Contents

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Continued on inside back cover

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Pappus's Hexagon Theorem in Real Projective Plane[1]

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Summary. In this article we prove, using Mizar [2], [1], the Pappus's hexagon theorem in the real projective plane: "Given one set of collinear points A, B, C, and another set of collinear points a, b, c, then the intersection points X, Y, Z of line pairs Ab and aB, Ac and aC, Bc and bC are collinear"[2].

More precisely, we prove that the structure `ProjectiveSpace TOP-REAL3` [10] (where `TOP-REAL3` is a metric space defined in [5]) satisfies the Pappus's axiom defined in [11] by Wojciech Leończuk and Krzysztof Prażmowski. Eugeniusz Kusak and Wojciech Leończuk formalized the Hessenberg theorem early in the MML [9]. With this result, the real projective plane is Desarguesian.

For proving the Pappus's theorem, two different proofs are given. First, we use the techniques developed in the section "Projective Proofs of Pappus's Theorem" in the chapter "Pappos's Theorem: Nine proofs and three variations" [12]. Secondly, Pascal's theorem [4] is used.

In both cases, to prove some lemmas, we use `Prover9`[3], the successor of the `Otter` prover and `ott2miz` by Josef Urban[4] [13], [8], [7].

In `Coq`, the Pappus's theorem is proved as the application of Grassmann-Cayley algebra [6] and more recently in Tarski's geometry [3].

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Keywords: Pappus's Hexagon Theorem; real projective plan; Grassmann-Plücker relation; Prover9

MML identifier: `PAPPUS`, version: `8.1.11 5.66.1402`

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[2]`https://en.wikipedia.org/wiki/Pappus's_hexagon_theorem`

[3]`https://www.cs.unm.edu/~mccune/prover9/`

[4]See its homepage `https://github.com/JUrban/ott2miz`

1. Preliminaries

From now on a, b, c, d, e, f, g, h, i denote real numbers and M denotes a square matrix over $\mathbb{R}$ of dimension 3.

Now we state the propositions:

(1) Suppose $M = \langle\langle a,b,c\rangle, \langle d,e,f\rangle, \langle g,h,i\rangle\rangle$. Then $\operatorname{Det} M = a\cdot e\cdot i - c\cdot e\cdot g - a\cdot f\cdot h + b\cdot f\cdot g - b\cdot d\cdot i + c\cdot d\cdot h$.

(2) Let us consider elements P_1, P_4, P_5 of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and elements p_1, p_2, p_3, p_4, p_5 of $\mathcal{E}^3_{\mathrm{T}}$. Suppose p_1 is not zero and $P_1 =$ the direction of p_1 and p_4 is not zero and $P_4 =$ the direction of p_4 and p_5 is not zero and $P_5 =$ the direction of p_5 and P_1, P_4 and P_5 are collinear. Then $\langle|p_1,p_2,p_4|\rangle\cdot\langle|p_1,p_3,p_5|\rangle = \langle|p_1,p_2,p_5|\rangle\cdot\langle|p_1,p_3,p_4|\rangle$.

(3) Let us consider non zero real numbers r_{416}, r_{415}, r_{413}, r_{418}, r_{419}, r_{412}, r_{712}, r_{746}, r_{716}, r_{742}, r_{715}, r_{743}, r_{713}, r_{745}, r_{749}, r_{718}, r_{719}, r_{748}. Suppose $(-r_{412})\cdot(-r_{713}) = (-r_{413})\cdot(-r_{712})$ and $(-r_{415})\cdot(-r_{719}) = (-r_{419})\cdot(-r_{715})$ and $(-r_{418})\cdot(-r_{716}) = (-r_{416})\cdot(-r_{718})$ and $(-r_{745})\cdot r_{416} = (-r_{746})\cdot r_{415}$ and $(-r_{748})\cdot r_{413} = (-r_{743})\cdot r_{418}$ and $(-r_{742})\cdot r_{419} = (-r_{749})\cdot r_{412}$ and $r_{712}\cdot r_{746} = r_{716}\cdot r_{742}$ and $r_{715}\cdot r_{743} = r_{713}\cdot r_{745}$. Then $r_{718}\cdot r_{749} = r_{719}\cdot r_{748}$.

2. Some Technical Lemmas Proved by `Prover9` and Translated with Help of `ott2miz`

From now on P_2 denotes a projective space defined in terms of collinearity and c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10} denote elements of P_2.

