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Formaliz. Math. 28 (**2**)

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FORMALIZED MATHEMATICS
Vol. 28, No. 2, Pages 129–135, 2020
DOI: 10.2478/forma-2020-0012

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Renamings and a Condition-free Formalization of Kronecker's Construction

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Summary. In [7], [9], [10] we presented a formalization of Kronecker's construction of a field extension E for a field F in which a given polynomial $p \in F[X] \backslash F$ has a root [5], [6], [3]. A drawback of our formalization was that it works only for polynomial-disjoint fields, that is for fields F with $F \cap F[X] = \emptyset$. The main purpose of Kronecker's construction is that by induction one gets a field extension of F in which p splits into linear factors. For our formalization this means that the constructed field extension E again has to be polynomial-disjoint.

In this article, by means of Mizar system [2], [1], we first analyze whether our formalization can be extended that way. Using the field of polynomials over F with degree smaller than the degree of p to construct the field extension E does not work: In this case E is polynomial-disjoint if and only if p is linear. Using $F[X]/{<}p{>}$ one can show that for $F = \mathbb{Q}$ and $F = \mathbb{Z}_n$ the constructed field extension E is again polynomial-disjoint, so that in particular algebraic number fields can be handled.

For the general case we then introduce renamings of sets X as injective functions f with $\mathrm{dom}(f) = X$ and $\mathrm{rng}(f) \cap (X \cup Z) = \emptyset$ for an arbitrary set Z. This, finally, allows to construct a field extension E of an arbitrary field F in which a given polynomial $p \in F[X] \backslash F$ splits into linear factors. Note, however, that to prove the existence of renamings we had to rely on the axiom of choice.

MSC: 12E05 12F05 68V20

Keywords: roots of polynomials; field extensions; Kronecker's construction

MML identifier: FIELD_5, version: 8.1.10 5.63.1382

ISSN 1426–2630(Print), 1898-9934(Online)

1. Preliminaries

Now we state the proposition:

(1) Let us consider sets X, Y. If $Y \subseteq X$, then $X \setminus Y \cup Y = X$.

Let us consider natural numbers n, m. Now we state the propositions:

(2) (i) $n + m = n + m$, and

(ii) $n \cdot m = n \cdot m$.

(3) (i) $n \subseteq m$ iff $n \leqslant m$, and

(ii) $n \in m$ iff $n < m$.

Let us consider a natural number n. Now we state the propositions:

(4) $2^n = 2^n$.

(5) If $n \geqslant 3$, then $n + n < 2^n$.

(6) If $n \geqslant 3$, then $n + n \in 2^n$. The theorem is a consequence of (2), (5), (3), and (4).

(7) $\mathbb{N}$ meets $2^{\mathbb{N}}$.

Let us consider a set X. Now we state the propositions:

(8) There exists an object o such that $o \notin X$.

(9) There exists a set Y such that

(i) $\overline{\overline{X}} \subseteq \overline{\overline{Y}}$, and

(ii) $X \cap Y = \emptyset$.

(10) Let us consider sets X, Y. Suppose $\overline{\overline{X}} \subseteq \overline{\overline{Y}}$. Then there exists a set Z such that

(i) $Z \subseteq Y$, and

(ii) $\overline{\overline{Z}} = \overline{\overline{X}}$.

(11) Let us consider a set X. Then there exists a set Y such that

(i) $\overline{\overline{X}} = \overline{\overline{Y}}$, and

(ii) $X \cap Y = \emptyset$.

The theorem is a consequence of (9) and (10).

(12) Let us consider a field E. Then every subfield of E is a subring of E.

(13) Let us consider a field F, and a subring R of F. Then R is a subfield of F if and only if R is a field.

Let F be a field and E be an extension of F. Note that there exists an extension of F which is E-extending. We introduce the notation E is F-infinite as an antonym for E is F-finite. Let us consider a field F, an extension E of F, and an E-extending extension K of F.

(14) $\mathrm{VecSp}(E, F)$ is a subspace of $\mathrm{VecSp}(K, F)$.

(15) (i) K is F-infinite, or

(ii) E is F-finite and $\deg(E, F) \leqslant \deg(K, F)$.

The theorem is a consequence of (14).

(16) Let us consider a field F, a polynomial p over F, and a non zero polynomial q over F. Then $\deg(p \bmod q) < \deg q$.

2. Linear Polynomials

Let R be a ring and p be a polynomial over R. We say that p is linear if and only if

(Def. 1) $\deg p = 1$.

Let R be a non degenerated ring. One can check that there exists a polynomial over R which is linear and there exists a polynomial over R which is non linear and there exists an element of the carrier of $\mathrm{PolyRing}(R)$ which is linear and there exists an element of the carrier of $\mathrm{PolyRing}(R)$ which is non linear and every polynomial over R which is zero is also non linear and every polynomial over R which is constant is also non linear.

Let F be a field. Let us note that every polynomial over F which is linear has also roots and every element of the carrier of $\mathrm{PolyRing}(F)$ which is linear is also irreducible and every element of the carrier of $\mathrm{PolyRing}(F)$ which is non linear and has roots is also reducible.

Let R be an integral domain, p be a linear polynomial over R, and q be a non constant polynomial over R. Let us note that $p * q$ is non linear.

Let F be a field, p be a linear polynomial over F, and q be a non constant polynomial over F. Let us note that $p * q$ has roots.

3. More on $\mathrm{PolyRing}(p)$

Let F be a field and p be a non constant element of the carrier of $\mathrm{PolyRing}(F)$. The functor $\mathrm{canHomP}(p)$ yielding a function from F into $\mathrm{PolyRing}(p)$ is defined by

(Def. 2) for every element a of F, $it(a) = a \restriction F$.

One can verify that $\mathrm{canHomP}(p)$ is additive, multiplicative, unity-preserving, and one-to-one and $\mathrm{PolyRing}(p)$ is F-homomorphic and F-monomorphic.

Let F be a polynomial-disjoint field and p be an irreducible element of the carrier of $\mathrm{PolyRing}(F)$. One can verify that $\mathrm{embField}(\mathrm{canHomP}(p))$ is add-associative, right complementable, associative, distributive, and almost left invertible and $\mathrm{embField}(\mathrm{canHomP}(p))$ is F-extending.

The functor KrRootP(p) yielding an element of embField(canHomP(p)) is defined by the term

(Def. 3) $((\text{emb-iso}(\text{canHomP}(p)))^{-1} \cdot ((\text{KroneckerIso}(p))^{-1}))(\text{KrRoot}(p))$.

Now we state the proposition:

(17) Let us consider a polynomial-disjoint field F, and an irreducible element p of the carrier of PolyRing(F). Then ExtEval(p, KrRootP(p)) $= 0_F$.
PROOF: Set K = KroneckerField(F, p). Set E = embField(canHomP(p)). Set h = (KroneckerIso(p)) $\cdot$ (emb-iso(canHomP(p))). Reconsider $P = K$ as an E-isomorphic field. Reconsider $i_1 = h$ as an isomorphism between E and P. Reconsider $i_2 = {i_1}^{-1}$ as a homomorphism from P to E. Reconsider $t = p_p$ as an element of the carrier of PolyRing(P). (PolyHom(i_2))(t) $= p$ by [4, (12)], [8, (17)]. □

4. On Embedding F into $F[X]/\langle p\rangle$ and PolyRing(p)

Now we state the propositions:

(18) Let us consider a field F, and a linear element p of the carrier of PolyRing(F). Then

(i) PolyRing(p) and F are isomorphic, and

(ii) the carrier of embField(canHomP(p)) = the carrier of F.

(19) Let us consider a strict field F, and a linear element p of the carrier of PolyRing(F). Then embField(canHomP(p)) $= F$. The theorem is a consequence of (18).

(20) Let us consider a field F, and a linear element p of the carrier of PolyRing(F). Then

(i) $\frac{\text{PolyRing}(F)}{\{p\}\text{–ideal}}$ and F are isomorphic, and

(ii) the carrier of embField(embedding(p)) = the carrier of F.

The theorem is a consequence of (18) and (16).

(21) Let us consider a strict field F, and a linear element p of the carrier of PolyRing(F). Then embField(embedding(p)) $= F$. The theorem is a consequence of (20).

(22) Let us consider a polynomial-disjoint field F, and an irreducible element p of the carrier of PolyRing(F). Then embField(canHomP(p)) is polynomial-disjoint if and only if p is linear. The theorem is a consequence of (18).

(23) Let us consider a field F, an irreducible element p of the carrier of PolyRing(F), and a polynomial-disjoint field E.
Suppose $E = \mathrm{embField}(\mathrm{embedding}(p))$. Then F is polynomial-disjoint.

Let n be a prime number and p be an irreducible element of the carrier of PolyRing($\mathbb{Z}/n$). Let us observe that embField(embedding(p)) is add-associative, right complementable, associative, distributive, and almost left invertible.

Let p be an irreducible element of the carrier of PolyRing($\mathbb{F}_{\mathbb{Q}}$). Let us note that embField(embedding(p)) is add-associative, right complementable, associative, distributive, and almost left invertible.

(24) Let us consider a prime number n, and a non constant element p of the carrier of PolyRing($\mathbb{Z}/n$). Then $\mathbb{Z}/n$ and $\frac{\mathrm{PolyRing}(\mathbb{Z}/n)}{\{p\}\text{–ideal}}$ are disjoint.

(25) Let us consider a non constant element p of the carrier of PolyRing($\mathbb{F}_{\mathbb{Q}}$). Then $\mathbb{F}_{\mathbb{Q}}$ and $\frac{\mathrm{PolyRing}(\mathbb{F}_{\mathbb{Q}})}{\{p\}\text{–ideal}}$ are disjoint.

Let n be a prime number and p be an irreducible element of the carrier of PolyRing($\mathbb{Z}/n$). Let us note that embField(embedding(p)) is polynomial-disjoint.

Let p be an irreducible element of the carrier of PolyRing($\mathbb{F}_{\mathbb{Q}}$). One can check that embField(embedding(p)) is polynomial-disjoint.

Let R be a ring. We say that R is strong polynomial disjoint if and only if

(Def. 4) for every element a of R and for every ring S and for every element p of the carrier of PolyRing(S), $a \neq p$.

Observe that $\mathbb{Z}^{\mathrm{R}}$ is strong polynomial disjoint and $\mathbb{F}_{\mathbb{Q}}$ is strong polynomial disjoint and $\mathbb{R}_{\mathrm{F}}$ is strong polynomial disjoint.

Let n be a non trivial natural number. Note that $\mathbb{Z}/n$ is strong polynomial disjoint and every ring which is strong polynomial disjoint is also polynomial-disjoint and there exists a field which is strong polynomial disjoint and there exists a field which is non strong polynomial disjoint.

(26) Let us consider a strong polynomial disjoint field F, an irreducible element p of the carrier of PolyRing(F), and a field E.
Suppose $E = \mathrm{embField}(\mathrm{embedding}(p))$. Then E is strong polynomial disjoint.

5. Renamings

Let X be a non empty set and Z be a set.

A Renaming of X and Z is a function defined by

(Def. 5) $\mathrm{dom}\, it = X$ and it is one-to-one and $\mathrm{rng}\, it \cap (X \cup Z) = \emptyset$.

Let r be a Renaming of X and Z. Let us note that $\mathrm{dom}\, r$ is non empty and $\mathrm{rng}\, r$ is non empty and every Renaming of X and Z is X-defined and one-to-one.

Let r be a Renaming of X and Z. Observe that the functor r^{-1} yields a function from $\operatorname{rng} r$ into X. Now we state the proposition:

(27) Let us consider a non empty set X, a set Z, and a Renaming r of X and Z. Then r^{-1} is onto.

Let F be a field, Z be a set, and r be a Renaming of the carrier of F and Z. The functor ren-add(r) yielding a binary operation on $\operatorname{rng} r$ is defined by

(Def. 6) for every elements a, b of $\operatorname{rng} r$, $it(a,b) = r((r^{-1})(a) + (r^{-1})(b))$.

The functor ren-mult(r) yielding a binary operation on $\operatorname{rng} r$ is defined by

(Def. 7) for every elements a, b of $\operatorname{rng} r$, $it(a,b) = r((r^{-1})(a) \cdot (r^{-1})(b))$.

The functor RenField(r) yielding a strict double loop structure is defined by

(Def. 8) the carrier of $it = \operatorname{rng} r$ and the addition of $it =$ ren-add(r) and the multiplication of $it =$ ren-mult(r) and the one of $it = r(1_F)$ and the zero of $it = r(0_F)$.

One can check that RenField(r) is non degenerated and RenField(r) is Abelian, add-associative, right zeroed, and right complementable and RenField(r) is commutative, associative, well unital, distributive, and almost left invertible.

One can check that the functor r^{-1} yields a function from RenField(r) into F. Now we state the propositions:

(28) Let us consider a field F, a set Z, and a Renaming r of the carrier of F and Z. Then r^{-1} is additive, multiplicative, unity-preserving, one-to-one, and onto. The theorem is a consequence of (27).

(29) Let us consider a field F, and a set Z. Then there exists a field E such that

(i) E and F are isomorphic, and

(ii) (the carrier of E) $\cap$ ((the carrier of F) $\cup$ Z) $= \emptyset$.

The theorem is a consequence of (28).

6. Kronecker's Construction

Let us consider a field F and a non constant element f of the carrier of PolyRing(F). Now we state the propositions:

(30) There exists an extension E of F such that f has a root in E.

(31) There exists an extension E of F such that f splits in E.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every field F for every non constant element f of the carrier of PolyRing(F) such that $\deg f = \$_1$ there exists an extension E of F such that f splits in E. $\mathcal{P}[1]$. For every non zero natural number k, $\mathcal{P}[k]$. Consider n being a natural number such that $\deg f = n$. $\square$

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Accepted May 19, 2020

FORMALIZED MATHEMATICS
Vol. 28, No. 2, Pages 137–154, 2020
DOI: 10.2478/forma-2020-0013

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Refined Finiteness and Degree Properties in Graphs

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Summary. In this article the finiteness of graphs is refined and the minimal and maximal degree of graphs are formalized in the Mizar system [3], based on the formalization of graphs in [4].

MSC: 68V20 05C07

Keywords: graph theory; vertex degree; maximum degree; minimum degree

MML identifier: `GLIB_013`, version: `8.1.10 5.63.1382`

0. Introduction

The first section introduces the attributes `vertex-finite` and `edge-finite`, which are a refinement of [4]'s `finite`. A notable result is the upper bound of the size of certain graphs in terms of their order, e.g. that a simple finite graph with order n and size m satisfies $m \leqslant \binom{n}{2}$.

Parametrized attributes for the order and size of a graph are introduced in the following section. The main purpose of this additional notation (e.g. `G is n-vertex` instead of `G.order() = n`) is to be used in clusterings and reservations in the future for easy access, e.g. `reserve K2 for simple complete 2-vertex _Graph`.

The third section formalizes locally finite graphs, which are well known (cf. [2], [5], [1]).

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The minimal and maximal degree of a graph are usually defined, together with the degree of a vertex, right at the beginning of general graph theory textbooks, often followed by the Handshaking lemma (cf. [1], [2], [7], [6]). While the Handshaking lemma is still not proven in this article, the last section introduces the minimal and supremal degree of a graph, the latter being called the maximal degree if a vertex attaining the supremal degree exists. This doesn't always have to be the case, of course: Take for example the sum of all complete graphs $\sum_{n=1}^{\infty} K_n$. Therefore the property of a graph having a maximal degree is formalized, too. All formalizations are done as well for in/out degrees and the relationship between them and the undirected degrees is taken into account.

1. Upper Size of Graphs without Parallel Edges

Let us consider a non-directed-multi graph G. Now we state the propositions:

(1) There exists a one-to-one function f such that

  (i) $\operatorname{dom} f =$ the edges of G, and

  (ii) $\operatorname{rng} f \subseteq$ (the vertices of G) $\times$ (the vertices of G), and

  (iii) for every object e such that $e \in \operatorname{dom} f$ holds $f(e) = \langle$(the source of $G)(e)$, (the target of $G)(e)\rangle$.

(2) G.size() $\subseteq G$.order() $\cdot$ G.order(). The theorem is a consequence of (1).

(3) Let us consider a directed-simple graph G. Then there exists a one-to-one function f such that

  (i) $\operatorname{dom} f =$ the edges of G, and

  (ii) $\operatorname{rng} f \subseteq$ ((the vertices of G) $\times$ (the vertices of G)) $\setminus$ (id_α), and

  (iii) for every object e such that $e \in \operatorname{dom} f$ holds $f(e) = \langle$(the source of $G)(e)$, (the target of $G)(e)\rangle$,

where α is the vertices of G. The theorem is a consequence of (1).

(4) Let us consider a non-multi graph G. Then there exists a one-to-one function f such that

  (i) $\operatorname{dom} f =$ the edges of G, and

  (ii) $\operatorname{rng} f \subseteq$ 2Set(the vertices of G) $\cup\, S_\alpha$, and

  (iii) for every object e such that $e \in \operatorname{dom} f$ holds $f(e) = \{$(the source of $G)(e),$ (the target of $G)(e)\}$,

where α is the vertices of G.

(5) Let us consider a simple graph G. Then there exists a one-to-one function f such that

(i) $\operatorname{dom} f =$ the edges of G, and

(ii) $\operatorname{rng} f \subseteq$ 2Set(the vertices of G), and

(iii) for every object e such that $e \in \operatorname{dom} f$ holds $f(e) = \{$(the source of $G)(e)$, (the target of $G)(e)\}$.

Proof: Consider f being a one-to-one function such that $\operatorname{dom} f =$ the edges of G and $\operatorname{rng} f \subseteq$ 2Set(the vertices of $G) \cup S_\alpha$, where α is the vertices of G and for every object e such that $e \in \operatorname{dom} f$ holds $f(e) = \{$(the source of $G)(e)$, (the target of $G)(e)\}$. $\operatorname{rng} f \cap S_\alpha = \emptyset$, where α is the vertices of G. □

2. Vertex- and Edge-finite Graphs

Let G be a graph. We say that G is vertex-finite if and only if

(Def. 1) the vertices of G is finite.

We say that G is edge-finite if and only if

(Def. 2) the edges of G is finite.

Let us consider a graph G. Now we state the propositions:

(6) G is vertex-finite if and only if G.order() is finite.

(7) G is edge-finite if and only if G.size() is finite.

(8) Let us consider graphs G_1, G_2. Suppose $G_1 \approx G_2$. Then

(i) if G_1 is vertex-finite, then G_2 is vertex-finite, and

(ii) if G_1 is edge-finite, then G_2 is edge-finite.

Let V be a non empty, finite set, E be a set, and S, T be functions from E into V. Observe that createGraph(V, E, S, T) is vertex-finite.

Let V be an infinite set. Let us observe that createGraph(V, E, S, T) is non vertex-finite.

Let V be a non empty set and E be a finite set. Let us observe that createGraph(V, E, S, T) is edge-finite.

Let E be an infinite set. One can verify that createGraph(V, E, S, T) is non edge-finite and every graph which is finite is also vertex-finite and edge-finite and every graph which is vertex-finite and edge-finite is also finite and every graph which is edgeless is also edge-finite and every graph which is trivial is also vertex-finite and every graph which is vertex-finite and non-directed-multi is also edge-finite and every graph which is non vertex-finite and loopfull is also non edge-finite and there exists a graph which is vertex-finite, edge-finite, and simple and there exists a graph which is vertex-finite and non edge-finite and there exists a graph which is non vertex-finite and edge-finite and there exists a graph which is non vertex-finite and non edge-finite.

Let G be a vertex-finite graph. Let us observe that G.order() is non zero and natural.

Let us observe that the functor G.order() yields a non zero natural number. Let G be an edge-finite graph. Let us note that G.size() is natural.

Now we state the propositions:

(9) Let us consider a vertex-finite, non-directed-multi graph G. Then $G.\mathrm{size}() \leqslant (G.\mathrm{order}())^{\mathbf{2}}$. The theorem is a consequence of (2).

(10) Let us consider a vertex-finite, directed-simple graph G. Then $G.\mathrm{size}() \leqslant (G.\mathrm{order}())^{\mathbf{2}} - G.\mathrm{order}()$. The theorem is a consequence of (3).

(11) Let us consider a vertex-finite, non-multi graph G. Then $G.\mathrm{size}() \leqslant \frac{(G.\mathrm{order}())^{\mathbf{2}}+G.\mathrm{order}()}{2}$. The theorem is a consequence of (4).

(12) Let us consider a vertex-finite, simple graph G. Then $G.\mathrm{size}() \leqslant \frac{(G.\mathrm{order}())^{\mathbf{2}}-G.\mathrm{order}()}{2}$. The theorem is a consequence of (5).

Let G be a vertex-finite graph. One can verify that the vertices of G is finite and every subgraph of G is vertex-finite and every directed graph complement of G with loops is vertex-finite and edge-finite and every undirected graph complement of G with loops is vertex-finite and edge-finite and every directed graph complement of G is vertex-finite and edge-finite and every graph complement of G is vertex-finite and edge-finite.

Let V be a finite set. One can check that every supergraph of G extended by the vertices from V is vertex-finite.

Let v be an object. One can check that every supergraph of G extended by v is vertex-finite.

Let e, w be objects. Note that every supergraph of G extended by e between vertices v and w is vertex-finite and every supergraph of G extended by v, w and e between them is vertex-finite.

Let E be a set. One can check that every graph given by reversing directions of the edges E of G is vertex-finite.

Let v be an object and V be a set. Note that every supergraph of G extended by vertex v and edges between v and V of G is vertex-finite and every graph by adding a loop to each vertex of G in V is vertex-finite.

Let G be a graph and V be an infinite set. One can verify that every supergraph of G extended by the vertices from V is non vertex-finite.

Let G be a non vertex-finite graph. Observe that the vertices of G is infinite and every supergraph of G is non vertex-finite and every subgraph of G which is spanning is also non vertex-finite and every directed graph complement of G with loops is non vertex-finite and every undirected graph complement of G

with loops is non vertex-finite and every directed graph complement of G is non vertex-finite and every graph complement of G is non vertex-finite.

Let E be a set. Let us note that every subgraph of G induced by V and E is non vertex-finite.

Let V be an infinite subset of the vertices of G. Note that every graph by adding a loop to each vertex of G in V is non edge-finite.

Let G be an edge-finite graph. One can check that the edges of G is finite and every subgraph of G is edge-finite.

Let V be a set. Note that every supergraph of G extended by the vertices from V is edge-finite.

Let E be a set. Note that every graph given by reversing directions of the edges E of G is edge-finite.

Let v be an object. Note that every supergraph of G extended by v is edge-finite.

Let e, w be objects. Let us note that every supergraph of G extended by e between vertices v and w is edge-finite and every supergraph of G extended by v, w and e between them is edge-finite.

Let V be a finite set. Note that every supergraph of G extended by vertex v and edges between v and V of G is edge-finite.

Let V be a finite subset of the vertices of G. Observe that every graph by adding a loop to each vertex of G in V is edge-finite.

Let G be a non vertex-finite, edge-finite graph. Let us observe that there exists a vertex of G which is isolated and every directed graph complement of G with loops is non edge-finite and every undirected graph complement of G with loops is non edge-finite and every directed graph complement of G is non edge-finite and every graph complement of G is non edge-finite.

Let G be a non edge-finite graph. One can verify that the edges of G is infinite and every supergraph of G is non edge-finite.

Let V be a set and E be an infinite subset of the edges of G. Let us observe that every subgraph of G induced by V and E is non edge-finite.

Let E be a finite set. One can verify that every subgraph of G with edges E removed is non edge-finite.

Let e be a set. Let us observe that every subgraph of G with edge e removed is non edge-finite.

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(13) Suppose F is weak subgraph embedding. Then

   (i) if G_2 is vertex-finite, then G_1 is vertex-finite, and

   (ii) if G_2 is edge-finite, then G_1 is edge-finite.

(14) If F is onto, then if G_1 is vertex-finite, then G_2 is vertex-finite and if G_1 is edge-finite, then G_2 is edge-finite.

(15) If F is isomorphism, then (G_1 is vertex-finite iff G_2 is vertex-finite) and (G_1 is edge-finite iff G_2 is edge-finite).

3. Order and Size of a Graph as Attributes

Let c be a cardinal number and G be a graph. We say that G is c-vertex if and only if

(Def. 3) G.order() $= c$.

We say that G is c-edge if and only if

(Def. 4) G.size() $= c$.

Let us consider a graph G. Now we state the propositions:

(16) G is vertex-finite if and only if there exists a non zero natural number n such that G is n-vertex.

(17) G is edge-finite if and only if there exists a natural number n such that G is n-edge.

Let us consider graphs G_1, G_2 and a cardinal number c. Now we state the propositions:

(18) Suppose the vertices of $G_1 =$ the vertices of G_2. Then if G_1 is c-vertex, then G_2 is c-vertex.

(19) Suppose the edges of $G_1 =$ the edges of G_2. Then if G_1 is c-edge, then G_2 is c-edge.

(20) If $G_1 \approx G_2$, then if G_1 is c-vertex, then G_2 is c-vertex and if G_1 is c-edge, then G_2 is c-edge.

(21) Every graph G is (G.order())-vertex and (G.size())-edge.

Let V be a non empty set, E be a set, and S, T be functions from E into V. Let us observe that createGraph(V, E, S, T) is $\overline{\overline{V}}$-vertex and $\overline{\overline{E}}$-edge.

Let a be a non zero cardinal number and b be a cardinal number. One can verify that there exists a graph which is a-vertex and b-edge.

Let c be a cardinal number. Let us observe that there exists a graph which is trivial and c-edge and every graph is non 0-vertex and every graph which is trivial is also 1-vertex and every graph which is 1-vertex is also trivial.

Let n be a non zero natural number. One can verify that every graph which is n-vertex is also vertex-finite.

Let c be a non zero cardinal number and G be a c-vertex graph. Observe that every subgraph of G which is spanning is also c-vertex and every directed graph complement of G with loops is c-vertex and every undirected graph complement of G with loops is c-vertex and every directed graph complement of G is c-vertex and every graph complement of G is c-vertex.

Let E be a set. One can verify that every graph given by reversing directions of the edges E of G is c-vertex.

Let V be a set. Let us note that every graph by adding a loop to each vertex of G in V is c-vertex.

Let v, e, w be objects. Observe that every supergraph of G extended by e between vertices v and w is c-vertex and every graph which is edgeless is also 0-edge and every graph which is 0-edge is also edgeless.

Let n be a natural number. Note that every graph which is n-edge is also edge-finite.

Let c be a cardinal number, G be a c-edge graph, and E be a set. Note that every graph given by reversing directions of the edges E of G is c-edge.

Let V be a set. Let us observe that every supergraph of G extended by the vertices from V is c-edge.

Now we state the proposition:

(22) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and a cardinal number c. Suppose F is isomorphism. Then

   (i) G_1 is c-vertex iff G_2 is c-vertex, and

   (ii) G_1 is c-edge iff G_2 is c-edge.

