem is a consequence of (74).

(77) Let us consider an interval A. Then $(\text{L-Meas})(A) = \varnothing A$. The theorem is a consequence of (73), (74), and (72).

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Developing Complementary Rough Inclusion Functions

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Summary. We continue the formal development of rough inclusion functions (RIFs), continuing the research on the formalization of rough sets [15] – a well-known tool of modelling of incomplete or partially unknown information. In this article we give the formal characterization of complementary RIFs, following a paper by Gomolińska [4]. We expand this framework introducing Jaccard index, Steinhaus generate metric, and Marczewski-Steinhaus metric space [1]. This is the continuation of [9]; additionally we implement also parts of [2], [3], and the details of this work can be found in [7].

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0. Introduction

In the paper, continuing our development of rough inclusion functions (RIFs), we deal with functions complementary to RIFs, and consider distance operators obtained from such functions.

Quite large part of the Mizar formalization of rough sets [5], [8] was done by means of the notion of a generalized approximation space understood as a pair $\langle U, \rho\rangle$, where ρ is an indiscernibility relation defined on the universe U. This reflects the standpoint of Skowron and Stepaniuk [16], based on tolerance relations instead of equivalence relations (claimed by Pawlak) and further generalized by Zhu [17], among many others. The framework build in a similar manner is contained in [10] and [11].

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In the alternative approach, used by Gomolińska [3], approximation spaces are treated as triples of the form $\mathcal{A} = (U, I, \kappa)$, where U is a non-empty set called the universe, $I : U \mapsto \wp U$ is an uncertainty mapping, and $\kappa : \wp U \times \wp U \mapsto [0, 1]$ is a rough inclusion function. The formalization of uncertainty mappings was discussed in [13], and the current submission goes further in this direction. Still however, we can merge our existing approaches via theory merging mechanism [6], having in mind that we should avoid duplications in the repository of Mizar texts as much as we can [12].

After filling some gaps in the Mizar Mathematical Library, proving preliminary facts needed later, in Sect. 2 we continue the development of functions complementary to RIFs. Given arbitrary preRIF f (where preRIF stands for a general mapping from the Cartesian square of the powerset of the universe into the unit interval, without any additional assumptions), we introduce the Mizar functor CMap f (see Def. 1), which is of much more general interest. Then we prove a list of properties of the complementary function on three well-known RIFs (see [9]): $\kappa^{£}$, κ_1, and κ_2.

Let us briefly recall these three mappings. The first one, standard rough inclusion function, $\kappa^{£}$ based on the ideas of Jan Łukasiewicz [14] is defined as follows:

$$\kappa^{£}(X, Y) = \begin{cases} \frac{|X \cap Y|}{|X|}, & \text{if } X \neq \emptyset \\ 1, & \text{otherwise} \end{cases}$$

Two others are

$$\kappa_1(X, Y) = \begin{cases} \frac{|Y|}{|X \cup Y|}, & \text{if } X \cup Y \neq \emptyset \\ 1, & \text{otherwise} \end{cases}$$

and

$$\kappa_2(X, Y) = \frac{|(U - X) \cup Y|}{|U|}.$$

Additionally, we introduce a new type for an object complementary to RIF, called just co-RIF.

Our testbed for chosen formal approach was Section 4, where full formalization of Proposition 4 from [3] was presented. This was also a step towards defining three metrics: δ_L, δ_1, and δ_2 (Def. 3, 4, and 5, respectively). It is worth noticing that even if we can deal with fixed rough approximation space, say R, we give this variable explicitly both in definitions of all three κ functions, and consequently in corresponding distances δ.

Section 5 contains the definition and very basic properties of Jaccard similarity coefficient J_s, widely used in data mining and information retrieval. We adopt the setting allowing both sets to be empty at the same time (then the value of Jaccard index is set to 1). Based on that, in Sect. 6 we define Jaccard distance (or Marczewski distance) for arbitrary subsets A, B of the universe as $1 - J_s(A, B)$.

We met some difficulties in proofs of the triangle inequality for such metrics, and in order to make it easier for us, we decided to implement some more ideas from the theory of distances. Namely, we introduced the symmetric difference metric (Def. 11). Then, using newly defined construction of Steinhaus generate metric (Def. 10), we can obtain from any distance a new one. The crucial fact was that the Jaccard distance is precisely Steinhaus generate metric from symmetric difference distance, hence all ordinary properties of a metric space can be easily obtained by means of this construction.

In the last section, we show that the value of Marczewski metric on two subsets A, B of given rough approximation space R is equal to $\delta_1(A, B)$. As δ_1 satisfies the triangle inequality, so does Marczewski metric.

1. Preliminaries

Let us consider finite sets x_1, x_2. Now we state the propositions:

(1) $\overline{\overline{x_1 \dot{-} x_2}} = \overline{\overline{x_1 \setminus x_2}} + \overline{\overline{x_2 \setminus x_1}}$.

(2) $\frac{2 \cdot \overline{\overline{x_1 \dot{-} x_2}}}{\overline{\overline{x_1}} + \overline{\overline{x_2}} + \overline{\overline{x_1 \dot{-} x_2}}} = \frac{\overline{\overline{x_1 \dot{-} x_2}}}{\overline{\overline{x_1 \cup x_2}}}$.

Now we state the propositions:

(3) Let us consider sets A, B, C. Then $A \dot{-} C = (A \dot{-} B) \dot{-} (B \dot{-} C)$.

(4) Let us consider finite sets A, B. Suppose $A \cup B \neq \emptyset$. Then $1 - \frac{\overline{\overline{A \cap B}}}{\overline{\overline{A \cup B}}} = \frac{\overline{\overline{A \dot{-} B}}}{\overline{\overline{A \cup B}}}$.