Now we state the propositions:

(4) Suppose $c_2 \neq c_1$ and $c_3 \neq c_1$ and $c_3 \neq c_2$ and $c_4 \neq c_2$ and $c_4 \neq c_3$ and $c_5 \neq c_1$ and $c_6 \neq c_1$ and $c_6 \neq c_5$ and $c_7 \neq c_5$ and $c_7 \neq c_6$ and c_1, c_4 and c_7 are not collinear and c_1, c_4 and c_2 are collinear and c_1, c_4 and c_3 are collinear and c_1, c_7 and c_5 are collinear and c_1, c_7 and c_6 are collinear and c_4, c_5 and c_8 are collinear and c_7, c_2 and c_8 are collinear and c_4, c_6 and c_9 are collinear and c_3, c_7 and c_9 are collinear and c_2, c_6 and c_{10} are collinear and c_3, c_5 and c_{10} are collinear. Then

(i) c_4, c_7 and c_2 are not collinear, and

(ii) c_4, c_{10} and c_3 are not collinear, and

(iii) c_4, c_7 and c_3 are not collinear, and

(iv) c_4, c_{10} and c_2 are not collinear, and

(v) c_4, c_7 and c_5 are not collinear, and

(vi) c_4, c_{10} and c_8 are not collinear, and

(vii) c_4, c_7 and c_8 are not collinear, and

(viii) c_4, c_{10} and c_5 are not collinear, and

(ix) c_4, c_7 and c_9 are not collinear, and

(x) c_4, c_{10} and c_6 are not collinear, and

(xi) c_4, c_7 and c_6 are not collinear, and

(xii) c_4, c_{10} and c_9 are not collinear, and

(xiii) c_7, c_{10} and c_5 are not collinear, and

(xiv) c_7, c_4 and c_6 are not collinear, and

(xv) c_7, c_{10} and c_9 are not collinear, and

(xvi) c_7, c_4 and c_3 are not collinear, and

(xvii) c_7, c_{10} and c_3 are not collinear, and

(xviii) c_7, c_4 and c_9 are not collinear, and

(xix) c_7, c_{10} and c_2 are not collinear, and

(xx) c_7, c_4 and c_8 are not collinear, and

(xxi) c_{10}, c_4 and c_2 are not collinear, and

(xxii) c_{10}, c_7 and c_6 are not collinear, and

(xxiii) c_{10}, c_4 and c_6 are not collinear, and

(xxiv) c_{10}, c_7 and c_2 are not collinear, and

(xxv) c_{10}, c_4 and c_5 are not collinear, and

(xxvi) c_{10}, c_7 and c_3 are not collinear, and

(xxvii) c_{10}, c_4 and c_3 are not collinear, and

(xxviii) c_{10}, c_7 and c_5 are not collinear.

(5) Suppose $c_2 \neq c_1$ and $c_3 \neq c_2$ and $c_5 \neq c_1$ and $c_7 \neq c_5$ and $c_7 \neq c_6$ and c_1, c_4 and c_7 are not collinear and c_1, c_4 and c_2 are collinear and c_1, c_4 and c_3 are collinear and c_1, c_7 and c_5 are collinear and c_1, c_7 and c_6 are collinear and c_4, c_5 and c_8 are collinear and c_7, c_2 and c_8 are collinear and c_2, c_6 and c_{10} are collinear and c_3, c_5 and c_{10} are collinear.
Then c_{10}, c_7 and c_8 are not collinear.

(6) Suppose c_1, c_4 and c_7 are not collinear and c_1, c_4 and c_2 are collinear and c_1, c_4 and c_3 are collinear and c_1, c_7 and c_5 are collinear and c_1, c_7 and c_6 are collinear and c_4, c_5 and c_8 are collinear and c_7, c_2 and c_8 are collinear and c_4, c_6 and c_9 are collinear and c_3, c_7 and c_9 are collinear and c_2, c_6 and c_{10} are collinear and c_3, c_5 and c_{10} are collinear. Then

(i) c_4, c_2 and c_3 are collinear, and
(ii) c_4, c_5 and c_8 are collinear, and
(iii) c_4, c_9 and c_6 are collinear, and
(iv) c_7, c_5 and c_6 are collinear, and
(v) c_7, c_9 and c_3 are collinear, and
(vi) c_7, c_2 and c_8 are collinear, and
(vii) c_{10}, c_2 and c_6 are collinear, and
(viii) c_{10}, c_5 and c_3 are collinear.