4. Locally Finite Graphs

Let G be a graph. We say that G is locally-finite if and only if

(Def. 5) for every vertex v of G, v.edgesInOut() is finite.

Now we state the propositions:

(23) Let us consider a graph G. Then G is locally-finite if and only if for every vertex v of G, v.degree() is finite.

(24) Let us consider graphs G_1, G_2. Suppose $G_1 \approx G_2$. If G_1 is locally-finite, then G_2 is locally-finite.

Let us consider a graph G. Now we state the propositions:

(25) G is locally-finite if and only if for every vertex v of G, v.edgesIn() is finite and v.edgesOut() is finite.

(26) G is locally-finite if and only if for every vertex v of G, v.inDegree() is finite and v.outDegree() is finite. The theorem is a consequence of (23).

Let us consider a non empty set V, a set E, and functions S, T from E into V. Now we state the propositions:

(27) Suppose for every element v of V, $S^{-1}(\{v\})$ is finite and $T^{-1}(\{v\})$ is finite. Then createGraph(V, E, S, T) is locally-finite. The theorem is a consequence of (25).

(28) Suppose there exists an element v of V such that $S^{-1}(\{v\})$ is infinite or $T^{-1}(\{v\})$ is infinite. Then createGraph(V, E, S, T) is not locally-finite. The theorem is a consequence of (25).

Let G be a non vertex-finite graph and V be an infinite subset of the vertices of G. One can verify that every supergraph of G extended by vertex the vertices of G and edges between the vertices of G and V of G is non locally-finite and every graph which is edge-finite is also locally-finite and there exists a graph which is locally-finite and there exists a graph which is non locally-finite.

Let G be a locally-finite graph. Note that every subgraph of G is locally-finite.

Let X be a finite set. One can check that G.edgesInto(X) is finite and G.edgesOutOf(X) is finite and G.edgesInOut(X) is finite and G.edgesBetween(X) is finite.

Let Y be a finite set. Note that G.edgesBetween(X, Y) is finite and
G.edgesDBetween(X, Y) is finite.

Let v be a vertex of G. One can verify that v.edgesIn() is finite and
v.edgesOut() is finite and v.edgesInOut() is finite and v.inDegree() is finite and v.outDegree() is finite and v.degree() is finite.

The functors: v.inDegree(), v.outDegree(), and v.degree() yield natural numbers. Let V be a set. Let us observe that every supergraph of G extended by the vertices from V is locally-finite and every graph by adding a loop to each vertex of G in V is locally-finite.

Let E be a set. Let us observe that every graph given by reversing directions of the edges E of G is locally-finite.

Let v, e, w be objects. Let us note that every supergraph of G extended by e between vertices v and w is locally-finite and every supergraph of G extended by v, w and e between them is locally-finite.

Now we state the proposition:

(29) Let us consider a graph G_2, an object v, a subset V of the vertices of G_2, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Suppose $v \notin$ the vertices of G_2. Then G_2 is locally-finite and V is finite if and only if G_1 is locally-finite. The theorem is a consequence of (23).

Let G be a locally-finite graph, v be an object, and V be a finite set. Let us note that every supergraph of G extended by vertex v and edges between v and V of G is locally-finite.

Let G be a non locally-finite graph. Let us observe that every supergraph of G is non locally-finite.

Let E be a finite set. Let us note that every subgraph of G with edges E removed is non locally-finite.

Let e be a set. Let us observe that every subgraph of G with edge e removed is non locally-finite.

Now we state the propositions:

(30) Let us consider a non locally-finite graph G_1, a finite subset V of the vertices of G_1, and a subgraph G_2 of G_1 with vertices V removed. Suppose for every vertex v of G_1 such that $v \in V$ holds v.edgesInOut() is finite. Then G_2 is not locally-finite. The theorem is a consequence of (24).

(31) Let us consider a non locally-finite graph G_1, a vertex v of G_1, and a subgraph G_2 of G_1 with vertex v removed. If v.edgesInOut() is finite, then G_2 is not locally-finite. The theorem is a consequence of (30).

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(32) If F is weak subgraph embedding and G_2 is locally-finite, then G_1 is locally-finite. The theorem is a consequence of (23).

(33) If F is onto and semi-directed-continuous and G_1 is locally-finite, then G_2 is locally-finite. The theorem is a consequence of (23).

(34) If F is isomorphism, then G_1 is locally-finite iff G_2 is locally-finite. The theorem is a consequence of (23) and (32).

5. Degree Properties in Graphs

Let G be a graph. The functors: $\bar{\Delta}(G)$, $\bar{\Delta}^-(G)$, $\bar{\Delta}^+(G)$, $\delta(G)$, $\delta^-(G)$, and $\delta^+(G)$ yielding cardinal numbers are defined by terms

(Def. 6) $\bigcup$ the set of all v.degree() where v is a vertex of G.

(Def. 7) $\bigcup$ the set of all v.inDegree() where v is a vertex of G,

(Def. 8) $\bigcup$ the set of all v.outDegree() where v is a vertex of G,

(Def. 9) $\bigcap$ the set of all v.degree() where v is a vertex of G,

(Def. 10) $\bigcap$ the set of all v.inDegree() where v is a vertex of G,

(Def. 11) $\bigcap$ the set of all v.outDegree() where v is a vertex of G,

respectively. Now we state the proposition:

(35) Let us consider a graph G, and a vertex v of G. Then

(i) $\delta(G) \subseteq v$.degree() $\subseteq \bar{\Delta}(G)$, and

(ii) $\delta^-(G) \subseteq v$.inDegree() $\subseteq \bar{\Delta}^-(G)$, and

(iii) $\delta^+(G) \subseteq v$.outDegree() $\subseteq \bar{\Delta}^+(G)$.

Let us consider a graph G and a cardinal number c. Now we state the propositions:

(36) $\delta(G) = c$ if and only if there exists a vertex v of G such that v.degree() $=$ c and for every vertex w of G, v.degree() $\subseteq w$.degree().

(37) $\delta^-(G) = c$ if and only if there exists a vertex v of G such that v.inDegree() $= c$ and for every vertex w of G, v.inDegree() $\subseteq w$.inDegree().

(38) $\delta^+(G) = c$ if and only if there exists a vertex v of G such that v.outDegree() $= c$ and for every vertex w of G, v.outDegree() $\subseteq w$.outDegree().

Let us consider a graph G. Now we state the propositions:

(39) $\bar{\Delta}^-(G) \subseteq \bar{\Delta}(G)$.

(40) $\bar{\Delta}^+(G) \subseteq \bar{\Delta}(G)$.

(41) $\delta^-(G) \subseteq \delta(G)$. The theorem is a consequence of (37) and (36).

(42) $\delta^+(G) \subseteq \delta(G)$. The theorem is a consequence of (38) and (36).

(43) $\delta(G) \subseteq \bar{\Delta}(G)$.

(44) $\delta^-(G) \subseteq \bar{\Delta}^-(G)$.

(45) $\delta^+(G) \subseteq \bar{\Delta}^+(G)$.

(46) If there exists a vertex v of G such that v is isolated, then $\delta(G) = 0$ and $\delta^-(G) = 0$ and $\delta^+(G) = 0$. The theorem is a consequence of (36), (37), and (38).

(47) If $\delta(G) = 0$, then there exists a vertex v of G such that v is isolated. The theorem is a consequence of (36).

Let us consider a graph G and a cardinal number c. Now we state the propositions:

(48) If there exists a vertex v of G such that v.degree() $= c$ and for every vertex w of G, w.degree() $\subseteq v$.degree(), then $\bar{\Delta}(G) = c$.

(49) If there exists a vertex v of G such that v.inDegree() $= c$ and for every vertex w of G, w.inDegree() $\subseteq v$.inDegree(), then $\bar{\Delta}^-(G) = c$.

(50) If there exists a vertex v of G such that v.outDegree() $= c$ and for every vertex w of G, w.outDegree() $\subseteq v$.outDegree(), then $\bar{\Delta}^+(G) = c$.

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(51) If F is weak subgraph embedding, then $\bar{\Delta}(G_1) \subseteq \bar{\Delta}(G_2)$.

(52) If F is weak subgraph embedding and rng $F_{\mathbb{V}} =$ the vertices of G_2, then $\delta(G_1) \subseteq \delta(G_2)$. The theorem is a consequence of (36).

(53) If F is onto and semi-directed-continuous, then $\bar{\Delta}(G_2) \subseteq \bar{\Delta}(G_1)$.

(54) Suppose F is onto and semi-directed-continuous and $\mathrm{dom}(F_{\mathbb{V}}) =$ the vertices of G_1. Then $\delta(G_2) \subseteq \delta(G_1)$. The theorem is a consequence of (36).

(55) If F is isomorphism, then $\bar{\Delta}(G_1) = \bar{\Delta}(G_2)$ and $\delta(G_1) = \delta(G_2)$. The theorem is a consequence of (51) and (52).

(56) If F is directed and weak subgraph embedding, then $\bar{\Delta}^-(G_1) \subseteq \bar{\Delta}^-(G_2)$ and $\bar{\Delta}^+(G_1) \subseteq \bar{\Delta}^+(G_2)$.

(57) Suppose F is directed and weak subgraph embedding and $\mathrm{rng}\, F_{\mathbb{V}} =$ the vertices of G_2. Then

(i) $\delta^-(G_1) \subseteq \delta^-(G_2)$, and

(ii) $\delta^+(G_1) \subseteq \delta^+(G_2)$.

The theorem is a consequence of (37) and (38).

(58) If F is onto and semi-directed-continuous, then $\bar{\Delta}^-(G_2) \subseteq \bar{\Delta}^-(G_1)$ and $\bar{\Delta}^+(G_2) \subseteq \bar{\Delta}^+(G_1)$.

(59) Suppose F is onto and semi-directed-continuous and $\mathrm{dom}(F_{\mathbb{V}}) =$ the vertices of G_1. Then

(i) $\delta^-(G_2) \subseteq \delta^-(G_1)$, and

(ii) $\delta^+(G_2) \subseteq \delta^+(G_1)$.

The theorem is a consequence of (37) and (38).

(60) Suppose F is directed-isomorphism. Then

(i) $\bar{\Delta}^-(G_1) = \bar{\Delta}^-(G_2)$, and

(ii) $\bar{\Delta}^+(G_1) = \bar{\Delta}^+(G_2)$, and

(iii) $\delta^-(G_1) = \delta^-(G_2)$, and

(iv) $\delta^+(G_1) = \delta^+(G_2)$.

The theorem is a consequence of (56), (57), (58), and (59).

(61) Let us consider a graph G_1, a set E, and a graph G_2 given by reversing directions of the edges E of G_1. Then

(i) $\bar{\Delta}(G_1) = \bar{\Delta}(G_2)$, and

(ii) $\delta(G_1) = \delta(G_2)$.

(62) Let us consider graphs G_1, G_2. Suppose $G_1 \approx G_2$. Then

(i) $\bar{\Delta}(G_1) = \bar{\Delta}(G_2)$, and

(ii) $\delta(G_1) = \delta(G_2)$, and

(iii) $\bar{\Delta}^-(G_1) = \bar{\Delta}^-(G_2)$, and

(iv) $\delta^-(G_1) = \delta^-(G_2)$, and

(v) $\bar{\Delta}^{+}(G_1) = \bar{\Delta}^{+}(G_2)$, and

(vi) $\delta^{+}(G_1) = \delta^{+}(G_2)$.

(63) Let us consider a graph G_1, and a subgraph G_2 of G_1. Then

(i) $\bar{\Delta}(G_2) \subseteq \bar{\Delta}(G_1)$, and

(ii) $\bar{\Delta}^{-}(G_2) \subseteq \bar{\Delta}^{-}(G_1)$, and

(iii) $\bar{\Delta}^{+}(G_2) \subseteq \bar{\Delta}^{+}(G_1)$.

The theorem is a consequence of (51) and (56).

(64) Let us consider a graph G_1, and a spanning subgraph G_2 of G_1. Then

(i) $\delta(G_2) \subseteq \delta(G_1)$, and

(ii) $\delta^{-}(G_2) \subseteq \delta^{-}(G_1)$, and

(iii) $\delta^{+}(G_2) \subseteq \delta^{+}(G_1)$.

The theorem is a consequence of (52) and (57).

Let us consider a graph G_2, a set V, and a supergraph G_1 of G_2 extended by the vertices from V. Now we state the propositions:

(65) (i) $\bar{\Delta}(G_1) = \bar{\Delta}(G_2)$, and

(ii) $\bar{\Delta}^{-}(G_1) = \bar{\Delta}^{-}(G_2)$, and

(iii) $\bar{\Delta}^{+}(G_1) = \bar{\Delta}^{+}(G_2)$.

The theorem is a consequence of (63).

(66) If $V \setminus$ (the vertices of G_2) $\neq \emptyset$, then $\delta(G_1) = 0$ and $\delta^{-}(G_1) = 0$ and $\delta^{+}(G_1) = 0$. The theorem is a consequence of (46).

Let G be a non edgeless graph. Observe that $\bar{\Delta}(G)$ is non empty and $\bar{\Delta}^{-}(G)$ is non empty and $\bar{\Delta}^{+}(G)$ is non empty.

Let G be a locally-finite graph. One can verify that $\delta(G)$ is natural and $\delta^{-}(G)$ is natural and $\delta^{+}(G)$ is natural.

The functors: $\delta(G)$, $\delta^{-}(G)$, and $\delta^{+}(G)$ yield natural numbers.

Let us consider a locally-finite graph G and a natural number n. Now we state the propositions:

(67) $\delta(G) = n$ if and only if there exists a vertex v of G such that v.degree() $= n$ and for every vertex w of G, v.degree() $\leqslant w$.degree(). The theorem is a consequence of (36).

(68) $\delta^{-}(G) = n$ if and only if there exists a vertex v of G such that v.inDegree() $= n$ and for every vertex w of G, v.inDegree() $\leqslant w$.inDegree(). The theorem is a consequence of (37).

(69) $\delta^{+}(G) = n$ if and only if there exists a vertex v of G such that v.outDegree() $= n$ and for every vertex w of G, v.outDegree() $\leqslant w$.outDegree(). The theorem is a consequence of (38).

Let us consider a graph G_2, vertices v, w of G_2, an object e, and a supergraph G_1 of G_2 extended by e between vertices v and w. Now we state the propositions:

(70) If $v \neq w$, then $\delta(G_1) = \delta(G_2)$ or $\delta(G_1) = v.\mathrm{degree}() \cap w.\mathrm{degree}() + 1$. The theorem is a consequence of (36) and (62).

(71) If $v \neq w$, then $\delta^-(G_1) = \delta^-(G_2)$ or $\delta^-(G_1) = w.\mathrm{inDegree}() + 1$. The theorem is a consequence of (37) and (62).

(72) If $v \neq w$, then $\delta^+(G_1) = \delta^+(G_2)$ or $\delta^+(G_1) = v.\mathrm{outDegree}() + 1$. The theorem is a consequence of (38) and (62).

Let us consider a locally-finite graph G_2, vertices v, w of G_2, an object e, and a supergraph G_1 of G_2 extended by e between vertices v and w. Now we state the propositions:

(73) If $v \neq w$, then $\delta(G_1) = \delta(G_2)$ or $\delta(G_1) = \min(v.\mathrm{degree}(), w.\mathrm{degree}())+1$. The theorem is a consequence of (70).

(74) If $v \neq w$, then $\delta^-(G_1) = \delta^-(G_2)$ or $\delta^-(G_1) = w.\mathrm{inDegree}() + 1$. The theorem is a consequence of (71).

(75) If $v \neq w$, then $\delta^+(G_1) = \delta^+(G_2)$ or $\delta^+(G_1) = v.\mathrm{outDegree}() + 1$. The theorem is a consequence of (72).

(76) Let us consider a graph G_2, an object v, and a supergraph G_1 of G_2 extended by vertex v and edges between v and the vertices of G_2. Suppose $v \notin$ the vertices of G_2. Then $\delta(G_1) = (\delta(G_2)+1) \cap G_2.\mathrm{order}()$. The theorem is a consequence of (36).

(77) Let us consider a finite graph G_2, an object v, and a supergraph G_1 of G_2 extended by vertex v and edges between v and the vertices of G_2. Suppose $v \notin$ the vertices of G_2. Then $\delta(G_1) = \min(\delta(G_2) + 1, G_2.\mathrm{order}())$. The theorem is a consequence of (76).

(78) Let us consider a graph G_2, a set V, and a graph G_1 by adding a loop to each vertex of G_2 in V. Then $\delta(G_1) \subseteq \delta(G_2) + 2$. The theorem is a consequence of (36) and (62).

Let G be an edge-finite graph. One can check that $\bar{\Delta}(G)$ is natural and $\bar{\Delta}^-(G)$ is natural and $\bar{\Delta}^+(G)$ is natural.

The functors: $\bar{\Delta}(G)$, $\bar{\Delta}^-(G)$, and $\bar{\Delta}^+(G)$ yield natural numbers. Let G be a graph. We say that G is with max degree if and only if

(Def. 12) there exists a vertex v of G such that for every vertex w of G, $w.\mathrm{degree}() \subseteq v.\mathrm{degree}()$.

We say that G is with max indegree if and only if

(Def. 13) there exists a vertex v of G such that for every vertex w of G, $w.\mathrm{inDegree}() \subseteq v.\mathrm{inDegree}()$.

We say that G is with max outdegree if and only if

(Def. 14) there exists a vertex v of G such that for every vertex w of G, w.outDegree() $\subseteq v$.outDegree().

Let us consider a graph G. Now we state the propositions:

(79) If G is with max degree, then there exists a vertex v of G such that

(i) v.degree() $= \bar{\Delta}(G)$, and

(ii) for every vertex w of G, w.degree() $\subseteq v$.degree().

The theorem is a consequence of (35).

(80) Suppose G is with max indegree. Then there exists a vertex v of G such that

(i) v.inDegree() $= \bar{\Delta}^{-}(G)$, and

(ii) for every vertex w of G, w.inDegree() $\subseteq v$.inDegree().

The theorem is a consequence of (35).

(81) Suppose G is with max outdegree. Then there exists a vertex v of G such that

(i) v.outDegree() $= \bar{\Delta}^{+}(G)$, and

(ii) for every vertex w of G, w.outDegree() $\subseteq v$.outDegree().

The theorem is a consequence of (35).

Let G be a graph. We introduce the notation G is without max degree as an antonym for G is with max degree. We introduce the notation G is without max indegree as an antonym for G is with max indegree. We introduce the notation G is without max outdegree as an antonym for G is with max outdegree.

Let us note that every graph which is with max indegree and with max outdegree is also with max degree and every graph which is vertex-finite is also with max degree, with max indegree, and with max outdegree and every graph which is edge-finite is also with max degree, with max indegree, and with max outdegree.

Now we state the proposition:

(82) Every with max degree graph is with max indegree or with max outdegree. The theorem is a consequence of (79), (40), (35), and (39).

Let G be a with max degree graph. We introduce the notation $\Delta(G)$ as a synonym of $\bar{\Delta}(G)$.

Let G be a with max indegree graph. We introduce the notation $\Delta^{-}(G)$ as a synonym of $\bar{\Delta}^{-}(G)$.

Let G be a with max outdegree graph. We introduce the notation $\Delta^{+}(G)$ as a synonym of $\bar{\Delta}^{+}(G)$.

Let G be a locally-finite, with max degree graph. Let us note that $\Delta(G)$ is natural.

Note that the functor $\Delta(G)$ yields a natural number. Let G be a locally-finite, with max indegree graph. Let us note that $\Delta^{-}(G)$ is natural.

Note that the functor $\Delta^{-}(G)$ yields a natural number. Let G be a locally-finite, with max outdegree graph. Let us note that $\Delta^{+}(G)$ is natural.

Note that the functor $\Delta^{+}(G)$ yields a natural number.

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(83) If F is isomorphism, then G_1 is with max degree iff G_2 is with max degree. The theorem is a consequence of (79) and (55).

(84) Suppose F is directed-isomorphism. Then

(i) G_1 is with max indegree iff G_2 is with max indegree, and

(ii) G_1 is with max outdegree iff G_2 is with max outdegree.

The theorem is a consequence of (80), (60), and (81).

(85) Let us consider graphs G_1, G_2. Suppose $G_1 \approx G_2$. Then

(i) if G_1 is with max degree, then G_2 is with max degree, and

(ii) if G_1 is with max indegree, then G_2 is with max indegree, and

(iii) if G_1 is with max outdegree, then G_2 is with max outdegree.

The theorem is a consequence of (83) and (84).

(86) Let us consider a graph G_1, a set E, and a graph G_2 given by reversing directions of the edges E of G_1. Then G_1 is with max degree if and only if G_2 is with max degree. The theorem is a consequence of (83).

Let G be a with max degree graph and E be a set. Observe that every graph given by reversing directions of the edges E of G is with max degree.

Let V be a set. Let us note that every supergraph of G extended by the vertices from V is with max degree and every graph by adding a loop to each vertex of G in V is with max degree.

Let v, e, w be objects. One can verify that every supergraph of G extended by e between vertices v and w is with max degree and every supergraph of G extended by v, w and e between them is with max degree.

Let v be an object and V be a set. One can verify that every supergraph of G extended by vertex v and edges between v and V of G is with max degree.

Let G be a with max indegree graph. Observe that every graph given by reversing directions of the edges of G is with max outdegree.

Let V be a set. One can verify that every supergraph of G extended by the vertices from V is with max indegree and every graph by adding a loop to each vertex of G in V is with max indegree.

Let v, e, w be objects. Let us note that every supergraph of G extended by e between vertices v and w is with max indegree and every supergraph of G extended by v, w and e between them is with max indegree.

Let v be an object and V be a set. Let us note that every supergraph of G extended by vertex v and edges between v and V of G is with max indegree.

Let G be a with max outdegree graph. One can check that every graph given by reversing directions of the edges of G is with max indegree.

Let V be a set. Let us note that every supergraph of G extended by the vertices from V is with max outdegree and every graph by adding a loop to each vertex of G in V is with max outdegree.

Let v, e, w be objects. One can verify that every supergraph of G extended by e between vertices v and w is with max outdegree and every supergraph of G extended by v, w and e between them is with max outdegree.

Let v be an object and V be a set. One can verify that every supergraph of G extended by vertex v and edges between v and V of G is with max outdegree.

Now we state the propositions:

(87) Let us consider a locally-finite, with max degree graph G, and a natural number n. Then $\Delta(G) = n$ if and only if there exists a vertex v of G such that v.degree() $= n$ and for every vertex w of G, w.degree() $\leqslant v$.degree(). The theorem is a consequence of (79) and (48).

(88) Let us consider a locally-finite, with max indegree graph G, and a natural number n. Then $\Delta^-(G) = n$ if and only if there exists a vertex v of G such that v.inDegree() $= n$ and for every vertex w of G, w.inDegree() $\leqslant v$.inDegree(). The theorem is a consequence of (80) and (49).

(89) Let us consider a locally-finite, with max outdegree graph G, and a natural number n. Then $\Delta^+(G) = n$ if and only if there exists a vertex v of G such that v.outDegree() $= n$ and for every vertex w of G, w.outDegree() $\leqslant v$.outDegree(). The theorem is a consequence of (81) and (50).

(90) Let us consider a cardinal number c, and a trivial, c-edge graph G. Then

(i) $\Delta^-(G) = c$, and

(ii) $\delta^-(G) = c$, and

(iii) $\Delta^+(G) = c$, and

(iv) $\delta^+(G) = c$, and

(v) $\Delta(G) = c + c$, and

(vi) $\delta(G) = c + c$.

The theorem is a consequence of (49), (37), (50), (38), (48), and (36).

Let G be a graph and v be a vertex of G. We say that v is with min degree if and only if

(Def. 15) v.degree() $= \delta(G)$.

We say that v is with min indegree if and only if

(Def. 16) v.inDegree() $= \delta^{-}(G)$.

We say that v is with min outdegree if and only if

(Def. 17) v.outDegree() $= \delta^{+}(G)$.

We say that v is with max degree if and only if

(Def. 18) v.degree() $= \bar{\Delta}(G)$.

We say that v is with max indegree if and only if

(Def. 19) v.inDegree() $= \bar{\Delta}^{-}(G)$.

We say that v is with max outdegree if and only if

(Def. 20) v.outDegree() $= \bar{\Delta}^{+}(G)$.

Let us consider a graph G and vertices v, w of G. Now we state the propositions:

(91) If v is with min degree, then v.degree() $\subseteq w$.degree(). The theorem is a consequence of (36).

(92) If v is with min indegree, then v.inDegree() $\subseteq w$.inDegree(). The theorem is a consequence of (37).

(93) If v is with min outdegree, then v.outDegree() $\subseteq w$.outDegree(). The theorem is a consequence of (38).

(94) If w is with max degree, then v.degree() $\subseteq w$.degree(). The theorem is a consequence of (79).

(95) If w is with max indegree, then v.inDegree() $\subseteq w$.inDegree(). The theorem is a consequence of (80).

(96) If w is with max outdegree, then v.outDegree() $\subseteq w$.outDegree(). The theorem is a consequence of (81).

Let G be a graph. Note that there exists a vertex of G which is with min degree and there exists a vertex of G which is with min indegree and there exists a vertex of G which is with min outdegree and every vertex of G which is with min indegree and with min outdegree is also with min degree and every vertex of G which is with max indegree and with max outdegree is also with max degree and every vertex of G which is isolated is also with min degree, with min indegree, and with min outdegree.

Let us consider a graph G. Now we state the propositions:

(97) G is with max degree if and only if there exists a vertex v of G such that v is with max degree. The theorem is a consequence of (79).

(98) G is with max indegree if and only if there exists a vertex v of G such that v is with max indegree. The theorem is a consequence of (80).

(99) G is with max outdegree if and only if there exists a vertex v of G such that v is with max outdegree. The theorem is a consequence of (81).

Let G be a with max degree graph. Observe that there exists a vertex of G which is with max degree.

Let G be a with max indegree graph. One can check that there exists a vertex of G which is with max indegree.

Let G be a with max outdegree graph. Observe that there exists a vertex of G which is with max outdegree.

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Accepted May 19, 2020

FORMALIZED MATHEMATICS
Vol. 28, No. 2, Pages 155–171, 2020
DOI: 10.2478/forma-2020-0014

https://www.sciendo.com/

About Graph Unions and Intersections

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Summary. In this article the union and intersection of a set of graphs are formalized in the Mizar system [5], based on the formalization of graphs in [7].