(5) Let us consider a finite set R, and subsets X, Y of R. Then $\overline{\overline{X \cup Y}} = \overline{\overline{X \cap Y}}$ if and only if $X = Y$.

Observe that there exists a metric space which is finite and non empty.

2. Complementary Rough Inclusion Functions

From now on R denotes a finite approximation space and X, Y, Z denote subsets of R.

Let R be a finite approximation space and f be a preRIF of R. The functor $\mathrm{CMap}\, f$ yielding a preRIF of R is defined by

(Def. 1) for every subsets x, y of R, $it(x, y) = 1 - f(x, y)$.

Now we state the propositions:

(6) Let us consider a preRIF f of R. Then $\mathrm{CMap}\,\mathrm{CMap}\,f = f$.
PROOF: Set $g = \mathrm{CMap}\,f$. For every element x of $2^\alpha \times 2^\alpha$, $(\mathrm{CMap}\,g)(x) = f(x)$, where α is the carrier of R. $\square$

(7) If $X \neq \emptyset$, then $(\mathrm{CMap}\,\kappa^{£}(R))(X, Y) = \frac{\overline{\overline{X \setminus Y}}}{\overline{\overline{X}}}$.

(8) If $X = \emptyset$, then $(\mathrm{CMap}\,\kappa^{£}(R))(X, Y) = 0$.

(9) If $X \neq \emptyset$, then $(\mathrm{CMap}\,\kappa^{£}(R))(X, Y) = \kappa^{£}(X, Y^{\mathrm{c}})$.

(10) If $X \cup Y \neq \emptyset$, then $(\mathrm{CMap}\,\kappa_1(R))(X, Y) = \frac{\overline{\overline{X \setminus Y}}}{\overline{\overline{X \cup Y}}}$.

(11) If $X \cup Y = \emptyset$, then $(\mathrm{CMap}\,\kappa_1(R))(X, Y) = 0$.

(12) $(\mathrm{CMap}\,\kappa_2(R))(X, Y) = \frac{\overline{\overline{X \setminus Y}}}{\overline{\overline{\Omega_R}}}$.

(13) Suppose $X \neq \emptyset$. Then $\kappa^{£}(X, Y) = \frac{(\mathrm{CMap}\,\kappa_1(R))(X, Y^{\mathrm{c}})}{\kappa_1(Y^{\mathrm{c}}, X)} = \frac{(\mathrm{CMap}\,\kappa_2(R))(X, Y^{\mathrm{c}})}{\kappa_2(\Omega_R, X)}$.

3. Introducing co-RIFs

Let us consider R. Let f be a preRIF of R. We say that f is co-RIF-like if and only if

(Def. 2) $\mathrm{CMap}\,f$ is a RIF of R.

Let f be a RIF of R. Let us observe that $\mathrm{CMap}\,f$ is co-RIF-like and there exists a preRIF of R which is co-RIF-like.

A co-RIF of R is a co-RIF-like preRIF of R.

4. Proposition 6 from [4]

From now on κ denotes a RIF of R. Now we state the propositions:

(14) $(\mathrm{CMap}\,\kappa)(X, Y) = 0$ if and only if $X \subseteq Y$.

(15) $(\mathrm{CMap}\,\kappa^{£}(R))(X, Y) = 0$ if and only if $X \subseteq Y$.
PROOF: If $(\mathrm{CMap}\,\kappa^{£}(R))(X, Y) = 0$, then $X \subseteq Y$. $\square$

(16) If $Y \subseteq Z$, then $(\mathrm{CMap}\,\kappa)(X, Z) \leqslant (\mathrm{CMap}\,\kappa)(X, Y)$.

(17) If $Y \subseteq Z$, then $(\mathrm{CMap}\,\kappa^{£}(R))(X, Z) \leqslant (\mathrm{CMap}\,\kappa^{£}(R))(X, Y)$.

(18) $(\mathrm{CMap}\,\kappa_2(R))(X, Y) \leqslant (\mathrm{CMap}\,\kappa_1(R))(X, Y) \leqslant (\mathrm{CMap}\,\kappa^{£}(R))(X, Y)$.

(19) Let us consider real numbers a, b, c. If $a \leqslant b$ and $0 \leqslant c < b$ and $0 < b$, then $\frac{a}{b} \geqslant \frac{a-c}{b-c}$.

(20) If $X \neq \emptyset$ and $Y = \emptyset$, then $(\mathrm{CMap}\,\kappa_1(R))(X, Y) = 1$. The theorem is a consequence of (10).

(21) If $X = \emptyset$ and $Y \neq \emptyset$, then $(\mathrm{CMap}\,\kappa_1(R))(X, Y) = 0$. The theorem is a consequence of (10).

(22) $(\mathrm{CMap}\,\kappa_1(R))(X,Y)+(\mathrm{CMap}\,\kappa_1(R))(Y,Z) \geqslant (\mathrm{CMap}\,\kappa_1(R))(X,Z)$. The theorem is a consequence of (14) and (20).

(23) $0 \leqslant (\mathrm{CMap}\,\kappa^{\pounds}(R))(X,Y) \leqslant 1$.

(24) $0 \leqslant (\mathrm{CMap}\,\kappa_1(R))(X,Y) + (\mathrm{CMap}\,\kappa_1(R))(Y,X) \leqslant 1$. The theorem is a consequence of (11) and (10).

(25) $0 \leqslant (\mathrm{CMap}\,\kappa_2(R))(X,Y) + (\mathrm{CMap}\,\kappa_2(R))(Y,X) \leqslant 1$. The theorem is a consequence of (12).

(26) Suppose $X = \emptyset$ and $Y \neq \emptyset$ or $X \neq \emptyset$ and $Y = \emptyset$.
Then $(\mathrm{CMap}\,\kappa^{\pounds}(R))(X,Y)+(\mathrm{CMap}\,\kappa^{\pounds}(R))(Y,X) = (\mathrm{CMap}\,\kappa_1(R))(X,Y)$ $+(\mathrm{CMap}\,\kappa_1(R))(Y,X) = 1$.

Let us consider R. The functors: $\delta_L(R)$, $\delta_1(R)$, and $\delta_2(R)$ yielding preRIFs of R are defined by conditions

(Def. 3) for every subsets x, y of R, $\delta_L(R)(x,y) = \frac{(\mathrm{CMap}\,\kappa^{\pounds}(R))(x,y)+(\mathrm{CMap}\,\kappa^{\pounds}(R))(y,x)}{2}$,

(Def. 4) for every subsets x, y of R, $\delta_1(R)(x,y) = (\mathrm{CMap}\,\kappa_1(R))(x,y) + (\mathrm{CMap}\,\kappa_1(R))(y,x)$,

(Def. 5) for every subsets x, y of R, $\delta_2(R)(x,y) = (\mathrm{CMap}\,\kappa_2(R))(x,y) + (\mathrm{CMap}\,\kappa_2(R))(y,x)$,

respectively. Now we state the propositions:

(27) $(\delta_L(R))(X,Y) = 0$ if and only if $X = Y$. The theorem is a consequence of (14).

(28) $(\delta_L(R))(X,Y) = (\delta_L(R))(Y,X)$.

(29) If $X \neq \emptyset$ and $Y = \emptyset$ or $X = \emptyset$ and $Y \neq \emptyset$, then $(\delta_L(R))(X,Y) = \frac{1}{2}$.

(30) Suppose $X \neq \emptyset$ and $Y \neq \emptyset$. Then $(\delta_L(R))(X,Y) = \frac{\frac{\overline{\overline{X\setminus Y}}}{\overline{\overline{X}}}+\frac{\overline{\overline{Y\setminus X}}}{\overline{\overline{Y}}}}{2}$. The theorem is a consequence of (7).

(31) $(\delta_1(R))(X,Y) = \frac{\overline{\overline{X \dot{-} Y}}}{\overline{\overline{X \cup Y}}}$. The theorem is a consequence of (10) and (14).

(32) $(\delta_2(R))(X,Y) = \frac{\overline{\overline{X \dot{-} Y}}}{\overline{\overline{\Omega_R}}}$. The theorem is a consequence of (12).

(33) $(\delta_1(R))(X,Y) + (\delta_1(R))(Y,Z) \geqslant (\delta_1(R))(X,Z)$. The theorem is a consequence of (22).

(34) $(\delta_1(R))(X,Y) = 0$ if and only if $X = Y$. The theorem is a consequence of (14).

(35) $(\delta_1(R))(X,Y) = (\delta_1(R))(Y,X)$.

(36) $(\delta_2(R))(X,Y) = 0$ if and only if $X = Y$. The theorem is a consequence of (14).

(37) $(\delta_2(R))(X,Y) = (\delta_2(R))(Y,X)$.

(38) $(\mathrm{CMap}\,\kappa_2(R))(X,Y)+(\mathrm{CMap}\,\kappa_2(R))(Y,Z) \geqslant (\mathrm{CMap}\,\kappa_2(R))(X,Z)$. The theorem is a consequence of (12).

(39) $(\delta_2(R))(X,Y)+(\delta_2(R))(Y,Z) \geqslant (\delta_2(R))(X,Z)$. The theorem is a consequence of (38).

5. Jaccard Index Measuring Similarity of Sets

Let R be a finite set and A, B be subsets of R. The functor JaccardIndex(A,B) yielding an element of $[0,1]$ is defined by the term

(Def. 6) $\begin{cases} \frac{\overline{\overline{A\cap B}}}{\overline{\overline{A\cup B}}}, & \textbf{if } A\cup B \neq \emptyset, \\ 1, & \textbf{otherwise}. \end{cases}$

Let us consider a finite set R and subsets A, B of R. Now we state the propositions:

(40) JaccardIndex$(A,B)=1$ if and only if $A=B$. The theorem is a consequence of (5).

(41) JaccardIndex$(A,B)=$ JaccardIndex(B,A).

6. Marczewski-Steinhaus Metric

Let X be a non empty set and f be a function from $X \times X$ into $\mathbb{R}$. Observe that f is non-negative yielding if and only if the condition (Def. 7) is satisfied.

(Def. 7) for every elements x, y of X, $f(x,y) \geqslant 0$.

One can verify that there exists a function from $X \times X$ into $\mathbb{R}$ which is discernible, symmetric, reflexive, and triangle and every function from $X \times X$ into $\mathbb{R}$ which is reflexive, symmetric, and triangle is also non-negative yielding.

Now we state the proposition:

(42) Let us consider a non empty set X, a non-negative yielding, discernible, triangle, reflexive function f from $X \times X$ into $\mathbb{R}$, and elements x, y of X. If $x \neq y$, then $f(x,y) > 0$.

Let R be a finite set. The functor JaccardDist R yielding a function from $2^R \times 2^R$ into $\mathbb{R}$ is defined by

(Def. 8) for every subsets A, B of R, $it(A,B) = 1 -$ JaccardIndex(A,B).