MSC: 68V20 05C76

Keywords: graph theory; graph union; graph intersection

MML identifier: `GLIB_014`, version: `8.1.10 5.63.1382`

0. Introduction

The union and intersection of two graphs are usually defined in any general graph theory textbook, although there are small differences between the authors from time to time. For example, Wilson [10] only allows two vertex- and edge-disjoint graphs to be united; his graph union is usually known as the disjoint union [2], [8] or sum [8] of two graphs, which will be formalized in in detail in another article. Bondy and Murty [2] as well as Diestel [4] allow unions of two arbitary simple graphs, but labelled the vertices in the graphical representation to avoid confusion. In both books it was silently assumed that edges between the same vertices in both graphs are the same, thereby securing the union to be a simple graph again. Wagner [9], while generalizing to the union and intersection of a family of graphs, explicitly states that condition and previously adds the condition, that on the other side identical edges in the graph family must have the same incident vertices. Naturally, in this paper union and intersection are generalized to families of multidigraphs, i.e. the graphs of [7]. Union and

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ISSN 1426–2630(Print), 1898-9934(Online)

intersection are defined as modes rather than functions in accordance with the style of the early `GLIB` articles and to leave this formalization extendable by graph decorators.

To denote the graph family, a `Graph-yielding Function` from [7] could have been used. But since sets of graphs would be needed sooner or later in the Mizar Mathematical Library [1] (e.g. to count all spanning trees of a graph), the set attribute `Graph-membered` is rigorously introduced in the first section.

In the second section, the first condition of Wagner is formalized. It simply means that for two graphs G and H from the family, their respective source and target function tolerate each other ($S(G) \approx S(H)$ and $T(G) \approx T(H)$, cf. [3]). As this property is indispensable for unions (or else in a union an edge could point to different vertices), the set attribute was named `\/-tolerating`. The graph union U for a $\cup$-tolerating set S is given by

$$U = (\bigcup_{G \in S} V(G), \bigcup_{G \in S} E(G), \bigcup_{G \in S} S(G), \bigcup_{G \in S} T(G)) .$$

While Wagner's second condition is useful to ensure the resulting graph union will be non-multi, it is not formalized in this article.

Since graphs without vertices are not allowed by the used definition [7], the difference between $\cup$-tolerating and `/\-tolerating` is the additional condition that $\bigcap_{G \in S} V(G)$ is non empty. Then the graph intersection I for a $\cap$-tolerating set S is given by

$$I = (\bigcap_{G \in S} V(G), \bigcap_{G \in S} E(G), \bigcap_{G \in S} S(G), \bigcap_{G \in S} T(G)) .$$

To avoid confusion with intersection graphs of any kind, the mode was named `GraphMeet`.

With this formalization the union of a graph with (any kind of) its complement will be complete and the intersection will be edgeless, just as intended by [6].

1. Sets of Graphs

Let X be a set. We say that X is graph-membered if and only if

(Def. 1) for every object x such that $x \in X$ holds x is a graph.

Observe that every set which is empty is also graph-membered.

Let F be a graph-yielding function. One can verify that $\operatorname{rng} F$ is graph-membered.

Let G_1 be a graph. Let us note that $\{G_1\}$ is graph-membered.

Let G_2 be a graph. Let us observe that $\{G_1, G_2\}$ is graph-membered and there exists a set which is empty and graph-membered and there exists a set which is trivial, finite, non empty, and graph-membered.

Let X be a graph-membered set. One can check that every subset of X is graph-membered.

Let Y be a set. Let us note that $X \cap Y$ is graph-membered and $X \setminus Y$ is graph-membered.

Let X, Y be graph-membered sets. Let us note that $X \cup Y$ is graph-membered and $X \dot{-} Y$ is graph-membered.

Let us consider a set X. Now we state the propositions:

(1) If for every object Y such that $Y \in X$ holds Y is a graph-membered set, then $\bigcup X$ is graph-membered.

(2) If there exists a graph-membered set Y such that $Y \in X$, then $\bigcap X$ is graph-membered.

Let X be a non empty, graph-membered set. Observe that every element of X is function-like and relation-like and every element of X is $\mathbb{N}$-defined and finite and every element of X is graph-like.

Let S be a graph-membered set. We say that S is plain if and only if

(Def. 2) for every graph G such that $G \in S$ holds G is plain.

We say that S is loopless if and only if

(Def. 3) for every graph G such that $G \in S$ holds G is loopless.

We say that S is non-multi if and only if

(Def. 4) for every graph G such that $G \in S$ holds G is non-multi.

We say that S is non-directed-multi if and only if

(Def. 5) for every graph G such that $G \in S$ holds G is non-directed-multi.

We say that S is simple if and only if

(Def. 6) for every graph G such that $G \in S$ holds G is simple.

We say that S is directed-simple if and only if

(Def. 7) for every graph G such that $G \in S$ holds G is directed-simple.

We say that S is acyclic if and only if

(Def. 8) for every graph G such that $G \in S$ holds G is acyclic.

We say that S is connected if and only if

(Def. 9) for every graph G such that $G \in S$ holds G is connected.

We say that S is tree-like if and only if

(Def. 10) for every graph G such that $G \in S$ holds G is tree-like.

We say that S is chordal if and only if

(Def. 11) for every graph G such that $G \in S$ holds G is chordal.

We say that S is edgeless if and only if

(Def. 12) for every graph G such that $G \in S$ holds G is edgeless.

We say that S is loopfull if and only if

(Def. 13) for every graph G such that $G \in S$ holds G is loopfull.

Let us observe that every graph-membered set which is empty is also plain, loopless, non-multi, non-directed-multi, simple, directed-simple, acyclic, connected, tree-like, chordal, edgeless, and loopfull and every graph-membered set which is non-multi is also non-directed-multi and every graph-membered set which is loopless and non-multi is also simple and every graph-membered set which is loopless and non-directed-multi is also directed-simple.

Every graph-membered set which is simple is also loopless and non-multi and every graph-membered set which is directed-simple is also loopless and non-directed-multi and every graph-membered set which is acyclic is also simple and every graph-membered set which is acyclic and connected is also tree-like and every graph-membered set which is tree-like is also acyclic and connected.

Let G_1 be a plain graph. Let us observe that $\{G_1\}$ is plain. Let G_2 be a plain graph. One can check that $\{G_1, G_2\}$ is plain.

Let G_1 be a loopless graph. One can verify that $\{G_1\}$ is loopless. Let G_2 be a loopless graph. Note that $\{G_1, G_2\}$ is loopless.

Let G_1 be a non-multi graph. One can check that $\{G_1\}$ is non-multi. Let G_2 be a non-multi graph. Let us note that $\{G_1, G_2\}$ is non-multi.

Let G_1 be a non-directed-multi graph. Note that $\{G_1\}$ is non-directed-multi. Let G_2 be a non-directed-multi graph. Observe that $\{G_1, G_2\}$ is non-directed-multi.

Let G_1 be a simple graph. Let us note that $\{G_1\}$ is simple. Let G_2 be a simple graph. One can verify that $\{G_1, G_2\}$ is simple.

Let G_1 be a directed-simple graph. Let us observe that $\{G_1\}$ is directed-simple. Let G_2 be a directed-simple graph. Note that $\{G_1, G_2\}$ is directed-simple.

Let G_1 be an acyclic graph. One can check that $\{G_1\}$ is acyclic. Let G_2 be an acyclic graph. Let us note that $\{G_1, G_2\}$ is acyclic.

Let G_1 be a connected graph. Note that $\{G_1\}$ is connected. Let G_2 be a connected graph. Observe that $\{G_1, G_2\}$ is connected.

Let G_1 be a tree-like graph. Let us note that $\{G_1\}$ is tree-like. Let G_2 be a tree-like graph. One can verify that $\{G_1, G_2\}$ is tree-like.

Let G_1 be a chordal graph. Let us observe that $\{G_1\}$ is chordal. Let G_2 be a chordal graph. One can check that $\{G_1, G_2\}$ is chordal.

Let G_1 be an edgeless graph. One can verify that $\{G_1\}$ is edgeless. Let G_2 be an edgeless graph. Note that $\{G_1, G_2\}$ is edgeless.

Let G_1 be a loopfull graph. One can check that $\{G_1\}$ is loopfull. Let G_2 be a loopfull graph. Let us note that $\{G_1, G_2\}$ is loopfull.

Let F be a plain, graph-yielding function. Observe that $\operatorname{rng} F$ is plain.

Let F be a loopless, graph-yielding function. One can verify that $\operatorname{rng} F$ is loopless.

Let F be a non-multi, graph-yielding function. Note that $\operatorname{rng} F$ is non-multi.

Let F be a non-directed-multi, graph-yielding function. Observe that $\operatorname{rng} F$ is non-directed-multi.

Let F be a simple, graph-yielding function. One can verify that $\operatorname{rng} F$ is simple.

Let F be a directed-simple, graph-yielding function. Observe that $\operatorname{rng} F$ is directed-simple.

Let F be an acyclic, graph-yielding function. Note that $\operatorname{rng} F$ is acyclic.

Let F be a connected, graph-yielding function. Observe that $\operatorname{rng} F$ is connected.

Let F be a tree-like, graph-yielding function. One can verify that $\operatorname{rng} F$ is tree-like.

Let F be a chordal, graph-yielding function. Observe that $\operatorname{rng} F$ is chordal.

Let F be an edgeless, graph-yielding function. One can verify that $\operatorname{rng} F$ is edgeless.

Let F be a loopfull, graph-yielding function. Note that $\operatorname{rng} F$ is loopfull.

Let X be a plain, graph-membered set. Observe that every subset of X is plain.

Let X be a loopless, graph-membered set. Note that every subset of X is loopless.

Let X be a non-multi, graph-membered set. One can verify that every subset of X is non-multi.

Let X be a non-directed-multi, graph-membered set. Observe that every subset of X is non-directed-multi.

Let X be a simple, graph-membered set. Note that every subset of X is simple.

Let X be a directed-simple, graph-membered set. One can check that every subset of X is directed-simple.

Let X be an acyclic, graph-membered set. One can verify that every subset of X is acyclic.

Let X be a connected, graph-membered set. Observe that every subset of X is connected.

Let X be a tree-like, graph-membered set. Note that every subset of X is tree-like.

Let X be a chordal, graph-membered set. One can check that every subset of X is chordal.

Let X be an edgeless, graph-membered set. Let us observe that every subset of X is edgeless.

Let X be a loopfull, graph-membered set. Let us note that every subset of X is loopfull.

Let X be a plain, graph-membered set and Y be a set. Note that $X \cap Y$ is plain and $X \setminus Y$ is plain.

Let X, Y be plain, graph-membered sets. Observe that $X \cup Y$ is plain and $X \dot{-} Y$ is plain.

Let X be a loopless, graph-membered set and Y be a set. Note that $X \cap Y$ is loopless and $X \setminus Y$ is loopless.

Let X, Y be loopless, graph-membered sets. Observe that $X \cup Y$ is loopless and $X \dot{-} Y$ is loopless.

Let X be a non-multi, graph-membered set and Y be a set. Note that $X \cap Y$ is non-multi and $X \setminus Y$ is non-multi.

Let X, Y be non-multi, graph-membered sets. Observe that $X \cup Y$ is non-multi and $X \dot{-} Y$ is non-multi.

Let X be a non-directed-multi, graph-membered set and Y be a set. Note that $X \cap Y$ is non-directed-multi and $X \setminus Y$ is non-directed-multi.

Let X, Y be non-directed-multi, graph-membered sets. Observe that $X \cup Y$ is non-directed-multi and $X \dot{-} Y$ is non-directed-multi.

Let X be a simple, graph-membered set and Y be a set. Note that $X \cap Y$ is simple and $X \setminus Y$ is simple.

Let X, Y be simple, graph-membered sets. Observe that $X \cup Y$ is simple and $X \dot{-} Y$ is simple.

Let X be a directed-simple, graph-membered set and Y be a set. Note that $X \cap Y$ is directed-simple and $X \setminus Y$ is directed-simple.

Let X, Y be directed-simple, graph-membered sets. Observe that $X \cup Y$ is directed-simple and $X \dot{-} Y$ is directed-simple.

Let X be an acyclic, graph-membered set and Y be a set. Note that $X \cap Y$ is acyclic and $X \setminus Y$ is acyclic.

Let X, Y be acyclic, graph-membered sets. Observe that $X \cup Y$ is acyclic and $X \dot{-} Y$ is acyclic.

Let X be a connected, graph-membered set and Y be a set. Note that $X \cap Y$ is connected and $X \setminus Y$ is connected.

Let X, Y be connected, graph-membered sets. Observe that $X \cup Y$ is connected and $X \dot{-} Y$ is connected.

Let X be a tree-like, graph-membered set and Y be a set. Note that $X \cap Y$ is tree-like and $X \setminus Y$ is tree-like.

Let X, Y be tree-like, graph-membered sets. Observe that $X \cup Y$ is tree-like and $X \dot{-} Y$ is tree-like.

Let X be a chordal, graph-membered set and Y be a set. Note that $X \cap Y$ is chordal and $X \setminus Y$ is chordal.

Let X, Y be chordal, graph-membered sets. Observe that $X \cup Y$ is chordal and $X \dot{-} Y$ is chordal.

Let X be an edgeless, graph-membered set and Y be a set. Note that $X \cap Y$ is edgeless and $X \setminus Y$ is edgeless.

Let X, Y be edgeless, graph-membered sets. Observe that $X \cup Y$ is edgeless and $X \dot{-} Y$ is edgeless.

Let X be a loopfull, graph-membered set and Y be a set. Note that $X \cap Y$ is loopfull and $X \setminus Y$ is loopfull.

Let X, Y be loopfull, graph-membered sets. Observe that $X \cup Y$ is loopfull and $X \dot{-} Y$ is loopfull. There exists a graph-membered set which is empty, plain, loopless, non-multi, non-directed-multi, simple, directed-simple, acyclic, connected, tree-like, chordal, edgeless, and loopfull. There exists a graph-membered set which is non empty, tree-like, acyclic, connected, simple, directed-simple, loopless, non-multi, and non-directed-multi.

There exists a graph-membered set which is non empty, edgeless, and chordal and there exists a graph-membered set which is non empty and loopfull and there exists a graph-membered set which is non empty and plain.

Let S be a non empty, plain, graph-membered set. One can verify that every element of S is plain.

Let S be a non empty, loopless, graph-membered set. Let us observe that every element of S is loopless.

Let S be a non empty, non-multi, graph-membered set. Observe that every element of S is non-multi.

Let S be a non empty, non-directed-multi, graph-membered set. Let us note that every element of S is non-directed-multi.

Let S be a non empty, simple, graph-membered set. Note that every element of S is simple.

Let S be a non empty, directed-simple, graph-membered set. Note that every element of S is directed-simple.

Let S be a non empty, acyclic, graph-membered set. Note that every element of S is acyclic.

Let S be a non empty, connected, graph-membered set. One can check that every element of S is connected.

Let S be a non empty, tree-like, graph-membered set. One can verify that every element of S is tree-like.

Let S be a non empty, chordal, graph-membered set. One can verify that every element of S is chordal.

Let S be a non empty, edgeless, graph-membered set. Let us observe that every element of S is edgeless.

Let S be a non empty, loopfull, graph-membered set. Observe that every element of S is loopfull.

Let S be a graph-membered set. The functors: the vertices of S, the edges of S, the source of S, and the target of S yielding sets are defined by conditions

(Def. 14) for every object V, $V \in$ the vertices of S iff there exists a graph G such that $G \in S$ and $V =$ the vertices of G,

(Def. 15) for every object E, $E \in$ the edges of S iff there exists a graph G such that $G \in S$ and $E =$ the edges of G,

(Def. 16) for every object s, $s \in$ the source of S iff there exists a graph G such that $G \in S$ and $s =$ the source of G,

(Def. 17) for every object t, $t \in$ the target of S iff there exists a graph G such that $G \in S$ and $t =$ the target of G,

respectively. Let S be a non empty, graph-membered set. The functors: the vertices of S, the edges of S, the source of S, and the target of S are defined by terms

(Def. 18) the set of all the vertices of G where G is an element of S,

(Def. 19) the set of all the edges of G where G is an element of S,

(Def. 20) the set of all the source of G where G is an element of S,

(Def. 21) the set of all the target of G where G is an element of S,

respectively. One can verify that $\bigcup$(the vertices of S) is non empty.

Let S be a graph-membered set. Note that the source of S is functional and the target of S is functional.

Let S be an empty, graph-membered set. Let us note that the vertices of S is empty and the edges of S is empty and the source of S is empty and the target of S is empty.

Let S be a non empty, graph-membered set. Let us observe that the vertices of S is non empty and the edges of S is non empty and the source of S is non empty and the target of S is non empty.

Let S be a trivial, graph-membered set. Note that the vertices of S is trivial and the edges of S is trivial and the source of S is trivial and the target of S is trivial.

Now we state the propositions:

(3) Let us consider a graph G. Then

(i) the vertices of $\{G\} = \{$the vertices of $G\}$, and

(ii) the edges of $\{G\} = \{$the edges of $G\}$, and

(iii) the source of $\{G\} = \{$the source of $G\}$, and

(iv) the target of $\{G\} = \{$the target of $G\}$.

(4) Let us consider graphs G, H. Then

(i) the vertices of $\{G, H\} = \{$the vertices of G, the vertices of $H\}$, and

(ii) the edges of $\{G, H\} = \{$the edges of G, the edges of $H\}$, and

(iii) the source of $\{G, H\} = \{$the source of G, the source of $H\}$, and

(iv) the target of $\{G, H\} = \{$the target of G, the target of $H\}$.

(5) Let us consider a graph-membered set S. Then

(i) $\overline{\overline{\alpha}} \subseteq \overline{\overline{S}}$, and

(ii) $\overline{\overline{\beta}} \subseteq \overline{\overline{S}}$, and

(iii) $\overline{\overline{\gamma}} \subseteq \overline{\overline{S}}$, and

(iv) $\overline{\overline{\delta}} \subseteq \overline{\overline{S}}$,

where α is the vertices of S, β is the edges of S, γ is the source of S, and δ is the target of S.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a graph G such that $\$_1 = G$ and $\$_2 =$ the vertices of G. For every object x such that $x \in S$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f_1 being a function such that $\operatorname{dom} f_1 = S$ and for every object x such that $x \in S$ holds $\mathcal{P}[x, f_1(x)]$. Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exists a graph G such that $\$_1 = G$ and $\$_2 =$ the edges of G. For every object x such that $x \in S$ there exists an object y such that $\mathcal{Q}[x, y]$. Consider f_2 being a function such that $\operatorname{dom} f_2 = S$ and for every object x such that $x \in S$ holds $\mathcal{Q}[x, f_2(x)]$.

Define $\mathcal{R}[\text{object}, \text{object}] \equiv$ there exists a graph G such that $\$_1 = G$ and $\$_2 =$ the source of G. For every object x such that $x \in S$ there exists an object y such that $\mathcal{R}[x, y]$. Consider f_3 being a function such that $\operatorname{dom} f_3 = S$ and for every object x such that $x \in S$ holds $\mathcal{R}[x, f_3(x)]$. Define $\mathcal{T}[\text{object}, \text{object}] \equiv$ there exists a graph G such that $\$_1 = G$ and $\$_2 =$ the target of G. For every object x such that $x \in S$ there exists an object y such that $\mathcal{T}[x, y]$. Consider f_4 being a function such that $\operatorname{dom} f_4 = S$ and for every object x such that $x \in S$ holds $\mathcal{T}[x, f_4(x)]$. $\square$

Let S be a finite, graph-membered set. Let us observe that the vertices of S is finite and the edges of S is finite and the source of S is finite and the target of S is finite.

Let S be an edgeless, graph-membered set. Note that $\bigcup$(the edges of S) is empty.

Let us consider graph-membered sets S_1, S_2. Now we state the propositions:

(6) (i) the vertices of $S_1 \cup S_2 =$ (the vertices of S_1) $\cup$ (the vertices of S_2), and

(ii) the edges of $S_1 \cup S_2 =$ (the edges of S_1) $\cup$ (the edges of S_2), and

(iii) the source of $S_1 \cup S_2 =$ (the source of S_1) $\cup$ (the source of S_2), and

(iv) the target of $S_1 \cup S_2 =$ (the target of S_1) $\cup$ (the target of S_2).

(7) (i) the vertices of $S_1 \cap S_2 \subseteq$ (the vertices of S_1) $\cap$ (the vertices of S_2), and

(ii) the edges of $S_1 \cap S_2 \subseteq$ (the edges of S_1) $\cap$ (the edges of S_2), and

(iii) the source of $S_1 \cap S_2 \subseteq$ (the source of S_1) $\cap$ (the source of S_2), and

(iv) the target of $S_1 \cap S_2 \subseteq$ (the target of S_1) $\cap$ (the target of S_2).

(8) (i) (the vertices of S_1) $\setminus$ (the vertices of S_2) $\subseteq$ the vertices of $S_1 \setminus S_2$, and

(ii) (the edges of S_1) $\setminus$ (the edges of S_2) $\subseteq$ the edges of $S_1 \setminus S_2$, and

(iii) (the source of S_1) $\setminus$ (the source of S_2) $\subseteq$ the source of $S_1 \setminus S_2$, and

(iv) (the target of S_1) $\setminus$ (the target of S_2) $\subseteq$ the target of $S_1 \setminus S_2$.

(9) (i) (the vertices of S_1)$\dot{-}$(the vertices of S_2) $\subseteq$ the vertices of $S_1 \dot{-} S_2$, and

(ii) (the edges of S_1)$\dot{-}$(the edges of S_2) $\subseteq$ the edges of $S_1 \dot{-} S_2$, and

(iii) (the source of S_1)$\dot{-}$(the source of S_2) $\subseteq$ the source of $S_1 \dot{-} S_2$, and

(iv) (the target of S_1)$\dot{-}$(the target of S_2) $\subseteq$ the target of $S_1 \dot{-} S_2$.

The theorem is a consequence of (8) and (6).

2. Union of Graphs

Let G_1, G_2 be graphs. We say that G_1 tolerates G_2 if and only if

(Def. 22) the source of G_1 tolerates the source of G_2 and the target of G_1 tolerates the target of G_2.

Let us observe that the predicate is reflexive and symmetric.

Let us consider graphs G_1, G_2. Now we state the propositions:

(10) If the edges of G_1 misses the edges of G_2, then G_1 tolerates G_2.

(11) Suppose the source of $G_1 \subseteq$ the source of G_2 and the target of $G_1 \subseteq$ the target of G_2. Then G_1 tolerates G_2.

(12) Let us consider a graph G_1, and subgraphs G_2, G_3 of G_1. Then G_2 tolerates G_3.

(13) Let us consider a graph G_1, and a subgraph G_2 of G_1. Then G_1 tolerates G_2. The theorem is a consequence of (12).

Let us consider graphs G_1, G_2. Now we state the propositions:

(14) If $G_1 \approx G_2$, then G_1 tolerates G_2. The theorem is a consequence of (13).

(15) G_1 tolerates G_2 if and only if for every objects e, v_1, w_1, v_2, w_2 such that e joins v_1 to w_1 in G_1 and e joins v_2 to w_2 in G_2 holds $v_1 = v_2$ and $w_1 = w_2$.

(16) Let us consider a graph G_1, a subset E of the edges of G_1, and a graph G_2 given by reversing directions of the edges E of G_1. Then G_1 tolerates G_2 if and only if $E \subseteq G_1$.loops(). The theorem is a consequence of (15).

Let S be a graph-membered set. We say that S is $\cup$-tolerating if and only if

(Def. 23) for every graphs G_1, G_2 such that $G_1, G_2 \in S$ holds G_1 tolerates G_2.

Let S be a non empty, graph-membered set. Observe that S is $\cup$-tolerating if and only if the condition (Def. 24) is satisfied.

(Def. 24) for every elements G_1, G_2 of S, G_1 tolerates G_2.

One can verify that every graph-membered set which is empty is also $\cup$-tolerating.

Let G be a graph. Observe that $\{G\}$ is $\cup$-tolerating and there exists a graph-membered set which is non empty and $\cup$-tolerating.

A graph union set is a non empty, $\cup$-tolerating, graph-membered set. Now we state the proposition:

(17) Let us consider graphs G_1, G_2. Then G_1 tolerates G_2 if and only if $\{G_1, G_2\}$ is $\cup$-tolerating.

Let S_1 be a $\cup$-tolerating, graph-membered set and S_2 be a set. Let us note that $S_1 \cap S_2$ is $\cup$-tolerating and $S_1 \setminus S_2$ is $\cup$-tolerating.

Now we state the proposition:

(18) Let us consider graph-membered sets S_1, S_2. Suppose $S_1 \cup S_2$ is $\cup$-tolerating. Then

(i) S_1 is $\cup$-tolerating, and

(ii) S_2 is $\cup$-tolerating.

Let S be a $\cup$-tolerating, graph-membered set. Let us note that the source of S is compatible and the target of S is compatible and $\bigcup$(the source of S) is function-like and relation-like and $\bigcup$(the target of S) is function-like and relation-like and $\bigcup$(the source of S) is ($\bigcup$(the edges of S))-defined and ($\bigcup$(the vertices of S))-valued and $\bigcup$(the target of S) is ($\bigcup$(the edges of S))-defined and ($\bigcup$(the vertices of S))-valued and $\bigcup$(the source of S) is total and $\bigcup$(the target of S) is total.

Let S be a graph union set.

A graph union of S is a graph defined by

(Def. 25) the vertices of *it* $= \bigcup$(the vertices of S) and the edges of *it* $= \bigcup$(the edges of S) and the source of *it* $= \bigcup$(the source of S) and the target of *it* $= \bigcup$(the target of S).

Now we state the propositions:

(19) Let us consider a graph union set S, and a graph union G of S. Then every element of S is a subgraph of G.

(20) Let us consider a graph union set S, a graph union G of S, and a graph G'. Then G' is a graph union of S if and only if $G \approx G'$.

Let S be a graph union set. One can check that there exists a graph union of S which is plain and there exists a graph union set which is loopless and there exists a graph union set which is edgeless and there exists a graph union set which is loopfull.

Let S be a loopless graph union set. Note that every graph union of S is loopless.

Let S be an edgeless graph union set. Observe that every graph union of S is edgeless.

Let S be a loopfull graph union set. One can check that every graph union of S is loopfull.

Now we state the proposition:

(21) Let us consider graphs G, H. Then G is a graph union of $\{H\}$ if and only if $G \approx H$. The theorem is a consequence of (3).

Let G_1, G_2 be graphs.

A graph union of G_1 and G_2 is a supergraph of G_1 defined by

(Def. 26) (i) there exists a graph union set S such that $S = \{G_1, G_2\}$ and *it* is a graph union of S, **if** G_1 tolerates G_2,

(ii) $it \approx G_1$, **otherwise**.

Now we state the proposition:

(22) Let us consider graphs G_1, G_2, G. Suppose G_1 tolerates G_2. Then G is a graph union of G_1 and G_2 if and only if the vertices of $G =$ (the vertices of G_1) $\cup$ (the vertices of G_2) and the edges of $G =$ (the edges of G_1) $\cup$ (the edges of G_2) and the source of $G =$ (the source of G_1)$+\cdot$(the source of G_2) and the target of $G =$ (the target of G_1)$+\cdot$(the target of G_2). The theorem is a consequence of (4) and (17).