Let R be a finite 1-sorted structure. The functor MarczewskiDistance R yielding a function from $2^{(\text{the carrier of } R)} \times 2^{(\text{the carrier of } R)}$ into $\mathbb{R}$ is defined by the term

(Def. 9) JaccardDist Ω_R.

7. Steinhaus Generate Metric

Let X be a non empty set, p be an element of X, and f be a function from $X \times X$ into $\mathbb{R}$. The functor SteinhausGen(f, p) yielding a function from $X \times X$ into $\mathbb{R}$ is defined by

(Def. 10) for every elements x, y of X, $it(x, y) = \frac{2 \cdot f(x,y)}{f(x,p)+f(y,p)+f(x,y)}$.

Let f be a non-negative yielding function from $X \times X$ into $\mathbb{R}$. Observe that SteinhausGen(f, p) is non-negative yielding.

Let f be a non-negative yielding, reflexive function from $X \times X$ into $\mathbb{R}$. One can verify that SteinhausGen(f, p) is reflexive.

Let f be a non-negative yielding, discernible function from $X \times X$ into $\mathbb{R}$. Let us observe that SteinhausGen(f, p) is discernible.

Let f be a non-negative yielding, symmetric function from $X \times X$ into $\mathbb{R}$. Let us note that SteinhausGen(f, p) is symmetric.

Let f be a discernible, symmetric, triangle, reflexive function from $X \times X$ into $\mathbb{R}$. Let us observe that SteinhausGen(f, p) is triangle.

8. Marczewski-Steinhaus Metric is Generated by Symmetric Difference Metric

Let X be a finite set. The functor SymmetricDiffDist X yielding a function from $2^X \times 2^X$ into $\mathbb{R}$ is defined by

(Def. 11) for every subsets x, y of X, $it(x, y) = \overline{\overline{x \dot{-} y}}$.

One can check that SymmetricDiffDist X is reflexive, discernible, symmetric, and triangle.

The functor SymDifMetrSpace X yielding a metric structure is defined by the term

(Def. 12) $\langle 2^X, \mathrm{SymmetricDiffDist}\, X \rangle$.

One can verify that SymDifMetrSpace X is non empty and SymDifMetrSpace X is reflexive, discernible, symmetric, and triangle.

Now we state the propositions:

(43) Let us consider a finite set R, and subsets A, B of R. Then $(\mathrm{JaccardDist}\, R)(A, B) = \frac{\overline{\overline{A \dot{-} B}}}{\overline{\overline{A \cup B}}}$. The theorem is a consequence of (4).

(44) Let us consider a finite set X. Then JaccardDist X = SteinhausGen(SymmetricDiffDist $X, \emptyset_X$). The theorem is a consequence of (43) and (2).

9. Steinhaus Metric Spaces

Let M be a finite, non empty metric space. One can check that the distance of M is symmetric, reflexive, discernible, and triangle.

Let M be a finite, non empty metric structure and p be an element of M. The functor SteinhausMetrSpace(M, p) yielding a metric structure is defined by the term

(Def. 13) $\langle$the carrier of M, SteinhausGen((the distance of M), p)$\rangle$.

Let M be a metric structure. We say that M is with nonnegative distance if and only if

(Def. 14) the distance of M is non-negative yielding.

Let A be a finite, non empty set. Note that the discrete metric of A is finite, non empty, and non-negative yielding and there exists a metric space which is finite, non empty, and with nonnegative distance.

Let M be a finite, non empty, with nonnegative distance metric structure and p be an element of M. Let us observe that SteinhausMetrSpace(M, p) is with nonnegative distance.

Let M be a finite, non empty, with nonnegative distance, discernible metric structure. Observe that SteinhausMetrSpace(M, p) is discernible.

Let M be a finite, non empty, with nonnegative distance, reflexive metric structure. Let us note that SteinhausMetrSpace(M, p) is reflexive.

Let M be a finite, non empty, with nonnegative distance, symmetric metric structure. Note that SteinhausMetrSpace(M, p) is symmetric.

Let M be a finite, non empty, discernible, symmetric, reflexive, triangle metric structure. Let us observe that SteinhausMetrSpace(M, p) is triangle.

Let R be a finite 1-sorted structure. Observe that MarczewskiDistance R is reflexive, discernible, and symmetric.

Now we state the proposition:

(45) Let us consider a finite approximation space R, and subsets A, B of R. Then (MarczewskiDistance R)$(A, B) = (\delta_1(R))(A, B)$. The theorem is a consequence of (43) and (31).

Let R be a finite 1-sorted structure. Note that MarczewskiDistance R is triangle.

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Elementary Number Theory Problems. Part I

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Summary. In this paper we demonstrate the feasibility of formalizing *recreational mathematics* in Mizar ([1], [2]) drawing examples from W. Sierpinski's book "250 Problems in Elementary Number Theory" [4]. The current work contains proofs of initial ten problems from the chapter devoted to the divisibility of numbers. Included are problems on several levels of difficulty.

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1. Problem 1

One can verify that there exists an integer which is positive.

Now we state the propositions:

(1) Let us consider a positive integer n. Then $n+1 \mid n^2+1$ if and only if $n=1$.

PROOF: If $n+1 \mid n^2+1$, then $n=1$ by [6, (2)]. □

(2) Let us consider integers i, n. If $|i|=n$, then $i=n$ or $i=-n$.

(3) Let us consider a natural number n. Suppose $n \mid 24$. Then

(i) $n=1$, or

(ii) $n=2$, or

(iii) $n=3$, or

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(iv) $n = 4$, or
(v) $n = 6$, or
(vi) $n = 8$, or
(vii) $n = 12$, or
(viii) $n = 24$.