Let us consider graphs G_1, G_2 and a graph union G of G_1 and G_2. Now we state the propositions:

(23) If G_1 tolerates G_2, then G is a supergraph of G_2. The theorem is a consequence of (19).

(24) If G_1 tolerates G_2, then G is a graph union of G_2 and G_1. The theorem is a consequence of (23).

(25) Let us consider graphs G_1, G_2, G', and a graph union G of G_1 and G_2. Then G' is a graph union of G_1 and G_2 if and only if $G \approx G'$. The theorem

is a consequence of (20).

Let G_1, G_2 be graphs. One can verify that there exists a graph union of G_1 and G_2 which is plain.

Now we state the proposition:

(26) Let us consider graphs G, G_1, and a subgraph G_2 of G_1. Then G is a graph union of G_1 and G_2 if and only if $G \approx G_1$. The theorem is a consequence of (13) and (22).

Let G_1, G_2 be loopless graphs. Observe that every graph union of G_1 and G_2 is loopless.

Let G_1, G_2 be edgeless graphs. Let us note that every graph union of G_1 and G_2 is edgeless.

Let G_1, G_2 be loopfull graphs. Note that every graph union of G_1 and G_2 is loopfull.

Now we state the proposition:

(27) Let us consider a graph G_1, a directed graph complement G_2 of G_1 with loops, a graph union G of G_1 and G_2, and vertices v, w of G. Then there exists an object e such that e joins v to w in G. The theorem is a consequence of (10), (22), and (23).

Let G_1 be a graph and G_2 be a directed graph complement of G_1 with loops. Let us observe that every graph union of G_1 and G_2 is loopfull and complete.

Now we state the proposition:

(28) Let us consider a graph G_1, an undirected graph complement G_2 of G_1 with loops, a graph union G of G_1 and G_2, and vertices v, w of G. Then there exists an object e such that e joins v and w in G. The theorem is a consequence of (10), (22), and (23).

Let G_1 be a graph and G_2 be an undirected graph complement of G_1 with loops. Let us note that every graph union of G_1 and G_2 is loopfull and complete.

Now we state the proposition:

(29) Let us consider a graph G_1, a directed graph complement G_2 of G_1, a graph union G of G_1 and G_2, and vertices v, w of G. If $v \neq w$, then there exists an object e such that e joins v to w in G. The theorem is a consequence of (10), (22), and (23).

Let G_1 be a graph and G_2 be a directed graph complement of G_1. One can check that every graph union of G_1 and G_2 is complete.

Now we state the proposition:

(30) Let us consider a graph G_1, a graph complement G_2 of G_1, a graph union G of G_1 and G_2, and vertices v, w of G. If $v \neq w$, then there exists an object e such that e joins v and w in G. The theorem is a consequence of (10), (22), and (23).

Let G_1 be a graph and G_2 be a graph complement of G_1. Let us note that every graph union of G_1 and G_2 is complete.

Let G_1 be a non-directed-multi graph and G_2 be a directed graph complement of G_1 with loops. One can verify that every graph union of G_1 and G_2 is non-directed-multi.

Let G_1 be a non-multi graph and G_2 be an undirected graph complement of G_1 with loops. Note that every graph union of G_1 and G_2 is non-multi.

Let G_1 be a non-directed-multi graph and G_2 be a directed graph complement of G_1. Observe that every graph union of G_1 and G_2 is non-directed-multi.

Let G_1 be a non-multi graph and G_2 be a graph complement of G_1. One can verify that every graph union of G_1 and G_2 is non-multi.

3. Intersection of Graphs

Let S be a graph-membered set. We say that S is $\cap$-tolerating if and only if

(Def. 27) $\bigcap$(the vertices of S) $\neq \emptyset$ and for every graphs G_1, G_2 such that G_1, $G_2 \in S$ holds G_1 tolerates G_2.

Let S be a non empty, graph-membered set. One can verify that S is $\cap$-tolerating if and only if the condition (Def. 28) is satisfied.

(Def. 28) $\bigcap$(the vertices of S) $\neq \emptyset$ and for every elements G_1, G_2 of S, G_1 tolerates G_2.

Now we state the proposition:

(31) Let us consider a graph-membered set S. Then S is $\cap$-tolerating if and only if S is $\cup$-tolerating and $\bigcap$(the vertices of S) $\neq \emptyset$.

Let G be a graph. Observe that $\{G\}$ is $\cap$-tolerating and every graph-membered set which is $\cap$-tolerating is also $\cup$-tolerating and non empty and there exists a graph-membered set which is $\cap$-tolerating.

A graph meet set is a $\cap$-tolerating, graph-membered set. Let S be a graph meet set. Note that $\bigcap$(the vertices of S) is non empty.

Now we state the propositions:

(32) Let us consider graphs G_1, G_2. Then G_1 tolerates G_2 and the vertices of G_1 meets the vertices of G_2 if and only if $\{G_1, G_2\}$ is $\cap$-tolerating. The theorem is a consequence of (4) and (17).

(33) Let us consider non empty, graph-membered sets S_1, S_2. Suppose $S_1 \cup S_2$ is $\cap$-tolerating. Then

(i) S_1 is $\cap$-tolerating, and

(ii) S_2 is $\cap$-tolerating.

The theorem is a consequence of (6) and (18).

Let S be a graph meet set. One can verify that $\bigcap$(the source of S) is function-like and relation-like and $\bigcap$(the target of S) is function-like and relation-like and $\bigcap$(the source of S) is ($\bigcap$(the edges of S))-defined and ($\bigcap$(the vertices of S))-valued and $\bigcap$(the target of S) is ($\bigcap$(the edges of S))-defined and ($\bigcap$(the vertices of S))-valued and $\bigcap$(the source of S) is total and $\bigcap$(the target of S) is total.

A graph meet of S is a graph defined by

(Def. 29) the vertices of $it = \bigcap$(the vertices of S) and the edges of $it = \bigcap$(the edges of S) and the source of $it = \bigcap$(the source of S) and the target of $it = \bigcap$(the target of S).

Now we state the propositions:

(34) Let us consider a graph meet set S, and a graph meet G of S. Then every element of S is a supergraph of G.

(35) Let us consider a graph meet set S, a graph meet G of S, and a graph G'. Then G' is a graph meet of S if and only if $G \approx G'$.

Let S be a graph meet set. Let us observe that there exists a graph meet of S which is plain.

Now we state the proposition:

(36) Let us consider graphs G, H. Then G is a graph meet of $\{H\}$ if and only if $G \approx H$. The theorem is a consequence of (3).

Let G_1, G_2 be graphs.

A graph meet of G_1 and G_2 is a subgraph of G_1 defined by

(Def. 30) (i) there exists a graph meet set S such that $S = \{G_1, G_2\}$ and it is a graph meet of S, **if** G_1 tolerates G_2 and the vertices of G_1 meets the vertices of G_2,

(ii) $it \approx G_1$, **otherwise**.

Now we state the proposition:

(37) Let us consider graphs G_1, G_2, G. Suppose G_1 tolerates G_2 and the vertices of G_1 meets the vertices of G_2. Then G is a graph meet of G_1 and G_2 if and only if the vertices of $G =$ (the vertices of G_1) $\cap$ (the vertices of G_2) and the edges of $G =$ (the edges of G_1) $\cap$ (the edges of G_2) and the source of $G =$ (the source of G_1)$\cap$(the source of G_2) and the target of $G =$ (the target of G_1)$\cap$(the target of G_2). The theorem is a consequence of (4) and (32).

Let us consider graphs G_1, G_2 and a graph meet G of G_1 and G_2. Now we state the propositions:

(38) If G_1 tolerates G_2 and the vertices of G_1 meets the vertices of G_2, then G is a subgraph of G_2. The theorem is a consequence of (34).

(39) If G_1 tolerates G_2 and the vertices of G_1 meets the vertices of G_2, then G is a graph meet of G_2 and G_1. The theorem is a consequence of (38).

(40) Let us consider graphs G_1, G_2, G', and a graph meet G of G_1 and G_2. Then G' is a graph meet of G_1 and G_2 if and only if $G \approx G'$. The theorem is a consequence of (35).

Let G_1, G_2 be graphs. One can check that there exists a graph meet of G_1 and G_2 which is plain.

Now we state the propositions:

(41) Let us consider graphs G, G_1, and a subgraph G_2 of G_1. Then G is a graph meet of G_1 and G_2 if and only if $G \approx G_2$. The theorem is a consequence of (13) and (37).

(42) Let us consider graphs G_1, G_2, and a graph meet G of G_1 and G_2. Suppose the vertices of G_1 meets the vertices of G_2 and the edges of G_1 misses the edges of G_2. Then G is edgeless. The theorem is a consequence of (10) and (37).

Let G_1 be a graph and G_2 be a directed graph complement of G_1 with loops. Let us observe that every graph meet of G_1 and G_2 is edgeless.

Let G_2 be an undirected graph complement of G_1 with loops. One can check that every graph meet of G_1 and G_2 is edgeless.

Let G_2 be a directed graph complement of G_1. Let us note that every graph meet of G_1 and G_2 is edgeless.

Let G_2 be a graph complement of G_1. Let us observe that every graph meet of G_1 and G_2 is edgeless.

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Accepted May 19, 2020

FORMALIZED MATHEMATICS
Vol. 28, No. 2, Pages 173–186, 2020
DOI: 10.2478/forma-2020-0015

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Unification of Graphs and Relations in Mizar

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Summary. A (di)graph without parallel edges can simply be represented by a binary relation of the vertices and on the other hand, any binary relation can be expressed as such a graph. In this article, this correspondence is formalized in the Mizar system [2], based on the formalization of graphs in [6] and relations in [11], [12]. Notably, a new definition of `createGraph` will be given, taking only a non empty set V and a binary relation $E \subseteq V \times V$ to create a (di)graph without parallel edges, which will provide to be very useful in future articles.

MSC: 05C62 68V20

Keywords: graph theory; binary relation

MML identifier: `GLUNIR00`, version: `8.1.10 5.63.1382`

0. Introduction

Digraphs without multiple edges can be represented by binary relations (cf. [4]) and this is in fact the way they are usually defined in textbooks which are primarily concerned about graphs without multiple edges (cf. [10], [3], [8]). While a mathematician can switch between these representations without problems, due to its pedantic nature the Mizar system [2] needs a formalization of this change of viewpoint, which is provided by this article. In the Mizar Mathematical Library [1] this problem hasn't been adressed yet, although the undirected analogon can be found as an alternative definition for simple graphs in [9] (which

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ISSN 1426–2630(Print), 1898-9934(Online)

isn't used anywhere else) and the friendship theorem was formalized in [7] using only relations.

In the first section the dominance and adjacency relation of a graph G are rigorously introduced. G isn't required to be without parallel edges for this, therefore the relations of G and the graph given by removing parallel edges (directed parallel for the dominance) as defined in [5] are the same.

The second section introduces the new functor definition for `createGraph`, taking a non empty set V and a relation $E \subseteq V \times V$ and returning a graph representing this relation. It is shown that the graph created this way from a dominance relation of a graph G without directed parallel edges is directed isomorphic to G itself.

Since undirected graphs are sometimes viewed as symmetric digraphs (cf. [3], [4], [8], the last section introduces a mode getting a graph without parallel edges of any kind by simply removing them from the functor result of the previous section. Similar to before, it is shown that the graph created this way from an adjacency relation of a graph G without parallel edges is isomorphic to G itself.

1. The Adjacency Relation

From now on G denotes a graph.

Let us consider G. The functor VertDomRel(G) yielding a binary relation on the vertices of G is defined by the term

(Def. 1) (the source of G **qua** binary relation)$^\smile \cdot$ (the target of G).

Let us consider objects v, w. Now we state the propositions:

(1) $\langle v, w \rangle \in$ VertDomRel(G) if and only if there exists an object e such that e joins v to w in G.

(2) $\langle v, w \rangle \in$ (VertDomRel(G))$^\smile$ if and only if there exists an object e such that e joins w to v in G. The theorem is a consequence of (1).

(3) G is loopless if and only if VertDomRel(G) is irreflexive.

Let G be a loopless graph. One can verify that VertDomRel(G) is irreflexive.

Let G be a non loopless graph. One can verify that VertDomRel(G) is non irreflexive.

Let G be a non-multi graph. One can verify that VertDomRel(G) is antisymmetric.

Let G be a simple graph. One can check that VertDomRel(G) is asymmetric.

Now we state the proposition:

(4) Let us consider a graph G. Suppose there exist objects e_1, e_2, x, y such that e_1 joins x to y in G and e_2 joins y to x in G. Then VertDomRel(G) is not asymmetric.

PROOF: Set $R = \mathrm{VertDomRel}(G)$. There exist objects x, y such that x, $y \in \operatorname{field} R$ and $\langle x, y\rangle$, $\langle y, x\rangle \in R$. $\square$

Let G be a non non-multi, non-directed-multi graph.

Note that $\mathrm{VertDomRel}(G)$ is non asymmetric.

Now we state the propositions:

(5) Let us consider a loopless graph G. Suppose $\operatorname{field} \mathrm{VertDomRel}(G) =$ the vertices of G. Then every component of G is not trivial. The theorem is a consequence of (1).

(6) Let us consider a graph G. Suppose every component of G is not trivial. Then $\operatorname{field} \mathrm{VertDomRel}(G) =$ the vertices of G. The theorem is a consequence of (1).

(7) Let us consider a non trivial, connected graph G. Then $\operatorname{field} \mathrm{VertDomRel}(G) =$ the vertices of G. The theorem is a consequence of (6).

Let G be a complete graph. One can verify that $\mathrm{VertDomRel}(G)$ is connected.

(8) G is edgeless if and only if $\mathrm{VertDomRel}(G)$ is empty. The theorem is a consequence of (1).

Let G be an edgeless graph. Let us observe that $\mathrm{VertDomRel}(G)$ is empty.

Let G be a non edgeless graph. One can verify that $\mathrm{VertDomRel}(G)$ is non empty.

Now we state the proposition:

(9) G is loopfull if and only if $\mathrm{VertDomRel}(G)$ is total and reflexive.

Let G be a loopfull graph. Note that $\mathrm{VertDomRel}(G)$ is reflexive and total.

Let G be a vertex-finite graph. Let us observe that $\mathrm{VertDomRel}(G)$ is finite.

(10) $\overline{\overline{\mathrm{VertDomRel}(G)}} = \overline{\overline{\mathrm{Classes}\,\mathrm{DEdgeParEqRel}(G)}}$.

PROOF: Set $R = \mathrm{VertDomRel}(G)$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an object e such that e joins $(\$_1)_{\mathbf{1}}$ to $(\$_1)_{\mathbf{2}}$ in G and $\$_2 = [e]_{\mathrm{DEdgeParEqRel}(G)}$. For every objects x, y_1, y_2 such that $x \in R$ and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. For every object x such that $x \in R$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = R$ and for every object x such that $x \in R$ holds $\mathcal{P}[x, f(x)]$. For every objects x_1, x_2 such that x_1, $x_2 \in \operatorname{dom} f$ and $f(x_1) = f(x_2)$ holds $x_1 = x_2$. $\square$

(11) $\overline{\overline{\mathrm{VertDomRel}(G)}} \subseteq G.\mathrm{size}()$. The theorem is a consequence of (10).

(12) Let us consider a non-directed-multi graph G. Then $G.\mathrm{size}() = \overline{\overline{\mathrm{VertDomRel}(G)}}$. The theorem is a consequence of (10).

Let us consider a vertex v of G. Now we state the propositions:

(13) $(\mathrm{VertDomRel}(G))^{\circ} v = v.\mathrm{outNeighbors}()$. The theorem is a consequence of (1).

(14) $\mathrm{Coim}(\mathrm{VertDomRel}(G), v) = v.\mathrm{inNeighbors}()$. The theorem is a consequence of (1).

(15) Let us consider a subgraph H of G. Then $\mathrm{VertDomRel}(H) \subseteq \mathrm{VertDomRel}(G)$. The theorem is a consequence of (1).

(16) Let us consider a subgraph H of G with directed-parallel edges removed. Then $\mathrm{VertDomRel}(H) = \mathrm{VertDomRel}(G)$. The theorem is a consequence of (15) and (1).

(17) Let us consider a subgraph H of G with loops removed. Then $\mathrm{VertDomRel}(H) = (\mathrm{VertDomRel}(G)) \setminus (\mathrm{id}_{\alpha})$, where α is the vertices of G. The theorem is a consequence of (1) and (15).

(18) Let us consider a directed-simple graph H of G. Then $\mathrm{VertDomRel}(H) = (\mathrm{VertDomRel}(G)) \setminus (\mathrm{id}_{\alpha})$, where α is the vertices of G. The theorem is a consequence of (17) and (16).

(19) Let us consider graphs G_1, G_2. If $G_1 \approx G_2$, then $\mathrm{VertDomRel}(G_1) = \mathrm{VertDomRel}(G_2)$. The theorem is a consequence of (1).

(20) Let us consider a graph H given by reversing directions of the edges of G. Then $\mathrm{VertDomRel}(H) = (\mathrm{VertDomRel}(G))^{\smile}$. The theorem is a consequence of (1).

(21) Let us consider a non empty subset V of the vertices of G, and a subgraph H of G induced by V. Then $\mathrm{VertDomRel}(H) = \mathrm{VertDomRel}(G) \cap (V \times V)$. The theorem is a consequence of (1) and (15).

(22) Let us consider a set V, and a subgraph H of G with vertices V removed. Suppose $V \subset$ the vertices of G. Then $\mathrm{VertDomRel}(H) = (\mathrm{VertDomRel}(G)) \setminus (V \times (\text{the vertices of } G) \cup (\text{the vertices of } G) \times V)$. The theorem is a consequence of (15) and (1).

Let us consider a non trivial graph G, a vertex v of G, and a subgraph H of G with vertex v removed. Now we state the propositions:

(23) $\mathrm{VertDomRel}(H) = (\mathrm{VertDomRel}(G)) \setminus (\{v\} \times (\text{the vertices of } G) \cup (\text{the vertices of } G) \times \{v\})$. The theorem is a consequence of (22).

(24) If v is isolated, then $\mathrm{VertDomRel}(H) = \mathrm{VertDomRel}(G)$.
PROOF: Set $V_1 = \{v\} \times (\text{the vertices of } G)$. Set $V_2 = (\text{the vertices of } G) \times \{v\}$. $(V_1 \cup V_2) \cap \mathrm{VertDomRel}(G) = \emptyset$. $\square$

(25) Let us consider a set V, and a supergraph H of G extended by the vertices from V. Then $\mathrm{VertDomRel}(H) = \mathrm{VertDomRel}(G)$. The theorem is a consequence of (15) and (1).

(26) Let us consider objects v, e, w, and a supergraph H of G extended by e between vertices v and w. Suppose there exists an object e_0 such that e_0 joins v to w in G. Then $\mathrm{VertDomRel}(H) = \mathrm{VertDomRel}(G)$. The theorem

is a consequence of (15), (1), and (19).

(27) Let us consider vertices v, w of G, an object e, and a supergraph H of G extended by e between vertices v and w. Suppose $e \notin$ the edges of G. Then VertDomRel(H) = VertDomRel(G) $\cup$ $\{\langle v, w\rangle\}$. The theorem is a consequence of (1) and (15).

(28) Let us consider a vertex v of G, objects e, w, and a supergraph H of G extended by v, w and e between them. Suppose $e \notin$ the edges of G and $w \notin$ the vertices of G. Then VertDomRel(H) = VertDomRel(G) $\cup$ $\{\langle v, w\rangle\}$. The theorem is a consequence of (27) and (25).

(29) Let us consider objects v, e, a vertex w of G, and a supergraph H of G extended by v, w and e between them. Suppose $e \notin$ the edges of G and $v \notin$ the vertices of G. Then VertDomRel(H) = VertDomRel(G) $\cup$ $\{\langle v, w\rangle\}$. The theorem is a consequence of (27) and (25).

(30) Let us consider a subset V of the vertices of G, and a graph H by adding a loop to each vertex of G in V. Then VertDomRel(H) = VertDomRel(G) $\cup$ id_V. The theorem is a consequence of (1) and (15).

(31) Let us consider a directed graph complement H of G with loops. Then VertDomRel(H) = ((the vertices of G) $\times$ (the vertices of G)) $\setminus$ (VertDomRel (G)). The theorem is a consequence of (1).

Let us consider G. The functor VertAdjSymRel(G) yielding a binary relation on the vertices of G is defined by the term

(Def. 2) VertDomRel(G) $\cup$ (VertDomRel(G))$^\smile$.

Now we state the propositions:

(32) Let us consider objects v, w. Then $\langle v, w\rangle \in$ VertAdjSymRel(G) if and only if there exists an object e such that e joins v and w in G. The theorem is a consequence of (1) and (2).

(33) Let us consider vertices v, w of G. Then $\langle v, w\rangle \in$ VertAdjSymRel(G) if and only if v and w are adjacent. The theorem is a consequence of (32).

(34) VertDomRel(G) $\subseteq$ VertAdjSymRel(G).

(35) VertAdjSymRel(G) = (the source of G **qua** binary relation)$^\smile$$\cdot$(the target of G) $\cup$ (the target of G **qua** binary relation)$^\smile$ $\cdot$ (the source of G).

Let us consider G. One can check that VertAdjSymRel(G) is symmetric.

Now we state the proposition:

(36) G is loopless if and only if VertAdjSymRel(G) is irreflexive.

Let G be a loopless graph. One can verify that VertAdjSymRel(G) is irreflexive.

Let G be a non loopless graph. One can check that VertAdjSymRel(G) is non irreflexive.

Now we state the propositions:

(37) Let us consider a loopless graph G. Suppose VertAdjSymRel(G) is total. Then every component of G is not trivial. The theorem is a consequence of (5).

(38) Let us consider a graph G. Suppose every component of G is not trivial. Then VertAdjSymRel(G) is total. The theorem is a consequence of (6).

Let G be a non trivial, connected graph. Note that VertAdjSymRel(G) is total.

Let G be a complete graph. Let us note that VertAdjSymRel(G) is connected.

Now we state the proposition:

(39) G is edgeless if and only if VertAdjSymRel(G) is empty.

Let G be an edgeless graph. One can check that VertAdjSymRel(G) is empty.

Let G be a non edgeless graph. Note that VertAdjSymRel(G) is non empty.

(40) G is loopfull if and only if VertAdjSymRel(G) is total and reflexive.

Let G be a loopfull graph. Let us observe that VertAdjSymRel(G) is reflexive and total.

Let G be a vertex-finite graph. Note that VertAdjSymRel(G) is finite.

Now we state the propositions:

(41) $\overline{\overline{\mathrm{Classes\,DEdgeParEqRel}(G)}} \subseteq \overline{\overline{\mathrm{VertAdjSymRel}(G)}}$. The theorem is a consequence of (34) and (10).

(42) $\overline{\overline{\mathrm{Classes\,EdgeParEqRel}(G)}} \subseteq \overline{\overline{\mathrm{VertAdjSymRel}(G)}}$.

Proof: Set $R = \mathrm{VertAdjSymRel}(G)$. Define $\mathcal{P}[\mathrm{object}, \mathrm{object}] \equiv$ there exists an object e such that e joins $(\$_1)_{\mathbf{1}}$ and $(\$_1)_{\mathbf{2}}$ in G and $\$_2 = [e]_{\mathrm{EdgeParEqRel}(G)}$. For every objects x, y_1, y_2 such that $x \in R$ and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. For every object x such that $x \in R$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\mathrm{dom}\, f = R$ and for every object x such that $x \in R$ holds $\mathcal{P}[x, f(x)]$. $\square$

(43) Let us consider a non-directed-multi graph G. Then $\overline{\overline{G.\mathrm{size}()}} \subseteq \overline{\overline{\mathrm{VertAdjSymRel}(G)}}$. The theorem is a consequence of (10), (12), and (41).

(44) Let us consider a vertex v of G. Then $(\mathrm{VertAdjSymRel}(G))^{\circ} v = v.\mathrm{allNeighbors}()$. The theorem is a consequence of (32).

(45) Let us consider a subgraph H of G. Then $\mathrm{VertAdjSymRel}(H) \subseteq \mathrm{VertAdjSymRel}(G)$. The theorem is a consequence of (15).

(46) Let us consider a subgraph H of G with parallel edges removed. Then $\mathrm{VertAdjSymRel}(H) = \mathrm{VertAdjSymRel}(G)$. The theorem is a consequence of (45) and (32).

(47) Let us consider a subgraph H of G with loops removed.

Then VertAdjSymRel(H) = (VertAdjSymRel(G))$\setminus$(id$_\alpha$), where α is the vertices of G. The theorem is a consequence of (17).

(48) Let us consider a simple graph H of G. Then VertAdjSymRel(H) = (VertAdjSymRel(G)) $\setminus$ (id$_\alpha$), where α is the vertices of G. The theorem is a consequence of (47) and (46).

(49) Let us consider graphs G_1, G_2. Suppose $G_1 \approx G_2$. Then VertAdjSymRel (G_1) = VertAdjSymRel(G_2). The theorem is a consequence of (19).

(50) Let us consider a set E, and a graph H given by reversing directions of the edges E of G. Then VertAdjSymRel(H) = VertAdjSymRel(G). The theorem is a consequence of (32).

(51) Let us consider a non empty subset V of the vertices of G, and a subgraph H of G induced by V. Then VertAdjSymRel(H) = VertAdjSymRel(G) $\cap$ ($V \times V$). The theorem is a consequence of (21).

(52) Let us consider a set V, and a subgraph H of G with vertices V removed. Suppose $V \subset$ the vertices of G. Then VertAdjSymRel(H) = (VertAdjSymRel(G)) $\setminus$ ($V \times$ (the vertices of G) $\cup$ (the vertices of G) $\times V$). The theorem is a consequence of (22).

Let us consider a non trivial graph G, a vertex v of G, and a subgraph H of G with vertex v removed. Now we state the propositions:

(53) VertAdjSymRel(H) = (VertAdjSymRel(G)) $\setminus$ ($\{v\} \times$ (the vertices of G) $\cup$ (the vertices of G) $\times \{v\}$). The theorem is a consequence of (52).

(54) If v is isolated, then VertAdjSymRel(H) = VertAdjSymRel(G). The theorem is a consequence of (24).

(55) Let us consider a set V, and a supergraph H of G extended by the vertices from V. Then VertAdjSymRel(H) = VertAdjSymRel(G). The theorem is a consequence of (25).

Let us consider vertices v, w of G, an object e, and a supergraph H of G extended by e between vertices v and w. Now we state the propositions:

(56) If v and w are adjacent, then VertAdjSymRel(H) = VertAdjSymRel(G). The theorem is a consequence of (26), (1), (27), and (49).