(4) Let us consider an integer t. Suppose $t \mid 24$. Then
(i) $t = -1$, or
(ii) $t = 1$, or
(iii) $t = -2$, or
(iv) $t = 2$, or
(v) $t = -3$, or
(vi) $t = 3$, or
(vii) $t = -4$, or
(viii) $t = 4$, or
(ix) $t = -6$, or
(x) $t = 6$, or
(xi) $t = -8$, or
(xii) $t = 8$, or
(xiii) $t = -12$, or
(xiv) $t = 12$, or
(xv) $t = -24$, or
(xvi) $t = 24$.

The theorem is a consequence of (3) and (2).

2. Problem 2

Now we state the proposition:

(5) Let us consider an integer x. Suppose $x - 3 \mid x^3 - 3$. Then
(i) $x = -21$, or
(ii) $x = -9$, or
(iii) $x = -5$, or
(iv) $x = -3$, or

(v) $x = -1$, or

(vi) $x = 0$, or

(vii) $x = 1$, or

(viii) $x = 2$, or

(ix) $x = 4$, or

(x) $x = 5$, or

(xi) $x = 6$, or

(xii) $x = 7$, or

(xiii) $x = 9$, or

(xiv) $x = 11$, or

(xv) $x = 15$, or

(xvi) $x = 27$.

The theorem is a consequence of (4).

3. Problem 3

Now we state the proposition:

(6) $\{n$, where n is a positive integer : $5 \mid 4 \cdot (n^2) + 1$ and $13 \mid 4 \cdot (n^2) + 1\}$ is infinite.

Proof: Set $S = \{n$, where n is a positive integer : $5 \mid 4 \cdot (n^2) + 1$ and $13 \mid 4 \cdot (n^2) + 1\}$. Define $\mathcal{F}(\text{natural number}) = 65 \cdot \$_1 + 56$. Consider f being a many sorted set indexed by $\mathbb{N}$ such that for every element n of $\mathbb{N}$, $f(n) = \mathcal{F}(n)$. Set $R = \operatorname{rng} f$. $R \subseteq S$. For every element m of $\mathbb{N}$, there exists an element n of $\mathbb{N}$ such that $n \geqslant m$ and $n \in R$. $\square$

4. Problem 4

Now we state the proposition:

(7) Let us consider a positive integer n. Then $169 \mid 3^{3 \cdot n+3} - 26 \cdot n - 27$.

Proof: Reconsider $k = n$ as a natural number. Define $\mathcal{P}[\text{natural number}] \equiv 169 \mid 3^{3 \cdot \$_1 + 3} - 26 \cdot \$_1 - 27$. For every natural number k such that $1 \leqslant k$ holds $\mathcal{P}[k]$. $\square$

5. Problem 5

Now we state the proposition:

(8) Let us consider a natural number k. Then $19 \mid 2^{2^{6 \cdot k+2}} + 3$.

6. Problem 6 (due to Kraitchik)

Now we state the proposition:

(9) $13 \mid 2^{70} + 3^{70}$.

7. Problem 7

Now we state the propositions:

(10) $11 \cdot 31 \cdot 61 \mid 20^{15} - 1$.

(11) Let us consider an integer a, and a natural number m. Then $a-1 \mid a^m-1$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv a - 1 \mid a^{\$_1} - 1$. For every natural number k, $\mathcal{P}[k]$. □

(12) Let us consider a natural number a, a positive integer m, and a finite 0-sequence f of $\mathbb{Z}$. Suppose $a > 1$ and $\operatorname{len} f = m - 1$ and for every natural number i such that $i \in \operatorname{dom} f$ holds $f(i) = a^{i+1} - 1$. Then $a^m - 1 \operatorname{div}(a - 1) = \sum f + m$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite 0-sequence f of $\mathbb{Z}$ such that $\operatorname{len} f = \$_1$ and for every natural number i such that $i \in \operatorname{dom} f$ holds $f(i) = a^{i+1} - 1$ holds $a^{\$_1+1} - 1 \operatorname{div}(a - 1) = \sum f + (\$_1 + 1)$. $\mathcal{P}[0]$. For every natural number k, $\mathcal{P}[k]$. □

8. Problem 8

Now we state the proposition:

(13) Let us consider a positive integer m, and a natural number a. Suppose $a > 1$. Then $\gcd(a^m - 1 \operatorname{div}(a - 1), a - 1) = \gcd(a - 1, m)$.
PROOF: Reconsider $m_0 = m$ as a natural number. Reconsider $m_1 = m_0 - 1$ as a natural number. Define $\mathcal{F}(\text{natural number}) = a^{\$_1+1} - 1$. Consider f being a finite 0-sequence such that $\operatorname{len} f = m_1$ and for every natural number i such that $i \in m_1$ holds $f(i) = \mathcal{F}(i)$ from [5, Sch.2]. $\operatorname{rng} f \subseteq \mathbb{Z}$. $a^m - 1 \operatorname{div}(a - 1) = \sum f + m$. □

9. Problem 9

Now we state the propositions:

(14) Let us consider finite 0-sequences s_1, s_2 of $\mathbb{N}$, and a natural number n. Suppose $\operatorname{len} s_1 = n+1$ and for every natural number i such that $i \in \operatorname{dom} s_1$ holds $s_1(i) = i^5$ and $\operatorname{len} s_2 = n+1$ and for every natural number i such that $i \in \operatorname{dom} s_2$ holds $s_2(i) = i^3$. Then $\sum s_2 \mid 3 \cdot (\sum s_1)$.

Proof: Define $\mathcal{F}(\text{natural number}) = {\$_1}^3$. Consider S_2 being a sequence of real numbers such that for every natural number i, $S_2(i) = \mathcal{F}(i)$. Define $\mathcal{G}(\text{natural number}) = {\$_1}^5$.

Consider S_1 being a sequence of real numbers such that for every natural number i, $S_1(i) = \mathcal{G}(i)$. $\square$

(15) Let us consider integers a, b, and a positive natural number m. Then $\sum\langle\binom{m}{0}a^0b^m, \ldots, \binom{m}{m}a^mb^0\rangle = a^m + b^m + \sum(\langle\binom{m}{0}a^0b^m, \ldots, \binom{m}{m}a^mb^0\rangle \restriction m)_{\downharpoonright 1}$.

(16) Let us consider natural numbers n, k. If n is odd, then $n \mid k^n + (n-k)^n$. The theorem is a consequence of (15).

10. Problem 10

Now we state the proposition:

(17) Let us consider a finite sequence s of elements of $\mathbb{N}$, and a natural number n. Suppose $n > 1$ and $\operatorname{len} s = n - 1$ and for every natural number i such that $i \in \operatorname{dom} s$ holds $s(i) = i^n$. If n is odd, then $n \mid \sum s$.

Proof: $\operatorname{rng}(s + \operatorname{Rev}(s)) \subseteq \mathbb{N}$. If n is odd, then $n \mid \sum s$ by [3, (3)]. $\square$

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On Fuzzy Negations Generated by Fuzzy Implications

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Summary. We continue in the Mizar system [2] the formalization of fuzzy implications according to the book of Baczyński and Jayaram "Fuzzy Implications" [1]. In this article we define fuzzy negations and show their connections with previously defined fuzzy implications [4] and [5] and triangular norms and conorms [6]. This can be seen as a step towards building a formal framework of fuzzy connectives [10]. We introduce formally Sugeno negation, boundary negations and show how these operators are pointwise ordered. This work is a continuation of the development of fuzzy sets [12], [3] in Mizar [7] started in [11] and partially described in [8]. This submission can be treated also as a part of a formal comparison of fuzzy and rough approaches to incomplete or uncertain information within the Mizar Mathematical Library [9].

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0. Introduction

The main aim of this Mizar article was to implement a formal counterpart of (the part of) Chapter 1.4, pp. 13–20 of Baczyński and Jayaram book "Fuzzy Implications" [1]. This is the fourth submission in the series formalizing this textbook, following [4], [5], and [6].

After filling some gaps – proving lemmas about monotone functions absent in the Mizar Mathematical Library, in Section 2 we recall the notion of conjugate

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fuzzy implications, and formally implement a method of generating a new fuzzy implication from a given one. We prove that I_f inherits corresponding properties of f, such as (NP) – the left neutrality property, (EP) – the exchange principle, (IP) – the identity principle, and (OP) – the ordering property, providing also registrations of clusters which guarantee the automatic handling of adjectives (their adjunction to the respective radix type), thus making a formalization work a bit easier.

Section 3, which is a fundamental part of this paper, contains elementary definitions needed to cope with fuzzy negations, and Sect. 4 provides a method of generating fuzzy negation from a given fuzzy implication. There are also concrete examples given in Section 5: the classical (standard) fuzzy complement N_{C} introduced at the beginning, two boundary (in the sense of the natural ordering of the functions) negations N_{D1} and N_{D2} (Def. 17 and 18, respectively). Section 6 shows which negations are generated from nine well-known fuzzy implications, so it can be treated as the formal counterpart of Table 1.7, p. 18 [1].

Fuzzy implication I	Fuzzy negation N_I
I_{LK}	N_{C}
I_{GD}	N_{D1}
I_{RC}	N_{C}
I_{KD}	N_{C}
I_{GG}	N_{D1}
I_{RS}	N_{D1}
I_{YG}	N_{D1}
I_{WB}	N_{D2}
I_{FD}	N_{C}

Section 7 is devoted to Sugeno negation (Def. 21), which can be used as a useful method of constructing examples of fuzzy negations (for example, substituting $\lambda = 0$ in the Sugeno negation, we obtain the standard fuzzy complementation). We conclude with some properties of conjugate fuzzy negations.

1. Preliminaries

Now we state the proposition:

(1) Let us consider real numbers x, r. If $0 \leqslant x \leqslant 1$ and $r > -1$, then $x \cdot r + 1 > 0$.

Let us consider a real number z. Now we state the propositions:

(2) If $z \in [0, 1]$ and $z \neq 0$, then there exists an element w of $[0, 1]$ such that $w < z$.

(3) If $z \in [0,1]$ and $z \neq 1$, then there exists an element w of $[0,1]$ such that $w > z$.

Note that there exists a unary operation on $[0,1]$ which is bijective and increasing and every unary operation on $[0,1]$ which is bijective and non-decreasing is also increasing and every unary operation on $[0,1]$ which is bijective and increasing is also non-decreasing. Let f be a bijective, increasing unary operation on $[0,1]$. One can check that f^{-1} is real-valued and function-like and $(f{\restriction}[0,1])^{-1}$ is real-valued. Now we state the propositions:

(4) Let us consider a one-to-one unary operation f on $[0,1]$, and an element d of $[0,1]$. If $d \in \operatorname{rng} f$, then $(f^{-1})(d) \in \operatorname{dom} f$.

(5) Let us consider a bijective, increasing unary operation f on $[0,1]$. Then f^{-1} is increasing.