(57) Suppose $e \notin$ the edges of G. Then VertAdjSymRel(H) = VertAdjSymRel (G) $\cup$ $\{\langle v, w\rangle, \langle w, v\rangle\}$. The theorem is a consequence of (27).

(58) Let us consider a vertex v of G, objects e, w, and a supergraph H of G extended by v, w and e between them. Suppose $e \notin$ the edges of G and $w \notin$ the vertices of G. Then VertAdjSymRel(H) = VertAdjSymRel(G) $\cup$ $\{\langle v, w\rangle, \langle w, v\rangle\}$. The theorem is a consequence of (57) and (55).

(59) Let us consider objects v, e, a vertex w of G, and a supergraph H of G extended by v, w and e between them. Suppose $e \notin$ the edges of G and $v \notin$

the vertices of G. Then VertAdjSymRel(H) = VertAdjSymRel$(G) \cup \{\langle v, w\rangle, \langle w, v\rangle\}$. The theorem is a consequence of (57) and (55).

(60) Let us consider an object v, a subset V of the vertices of G, and a supergraph H of G extended by vertex v and edges between v and V of G. Suppose $v \notin$ the vertices of G. Then VertAdjSymRel(H) = (VertAdjSymRel$(G) \cup \{v\} \times V) \cup V \times \{v\}$. The theorem is a consequence of (32) and (45).

(61) Let us consider a subset V of the vertices of G, and a graph H by adding a loop to each vertex of G in V. Then VertAdjSymRel(H) = VertAdjSymRel$(G) \cup \mathrm{id}_V$. The theorem is a consequence of (30).

(62) Let us consider an undirected graph complement H of G with loops. Then VertAdjSymRel(H) = ((the vertices of G) $\times$ (the vertices of G)) $\setminus$ (VertAdjSymRel(G)). The theorem is a consequence of (32).

2. Create non-Directed-Multi Graphs from Relations

In the sequel V denotes a non empty set and E denotes a binary relation on V.

Let us consider V and E. The functor createGraph(V, E) yielding a graph is defined by the term

(Def. 3) createGraph$(V, E, \pi_1(V \boxtimes V){\restriction}E, \pi_2(V \boxtimes V){\restriction}E)$.

Let us note that the edges of createGraph(V, E) is relation-like.

Now we state the propositions:

(63) Let us consider objects v, w. Then $\langle v, w\rangle \in E$ if and only if $\langle v, w\rangle$ joins v to w in createGraph(V, E).

(64) Let us consider objects e, v, w. Suppose e joins v to w in createGraph(V, E). Then $e = \langle v, w\rangle$. The theorem is a consequence of (63).

(65) VertDomRel(createGraph(V, E)) $= E$. The theorem is a consequence of (1) and (63).

Let us consider V and E. One can verify that createGraph(V, E) is plain and non-directed-multi.

Now we state the proposition:

(66) V is trivial if and only if createGraph(V, E) is trivial.

Let V be a trivial, non empty set and E be a binary relation on V. One can check that createGraph(V, E) is trivial.

Let V be a non trivial set. Let us observe that createGraph(V, E) is non trivial.

Now we state the proposition:

(67) E is irreflexive if and only if createGraph(V, E) is loopless. The theorem is a consequence of (65).

Let us consider V. Let E be an irreflexive binary relation on V. Let us note that createGraph(V, E) is loopless.

Let E be a non irreflexive binary relation on V. Observe that createGraph(V, E) is non loopless.

Now we state the proposition:

(68) E is antisymmetric if and only if createGraph(V, E) is non-multi. The theorem is a consequence of (64) and (65).

Let us consider V. Let E be an antisymmetric binary relation on V. One can check that createGraph(V, E) is non-multi.

Let V be a non trivial set and E be a non antisymmetric binary relation on V. Note that createGraph(V, E) is non non-multi.

Let us consider V. Let E be an asymmetric binary relation on V. One can verify that createGraph(V, E) is simple.

Now we state the proposition:

(69) If createGraph(V, E) is complete, then E is connected. The theorem is a consequence of (65).

Let V be a non trivial set and E be a non connected binary relation on V. Note that createGraph(V, E) is non complete.

Now we state the proposition:

(70) E is empty if and only if createGraph(V, E) is edgeless. The theorem is a consequence of (65).

Let us consider V. Let E be an empty binary relation on V. One can verify that createGraph(V, E) is edgeless.

Let E be a non empty binary relation on V. Note that createGraph(V, E) is non edgeless.

Now we state the proposition:

(71) E is total and reflexive if and only if createGraph(V, E) is loopfull. The theorem is a consequence of (65).

Let us consider V. Let E be a total, reflexive binary relation on V. Let us note that createGraph(V, E) is loopfull.

Let E be a non total binary relation on V. Observe that createGraph(V, E) is non loopfull.

Let V be a finite, non empty set and E be a binary relation on V. One can check that createGraph(V, E) is finite.

Let us consider V. Let E be a finite binary relation on V. One can check that createGraph(V, E) is edge-finite.

Let us consider a vertex v of createGraph(V, E). Now we state the propositions:

(72) $E^\circ v = v$.outNeighbors(). The theorem is a consequence of (63) and (64).

(73) $\mathrm{Coim}(E, v) = v$.inNeighbors(). The theorem is a consequence of (63) and (64).

(74) Let us consider a set X. Then $E{\restriction}X =$ (createGraph(V, E)).edgesOutOf (X). The theorem is a consequence of (63) and (64).

(75) Let us consider a set Y. Then $Y{\upharpoonleft}E =$ (createGraph(V, E)).edgesInto(Y). The theorem is a consequence of (63) and (64).

Let us consider sets X, Y. Now we state the propositions:

(76) $(Y{\upharpoonleft}E){\restriction}X =$ (createGraph(V, E)).edgesDBetween(X, Y). The theorem is a consequence of (75) and (74).

(77) $(Y{\upharpoonleft}E){\restriction}X \cup (X{\upharpoonleft}E){\restriction}Y =$ (createGraph(V, E)).edgesBetween(X, Y). The theorem is a consequence of (76).

Let us consider a vertex v of createGraph(V, E). Now we state the propositions:

(78) $E{\restriction}\{v\} = v$.edgesOut(). The theorem is a consequence of (74).

(79) $\{v\}{\upharpoonleft}E = v$.edgesIn(). The theorem is a consequence of (75).

(80) Let us consider a set X. Then $E{\restriction}X \cup X{\upharpoonleft}E =$ (createGraph(V, E)) .edgesInOut(X). The theorem is a consequence of (74) and (75).

(81) $\mathrm{dom}\, E = \mathrm{rng}$(the source of createGraph(V, E)). The theorem is a consequence of (63) and (64).

(82) $\mathrm{rng}\, E = \mathrm{rng}$(the target of createGraph(V, E)). The theorem is a consequence of (63) and (64).

(83) Let us consider a vertex v of createGraph(V, E). Then v is isolated if and only if $v \notin \mathrm{field}\, E$. The theorem is a consequence of (63) and (64).

(84) E is symmetric if and only if VertAdjSymRel(createGraph(V, E)) $= E$. The theorem is a consequence of (65).

(85) Let us consider a non empty set V_1, a non empty subset V_2 of V_1, a binary relation E_1 on V_1, and a binary relation E_2 on V_2. Suppose $E_2 \subseteq E_1$. Then createGraph(V_2, E_2) is a subgraph of createGraph(V_1, E_1) induced by V_2 and E_2.

Let us consider a non-directed-multi graph G. Now we state the propositions:

(86) There exists a partial graph mapping F from G to createGraph(the vertices of G, VertDomRel(G)) such that

(i) F is directed-isomorphism, and

(ii) $F_{\mathbb{V}} = \mathrm{id}_\alpha$, and

(iii) for every object e such that $e \in$ the edges of G holds $(F_{\mathbb{E}})(e) = \langle$(the source of $G)(e)$, (the target of $G)(e)\rangle$,

where α is the vertices of G.

(87) createGraph(the vertices of G, VertDomRel(G)) is G-directed-isomorphic. The theorem is a consequence of (86).

3. Create non-Multi Graphs from Symmetric Relations

In the sequel E denotes a symmetric binary relation on V.

Let us consider V and E.

A graph created from the symmetric relation V on E is a subgraph of createGraph(V, E) with parallel edges removed. From now on G denotes a graph created from the symmetric relation V on E.

Now we state the propositions:

(88) Let us consider objects v, w. Then $\langle v, w\rangle \in E$ if and only if $\langle v, w\rangle$ joins v to w in G or $\langle w, v\rangle$ joins w to v in G. The theorem is a consequence of (63).

(89) Let us consider vertices v, w of G. Then $\langle v, w\rangle \in E$ if and only if v and w are adjacent. The theorem is a consequence of (88) and (63).

Let us consider V and E. Let us observe that every graph created from the symmetric relation V on E is non-multi.

Now we state the proposition:

(90) The edges of $G \subseteq E$.

Let us consider graphs G_1, G_2 created from the symmetric relation V on E. Now we state the propositions:

(91) The vertices of $G_1 =$ the vertices of G_2.

(92) G_2 is G_1-isomorphic.

(93) V is trivial if and only if G is trivial.

Let V be a trivial, non empty set and E be a symmetric binary relation on V. Observe that every graph created from the symmetric relation V on E is trivial.

Let V be a non trivial set. Let us note that every graph created from the symmetric relation V on E is non trivial.

Now we state the proposition:

(94) E is irreflexive if and only if G is loopless.

Let us consider V. Let E be a symmetric, irreflexive binary relation on V. One can verify that every graph created from the symmetric relation V on E is loopless.

Let E be a symmetric, non irreflexive binary relation on V. Observe that every graph created from the symmetric relation V on E is non loopless.

Now we state the proposition:

(95) If G is complete, then E is connected. The theorem is a consequence of (69).

Let V be a non trivial set and E be a symmetric, non connected binary relation on V. Note that every graph created from the symmetric relation V on E is non complete.

Now we state the proposition:

(96) E is empty if and only if G is edgeless.

Let us consider V. Let E be an empty binary relation on V. Let us note that every graph created from the symmetric relation V on E is edgeless.

Let E be a symmetric, non empty binary relation on V. One can check that every graph created from the symmetric relation V on E is non edgeless.

Now we state the proposition:

(97) E is total and reflexive if and only if G is loopfull. The theorem is a consequence of (71).

Let us consider V. Let E be a total, reflexive, symmetric binary relation on V. Observe that every graph created from the symmetric relation V on E is loopfull.

Let E be a symmetric, non total binary relation on V. Note that every graph created from the symmetric relation V on E is non loopfull.

Let V be a finite, non empty set and E be a symmetric binary relation on V. One can verify that every graph created from the symmetric relation V on E is finite.

Now we state the propositions:

(98) Let us consider a vertex v of G. Then $E^\circ v = v.\mathrm{allNeighbors}()$. The theorem is a consequence of (72) and (73).

(99) Let us consider a set X. Then $G.\mathrm{edgesInOut}(X) \subseteq E{\restriction}X \cup X{\upharpoonleft}E$. The theorem is a consequence of (80).

(100) Let us consider sets X, Y. Then $G.\mathrm{edgesBetween}(X, Y) \subseteq (Y{\upharpoonleft}E){\restriction}X \cup (X{\upharpoonleft}E){\restriction}Y$. The theorem is a consequence of (77).

Let us consider a vertex v of G. Now we state the propositions:

(101) $v.\mathrm{edgesOut}() \subseteq E{\restriction}\{v\}$. The theorem is a consequence of (78).

(102) $v.\mathrm{edgesIn}() \subseteq \{v\}{\upharpoonleft}E$. The theorem is a consequence of (79).

(103) v is isolated if and only if $v \notin \mathrm{field}\, E$. The theorem is a consequence of (83).

(104) Let us consider a graph G created from the symmetric relation V on E. Then $\mathrm{VertAdjSymRel}(G) = E$. The theorem is a consequence of (33) and (89).

(105) Let us consider a non empty set V_1, a non empty subset V_2 of V_1, a symmetric binary relation E_1 on V_1, a symmetric binary relation E_2 on V_2, a graph G_1 created from the symmetric relation V_1 on E_1, and a graph G_2 created from the symmetric relation V_2 on E_2. Suppose $E_2 \subseteq E_1$. Then there exists a partial graph mapping F from G_2 to G_1 such that

(i) F is weak subgraph embedding, and

(ii) $F_{\mathbb{V}} = \mathrm{id}_{V_2}$, and

(iii) for every objects v, w such that $\langle v, w\rangle \in$ the edges of G_2 holds $(F_{\mathbb{E}})(\langle v, w\rangle) = \langle v, w\rangle$ or $(F_{\mathbb{E}})(\langle v, w\rangle) = \langle w, v\rangle$.

Proof: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exist objects v, w such that $\$_1 = \langle v, w\rangle$ and $\$_2 \in$ the edges of G_1 and ($\$_2 = \langle v, w\rangle$ or $\$_2 = \langle w, v\rangle$). For every objects x, y_1, y_2 such that $x \in$ the edges of G_2 and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. For every object x such that $x \in$ the edges of G_2 there exists an object y such that $\mathcal{P}[x, y]$. Consider g being a function such that $\mathrm{dom}\, g =$ the edges of G_2 and for every object x such that $x \in$ the edges of G_2 holds $\mathcal{P}[x, g(x)]$. For every objects x_1, x_2 such that x_1, $x_2 \in \mathrm{dom}\, g$ and $g(x_1) = g(x_2)$ holds $x_1 = x_2$. Consider v_0, w_0 being objects such that $\langle v, w\rangle = \langle v_0, w_0\rangle$ and $g(\langle v, w\rangle) \in$ the edges of G_1 and $g(\langle v, w\rangle) = \langle v_0, w_0\rangle$ or $g(\langle v, w\rangle) = \langle w_0, v_0\rangle$. □

(106) Let us consider a non-multi graph G_1, and a graph G_2 created from the symmetric relation the vertices of G_1 on $\mathrm{VertAdjSymRel}(G_1)$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) F is isomorphism, and

(ii) $F_{\mathbb{V}} = \mathrm{id}_{\alpha}$, and

(iii) for every object e such that $e \in$ the edges of G_1 holds $(F_{\mathbb{E}})(e) = \langle$(the source of $G_1)(e)$, (the target of $G_1)(e)\rangle$ or $(F_{\mathbb{E}})(e) = \langle$(the target of $G_1)(e)$, (the source of $G_1)(e)\rangle$,

where α is the vertices of G_1.

Proof: Set $E_0 = \mathrm{VertAdjSymRel}(G)$. Set $G_0 = \mathrm{createGraph}($the vertices of $G, E_0)$. Consider E' being a representative selection of the parallel edges of G_0 such that G' is a subgraph of G_0 induced by the vertices of G_0 and E'. Define $\mathcal{P}[\text{object}, \text{object}] \equiv \$_2 \in E'$ and ($\$_2 = \langle$(the source of $G)(\$_1)$, (the target of $G)(\$_1)\rangle$ or $\$_2 = \langle$(the target of $G)(\$_1)$, (the source of $G)(\$_1)\rangle$). For every objects x, y_1, y_2 such that $x \in$ the edges of G and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. For every object x such that

$x \in$ the edges of G there exists an object y such that $\mathcal{P}[x, y]$. Consider g being a function such that $\operatorname{dom} g =$ the edges of G and for every object x such that $x \in$ the edges of G holds $\mathcal{P}[x, g(x)]$. $\square$

(107) Let us consider a non-multi graph G_1. Then every graph created from the symmetric relation the vertices of G_1 on VertAdjSymRel(G_1) is G_1-isomorphic. The theorem is a consequence of (106).

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Accepted May 31, 2020

FORMALIZED MATHEMATICS
Vol. 28, No. 2, Pages 187–196, 2020
DOI: 10.2478/forma-2020-0016

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Partial Correctness of a Fibonacci Algorithm

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Summary. In this paper we introduce some notions to facilitate formulating and proving properties of iterative algorithms encoded in nominative data language [19] in the Mizar system [3], [1]. It is tested on verification of the partial correctness of an algorithm computing n-th Fibonacci number:

```
i := 0
s := 0
b := 1
c := 0
while (i <> n)
   c := s
   s := b
   b := c + s
   i := i + 1
return s
```

This paper continues verification of algorithms [10], [13], [12] written in terms of simple-named complex-valued nominative data [6], [8], [17], [11], [14], [15]. The validity of the algorithm is presented in terms of semantic Floyd-Hoare triples over such data [9]. Proofs of the correctness are based on an inference system for an extended Floyd-Hoare logic [2], [4] with partial pre- and post-conditions [16], [18], [7], [5].

MSC: 68Q60 03B70 03B35

Keywords: nominative data; program verification; Fibonacci sequence

MML identifier: NOMIN_7, version: 8.1.10 5.63.1382

ISSN 1426–2630(Print), 1898-9934(Online)

1. Introduction

From now on D denotes a non empty set, m, n, N denote natural numbers, z_2 denotes a non zero natural number, f_1, f_2, f_3, f_4, f_5, f_6 denote binominative functions of D, p_1, p_2, p_3, p_4, p_5, p_6, p_7 denote partial predicates of D, d, v denote objects.

Observe that V, A denote sets, z denotes an element of V, val denotes a function, loc denotes a V-valued function, d_1 denotes a non-atomic nominative data of V and A, and T denotes a nominative data with simple names from V and complex values from A.

Let R_1, R_2 be binary relations. We say that R_1 is valid w.r.t. R_2 if and only if

(Def. 1) $\operatorname{rng} R_1 \subseteq \operatorname{dom} R_2$.

Let us consider V, loc, val, and N. We say that loc and val are different w.r.t. N if and only if

(Def. 2) for every natural numbers m, n such that $1 \leqslant m \leqslant N$ and $1 \leqslant n \leqslant N$ holds $val(m) \neq loc_{/n}$.

Now we state the propositions:

(1) Suppose $loc \restriction \operatorname{Seg} N$ is one-to-one and $\operatorname{Seg} N \subseteq \operatorname{dom} loc$. Let us consider natural numbers i, j. Suppose $1 \leqslant i \leqslant N$ and $1 \leqslant j \leqslant N$ and $i \neq j$. Then $loc_{/i} \neq loc_{/j}$.

(2) If V is not empty and $v \in \operatorname{dom} d_1$, then $(d_1 \nabla_a^z (v \Rightarrow_a)(d_1))(z) = d_1(v)$.

Let us consider D, f_1, f_2, f_3, f_4, f_5, and f_6. The functor PP-composition(f_1, f_2, f_3, f_4, f_5, f_6) yielding a binominative function of D is defined by the term

(Def. 3) $\text{PP-composition}(f_1, f_2, f_3, f_4, f_5) \bullet f_6$.

Now we state the proposition:

(3) Unconditional composition rule for 6 programs:
Suppose $\langle p_1, f_1, p_2\rangle$ is an SFHT of D and $\langle p_2, f_2, p_3\rangle$ is an SFHT of D and $\langle p_3, f_3, p_4\rangle$ is an SFHT of D and $\langle p_4, f_4, p_5\rangle$ is an SFHT of D and $\langle p_5, f_5, p_6\rangle$ is an SFHT of D and $\langle p_6, f_6, p_7\rangle$ is an SFHT of D and $\langle \sim p_2, f_2, p_3\rangle$ is an SFHT of D and $\langle \sim p_3, f_3, p_4\rangle$ is an SFHT of D and $\langle \sim p_4, f_4, p_5\rangle$ is an SFHT of D and $\langle \sim p_5, f_5, p_6\rangle$ is an SFHT of D and $\langle \sim p_6, f_6, p_7\rangle$ is an SFHT of D. Then $\langle p_1, \text{PP-composition}(f_1, f_2, f_3, f_4, f_5, f_6), p_7\rangle$ is an SFHT of D.

Let us consider V, A, loc, val, and d_1. Let z_2 be a natural number. Assume $z_2 > 0$. The functor $\text{LocalOverlapSeq}(A, loc, val, d_1, z_2)$ yielding a finite sequence of elements of $\text{ND}_{\text{SC}}(V, A)$ is defined by

(Def. 4) $\operatorname{len} it = z_2$ and $it(1) = d_1 \nabla_a^{(loc_{/1})}(val(1) \Rightarrow_a)(d_1)$ and for every natural number n such that $1 \leqslant n < \operatorname{len} it$ holds $it(n+1) = it(n)\nabla_a^{(loc_{/n+1})}(val(n+1) \Rightarrow_a)(it(n))$.

Let f be a function. We say that f is (V,A)-nonatomicND yielding if and only if

(Def. 5) for every object n such that $n \in \operatorname{dom} f$ holds $f(n)$ is a non-atomic nominative data of V and A.

Let f be a finite sequence. Let us observe that f is (V,A)-nonatomicND yielding if and only if the condition (Def. 6) is satisfied.

(Def. 6) for every natural number n such that $1 \leqslant n \leqslant \operatorname{len} f$ holds $f(n)$ is a non-atomic nominative data of V and A.

Let us consider d_1. Observe that $\langle d_1 \rangle$ is (V,A)-nonatomicND yielding and there exists a finite sequence which is (V,A)-nonatomicND yielding.

Now we state the proposition:

(4) Let us consider a (V,A)-nonatomicND yielding finite sequence f. If $n \in \operatorname{dom} f$, then $f(n)$ is a non-atomic nominative data of V and A.

Let us consider V, A, loc, val, d_1, and z_2. One can check that LocalOverlapSeq (A, loc, val, d_1, z_2) is (V,A)-nonatomicND yielding.

Let us consider n. Let us observe that $(\text{LocalOverlapSeq}(A, loc, val, d_1, z_2))(n)$ is function-like and relation-like.

Let us consider a natural number n. Now we state the propositions:

(5) Suppose V is not empty and V is without nonatomic nominative data w.r.t. A. Then suppose $1 \leqslant n < z_2$ and $val(n+1) \in \operatorname{dom}((\text{LocalOverlapSeq}(A, loc, val, d_1, z_2))(n))$. Then $\operatorname{dom}((\text{LocalOverlapSeq}(A, loc, val, d_1, z_2))(n+1)) = \{loc_{/n+1}\} \cup \operatorname{dom}((\text{LocalOverlapSeq}(A, loc, val, d_1, z_2))(n))$.

(6) Suppose V is not empty and V is without nonatomic nominative data w.r.t. A. Then suppose $1 \leqslant n < z_2$ and $val(n+1) \in \operatorname{dom}((\text{LocalOverlapSeq}(A, loc, val, d_1, z_2))(n))$. Then $\operatorname{dom}((\text{LocalOverlapSeq}(A, loc, val, d_1, z_2))(n)) \subseteq \operatorname{dom}((\text{LocalOverlapSeq}(A, loc, val, d_1, z_2))(n+1))$. The theorem is a consequence of (5).

Let us consider V, A, loc, val, d_1, and z_2. We say that loc, val and z_2 are correct w.r.t. d_1 if and only if

(Def. 7) V is not empty and V is without nonatomic nominative data w.r.t. A and val is valid w.r.t. d_1 and $\operatorname{dom}(\text{LocalOverlapSeq}(A, loc, val, d_1, z_2)) \subseteq \operatorname{dom} val$.

Now we state the proposition:

(7) Suppose loc, val and z_2 are correct w.r.t. d_1. Let us consider a natural number n. Suppose $1 \leqslant n \leqslant z_2$. Then $\operatorname{dom} d_1 \subseteq \operatorname{dom}((\text{LocalOverlapSeq}(A,$

$loc, val, d_1, z_2))(n))$.

Proof: Set $F = \mathrm{LocalOverlapSeq}(A, loc, val, d_1, z_2)$. Define $\mathcal{P}$[natural number] $\equiv$ if $1 \leqslant \$_1 \leqslant z_2$, then $\mathrm{dom}\, d_1 \subseteq \mathrm{dom}(F(\$_1))$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. $\square$

Let us consider natural numbers m, n. Now we state the propositions:

(8) Suppose loc, val and z_2 are correct w.r.t. d_1. Then suppose $1 \leqslant n \leqslant m \leqslant z_2$. Then $\mathrm{dom}((\mathrm{LocalOverlapSeq}(A, loc, val, d_1, z_2))(n)) \subseteq \mathrm{dom}$ $((\mathrm{LocalOverlapSeq}(A, loc, val, d_1, z_2))(m))$. The theorem is a consequence of (7) and (6).

(9) Suppose loc, val and z_2 are correct w.r.t. d_1. Then if $1 \leqslant n \leqslant m \leqslant z_2$, then $loc_{/n} \in \mathrm{dom}$ $((\mathrm{LocalOverlapSeq}(A, loc, val, d_1, z_2))(m))$. The theorem is a consequence of (8) and (7).

(10) Suppose loc, val and z_2 are correct w.r.t. d_1. Then if ($n \in \mathrm{dom}\, val$ or $1 \leqslant n \leqslant z_2$) and $1 \leqslant m \leqslant z_2$, then $val(n) \in \mathrm{dom}((\mathrm{LocalOverlapSeq}(A, loc, val, d_1, z_2))(m))$. The theorem is a consequence of (7).

Let us consider natural numbers j, m, n. Now we state the propositions:

(11) Suppose loc, val and z_2 are correct w.r.t. d_1 and loc and val are different w.r.t. z_2. Then suppose $1 \leqslant n \leqslant m < j \leqslant z_2$. Then $((\mathrm{LocalOverlapSeq}(A, loc, val, d_1, z_2))(n))(val(j)) = (\mathrm{LocalOverlapSeq}(A, loc, val, d_1, z_2))(m)$ $(val(j))$.

Proof: Set $F = \mathrm{LocalOverlapSeq}(A, loc, val, d_1, z_2)$. Set $l_1 = val(j)$. Define $\mathcal{P}$[natural number] $\equiv$ if $n \leqslant \$_1 < j \leqslant z_2$, then $F(n)(l_1) = F(\$_1)(l_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. $\square$

(12) Suppose loc, val and z_2 are correct w.r.t. d_1 and $\mathrm{Seg}\, z_2 \subseteq \mathrm{dom}\, loc$ and $loc \restriction \mathrm{Seg}\, z_2$ is one-to-one. Then suppose $1 \leqslant j \leqslant n \leqslant m \leqslant z_2$. Then $(\mathrm{LocalOverlapSeq}(A, loc, val, d_1, z_2))(n)(loc_{/j}) =$ $(\mathrm{LocalOverlapSeq}(A, loc, val, d_1, z_2))(m)(loc_{/j})$.

Proof: Set $F = \mathrm{LocalOverlapSeq}(A, loc, val, d_1, z_2)$. Set $l_1 = loc_{/j}$. Define $\mathcal{P}$[natural number] $\equiv$ if $n \leqslant \$_1 \leqslant z_2$, then $F(n)(l_1) = F(\$_1)(l_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. $\square$

(13) Let us consider a z_2-element finite sequence val. Suppose $\mathrm{Seg}\, z_2 \subseteq \mathrm{dom}\, loc$ and $loc \restriction \mathrm{Seg}\, z_2$ is one-to-one and loc and val are different w.r.t. z_2 and loc, val and z_2 are correct w.r.t. d_1. If $1 \leqslant n \leqslant m \leqslant z_2$, then $((\mathrm{LocalOverlapSeq}$ $(A, loc, val, d_1, z_2))(m))(loc_{/n}) = d_1(val(n))$.