Let f be a bijective, increasing unary operation on $[0,1]$. Let us note that f^{-1} is increasing. Let us consider a unary operation f on $[0,1]$. Now we state the propositions:

(6) f is non-decreasing if and only if for every elements a, b of $[0,1]$ such that $a \leqslant b$ holds $f(a) \leqslant f(b)$.

(7) f is non-increasing if and only if for every elements a, b of $[0,1]$ such that $a \leqslant b$ holds $f(a) \geqslant f(b)$.

(8) f is decreasing if and only if for every elements a, b of $[0,1]$ such that $a < b$ holds $f(a) > f(b)$.

(9) f is increasing if and only if for every elements a, b of $[0,1]$ such that $a < b$ holds $f(a) < f(b)$.

(10) Let us consider an increasing, bijective unary operation f on $[0,1]$. Then

(i) $f(0) = 0$, and

(ii) $f(1) = 1$.

Let f be a bijective, increasing unary operation on $[0,1]$. Observe that f^{-1} is bijective and increasing as a unary operation on $[0,1]$.

2. Conjugate Fuzzy Implications

The functor Φ yielding a set is defined by the term

(Def. 1) the set of all f where f is a bijective, increasing unary operation on $[0,1]$.

Let f be a binary operation on $[0,1]$ and φ be a bijective, increasing unary operation on $[0,1]$. The functor f_φ yielding a binary operation on $[0,1]$ is defined by

(Def. 2) for every elements x_1, x_2 of $[0,1]$, $it(x_1, x_2) = (\varphi^{-1})(f(\varphi(x_1), \varphi(x_2)))$.

Let f, g be binary operations on $[0,1]$. We say that f, g are conjugate if and only if

(Def. 3) there exists a bijective, increasing unary operation φ on $[0,1]$ such that $g = f_{\varphi}$.

Let I be a fuzzy implication and f be a bijective, non-decreasing unary operation on $[0,1]$. Let us note that I_f is antitone w.r.t. 1st coordinate, isotone w.r.t. 2nd coordinate, 00-dominant, 11-dominant, and 10-weak.

(11) Let us consider a fuzzy implication I, and a bijective, increasing unary operation f on $[0,1]$. Then I_f is a fuzzy implication.

Let us note that there exists a fuzzy implication which satisfies (NP), (OP), (EP), and (IP). Let us consider a fuzzy implication I and a bijective, increasing unary operation f on $[0,1]$. Now we state the propositions:

(12) If I satisfies (NP), then I_f satisfies (NP). The theorem is a consequence of (10).

(13) If I satisfies (EP), then I_f satisfies (EP).

(14) If I satisfies (IP), then I_f satisfies (IP). The theorem is a consequence of (10).

(15) If I satisfies (OP), then I_f satisfies (OP).
PROOF: Set $g = I_f$. If $g(x, y) = 1$, then $x \leqslant y$. $f(x) \leqslant f(y)$. $(f^{-1})(I(f(x), f(y))) = 1$. $\square$

Let I be fuzzy implication satisfying (NP) and f be a bijective, increasing unary operation on $[0,1]$. Let us observe that I_f satisfies (NP). Let I be fuzzy implication satisfying (EP). Observe that I_f satisfies (EP). Let I be fuzzy implication satisfying (IP). Let us note that I_f satisfies (IP). Let I be fuzzy implication satisfying (OP). Note that I_f satisfies (OP). Now we state the proposition:

(16) Let us consider a fuzzy implication I, and a bijective, increasing unary operation f on $[0,1]$. Then $I_f = f^{-1} \cdot I \cdot (f \times f)$.
PROOF: Set $g = I_f$. For every object x such that $x \in \operatorname{dom} g$ holds $g(x) = (f^{-1} \cdot I \cdot (f \times f))(x)$. $\square$

3. Fuzzy Negations

Let N be a unary operation on $[0,1]$. We say that N is satisfying (N1) if and only if

(Def. 4) $N(0) = 1$ and $N(1) = 0$.

We say that N is satisfying (N2) if and only if

(Def. 5) N is non-increasing.

The functor N_C yielding a unary operation on $[0,1]$ is defined by

(Def. 6) for every element x of $[0,1]$, $it(x) = 1 - x$.

Note that N_C is satisfying (N1) and satisfying (N2) and N_C is bijective and decreasing and there exists a unary operation on $[0,1]$ which is bijective and decreasing and there exists a unary operation on $[0,1]$ which is satisfying (N1) and satisfying (N2).

A fuzzy negation is a satisfying (N1), satisfying (N2) unary operation on $[0,1]$. Let f be a unary operation on $[0,1]$. We say that f is continuous if and only if

(Def. 7) there exists a function g from $\mathbb{I}$ into $\mathbb{I}$ such that $f = g$ and g is continuous.

Let N be a unary operation on $[0,1]$. We say that N is involutive if and only if

(Def. 8) for every element x of $[0,1]$, $N(N(x)) = x$.

We say that N is satisfying (N3) if and only if

(Def. 9) N is decreasing.

We say that N is satisfying (N4) if and only if

(Def. 10) N is continuous.

We say that N is satisfying (N5) if and only if

(Def. 11) N is involutive.

We say that N is strict if and only if

(Def. 12) N is satisfying (N3) and satisfying (N4).

We say that N is strong if and only if

(Def. 13) N is satisfying (N5).

We say that N is non-vanishing if and only if

(Def. 14) for every element x of $[0,1]$, $N(x) = 0$ iff $x = 1$.

We say that N is non-filling if and only if

(Def. 15) for every element x of $[0,1]$, $N(x) = 1$ iff $x = 0$.

4. Generating Fuzzy Negations from Fuzzy Implications

Now we state the proposition:

(17) Let us consider a decreasing, bijective unary operation f on $[0,1]$. Then

(i) $f(0) = 1$, and

(ii) $f(1) = 0$.

Let I be a binary operation on $[0,1]$. The functor N_I yielding a unary operation on $[0,1]$ is defined by

(Def. 16) for every element x of $[0,1]$, $it(x) = I(x,0)$.

Let I be binary operation on $[0,1]$ satisfying (I1), (I3), and (I5). Note that N_I is satisfying (N1) and satisfying (N2).

Now we state the proposition:

(18) Let us consider a fuzzy implication I. Then N_I is a fuzzy negation.

5. Boundary Fuzzy Negations

The functors: N_{D1} and N_{D2} yielding unary operations on $[0,1]$ are defined by conditions

(Def. 17) for every element x of $[0,1]$, if $x = 0$, then $N_{\mathrm{D1}}(x) = 1$ and if $x \neq 0$, then $N_{\mathrm{D1}}(x) = 0$,

(Def. 18) for every element x of $[0,1]$, if $x = 1$, then $N_{\mathrm{D2}}(x) = 0$ and if $x \neq 1$, then $N_{\mathrm{D2}}(x) = 1$,

respectively. Let f_1, f_2 be unary operations on $[0,1]$. We say that $f_1 \leqslant f_2$ if and only if

(Def. 19) for every element a of $[0,1]$, $f_1(a) \leqslant f_2(a)$.

Let us note that N_{D1} is satisfying (N1) and satisfying (N2) and N_{D2} is satisfying (N1) and satisfying (N2).

Now we state the proposition:

(19) Let us consider a fuzzy negation N. Then $N_{\mathrm{D1}} \leqslant N \leqslant N_{\mathrm{D2}}$.

6. Fuzzy Negations Generated by Nine Fuzzy Implications

Now we state the propositions:

(20) $N_{I_{\mathrm{LK}}} = N_C$.

Proof: Set $I = I_{\mathrm{LK}}$. Set $f = N_I$. Set $g = N_C$. For every element x of $[0,1]$, $f(x) = g(x)$. $\square$

(21) $N_{I_{\mathrm{GD}}} = N_{\mathrm{D1}}$.

(22) $N_{I_{\mathrm{RC}}} = N_C$.

(23) $N_{I_{\mathrm{KD}}} = N_C$.

Proof: Set $I = I_{\mathrm{KD}}$. Set $f = N_I$. Set $g = N_C$. For every element x of $[0,1]$, $f(x) = g(x)$. $\square$

(24) $N_{I_{\mathrm{GG}}} = N_{\mathrm{D1}}$.

(25) $N_{I_{\mathrm{RS}}} = N_{\mathrm{D1}}$.

(26) $N_{I_{\mathrm{YG}}} = N_{\mathrm{D1}}$.

(27) $N_{I_{\mathrm{WB}}} = N_{\mathrm{D2}}$.

(28) $N_{I_{\rm FD}} = N_C$.

Proof: Set $I = I_{\rm FD}$. Set $f = N_I$. Set $g = N_C$. For every element x of $[0,1]$, $f(x) = g(x)$. □

(29) Let us consider binary operation I on $[0,1]$ satisfying (EP) and (OP). Then N_I is a fuzzy negation.

(30) Let us consider binary operation I on $[0,1]$ satisfying (EP) and (OP), and an element x of $[0,1]$. Then $x \leqslant (N_I)((N_I)(x))$.

(31) Let us consider binary operation I on $[0,1]$ satisfying (EP) and (OP). Then $(N_I) \cdot (N_I) \cdot (N_I) = N_I$. The theorem is a consequence of (7) and (30).

7. Sugeno Negation

Let x, λ be real numbers. We say that λ is greater than x if and only if

(Def. 20) $\lambda > x$.

One can verify that there exists a real number which is greater than (-1).

Let λ be a real number. Assume $\lambda > -1$. The functor SugenoNegation λ yielding a unary operation on $[0,1]$ is defined by

(Def. 21) for every element x of $[0,1]$, $it(x) = \frac{1-x}{1+\lambda \cdot x}$.

Now we state the proposition:

(32) $N_C =$ SugenoNegation 0.

Let λ be a greater than (-1) real number. Note that SugenoNegation λ is satisfying (N1) and satisfying (N2).

8. Conjugate Fuzzy Negations

Let f be a unary operation on $[0,1]$ and φ be a bijective, increasing unary operation on $[0,1]$. The functor f_φ yielding a unary operation on $[0,1]$ is defined by

(Def. 22) for every element x of $[0,1]$, $it(x) = (\varphi^{-1})(f(\varphi(x)))$.

Now we state the proposition:

(33) Let us consider a fuzzy negation I, and a bijective, increasing unary operation f on $[0,1]$. Then $I_f = f^{-1} \cdot I \cdot f$.

Proof: Set $g = I_f$. For every object x such that $x \in \operatorname{dom} g$ holds $g(x) = (f^{-1} \cdot I \cdot f)(x)$. □

Let f, g be unary operations on $[0,1]$. We say that f, g are conjugate if and only if

(Def. 23) there exists a bijective, increasing unary operation φ on $[0,1]$ such that $g = f_\varphi$.

Let N be a fuzzy negation and φ be a bijective, increasing unary operation on $[0,1]$. One can check that N_φ is satisfying (N1) and satisfying (N2).

Now we state the proposition:

(34) Let us consider a fuzzy implication I, and a bijective, increasing unary operation φ on $[0,1]$. Then $(N_I)_\varphi = N_{I_\varphi}$. The theorem is a consequence of (10).

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