Proof: Set F = LocalOverlapSeq(A, loc, val, d_1, z_2). Define $\mathcal{P}$[natural number] $\equiv$ if $n \leqslant \$_1 \leqslant z_2$, then $(F(\$_1))(loc_{/n}) = d_1(val(n))$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

(14) Let us consider a z_2-element finite sequence val. Suppose loc and val are different w.r.t. z_2 and loc, val and z_2 are correct w.r.t. d_1. Let us consider natural numbers m, n. Suppose $1 \leqslant m \leqslant z_2$ and $1 \leqslant n \leqslant z_2$. Then ((LocalOverlapSeq$(A, loc, val, d_1, z_2))(m))(val(n)) = d_1(val(n))$.

Proof: Set F = LocalOverlapSeq(A, loc, val, d_1, z_2). Define $\mathcal{P}$[natural number] $\equiv$ if $1 \leqslant \$_1 \leqslant z_2$, then $(F(\$_1))(val(n)) = d_1(val(n))$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

(15) Let us consider a z_2-element finite sequence val. Suppose loc, val and z_2 are correct w.r.t. d_1 and Seg $z_2 \subseteq$ dom loc and $loc \restriction$ Seg z_2 is one-to-one and loc and val are different w.r.t. z_2. Let us consider natural numbers j, m, n. Suppose $1 \leqslant j < m \leqslant n \leqslant z_2$. Then ((LocalOverlapSeq$(A, loc, val, d_1, z_2))$ $(n))(loc_{/m})$ = (LocalOverlapSeq$(A, loc, val, d_1, z_2))(j)(val(m))$.

Proof: Set F = LocalOverlapSeq(A, loc, val, d_1, z_2). Define $\mathcal{P}$[natural number] $\equiv$ if $m \leqslant \$_1 \leqslant z_2$, then $(F(\$_1))(loc_{/m}) = F(j)(val(m))$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

Let us consider V, A, loc, and val. Let z_2 be a natural number. Assume $0 < z_2$. The functor initial-assignments-Seq(A, loc, val, z_2) yielding a finite sequence of elements of $\mathrm{ND_{SC}}(V, A) \dot{\rightarrow} \mathrm{ND_{SC}}(V, A)$ is defined by

(Def. 8) len $it = z_2$ and $it(1) = \mathrm{Asg}^{(loc_{/1})}(val(1) \Rightarrow_a)$ and for every natural number n such that $1 \leqslant n < z_2$ holds $it(n+1) = it(n) \bullet (\mathrm{Asg}^{(loc_{/n+1})}(val(n+1) \Rightarrow_a))$.

The functor initial-assignments(A, loc, val, z_2) yielding a binominative function over simple-named complex-valued nominative data of V and A is defined by the term

(Def. 9) (initial-assignments-Seq$(A, loc, val, z_2))(z_2)$.

2. Main Algorithm

Let us consider V, A, and loc. The functor Fibonacci-loop-body(A, loc) yielding a binominative function over simple-named complex-valued nominative data of V and A is defined by the term

(Def. 10) PP-composition$(\mathrm{Asg}^{(loc_{/6})}((loc_{/4}) \Rightarrow_a), \mathrm{Asg}^{(loc_{/4})}((loc_{/5}) \Rightarrow_a), \mathrm{Asg}^{(loc_{/5})}$ $(\mathrm{addition}(A, loc_{/6}, loc_{/4})), \mathrm{Asg}^{(loc_{/1})}(\mathrm{addition}(A, loc_{/1}, loc_{/2})))$.

The functor Fibonacci-main-loop(A, loc) yielding a binominative function over simple-named complex-valued nominative data of V and A is defined by the term

(Def. 11) $\mathrm{WH}(\neg\,\mathrm{Equality}(A, loc_{/1}, loc_{/3}), \mathrm{Fibonacci\text{-}loop\text{-}body}(A, loc))$.

Let us consider val. The functor Fibonacci-main-part(A, loc, val) yielding a binominative function over simple-named complex-valued nominative data of V and A is defined by the term

(Def. 12) $\mathrm{initial\text{-}assignments}(A, loc, val, 6) \bullet (\mathrm{Fibonacci\text{-}main\text{-}loop}(A, loc))$.

Let us consider z. The functor Fibonacci-program(A, loc, val, z) yielding a binominative function over simple-named complex-valued nominative data of V and A is defined by the term

(Def. 13) $\mathrm{Fibonacci\text{-}main\text{-}part}(A, loc, val) \bullet (\mathrm{Asg}^{z}((loc_{/4}) \Rightarrow_{a}))$.

From now on n_0 denotes a natural number.

Let us consider V, A, val, n_0, and d. We say that val, n_0, and d constitute a valid input for the Fibonacci algorithm w.r.t. V and A if and only if

(Def. 14) there exists a non-atomic nominative data d_1 of V and A such that $d = d_1$ and $\{val(1), val(2), val(3), val(4), val(5), val(6)\} \subseteq \mathrm{dom}\, d_1$ and $d_1(val(1)) = 0$ and $d_1(val(2)) = 1$ and $d_1(val(3)) = n_0$ and $d_1(val(4)) = 0$ and $d_1(val(5)) = 1$ and $d_1(val(6)) = 0$.

The functor valid-Fibonacci-input(V, A, val, n_0) yielding a partial predicate over simple-named complex-valued nominative data of V and A is defined by

(Def. 15) $\mathrm{dom}\, it = \mathrm{ND_{SC}}(V, A)$ and for every object d such that $d \in \mathrm{dom}\, it$ holds if val, n_0, and d constitute a valid input for the Fibonacci algorithm w.r.t. V and A, then $it(d) = \mathit{true}$ and if val, n_0, and d do not constitute a valid input for the Fibonacci algorithm w.r.t. V and A, then $it(d) = \mathit{false}$.

One can check that valid-Fibonacci-input(V, A, val, n_0) is total.

Let us consider z and d. We say that z, n_0, and d constitute a valid output for the Fibonacci algorithm w.r.t. A if and only if

(Def. 16) there exists a non-atomic nominative data d_1 of V and A such that $d = d_1$ and $z \in \mathrm{dom}\, d_1$ and $d_1(z) = \mathrm{Fib}(n_0)$.

The functor valid-Fibonacci-output(A, z, n_0) yielding a partial predicate over simple-named complex-valued nominative data of V and A is defined by

(Def. 17) $\mathrm{dom}\, it = \{d$, where d is a nominative data with simple names from V and complex values from $A : d \in \mathrm{dom}(z \Rightarrow_{a})\}$ and for every object d such that $d \in \mathrm{dom}\, it$ holds if z, n_0, and d constitute a valid output for the Fibonacci algorithm w.r.t. A, then $it(d) = \mathit{true}$ and if z, n_0, and d do not constitute a valid output for the Fibonacci algorithm w.r.t. A, then $it(d) = \mathit{false}$.

Let us consider *loc* and d. We say that *loc*, n_0, and d constitute an invariant for the Fibonacci algorithm w.r.t. A if and only if

(Def. 18) there exists a non-atomic nominative data d_1 of V and A such that $d = d_1$ and $\{loc_{/1}, loc_{/2}, loc_{/3}, loc_{/4}, loc_{/5}, loc_{/6}\} \subseteq \mathrm{dom}\, d_1$ and $d_1(loc_{/2}) = 1$ and $d_1(loc_{/3}) = n_0$ and there exists a natural number I such that $I = d_1(loc_{/1})$ and $d_1(loc_{/4}) = \mathrm{Fib}(I)$ and $d_1(loc_{/5}) = \mathrm{Fib}(I + 1)$.

The functor Fibonacci-inv(A, loc, n_0) yielding a partial predicate over simple-named complex-valued nominative data of V and A is defined by

(Def. 19) $\mathrm{dom}\, it = \mathrm{ND_{SC}}(V, A)$ and for every object d such that $d \in \mathrm{dom}\, it$ holds if *loc*, n_0, and d constitute an invariant for the Fibonacci algorithm w.r.t. A, then $it(d) = true$ and if *loc*, n_0, and d do not constitute an invariant for the Fibonacci algorithm w.r.t. A, then $it(d) = false$.

Let us observe that Fibonacci-inv(A, loc, n_0) is total.

Now we state the propositions:

(16) Let us consider a 6-element finite sequence *val*. Suppose V is not empty and V is without nonatomic nominative data w.r.t. A and $\mathrm{Seg}\, 6 \subseteq \mathrm{dom}\, loc$ and $loc \restriction \mathrm{Seg}\, 6$ is one-to-one and *loc* and *val* are different w.r.t. 6. Then $\langle$valid-Fibonacci-input(V, A, val, n_0), initial-assignments$(A, loc, val, 6)$, Fibonacci-inv$(A, loc, n_0)\rangle$ is an SFHT of $\mathrm{ND_{SC}}(V, A)$.

Proof: Set $i = loc_{/1}$. Set $j = loc_{/2}$. Set $n = loc_{/3}$. Set $s = loc_{/4}$. Set $b = loc_{/5}$. Set $c = loc_{/6}$. Set $i_1 = val(1)$. Set $j_1 = val(2)$. Set $n_1 = val(3)$. Set $s_1 = val(4)$. Set $b_1 = val(5)$. Set $c_1 = val(6)$. Set $I =$ valid-Fibonacci-input(V, A, val, n_0). Set $i_2 =$ Fibonacci-inv(A, loc, n_0). Set $D_3 = i_1 \Rightarrow_a$. Set $D_4 = j_1 \Rightarrow_a$. Set $D_5 = n_1 \Rightarrow_a$. Set $D_6 = s_1 \Rightarrow_a$. Set $D_1 = b_1 \Rightarrow_a$. Set $D_2 = c_1 \Rightarrow_a$. Set $U_1 = \mathrm{S_P}(i_2, D_2, c)$. Set $T_1 = \mathrm{S_P}(U_1, D_1, b)$. Set $S_1 = \mathrm{S_P}(T_1, D_6, s)$. Set $R_1 = \mathrm{S_P}(S_1, D_5, n)$. Set $Q_1 = \mathrm{S_P}(R_1, D_4, j)$. Set $P_1 = \mathrm{S_P}(Q_1, D_3, i)$. $I \models P_1$. $\square$

(17) Suppose V is not empty and A is complex containing and V is without nonatomic nominative data w.r.t. A and for every T, T is a value on $loc_{/1}$ and T is a value on $loc_{/2}$ and T is a value on $loc_{/4}$ and T is a value on $loc_{/6}$ and $\mathrm{Seg}\, 6 \subseteq \mathrm{dom}\, loc$ and $loc \restriction \mathrm{Seg}\, 6$ is one-to-one. Then $\langle$Fibonacci-inv(A, loc, n_0), Fibonacci-loop-body(A, loc), Fibonacci-inv$(A, loc, n_0)\rangle$ is an SFHT of $\mathrm{ND_{SC}}(V, A)$. The theorem is a consequence of (1) and (2).

(18) Suppose V is not empty and A is complex containing and V is without nonatomic nominative data w.r.t. A and for every T, T is a value on $loc_{/1}$ and T is a value on $loc_{/2}$ and T is a value on $loc_{/4}$ and T is a value on $loc_{/6}$ and $\mathrm{Seg}\, 6 \subseteq \mathrm{dom}\, loc$ and $loc \restriction \mathrm{Seg}\, 6$ is one-to-one. Then $\langle$Fibonacci-inv(A, loc, n_0), Fibonacci-main-loop(A, loc), Equality$(A, loc_{/1},$

$loc_{/3}) \wedge \text{Fibonacci-inv}(A, loc, n_0))\rangle$ is an SFHT of $\text{ND}_{\text{SC}}(V, A)$. The theorem is a consequence of (17).

(19) Let us consider a 6-element finite sequence val. Suppose V is not empty and A is complex containing and V is without nonatomic nominative data w.r.t. A and for every T, T is a value on $loc_{/1}$ and T is a value on $loc_{/2}$ and T is a value on $loc_{/4}$ and T is a value on $loc_{/6}$ and $\text{Seg}\,6 \subseteq \text{dom}\,loc$ and $loc{\restriction}\text{Seg}\,6$ is one-to-one and loc and val are different w.r.t. 6. Then $\langle \text{valid-Fibonacci-input}(V, A, val, n_0), \text{Fibonacci-main-part}(A, loc, val)$, $\text{Equality}(A, loc_{/1}, loc_{/3}) \wedge \text{Fibonacci-inv}(A, loc, n_0))\rangle$ is an SFHT of $\text{ND}_{\text{SC}}(V, A)$. The theorem is a consequence of (16) and (18).

(20) Suppose V is not empty and V is without nonatomic nominative data w.r.t. A and for every T, T is a value on $loc_{/1}$ and T is a value on $loc_{/3}$. Then $\text{Equality}(A, loc_{/1}, loc_{/3}) \wedge \text{Fibonacci-inv}(A, loc, n_0) \models \text{S}_{\text{P}}$ $(\text{valid-Fibonacci-output}(A, z, n_0), (loc_{/4}) \Rightarrow_a, z)$.

Proof: Set $i = loc_{/1}$. Set $j = loc_{/2}$. Set $n = loc_{/3}$. Set $s = loc_{/4}$. Set $b = loc_{/5}$. Set $c = loc_{/6}$. Set $D_6 = s \Rightarrow_a$. Set $E_1 = \{i, j, n, s, b, c\}$. Consider d_1 being a non-atomic nominative data of V and A such that $d = d_1$ and $E_1 \subseteq \text{dom}\,d_1$ and $d_1(j) = 1$ and $d_1(n) = n_0$ and there exists a natural number I such that $I = d_1(i)$ and $d_1(s) = \text{Fib}(I)$ and $d_1(b) = \text{Fib}(I+1)$. Reconsider $d_3 = d$ as a nominative data with simple names from V and complex values from A. Set $L = d_3 \nabla_a^z D_6(d_3)$. z, n_0, and L constitute a valid output for the Fibonacci algorithm w.r.t. A. $\square$

(21) Suppose V is not empty and V is without nonatomic nominative data w.r.t. A and for every T, T is a value on $loc_{/1}$ and T is a value on $loc_{/3}$. Then $\langle \text{Equality}(A, loc_{/1}, loc_{/3}) \wedge \text{Fibonacci-inv}(A, loc, n_0), \text{Asg}^z((loc_{/4}) \Rightarrow_a$ $), \text{valid-Fibonacci-output}(A, z, n_0))\rangle$ is an SFHT of $\text{ND}_{\text{SC}}(V, A)$. The theorem is a consequence of (20).

(22) Suppose for every T, T is a value on $loc_{/1}$ and T is a value on $loc_{/3}$. Then $\langle \sim (\text{Equality}(A, loc_{/1}, loc_{/3}) \wedge \text{Fibonacci-inv}(A, loc, n_0)), \text{Asg}^z((loc_{/4}) \Rightarrow_a)$, $\text{valid-Fibonacci-output}(A, z, n_0))\rangle$ is an SFHT of $\text{ND}_{\text{SC}}(V, A)$.

(23) Partial correctness of a Fibonacci algorithm:
Let us consider a 6-element finite sequence val. Suppose V is not empty and A is complex containing and V is without nonatomic nominative data w.r.t. A and for every T, T is a value on $loc_{/1}$ and T is a value on $loc_{/2}$ and T is a value on $loc_{/3}$ and T is a value on $loc_{/4}$ and T is a value on $loc_{/6}$ and $\text{Seg}\,6 \subseteq \text{dom}\,loc$ and $loc{\restriction}\text{Seg}\,6$ is one-to-one and loc and val are different w.r.t. 6. Then $\langle \text{valid-Fibonacci-input}(V, A, val, n_0)$, $\text{Fibonacci-program}(A, loc, val, z), \text{valid-Fibonacci-output}(A, z, n_0))\rangle$ is an SFHT of $\text{ND}_{\text{SC}}(V, A)$. The theorem is a consequence of (19), (21), and (22).

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Accepted May 31, 2020

FORMALIZED MATHEMATICS
Vol. 28, No. 2, Pages 197–210, 2020
DOI: 10.2478/forma-2020-0017

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Multiplication-Related Classes of Complex Numbers

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Summary. The use of registrations is useful in shortening Mizar proofs [1], [2], both in terms of formalization time and article space. The proposed system of classes for complex numbers aims to facilitate proofs involving basic arithmetical operations and order checking. It seems likely that the use of self-explanatory adjectives could also improve legibility of these proofs, which would be an important achievement [3]. Additionally, some potentially useful definitions, following those defined for real numbers, are introduced.

MSC: 40-04 68V20

Keywords: complex numbers; multiplication; order

MML identifier: COMPLEX3, version: 8.1.10 5.63.1382

Let a be a complex number. One can check that $(a^{-1})^{-1}$ reduces to a.

We say that a is heavy if and only if

(Def. 1) $|a| > 1$.

We say that a is light if and only if

(Def. 2) $|a| < 1$.

We say that a is weightless if and only if

(Def. 3) $|a| = 0$ or $|a| = 1$.

Let us consider a real number a. Now we state the propositions:

(1) (i) a is heavy and negative iff $a < -1$, and

(ii) a is light and negative iff $-1 < a < 0$, and

(iii) a is light and positive iff $0 < a < 1$, and

(iv) a is heavy and positive iff $a > 1$, and

ISSN 1426–2630(Print), 1898-9934(Online)

(v) a is weightless and positive iff $a = 1$, and

(vi) a is weightless and negative iff $a = -1$.

(2) (i) a is non light and negative iff $a \leqslant -1$, and

(ii) a is non heavy and negative iff $-1 \leqslant a < 0$, and

(iii) a is non heavy and positive iff $0 < a \leqslant 1$, and

(iv) a is non light and positive iff $1 \leqslant a$.

(3) a is weightless if and only if $a = \mathrm{sgn}(a)$.

PROOF: If a is weightless, then $a = \mathrm{sgn}(a)$. If $a = \mathrm{sgn}(a)$, then a is weightless. □

Let us note that every complex number which is zero is also weightless and every complex number which is heavy is also non light and every complex number which is non light is also non zero and every complex number which is heavy is also non weightless and every non zero complex number which is light is also non weightless and every integer which is light is also zero.

Every natural number which is trivial is also weightless and every natural number which is non heavy is also trivial and every natural number which is non zero is also non light and every natural number which is non trivial is also heavy and every complex number which is weightless is also non heavy and every complex number which is light is also non heavy and every non negative real number which is non light is also positive.

There exists a positive real number which is heavy and there exists a negative real number which is heavy and there exists a positive real number which is light and there exists a negative real number which is light and there exists a weightless integer which is positive and there exists a weightless integer which is negative.

Let us consider a complex number a. Now we state the propositions:

(4) $\Re(a) \geqslant -|a|$.

(5) $\Im(a) \geqslant -|a|$.

(6) $|\Re(a)| + |\Im(a)| \geqslant |a|$.

Let a be a complex number. Let us observe that $a \cdot (a^{-1})$ is trivial and $a \cdot \overline{a}$ is real and $a \cdot \overline{a}^2$ is non negative and $\frac{a}{|a|}$ is weightless.

The functor director(a) yielding a weightless complex number is defined by the term

(Def. 4) $\frac{a}{|a|}$.

Let us consider a complex number a. Now we state the propositions:

(7) $a = |a| \cdot \mathrm{director}(a)$.

(8) $\mathrm{director}(-a) = -\mathrm{director}(a)$.

Let a be a real number. We identify $\mathrm{sgn}(a)$ with $\mathrm{director}(a)$. Observe that $\mathrm{director}(a)$ is integer.

Let a be a negative real number. One can verify that $\mathrm{director}(a)$ is negative.

Let a be a positive real number. Note that $\mathrm{director}(a)$ is positive.

Let us note that $\mathrm{director}(0)$ reduces to 0.

Let a be a non weightless complex number. Let us note that $|a|$ is positive and $-a$ is non weightless and $\overline{a}$ is non weightless and a^{-1} is non weightless.

Let a be a weightless complex number. Observe that $-a$ is weightless and $\overline{a}$ is weightless and a^{-1} is weightless and $a \cdot \overline{a}$ is weightless and $|\Re(a)|$ is non heavy and $|\Im(a)|$ is non heavy and $|a| - 1$ is weightless and $1 - |a|$ is weightless.

Let a be a weightless real number. One can verify that $\mathrm{sgn}(a)$ reduces to a.

Let a be a heavy complex number. One can verify that $-a$ is heavy and $\overline{a}$ is heavy and a^{-1} is light and $a \cdot \overline{a}$ is heavy and $|\Re(a)| + |\Im(a)|$ is heavy and $|a| - 1$ is positive and $1 - |a|$ is negative.

Let a be a non light complex number. Note that $-a$ is non light and $\overline{a}$ is non light and a^{-1} is non heavy and $a \cdot \overline{a}$ is non light and $|\Re(a)| + |\Im(a)|$ is non light and $|a| - 1$ is non negative and $1 - |a|$ is non positive.

Let a be a light complex number. Observe that $-a$ is light and $\overline{a}$ is light and $a \cdot \overline{a}$ is light and $|a| - 1$ is negative and $1 - |a|$ is positive and $\Re(a)$ is light and $\Im(a)$ is light and $\Re(a) - 1$ is negative and $\Re(a) - 2$ is heavy and $\Im(a) - 1$ is negative and $\Im(a) - 2$ is heavy.

Let a be a non zero, light complex number. Note that a^{-1} is heavy.

Let a be a non heavy complex number. Let us note that $-a$ is non heavy and $\overline{a}$ is non heavy and $a \cdot \overline{a}$ is non heavy and $|a| - 1$ is non positive and $1 - |a|$ is non negative and $\Re(a)$ is non heavy and $\Im(a)$ is non heavy and $\Re(a) - 1$ is non positive and $\Im(a) - 1$ is non positive.

Let a be a non zero, non heavy complex number. Let us observe that a^{-1} is non light.

Let a be a complex number. The functor $\mathrm{rsgn}(a)$ yielding a non heavy complex number is defined by the term

(Def. 5) $\Re(\mathrm{director}(a))$.

The functor $\mathrm{isgn}(a)$ yielding a non heavy complex number is defined by the term

(Def. 6) $\Im(\mathrm{director}(a))$.

Let a be a real number. We identify $\mathrm{sgn}(a)$ with $\mathrm{rsgn}(a)$. One can check that $\mathrm{isgn}(a)$ is zero and $\mathrm{frac}\, a$ is light and $|a| + a$ is non negative and $|a| - a$ is non negative.

Let a be a heavy, positive real number. Observe that $a - 1$ is positive and $1 - a$ is negative.

Let a be a light, positive real number. One can check that $a-1$ is negative and $1-a$ is positive.

Now we state the propositions:

(9) Every non heavy complex number is light or weightless.

(10) Every non light complex number is heavy or weightless.

(11) Let us consider a heavy, positive real number a, and a non heavy real number b. Then $a > b > -a$. The theorem is a consequence of (1).

(12) Let us consider a non light, positive real number a, and a light real number b. Then $a > b > -a$. The theorem is a consequence of (1).

Let a be a heavy complex number and b be a non light complex number. Observe that $a \cdot b$ is heavy.

Let a, b be non light complex numbers. Note that $a \cdot b$ is non light.

Let a be a light complex number and b be a non heavy complex number. One can check that $a \cdot b$ is light.

Let a, b be non heavy complex numbers. Let us observe that $a \cdot b$ is non heavy.

Let a, b be weightless complex numbers. Let us note that $a \cdot b$ is weightless.

Let a be a complex number. The functor cfrac(a) yielding a light complex number is defined by the term

(Def. 7) $\operatorname{director}(a) \cdot \operatorname{frac}|a|$.

Now we state the proposition:

(13) Let us consider a complex number a. Then $\operatorname{cfrac}(-a) = -\operatorname{cfrac}(a)$. The theorem is a consequence of (8).

Let a be a non negative real number. We identify cfrac(a) with $\operatorname{frac} a$. Now we state the proposition:

(14) Let us consider a complex number a, and a natural number n. Then $|a|^n = |a^n|$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv |a|^{\$_1} = |a^{\$_1}|$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number l, $\mathcal{P}[l]$. □

Let a be a weightless complex number and n be a natural number. One can check that a^n is weightless.

Let a be a weightless real number. One can verify that $a^{2 \cdot n} - 1$ is weightless.

Let a be a non light complex number. Let us note that a^n is non light.

Let a be a non light real number. One can check that $a^{2 \cdot n} - 1$ is non negative.

Let a be a light complex number and n be a non zero natural number. Note that a^n is light and $\sqrt[n]{a}$ is light.

Let a be a light real number. Let us observe that $a^{2 \cdot n} - 1$ is negative.

Let a be a non heavy complex number and n be a natural number. One can check that a^n is non heavy.

Let a be a non heavy real number. Observe that $a^{2 \cdot n} - 1$ is non positive.

Let a be a heavy complex number and n be a non zero natural number. Let us observe that a^n is heavy and $\sqrt[n]{a}$ is heavy.

Let a be a non weightless complex number. One can check that a^n is non weightless.

Let a be a weightless complex number. Let us observe that $\sqrt[n]{a}$ is weightless.

Let a be a non weightless complex number. Observe that $\sqrt[n]{a}$ is non weightless.

Let a be a non light complex number. Note that $\sqrt[n]{a}$ is non light.

Let a be a non heavy complex number. One can verify that $\sqrt[n]{a}$ is non heavy.

Let a, b be weightless complex numbers. Observe that $\frac{a}{b}$ is weightless.

Let a be a non heavy complex number and b be a heavy complex number. Observe that $\frac{a}{b}$ is light.

Let a be a light complex number and b be a non light complex number. Observe that $\frac{a}{b}$ is light.

Let a be a non light complex number and b be a non zero, light complex number. Let us observe that $\frac{a}{b}$ is heavy.

Let a be a heavy complex number and b be a non zero, non heavy complex number. One can verify that $\frac{a}{b}$ is heavy.

Let a be a heavy, positive real number and b be a non negative real number. Note that $a + b$ is heavy.

Let a be a heavy, negative real number and b be a non positive real number. Let us observe that $a + b$ is heavy.

Let a be a non light, positive real number and b be a positive real number. One can check that $a + b$ is heavy.

Let a be a non light, negative real number and b be a negative real number. Let us note that $a + b$ is heavy.

Let a be a non heavy real number and b be a heavy, positive real number. Let us observe that $a + b$ is positive.

Let a be a light real number and b be a non light, positive real number. Note that $a + b$ is positive.

Let a be a non heavy real number. Note that $a + b$ is non negative.

Let b be a heavy, negative real number. Observe that $a + b$ is negative.

Let a be a light real number and b be a non light, negative real number. One can check that $a + b$ is negative.

Let a be a non heavy real number. One can check that $a + b$ is non positive.

Let a be a light, positive real number and c be a light, negative real number. One can verify that $a + c$ is light.

Let a be a non heavy, positive real number and c be a non heavy, negative real number. Let us note that $a + c$ is non heavy.

Let a, b be real numbers. One can check that $a - \min(a, b)$ is non negative.

Let a, b be weightless real numbers. Observe that $\min(a, b)$ is weightless and $\max(a, b)$ is weightless.

Let a, b be light real numbers. Note that $\min(a, b)$ is light and $\max(a, b)$ is light.

Let a, b be heavy real numbers. One can verify that $\min(a, b)$ is heavy and $\max(a, b)$ is heavy.

Let a, b be positive real numbers. Observe that $\frac{\min(a,b)}{\max(a,b)}$ is non heavy and $\frac{\max(a,b)}{\min(a,b)}$ is non light and $\frac{a+b}{a}$ is heavy and $\frac{a}{a+b}$ is light.

Let us consider real numbers a, b. Now we state the propositions:

(15) If $a \cdot b$ is positive, then $|a - b| < |a + b|$.

(16) If $a \cdot b$ is negative, then $|a - b| > |a + b|$.

(17) Let us consider non zero real numbers a, b. Then $|a^2 - b^2| < |a^2 + b^2|$. The theorem is a consequence of (15).

(18) Let us consider positive real numbers a, b, c. If $a < b$, then $\frac{b+c}{a+c}$ is heavy.

(19) Let us consider positive real numbers a, b. Then $\frac{\frac{a}{b}+\frac{b}{a}}{2} \geqslant 1$.

(20) Let us consider negative real numbers a, b. Then $\frac{\frac{a}{b}+\frac{b}{a}}{2} \geqslant 1$.

(21) Let us consider a negative real number a, and a positive real number b. Then $\frac{\frac{a}{b}+\frac{b}{a}}{2} \leqslant -1$.

Let a, b be non zero real numbers. Let us note that $\frac{\frac{a}{b}+\frac{b}{a}}{2}$ is non light and $\frac{a}{b} + \frac{b}{a}$ is heavy.

Now we state the proposition:

(22) Let us consider non zero real numbers a, b. Then $(\frac{a}{b} + \frac{b}{a})^2 \geqslant 4$. The theorem is a consequence of (1).

Let a, b be positive real numbers. Note that $\frac{(a+2\cdot b)\cdot a}{(a+b)^2}$ is non heavy and $\frac{b}{a}+\frac{a}{b}-1$ is non light and $\frac{(a+b)\cdot(a^{-1}+b^{-1})}{4}$ is non light.

Let a, b be light real numbers. Let us note that $\frac{a+b}{1+a\cdot b}$ is non heavy.

Let a, b, c, d be positive real numbers. Note that $\frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$ is heavy.

Let a be a non negative real number. Observe that $|-a|$ reduces to a.

Observe that there exists a natural number which is trivial and non zero and there exists a natural number which is trivial.

Let a, b be non zero real numbers. One can verify that $\min(a, b)$ is non zero and $\max(a, b)$ is non zero.

Let a be a non negative real number and b be a real number. Let us note that $\max(a,b)$ is non negative.

Let a be a non positive real number. One can check that $\min(a,b)$ is non positive.

Let a be a positive real number. One can verify that $\max(a,b)$ is positive.

Let a be a negative real number. One can verify that $\min(a,b)$ is negative.

Let a, b be non negative real numbers. Observe that $\min(a,b)$ is non negative.

Let a, b be non positive real numbers. One can verify that $\max(a,b)$ is non positive.

Let a be a positive real number and b be a non negative real number. Observe that $\frac{a}{a+b}$ is non heavy and $\frac{a+b}{a}$ is non light.

Let a, b be positive real numbers. One can verify that $\frac{a}{\max(a,b)}$ is non heavy and $\frac{a}{\min(a,b)}$ is non light. Now we state the propositions:

(23) Let us consider real numbers a, b. If $\operatorname{sgn}(a) > \operatorname{sgn}(b)$, then $a > b$.

(24) Let us consider non zero real numbers a, b. Suppose $\operatorname{sgn}(a) > \operatorname{sgn}(b)$. Then

   (i) a is positive, and

   (ii) b is negative.

Let a, b be real numbers. Let us note that $\max(a,b) - \min(a,b)$ is non negative.

One can check that $(\operatorname{sgn}(a-b)) \cdot (\max(a,b) - \min(a,b))$ reduces to $a-b$.

Let a be a real number. Note that a^1 reduces to a and 1^a reduces to 1. One can check that a^0 is natural and a^0 is weightless.

Let a be a positive real number and b be a real number. One can check that a^b is positive.

Let a be a weightless, positive real number and b be a positive real number. Let us note that b^a reduces to b.

Let a be a heavy, positive real number. Observe that a^b is heavy.

Let b be a negative real number. Note that a^b is light.

Let a be a light, positive real number and b be a positive real number. Note that a^b is light.

Let b be a negative real number. Note that a^b is heavy.

Let a be a non weightless, positive real number and b be a real number. Observe that $\log_a(a^b)$ reduces to b.

Let b be a positive real number. Observe that $a^{\log_a b}$ reduces to b.

Now we state the propositions:

(25) Let us consider positive real numbers a, b. Then $a > b$ if and only if $\frac{1}{a} < \frac{1}{b}$.

(26) Let us consider negative real numbers a, b. Then $a > b$ if and only if $\frac{1}{a} < \frac{1}{b}$.

(27) Let us consider positive real numbers a, b. Then $\frac{1}{a} > \frac{1}{b}$ if and only if $-a > -b$.

(28) Let us consider negative real numbers a, b. Then $\frac{1}{a} > \frac{1}{b}$ if and only if $-a > -b$.

(29) Let us consider positive real numbers a, b. Then $\mathrm{sgn}(\frac{1}{a} - \frac{1}{b}) = \mathrm{sgn}(b-a)$.

(30) Let us consider negative real numbers a, b. Then $\mathrm{sgn}(\frac{1}{a} - \frac{1}{b}) = \mathrm{sgn}(b-a)$.

Let us consider non zero real numbers a, b. Now we state the propositions:

(31) $\mathrm{sgn}(\frac{1}{a} - \frac{1}{b}) = \mathrm{sgn}(b - a)$ if and only if $\mathrm{sgn}(b) = \mathrm{sgn}(a)$. The theorem is a consequence of (29), (30), and (24).

(32) $a + b = a \cdot b$ if and only if $\frac{1}{a} + \frac{1}{b} = 1$.

Let us consider positive real numbers a, b. Now we state the propositions:

(33) $a + b > a \cdot b$ if and only if $\frac{1}{a} + \frac{1}{b} > 1$.

(34) $a + b < a \cdot b$ if and only if $\frac{1}{a} + \frac{1}{b} < 1$. The theorem is a consequence of (32) and (33).

(35) Let us consider a non heavy, positive real number a, and a positive real number b. Then $a + b > a \cdot b$. The theorem is a consequence of (33).

(36) Let us consider non zero real numbers a, b. Then $a - b = a \cdot b$ if and only if $\frac{1}{b} - \frac{1}{a} = 1$.

(37) Let us consider positive real numbers a, b. If $a - b = a \cdot b$, then b is light. The theorem is a consequence of (1) and (36).

Let us consider positive real numbers a, b, c, d. Now we state the propositions:

(38) If $a + b = c + d$, then $\max(a, b) - \max(c, d) = \min(c, d) - \min(a, b)$.

(39) If $a + b = c + d$, then $\max(a, b) = \max(c, d)$ iff $\min(a, b) = \min(c, d)$.

(40) If $a + b = c + d$, then $\max(a, b) > \max(c, d)$ iff $\min(a, b) < \min(c, d)$. The theorem is a consequence of (38).

(41) If $a + b = c + d$ and $a \cdot b = c \cdot d$, then $\max(a, b) = \max(c, d)$. The theorem is a consequence of (38).

Let us consider positive real numbers a, b, c, d and a real number n. Now we state the propositions:

(42) If $a + b = c + d$ and $a \cdot b = c \cdot d$, then $a^n + b^n = c^n + d^n$. The theorem is a consequence of (41).

(43) If $a + b = c + d$ and $a^n + b^n \neq c^n + d^n$, then $a \cdot b \neq c \cdot d$.

Let us consider positive real numbers a, b, c, d. Now we state the propositions:

(44) If $a+b=c+d$, then $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}+\frac{1}{d}$ iff $a\cdot b=c\cdot d$.

(45) If $a+b=c+d$, then $\frac{1}{a}+\frac{1}{b}>\frac{1}{c}+\frac{1}{d}$ iff $a\cdot b<c\cdot d$.

(46) If $a+b\geqslant c+d$ and $a\cdot b<c\cdot d$, then $\frac{1}{a}+\frac{1}{b}>\frac{1}{c}+\frac{1}{d}$.

(47) If $a\cdot b<c\cdot d$ and $\frac{1}{a}+\frac{1}{b}\leqslant\frac{1}{c}+\frac{1}{d}$, then $a+b<c+d$.

(48) If $a+b\leqslant c+d$ and $\frac{1}{a}+\frac{1}{b}>\frac{1}{c}+\frac{1}{d}$, then $a\cdot b<c\cdot d$.

(49) If $a\cdot b\geqslant c\cdot d$, then $a+b>c+d$ or $\frac{1}{a}+\frac{1}{b}\leqslant\frac{1}{c}+\frac{1}{d}$.

(50) Let us consider positive real numbers a, b, and real numbers n, m. Then

(i) $a^{m+n}+b^{m+n}=\frac{(a^m+b^m)\cdot(a^n+b^n)+(a^n-b^n)\cdot(a^m-b^m)}{2}$, and

(ii) $a^{m+n}-b^{m+n}=\frac{(a^m+b^m)\cdot(a^n-b^n)+(a^n+b^n)\cdot(a^m-b^m)}{2}$.

(51) Let us consider positive real numbers a, b, and a real number n. Then $a^{n+1}+b^{n+1}=\frac{(a^n+b^n)\cdot(a+b)+(a-b)\cdot(a^n-b^n)}{2}$. The theorem is a consequence of (50).

Let us consider positive real numbers a, b and positive real numbers n, m. Now we state the propositions:

(52) $a^{n+m}+b^{n+m}\geqslant\frac{(a^n+b^n)\cdot(a^m+b^m)}{2}$.

Proof: $(a^n-b^n)\cdot(a^m-b^m)\geqslant 0$. □

(53) $a^{n+m}+b^{n+m}=\frac{(a^n+b^n)\cdot(a^m+b^m)}{2}$ if and only if $a=b$.

Proof: If $a=b$, then $a^{n+m}+b^{n+m}=\frac{(a^n+b^n)\cdot(a^m+b^m)+0}{2}$. If $a\neq b$, then $(a^n-b^n)\cdot(a^m-b^m)>0$. □

Let us consider positive real numbers a, b, c, d. Now we state the propositions:

(54) If $a+b\leqslant c+d$ and $\max(a,b)>\max(c,d)$, then $a\cdot b<c\cdot d$.

(55) If $a+b\leqslant c+d$ and $a\cdot b>c\cdot d$, then $\max(a,b)<\max(c,d)$ and $\min(a,b)>\min(c,d)$. The theorem is a consequence of (54).

(56) $\max(a,b)=\max(c,d)$ and $\min(a,b)=\min(c,d)$ if and only if $a\cdot b=c\cdot d$ and $a+b=c+d$. The theorem is a consequence of (41).

(57) Let us consider non negative real numbers a, b, and a positive real number c. Then $a\geqslant b$ if and only if $a^c\geqslant b^c$.

(58) Let us consider non negative real numbers a, b, n. Then

(i) $\max(a^n,b^n)=(\max(a,b))^n$, and

(ii) $\min(a^n,b^n)=(\min(a,b))^n$.

The theorem is a consequence of (57).

(59) Let us consider positive real numbers a, b, c, d. Suppose $a\cdot b>c\cdot d$ and $\frac{a}{b}\geqslant\frac{c}{d}$ or $a\cdot b\geqslant c\cdot d$ and $\frac{a}{b}>\frac{c}{d}$. Then $a>c$.

(60) Let us consider a positive real number a. Then $1-a<\frac{1}{1+a}$.

(61) Let us consider a light, positive real number a. Then $1 + a < \frac{1}{1-a}$.

(62) Let us consider positive real numbers a, b, a non negative real number m, and a positive real number n. If $a^m + b^m \leqslant 1$, then $a^{m+n} + b^{m+n} < 1$. The theorem is a consequence of (1).

(63) Let us consider positive real numbers a, b, a non positive real number m, and a negative real number n. If $a^m + b^m \leqslant 1$, then $a^{m+n} + b^{m+n} < 1$. The theorem is a consequence of (62).

(64) Let us consider positive real numbers a, b, c, n, and a non negative real number m. If $a^m + b^m \leqslant c^m$, then $a^{m+n} + b^{m+n} < c^{m+n}$. The theorem is a consequence of (62).

(65) Let us consider positive real numbers a, b, and a heavy, positive real number n. Then $a^n + b^n < (a+b)^n$. The theorem is a consequence of (64).

Let k be a positive real number and n be a heavy, positive real number. Let us observe that $(k + 1)^n - k^n$ is heavy and positive.

Let k be a heavy, positive real number and n be a non negative real number. One can verify that $k^{n+1} - k^n$ is positive.

Now we state the propositions:

(66) Let us consider a positive real number k, and a heavy, positive real number n. Then $(k+1)^n > k^n + 1$. The theorem is a consequence of (65).

(67) Let us consider positive real numbers a, b, and a light, positive real number n. Then $a^n + b^n > (a+b)^n$. The theorem is a consequence of (64).

(68) Let us consider a positive real number k, and a light, positive real number n. Then $(k + 1)^n < k^n + 1$. The theorem is a consequence of (67).

(69) Let us consider a positive real number k, and a non positive real number n. Then $(k + 1)^n < k^n + 1$.

(70) Let us consider positive real numbers a, b, and a non positive real number n. Then $a^n + b^n > (a + b)^n$. The theorem is a consequence of (69).

Let us consider positive real numbers a, b and a real number n. Now we state the propositions:

(71) $(a+b)^n > a^n + b^n$ if and only if n is heavy and positive. The theorem is a consequence of (1), (67), (70), and (65).

(72) $(a+b)^n = a^n + b^n$ if and only if $n = 1$. The theorem is a consequence of (71), (70), and (67).

(73) $(a+b)^n < a^n + b^n$ if and only if $n < 1$. The theorem is a consequence of (1), (71), and (72).

Let us consider positive real numbers a, b, c. Now we state the propositions:

(74) $(a + b) \cdot (a + c) > a \cdot (a + b + c)$.

(75) $\frac{a+b+c}{a+b} < \frac{a+c}{a}$. The theorem is a consequence of (74).

(76) Let us consider positive real numbers a, b, c, and a positive real number n. Then $\frac{(a+b+c)^n}{(a+b)^n} < \frac{(a+c)^n}{a^n}$. The theorem is a consequence of (75).

(77) Let us consider heavy, positive real numbers a, b. Then $a+b-1 > \frac{a}{b} > \frac{1}{a+b-1}$. The theorem is a consequence of (1).

(78) Let us consider positive real numbers a, b, c. Then $\frac{a+b+c}{a} > \frac{a+b}{a+c} > \frac{a}{a+b+c}$. The theorem is a consequence of (77).

Let us consider a light, positive real number a and a heavy, positive real number n. Now we state the propositions:

(79) $(1+a)^n \cdot (1-a)^n < (1+a^n) \cdot (1-a^n)$. The theorem is a consequence of (65).

(80) $\frac{(1+a)^n}{1+a^n} < \frac{1-a^n}{(1-a)^n}$. The theorem is a consequence of (79).

Let us consider a light, positive real number a. Now we state the propositions:

(81) (i) $\max(a, 1-a) \geqslant \frac{1}{2}$, and

(ii) $\min(a, 1-a) \leqslant \frac{1}{2}$.

(82) $\frac{1}{1+a} + \frac{1}{1-a} > 2$.

(83) Let us consider a heavy, positive real number a. Then $\frac{1}{a+1} + \frac{1}{a-1} > \frac{2}{a}$.

(84) Let us consider positive real numbers a, b, and a heavy, positive real number n. Then $(2 \cdot a+b)^n + b^n < 2 \cdot (a+b)^n$. The theorem is a consequence of (65).

(85) Let us consider heavy, positive real numbers a, n. Then $(a+1)^n - (a-1)^n > 2^n$. The theorem is a consequence of (65).

(86) Let us consider a light, positive real number a, and a heavy, positive real number n. Then $2^n > (1+a)^n - (1-a)^n > 2 \cdot a^n$. The theorem is a consequence of (1) and (65).

(87) Let us consider heavy, positive real numbers a, n, and a light, positive real number b. Then $(a+1)^n - (a-1)^n > (a+b)^n - (a-b)^n > 2 \cdot b^n$. The theorem is a consequence of (1) and (65).

(88) Let us consider positive real numbers a, b, and a positive real number n. Then $2 \cdot (a+b)^n > (a+b)^n + a^n > 2 \cdot (a^n)$.

Let us consider positive real numbers a, b. Now we state the propositions:

(89) If $a \neq b$, then there exist real numbers n, m such that $a = \frac{a}{b}^n$ and $b = \frac{a}{b}^m$.

(90) If $a \neq b$, then there exist real numbers n, m such that $a-b = \frac{a}{b}^n \cdot (\frac{a}{b}^m - 1)$. The theorem is a consequence of (89).

(91) Let us consider positive real numbers a, m, n. Then $a^n + a^m = a^{\min(n,m)} \cdot (1 + a^{|m-n|})$.

(92) Let us consider non weightless, positive real numbers a, b. Then $\log_a b = \frac{1}{\log_b a}$. The theorem is a consequence of (1).

Let a be a heavy, positive real number and b be a positive real number. One can check that $\log_a(a+b)$ is heavy and $\log_{a+b} a$ is light.

Now we state the propositions:

(93) Let us consider a positive, non weightless real number a, and a positive real number b. Then $\log_a b = 0$ if and only if $b = 1$.
PROOF: $|a| \neq 1$. If $\log_a b = 0$, then $b = 1$. □

(94) Let us consider a non weightless, positive real number a, and a positive real number b. Then $\log_a b = 1$ if and only if $a = b$. The theorem is a consequence of (1).

(95) Let us consider positive real numbers a, b, and a non zero real number n. Then $a^n = b^n$ if and only if $a = b$.
PROOF: If $a \neq b$, then $a^n \neq b^n$. □

(96) Let us consider a non weightless, positive real number a, and a positive real number b. Then

(i) $\log_a b = -\log_{\frac{1}{a}} b$, and

(ii) $\log_{\frac{1}{a}} b = \log_a \frac{1}{b}$, and

(iii) $\log_a b = -\log_a \frac{1}{b}$, and

(iv) $\log_a b = \log_{\frac{1}{a}} \frac{1}{b}$.

The theorem is a consequence of (1).

(97) Let us consider a heavy, positive real number a, and a positive real number b. Then $a > b$ if and only if $\log_a b < 1$.
PROOF: $a > 1$. If $\log_a b < 1$, then $a > b$. If $a > b$, then $\log_a b < 1$. □

(98) Let us consider a light, positive real number a, and a positive real number b. Then $a < b$ if and only if $\log_a b < 1$. The theorem is a consequence of (97) and (96).

(99) Let us consider a heavy, positive real number a, and a positive real number b. Then $a < b$ if and only if $\log_a b > 1$. The theorem is a consequence of (97) and (94).

(100) Let us consider a light, positive real number a, and a positive real number b. Then $a > b$ if and only if $\log_a b > 1$. The theorem is a consequence of (99) and (96).

Let us consider non weightless, positive real numbers a, b. Now we state the propositions:

(101) If $\log_a b \geqslant 1$, then $0 < \log_b a \leqslant 1$. The theorem is a consequence of (92).

(102) If $\log_a b \leqslant -1$, then $0 > \log_b a \geqslant -1$. The theorem is a consequence of (92).

Let us consider heavy, positive real numbers a, b. Now we state the propositions:

(103) If $\log_a b > \log_b a \geqslant 1$, then $a > b$. The theorem is a consequence of (1).

(104) If $\log_b a < 1$, then $a < b$. The theorem is a consequence of (1) and (94).

Let us consider heavy, positive real numbers a, c and positive real numbers b, d. Now we state the propositions:

(105) If $\log_a b \leqslant \log_c d$ and $a < b$, then $c < d$. The theorem is a consequence of (99).

(106) If $\log_a b \geqslant \log_c d$ and $a > b$, then $c > d$. The theorem is a consequence of (97).

Let us consider a heavy, positive real number a, a light, positive real number c, and positive real numbers b, d. Now we state the propositions:

(107) If $\log_a b \leqslant \log_c d$ and $a < b$, then $c > d$. The theorem is a consequence of (99) and (100).

(108) If $\log_a b \geqslant \log_c d$ and $a > b$, then $c < d$. The theorem is a consequence of (97) and (98).

Let us consider light, positive real numbers a, c and positive real numbers b, d. Now we state the propositions:

(109) If $\log_a b \leqslant \log_c d$ and $a > b$, then $c > d$. The theorem is a consequence of (96) and (105).

(110) If $\log_a b \geqslant \log_c d$ and $a < b$, then $c < d$. The theorem is a consequence of (96) and (106).

Let us consider a light, positive real number a, a heavy, positive real number c, and positive real numbers b, d. Now we state the propositions:

(111) If $\log_a b \leqslant \log_c d$ and $a > b$, then $c < d$. The theorem is a consequence of (100) and (99).

(112) If $\log_a b \geqslant \log_c d$ and $a < b$, then $c > d$. The theorem is a consequence of (98) and (97).

Let us consider heavy, positive real numbers a, c and positive real numbers b, d. Now we state the propositions:

(113) If $\log_a b < \log_c d$ and $a \leqslant b$, then $c < d$. The theorem is a consequence of (97) and (99).

(114) If $\log_a b \leqslant \log_c d$ and $a \leqslant b$, then $c \leqslant d$. The theorem is a consequence of (97).

(115) Let us consider positive real numbers a, b. If $a > b$, then $\log_{\frac{a}{b}} a > \log_{\frac{a}{b}} b$.

ACKNOWLEDGEMENT: Ad Maiorem Dei Gloriam

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Accepted May 31, 2020

FORMALIZED MATHEMATICS
Vol. 28, No. 2, Pages 211–215, 2020
DOI: 10.2478/forma-2020-0018
https://www.sciendo.com/

Grothendieck Universes[1]

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Summary. The foundation of the Mizar Mathematical Library [2], is first-order Tarski-Grothendieck set theory. However, the foundation explicitly refers only to Tarski's Axiom A, which states that for every set X there is a Tarski universe U such that $X \in U$. In this article, we prove, using the Mizar [3] formalism, that the Grothendieck name is justified. We show the relationship between Tarski and Grothendieck universe.

First we prove in Theorem (17) that every Grothendieck universe satisfies Tarski's Axiom A. Then in Theorem (18) we prove that every Grothendieck universe that contains a given set X, even the least (with respect to inclusion) denoted by `GrothendieckUniverse` X, has as a subset the least (with respect to inclusion) Tarski universe that contains X, denoted by the `Tarski-Class` X. Since Tarski universes, as opposed to Grothendieck universes [5], might not be transitive (called `epsilon-transitive` in the Mizar Mathematical Library [1]) we focused our attention to demonstrate that `Tarski-Class` $X \subsetneq$ `GrothendieckUniverse` X for some X.

Then we show in Theorem (19) that `Tarski-Class` X where X is the singleton of any infinite set is a proper subset of `GrothendieckUniverse` X. Finally we show that `Tarski-Class` $X =$ `GrothendieckUniverse` X holds under the assumption that X is a transitive set.

The formalisation is an extension of the formalisation used in [4].

MSC: 03E70 68V20

Keywords: Tarski-Grothendieck set theory; Tarski's Axiom A; Grothendieck universe

MML identifier: CLASSES3, version: 8.1.10 5.63.1382

[1]This work has been supported by the Polish National Science Centre granted by decision no. DEC-2015/19/D/ST6/01473.

ISSN 1426–2630(Print), 1898-9934(Online)

1. Grothendieck Universes Axioms

From now on X, Y, Z denote sets, x, y, z denote objects, and A, B, C denote ordinal numbers.

Let us consider X. We say that X is power-closed if and only if

(Def. 1) if $Y \in X$, then $2^Y \in X$.

We say that X is union-closed if and only if

(Def. 2) if $Y \in X$, then $\bigcup Y \in X$.

We say that X is Family-Union-closed if and only if

(Def. 3) for every Y and for every function f such that $\operatorname{dom} f = Y$ and $\operatorname{rng} f \subseteq X$ and $Y \in X$ holds $\bigcup \operatorname{rng} f \in X$.

Note that every set which is Tarski is also power-closed and subset-closed and every set which is transitive and Tarski is also union-closed and Family-Union-closed and every set which is transitive and Family-Union-closed is also union-closed and every set which is transitive and power-closed is also subset-closed.

A Grothendieck is a transitive, power-closed, Family-Union-closed set.

2. Grothendieck Universe Operator

Let X be a set. A Grothendieck of X is a Grothendieck defined by

(Def. 4) $X \in it$.

Let G_1, G_2 be Grothendiecks. One can verify that $G_1 \cap G_2$ is transitive, power-closed, and Family-Union-closed.

Now we state the proposition:

(1) Let us consider Grothendiecks G_1, G_2 of X. Then $G_1 \cap G_2$ is a Grothendieck of X.

Let X be a set. The functor GrothendieckUniverse(X) yielding a Grothendieck of X is defined by

(Def. 5) for every Grothendieck G of X, $it \subseteq G$.

The scheme *ClosedUnderReplacement* deals with a set $\mathcal{X}$ and a Grothendieck $\mathcal{U}$ of $\mathcal{X}$ and a unary functor $\mathcal{F}$ yielding a set and states that

(Sch. 1) $\{\mathcal{F}(x)$, where x is an element of $\mathcal{X} : x \in \mathcal{X}\} \in \mathcal{U}$

provided

- if $Y \in \mathcal{X}$, then $\mathcal{F}(Y) \in \mathcal{U}$.

In the sequel U denotes a Grothendieck. Now we state the proposition:

(2) Let us consider a function f. If $\mathrm{dom}\, f \in U$ and $\mathrm{rng}\, f \subseteq U$, then $\mathrm{rng}\, f \in U$.
PROOF: Set $A = \mathrm{dom}\, f$. Define $\mathcal{S}(\text{set}) = \{f(\$_1)\}$. Consider s being a function such that $\mathrm{dom}\, s = A$ and for every X such that $X \in A$ holds $s(X) = \mathcal{S}(X)$. $\mathrm{rng}\, s \subseteq U$. $\bigcup s \subseteq \mathrm{rng}\, f$. $\mathrm{rng}\, f \subseteq \bigcup s$. $\square$

3. SET OF ALL SETS UP TO GIVEN RANK

Let x be an object. The functor $\mathbf{R}\mathrm{rank}(x)$ yielding a transitive set is defined by the term

(Def. 6) $\mathbf{R}_{\mathrm{rk}(x)}$.

Now we state the propositions:

(3) $X \in \mathbf{R}_A$ if and only if there exists B such that $B \in A$ and $X \in 2^{\mathbf{R}_B}$.
PROOF: If $X \in \mathbf{R}_A$, then there exists B such that $B \in A$ and $X \in 2^{\mathbf{R}_B}$. $\square$

(4) $Y \in \mathbf{R}\mathrm{rank}(X)$ if and only if there exists Z such that $Z \in X$ and $Y \in 2^{\mathbf{R}\mathrm{rank}(Z)}$.
PROOF: If $Y \in \mathbf{R}\mathrm{rank}(X)$, then there exists Z such that $Z \in X$ and $Y \in 2^{\mathbf{R}\mathrm{rank}(Z)}$. $\square$

(5) If $x \in X$ and $y \in \mathbf{R}\mathrm{rank}(x)$, then $y \in \mathbf{R}\mathrm{rank}(X)$.

(6) If $Y \in \mathbf{R}\mathrm{rank}(X)$, then there exists x such that $x \in X$ and $Y \subseteq \mathbf{R}\mathrm{rank}(x)$. The theorem is a consequence of (4).

(7) $X \subseteq \mathbf{R}\mathrm{rank}(X)$.

(8) If $X \subseteq \mathbf{R}\mathrm{rank}(Y)$, then $\mathbf{R}\mathrm{rank}(X) \subseteq \mathbf{R}\mathrm{rank}(Y)$.

(9) If $X \in \mathbf{R}\mathrm{rank}(Y)$, then $\mathbf{R}\mathrm{rank}(X) \in \mathbf{R}\mathrm{rank}(Y)$.

(10) (i) $X \in \mathbf{R}\mathrm{rank}(Y)$, or

(ii) $\mathbf{R}\mathrm{rank}(Y) \subseteq \mathbf{R}\mathrm{rank}(X)$.

(11) (i) $\mathbf{R}\mathrm{rank}(X) \in \mathbf{R}\mathrm{rank}(Y)$, or

(ii) $\mathbf{R}\mathrm{rank}(Y) \subseteq \mathbf{R}\mathrm{rank}(X)$.

(12) If $X \in U$ and $X \approx A$, then $A \in U$.
PROOF: Define $\mathcal{P}[\text{ordinal number}] \equiv$ for every X such that $X \approx \$_1$ and $X \in U$ holds $\$_1 \in U$. For every ordinal number A such that for every ordinal number C such that $C \in A$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[A]$. For every ordinal number O, $\mathcal{P}[O]$. $\square$

(13) If $X \in Y \in U$, then $X \in U$.

(14) If $X \in U$, then $\mathbf{R}\mathrm{rank}(X) \in U$.

Proof: Define $\mathcal{P}$[ordinal number] $\equiv$ for every set A such that $\mathrm{rk}(A) \in \$_1$ and $A \in U$ holds $\mathbf{R}\mathrm{rank}(A) \in U$. For every A such that for every C such that $C \in A$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[A]$. For every ordinal number O, $\mathcal{P}[O]$. $\square$

(15) If $A \in U$, then $\mathbf{R}_A \in U$.
Proof: Define $\mathcal{P}$[ordinal number] $\equiv$ if $\$_1 \in U$, then $\mathbf{R}_{\$_1} \in U$. For every A such that for every C such that $C \in A$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[A]$. For every ordinal number O, $\mathcal{P}[O]$. $\square$

4. Tarski vs. Grothendieck Universe

Now we state the propositions:

(16) If $X \subseteq U$ and $X \notin U$, then there exists a function f such that f is one-to-one and $\mathrm{dom}\, f = \mathrm{On}\, U$ and $\mathrm{rng}\, f = X$.
Proof: For every set x such that $x \in \mathrm{On}\, U$ holds x is an ordinal number and $x \subseteq \mathrm{On}\, U$. Reconsider $\Lambda = \mathrm{On}\, U$ as an ordinal number. There exists a function *THE* such that for every set x such that $\emptyset \neq x \subseteq X$ holds $\mathit{THE}(x) \in x$. Consider *THE* being a function such that for every set x such that $\emptyset \neq x \subseteq X$ holds $\mathit{THE}(x) \in x$. Define $\mathcal{R}(\text{set}) = \{\mathrm{rk}(x)$, where x is an element of $\$_1 : x \in \$_1\}$. For every set A and for every object x, $x \in \mathcal{R}(A)$ iff there exists a set a such that $a \in A$ and $x = \mathrm{rk}(a)$.

Define $\mathcal{Q}[\text{set}, \text{object}] \equiv \$_2 \in X \setminus \$_1$ and for every ordinal number B such that $B \in \mathcal{R}(X \setminus \$_1)$ holds $\mathrm{rk}(\$_2) \subseteq B$. Define $\mathcal{F}(\text{transfinite sequence}) = \mathit{THE}(\{x$, where x is an element of $X : \mathcal{Q}[\mathrm{rng}\, \$_1, x]\})$. Consider f being a transfinite sequence such that $\mathrm{dom}\, f = \Lambda$ and for every ordinal number A and for every transfinite sequence L such that $A \in \Lambda$ and $L = f{\restriction}A$ holds $f(A) = \mathcal{F}(L)$. For every ordinal number A such that $A \in \Lambda$ holds $\mathcal{Q}[\mathrm{rng}(f{\restriction}A), f(A)]$. f is one-to-one. $\mathrm{rng}\, f \subseteq X$. $X \subseteq \mathrm{rng}\, f$. $\square$

(17) Every Grothendieck is Tarski.
Proof: If $X \notin U$, then $X \approx U$. $\square$

Let us note that every set which is transitive, power-closed, and Family-Union-closed is also universal and every set which is universal is also transitive, power-closed, and Family-Union-closed.

Now we state the propositions:

(18) Let us consider a Grothendieck G of X. Then $\mathbf{T}(X) \subseteq G$.

(19) Let us consider an infinite set X. Then $X \notin \mathbf{T}(\{X\})$.
Proof: Define $\mathcal{B}(\text{set}, \text{set}) = \$_2 \cup 2^{\$_2}$. Consider f being a function such that $\mathrm{dom}\, f = \mathbb{N}$ and $f(0) = \{\{A\}, \emptyset\}$ and for every natural number n, $f(n+1) = \mathcal{B}(n, f(n))$. Set $U = \bigcup f$. Define $\mathcal{M}[\text{object}, \text{object}] \equiv \$_1 \in f(\$_2)$ and $\$_2 \in \mathrm{dom}\, f$ and for every natural numbers i, j such that $i < j = \$_2$

holds $\$_1 \notin f(i)$. For every object x such that $x \in U$ there exists an object y such that $\mathcal{M}[x, y]$.

Consider M being a function such that $\operatorname{dom} M = U$ and for every object x such that $x \in U$ holds $\mathcal{M}[x, M(x)]$. U is subset-closed. For every X such that $X \in U$ holds $2^X \in U$. Define $\mathcal{D}[\text{natural number}] \equiv f(\$_1)$ is finite. For every natural number n such that $\mathcal{D}[n]$ holds $\mathcal{D}[n+1]$. For every natural number n, $\mathcal{D}[n]$. For every set x such that $x \in \operatorname{dom} f$ holds $f(x)$ is countable. For every X such that $X \subseteq U$ holds $X \approx U$ or $X \in U$. $A \notin U$. $\square$

(20) Let us consider an infinite set X. Then $\mathbf{T}(\{X\}) \subset$ GrothendieckUniverse $(\{X\})$. The theorem is a consequence of (18) and (19).

(21) (i) GrothendieckUniverse(X) is a universal class, and

(ii) for every universal class U such that $X \in U$ holds GrothendieckUniverse$(X) \subseteq U$.

(22) Let us consider a transitive set X. Then $\mathbf{T}(X) =$ GrothendieckUniverse(X). The theorem is a consequence of (18).

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Accepted May 31, 2020

FORMALIZED MATHEMATICS
Vol. 28, No. 2, Pages 217–225, 2020
DOI: 10.2478/forma-2020-0019

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Formalization of Quasilattices

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Summary. The main aim of this article is to introduce formally one of the generalizations of lattices, namely quasilattices, which can be obtained from the axiomatization of the former class by certain weakening of ordinary absorption laws. We show propositions QLT-1 to QLT-7 from [15], presenting also some short variants of corresponding axiom systems. Some of the results were proven in the Mizar [1], [2] system with the help of Prover9 [14] proof assistant.

MSC: 68V20 06B05 06B75

Keywords: lattice theory; quasilattice; absorption law

MML identifier: `LATQUASI`, version: `8.1.10 5.63.1382`

0. Introduction

For years, lattice theory was quite dynamically developed area of mathematics represented formally in the Mizar Mathematical Library. The first Mizar article in this topic was [18], and the monographs of two authors were stimulating source for formalization efforts: Birkhoff [3] (especially at the very beginning), and then Grätzer [12], [13]. The chosen approach was just the algebraic one, with two operation of binary supremum and infimum, and the induced ordering relation as a generated Mizar predicate.

Initially, the formalization efforts within lattice theory were not very systematic, but during the project of translating "Compendium of Continuous Lattices" [5] into Mizar formalism with a number of people involved, a lot of work was done to provide the alternative approach for lattices, with relational structures as the starting point (as it was claimed in [4]).

The series of Mizar articles with MML identifiers beginning with `YELLOW` (with numerals), e.g. [7] was written to explore this specific field in a more detailed way, but the structures behind both approaches are different (although from the informal viewpoint the difference is meaningless [10]). Still however, the correspondence between relational structures and lattices in the form of the Mizar structure `LattRelStr` with binary operations and the underlying ordering relation available as parallel selectors in the merged structure was studied [8]. An overview of the mechanization of lattice theory in the repository of Mizar texts can be found in [6]. Most of described efforts were done more or less manually.

Our work can be seen as a step towards a Mizar support for [15] or [16], where original proof objects by Otter/Prover9 were used. Some preliminary works in this direction were already done in [9] by present authors. We use the interface `ott2miz` [17] which allows for the automated translation of proofs; these automatically generated proofs are usually quite lenghty, even after native enhancements done by internal Mizar software for library revisions.

In the present development, we deal with the parts of Chap. 6 "Lattice-like algebras" of [15], pp. 111–135, devoted to quasilattices.

The class of quasilattices (QLT) can be characterized from the standard set of axioms for lattices (with idempotence for the join and meet operations included), where absorption laws are replaced by the pair of link laws (called `QLT1` and `QLT2` in the Mizar source – compare Def. 1 and Def. 2). Def. 8 and Def. 9 provide standard examples of structures which are quasilatices, but not necessarily lattices (absorption laws do not hold). In the latter one, the lattice operations are given by

$\sqcup$	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

$\sqcap$	0	1	2
0	0	1	0
1	1	1	1
2	0	1	2

Then we prove, using Mizar formalism, the new form of distributivity for QLT, that the standard distributivity implies its dual, and self-dual, a bit longer, form of distributivity (QLT-1, QLT-2, QLT-3). Later we characterize Bowden's inequality (which forces quasilattices, and hence lattices, to be distributive – QLT-4) and some modularity conditions (QLT-5 and QLT-6) – both in the form of the equations (taking into account automatic treatment of the equality predicate in Mizar [11] and the design of Prover9 this is more feasible), and in the more common (at least from informal point of view) form of implication with inequality. The final section shows that the meet operation need not be unique in QLT (although in the class of lattices, starting with the same join operation, the other operation is uniquely defined).

1. PRELIMINARIES

From now on L denotes a non empty lattice structure and v_3, v_{101}, v_{100}, v_{102}, v_{103}, v_2, v_1, v_0 denote elements of L.

Let L be a non empty lattice structure. We say that L satisfies QLT1 if and only if

(Def. 1) for every elements v_0, v_2, v_1 of L, $(v_0 \sqcap (v_1 \sqcup v_2)) \sqcup (v_0 \sqcap v_1) = v_0 \sqcap (v_1 \sqcup v_2)$.

We say that L satisfies QLT2 if and only if

(Def. 2) for every elements v_0, v_2, v_1 of L, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcap (v_0 \sqcup v_1) = v_0 \sqcup (v_1 \sqcap v_2)$.

We say that L is QLT-distributive if and only if

(Def. 3) for every elements v_1, v_2, v_0 of L, $v_0 \sqcap (v_1 \sqcup (v_0 \sqcap v_2)) = v_0 \sqcap (v_1 \sqcup v_2)$.

Observe that every non empty lattice structure which is trivial is also QLT-distributive and satisfies also QLT1 and QLT2 and every non empty lattice structure which is trivial is also join-idempotent and meet-idempotent and there exists a non empty lattice structure which is join-commutative, join-associative, join-idempotent, meet-commutative, meet-associative, and meet-idempotent and satisfies QLT1 and QLT2.

Let L be a join-commutative, non empty lattice structure. One can verify that L satisfies QLT1 if and only if the condition (Def. 4) is satisfied.

(Def. 4) for every elements v_0, v_1, v_2 of L, $v_0 \sqcap v_1 \sqsubseteq v_0 \sqcap (v_1 \sqcup v_2)$.

Note that $\{0, 1, 2\}$ is real-membered and every element of $\{0, 1, 2\}$ is real.

Let x, y be elements of $\{0, 1, 2\}$. The functor $\mathrm{OpEx2}(x, y)$ yielding an element of $\{0, 1, 2\}$ is defined by the term

(Def. 5) $\begin{cases} 1, & \textbf{if } x = 1 \text{ or } y = 1, \\ \min(x, y), & \textbf{if } x \neq 1 \text{ and } y \neq 1. \end{cases}$

The functors: QLTEx1 and QLTEx2 yielding binary operations on $\{0, 1, 2\}$ are defined by conditions

(Def. 6) for every elements x, y of $\{0, 1, 2\}$, if $x = y$, then $\mathrm{QLTEx1}(x, y) = x$ and if $x \neq y$, then $\mathrm{QLTEx1}(x, y) = 0$,

(Def. 7) for every elements x, y of $\{0, 1, 2\}$, if $x = 1$ or $y = 1$, then $\mathrm{QLTEx2}(x, y) = 1$ and if $x \neq 1$ and $y \neq 1$, then $\mathrm{QLTEx2}(x, y) = \min(x, y)$,

respectively. Now we state the proposition:

(1) $\mathrm{QLTEx1} \neq \mathrm{QLTEx2}$.

The functors: QLTLattice1 and QLTLattice2 yielding strict, non empty lattice structures are defined by terms

(Def. 8) $\langle \{0, 1, 2\}, \mathrm{QLTEx1}, \mathrm{QLTEx1} \rangle$,

(Def. 9) $\langle \{0, 1, 2\}, \mathrm{QLTEx1}, \mathrm{QLTEx2} \rangle$, respectively. Let us note that QLTEx1 is commutative, associative, and idempotent and QLTEx2 is commutative, associative, and idempotent and QLTLattice1 is join-commutative, join-associative, and join-idempotent and QLTLattice1 is meet-commutative, meet-associative, and meet-idempotent.

Let us consider elements v_0, v_1 of QLTLattice1. Now we state the propositions:

(2) If $v_1 = 0$, then $v_0 \sqcap v_1 = v_1$.

(3) If $v_1 = 0$, then $v_0 \sqcup v_1 = v_1$.

Observe that QLTLattice1 satisfies QLT1 and QLTLattice1 satisfies QLT2 and every element of QLTLattice2 is real and QLTLattice2 is join-commutative, join-associative, and join-idempotent and QLTLattice2 is meet-commutative, meet-associative, and meet-idempotent.

Observe also that QLTLattice2 satisfies QLT1 and QLTLattice2 satisfies QLT2 and QLTLattice2 is non join-absorbing and QLTLattice2 is non meet-absorbing and QLTLattice1 is non join-absorbing and QLTLattice1 is non meet-absorbing.

A quasilattice is a join-commutative, join-associative, meet-commutative, meet-associative, join-idempotent, meet-idempotent, non empty lattice structure satisfying QLT1 and QLT2.

2. Properties of Quasilattices: QLT-1

Now we state the propositions:

(4) Suppose for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcap (v_1 \sqcup v_2)) \sqcup (v_0 \sqcap v_1) = v_0 \sqcap (v_1 \sqcup v_2)$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$ and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcap (v_0 \sqcup v_1) = v_0 \sqcup (v_1 \sqcap v_2)$ and for every v_1, v_2, and v_0, $v_0 \sqcap (v_1 \sqcup (v_0 \sqcap v_2)) = v_0 \sqcap (v_1 \sqcup v_2)$. $(v_1 \sqcap v_2) \sqcup (v_1 \sqcap v_3) = v_1 \sqcap (v_2 \sqcup v_3)$.

(5) If L is meet-commutative, join-idempotent, join-associative, join-commutative, and QLT-distributive and satisfies QLT1 and QLT2, then L is distributive. The theorem is a consequence of (4).

Observe that every non empty lattice structure which is meet-commutative, join-idempotent, join-associative, join-commutative, and QLT-distributive and satisfies QLT1 and QLT2 is also distributive.

3. QLT-2

Now we state the propositions:

(6) Suppose for every v_0, $v_0 \sqcap v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcap v_1) \sqcap v_2 = v_0 \sqcap (v_1 \sqcap v_2)$ and for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$ and for every v_0, v_2, and v_1, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcap (v_0 \sqcup v_1) = v_0 \sqcup (v_1 \sqcap v_2)$ and for every v_0, v_2, and v_1, $v_0 \sqcap (v_1 \sqcup v_2) = (v_0 \sqcap v_1) \sqcup (v_0 \sqcap v_2)$. $v_1 \sqcup (v_2 \sqcap v_3) = (v_1 \sqcup v_2) \sqcap (v_1 \sqcup v_3)$.

(7) If L is meet-idempotent, meet-associative, meet-commutative, join-idempotent, join-associative, and distributive and satisfies QLT2, then L is distributive'. The theorem is a consequence of (6).

Let us observe that every non empty lattice structure which is meet-idempotent, meet-associative, meet-commutative, join-idempotent, join-associative, and distributive and satisfies QLT2 is also distributive'.

4. QLT-3

Let us consider L. We say that L is QLT-selfdistributive if and only if

(Def. 10) for every v_2, v_1, and v_0, $(((v_0 \sqcap v_1) \sqcup v_2) \sqcap v_1) \sqcup (v_2 \sqcap v_0) = (((v_0 \sqcup v_1) \sqcap v_2) \sqcup v_1) \sqcap (v_2 \sqcup v_0)$.

Now we state the proposition:

(8) Suppose for every v_0, $v_0 \sqcap v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcap v_1) \sqcap v_2 = v_0 \sqcap (v_1 \sqcap v_2)$ and for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcap (v_1 \sqcup v_2)) \sqcup (v_0 \sqcap v_1) = v_0 \sqcap (v_1 \sqcup v_2)$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$ and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcap (v_0 \sqcup v_1) = v_0 \sqcup (v_1 \sqcap v_2)$ and for every v_2, v_1, and v_0, $(((v_0 \sqcap v_1) \sqcup v_2) \sqcap v_1) \sqcup (v_2 \sqcap v_0) = (((v_0 \sqcup v_1) \sqcap v_2) \sqcup v_1) \sqcap (v_2 \sqcup v_0)$. $v_1 \sqcup (v_2 \sqcap v_3) = (v_1 \sqcup v_2) \sqcap (v_1 \sqcup v_3)$.

Let us note that every non empty lattice structure which is meet-idempotent, meet-associative, meet-commutative, join-idempotent, join-associative, join-commutative, and QLT-selfdistributive and satisfies QLT1 and QLT2 is also distributive'.

5. QLT-4: Bowden Inequality

Let us consider L. We say that L satisfies Bowden inequality if and only if

(Def. 11) for every elements x, y, z of L, $(x \sqcup y) \sqcap z \sqsubseteq x \sqcup (y \sqcap z)$.

Let L be a join-commutative, non empty lattice structure. Observe that L satisfies Bowden inequality if and only if the condition (Def. 12) is satisfied.

(Def. 12) for every elements v_0, v_2, v_1 of L, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcup ((v_0 \sqcup v_1) \sqcap v_2) = v_0 \sqcup (v_1 \sqcap v_2)$.

Now we state the proposition:

(9) Suppose for every v_0, $v_0 \sqcap v_0 = v_0$ and for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcap (v_1 \sqcup v_2)) \sqcup (v_0 \sqcap v_1) = v_0 \sqcap (v_1 \sqcup v_2)$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$ and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcap (v_0 \sqcup v_1) = v_0 \sqcup (v_1 \sqcap v_2)$ and for every v_0, v_2, and v_1, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcup ((v_0 \sqcup v_1) \sqcap v_2) = v_0 \sqcup (v_1 \sqcap v_2)$. $v_1 \sqcup (v_2 \sqcap v_3) = (v_1 \sqcup v_2) \sqcap (v_1 \sqcup v_3)$.

Note that every non empty lattice structure which is meet-idempotent, meet-associative, meet-commutative, join-idempotent, join-associative, and join-commutative and satisfies QLT1, QLT2, and Bowden inequality is also distributive'.

6. Preliminaries to QLT-5: Modularity for Quasilattices

Let us consider L. We say that L is QLT-selfmodular if and only if

(Def. 13) for every v_2, v_1, and v_0, $(v_0 \sqcap v_1) \sqcup (v_2 \sqcap (v_0 \sqcup v_1)) = (v_0 \sqcup v_1) \sqcap (v_2 \sqcup (v_0 \sqcap v_1))$.

Let L be a join-idempotent, non empty lattice structure and a, b be elements of L. Let us note that the predicate $a \sqsubseteq b$ is reflexive.

Let us consider v_1, v_2, and v_3. Now we state the propositions:

(10) Suppose for every v_0, $v_0 \sqcap v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcap v_1) \sqcap v_2 = v_0 \sqcap (v_1 \sqcap v_2)$ and for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcap (v_1 \sqcup v_2)) \sqcup (v_0 \sqcap v_1) = v_0 \sqcap (v_1 \sqcup v_2)$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$ and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcap (v_0 \sqcup v_1) = v_0 \sqcup (v_1 \sqcap v_2)$ and for every v_0, v_1, and v_2 such that $v_0 \sqcup v_1 = v_1$ holds $v_0 \sqcup (v_2 \sqcap v_1) = (v_0 \sqcup v_2) \sqcap v_1$. Then $(v_1 \sqcap v_2) \sqcup (v_1 \sqcap v_3) = v_1 \sqcap (v_2 \sqcup (v_1 \sqcap v_3))$.

(11) Suppose for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcap (v_1 \sqcup v_2)) \sqcup (v_0 \sqcap v_1) = v_0 \sqcap (v_1 \sqcup v_2)$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$

and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcap (v_0 \sqcup v_1) = v_0 \sqcup (v_1 \sqcap v_2)$ and for every v_2, v_1, and v_0, $(v_0 \sqcap v_1) \sqcup (v_0 \sqcap v_2) = v_0 \sqcap (v_1 \sqcup (v_0 \sqcap v_2))$. Then if $v_1 \sqcup v_2 = v_2$, then $v_1 \sqcup (v_3 \sqcap v_2) = (v_1 \sqcup v_3) \sqcap v_2$.

Let L be a meet-idempotent, join-idempotent, meet-commutative, join-commutative, meet-associative, join-associative, non empty lattice structure satisfying QLT1 and QLT2. Observe that L is modular if and only if the condition (Def. 14) is satisfied.

(Def. 14) for every elements v_1, v_2, v_3 of L, $(v_1 \sqcap v_2) \sqcup (v_1 \sqcap v_3) = v_1 \sqcap (v_2 \sqcup (v_1 \sqcap v_3))$.

7. QLT-5

Now we state the proposition:

(12) Suppose for every v_0, $v_0 \sqcap v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcap v_1) \sqcap v_2 = v_0 \sqcap (v_1 \sqcap v_2)$ and for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcap (v_1 \sqcup v_2)) \sqcup (v_0 \sqcap v_1) = v_0 \sqcap (v_1 \sqcup v_2)$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$ and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcap (v_0 \sqcup v_1) = v_0 \sqcup (v_1 \sqcap v_2)$ and for every v_2, v_1, and v_0, $(v_0 \sqcap v_1) \sqcup (v_2 \sqcap (v_0 \sqcup v_1)) = (v_0 \sqcup v_1) \sqcap (v_2 \sqcup (v_0 \sqcap v_1))$. $(v_1 \sqcap v_2) \sqcup (v_1 \sqcap v_3) = v_1 \sqcap (v_2 \sqcup (v_1 \sqcap v_3))$.

Let us note that every non empty lattice structure which is meet-idempotent, meet-associative, meet-commutative, join-idempotent, join-associative, join-commutative, and QLT-selfmodular and satisfies QLT1 and QLT2 is also modular.

8. QLT-6

Now we state the proposition:

(13) Suppose for every v_0, $v_0 \sqcap v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcap v_1) \sqcap v_2 = v_0 \sqcap (v_1 \sqcap v_2)$ and for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcap (v_1 \sqcup v_2)) \sqcup (v_0 \sqcap v_1) = v_0 \sqcap (v_1 \sqcup v_2)$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$ and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcap (v_0 \sqcup v_1) = v_0 \sqcup (v_1 \sqcap v_2)$ and for every v_2, v_1, and v_0, $((v_0 \sqcup v_1) \sqcap v_2) \sqcup v_1 = ((v_2 \sqcup v_1) \sqcap v_0) \sqcup v_1$. $(v_1 \sqcap v_2) \sqcup (v_1 \sqcap v_3) = v_1 \sqcap (v_2 \sqcup (v_1 \sqcap v_3))$.

Let us consider L. We say that L is QLT-selfmodular' if and only if

(Def. 15) for every v_2, v_1, and v_0, $((v_0 \sqcup v_1) \sqcap v_2) \sqcup v_1 = ((v_2 \sqcup v_1) \sqcap v_0) \sqcup v_1$.

Observe that every non empty lattice structure which is meet-idempotent, meet-associative, meet-commutative, join-idempotent, join-associative, join-commutative, and QLT-selfmodular' and satisfies QLT1 and QLT2 is also modular.

9. The Counterexample Needed to Prove QLT-7

Now we state the proposition:

(14) There exist quasilattices L_1, L_2 such that

(i) the carrier of L_1 = the carrier of L_2, and

(ii) the join operation of L_1 = the join operation of L_2, and

(iii) the meet operation of $L_1 \neq$ the meet operation of L_2.

The theorem is a consequence of (1).

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Accepted May 31, 2020