(7) Suppose $c_3 \neq c_1$ and $c_3 \neq c_2$ and $c_6 \neq c_1$ and $c_6 \neq c_5$ and c_1, c_2 and c_5 are not collinear and c_1, c_2 and c_3 are collinear and c_1, c_5 and c_6 are collinear. Then

(i) c_2, c_3 and c_5 are not collinear, and
(ii) c_2, c_3 and c_6 are not collinear, and
(iii) c_2, c_5 and c_6 are not collinear, and
(iv) c_3, c_5 and c_6 are not collinear.

(8) Suppose $c_3 \neq c_1$ and $c_4 \neq c_1$ and $c_4 \neq c_3$ and $c_3 \neq c_2$ and $c_4 \neq c_2$ and $c_6 \neq c_1$ and $c_7 \neq c_1$ and $c_7 \neq c_6$ and $c_6 \neq c_5$ and $c_7 \neq c_5$ and c_1, c_2 and c_5 are not collinear and c_1, c_2 and c_3 are collinear and c_1, c_2 and c_4 are collinear and c_1, c_5 and c_6 are collinear and c_1, c_5 and c_7 are collinear. Then

(i) c_1, c_3 and c_6 are not collinear, and
(ii) c_1, c_3 and c_4 are collinear, and
(iii) c_1, c_6 and c_7 are collinear, and
(iv) $c_3 \neq c_1$, and
(v) $c_2 \neq c_1$, and
(vi) $c_3 \neq c_2$, and
(vii) $c_4 \neq c_3$, and
(viii) $c_4 \neq c_2$, and
(ix) $c_6 \neq c_1$, and
(x) $c_5 \neq c_1$, and
(xi) $c_6 \neq c_5$, and
(xii) $c_7 \neq c_6$, and
(xiii) $c_7 \neq c_5$, and

(xiv) c_1, c_4 and c_7 are not collinear, and

(xv) c_1, c_4 and c_3 are collinear, and

(xvi) c_1, c_4 and c_2 are collinear, and

(xvii) c_1, c_7 and c_6 are collinear, and

(xviii) c_1, c_7 and c_5 are collinear.

(9) Suppose $c_4 \neq c_2$ and $c_4 \neq c_3$ and $c_8 \neq c_2$ and c_2, c_3 and c_6 are not collinear. Then

(i) c_2, c_3 and c_4 are not collinear, or

(ii) c_2, c_6 and c_8 are not collinear, or

(iii) c_3, c_4 and c_8 are not collinear.

(10) Suppose $c_4 \neq c_1$ and $c_6 \neq c_5$ and c_1, c_2 and c_5 are not collinear. Then

(i) c_1, c_2 and c_4 are not collinear, or

(ii) c_1, c_5 and c_6 are not collinear, or

(iii) c_4, c_6 and c_8 are not collinear, or

(iv) $c_8 \neq c_5$.

(11) Suppose $c_4 \neq c_2$ and $c_6 \neq c_1$ and c_1, c_2 and c_5 are not collinear and c_1, c_2 and c_4 are collinear and c_1, c_5 and c_6 are collinear and c_4, c_6 and c_8 are collinear. Then $c_8 \neq c_2$.

(12) If c_1, c_2 and c_5 are not collinear and c_1, c_2 and c_3 are collinear and c_1, c_2 and c_4 are collinear, then c_2, c_3 and c_4 are collinear.

(13) If c_1, c_2 and c_5 are not collinear and c_1, c_5 and c_6 are collinear and c_1, c_5 and c_7 are collinear, then c_5, c_6 and c_7 are collinear.

(14) If $c_3 \neq c_1$ and c_1, c_2 and c_5 are not collinear and c_1, c_2 and c_3 are collinear and c_1, c_5 and c_7 are collinear, then $c_7 \neq c_3$.

(15) Suppose $c_4 \neq c_1$ and $c_4 \neq c_3$ and c_1, c_2 and c_5 are not collinear and c_1, c_2 and c_3 are collinear and c_1, c_2 and c_4 are collinear and c_4, c_5 and c_9 are collinear. Then $c_9 \neq c_3$.

(16) Suppose $c_4 \neq c_1$ and $c_4 \neq c_2$ and $c_6 \neq c_1$ and $c_7 \neq c_6$ and $c_7 \neq c_5$ and c_1, c_2 and c_5 are not collinear and c_1, c_2 and c_4 are collinear and c_1, c_5 and c_6 are collinear and c_1, c_5 and c_7 are collinear and c_2, c_7 and c_9 are collinear and c_4, c_5 and c_9 are collinear. Then c_9, c_2 and c_5 are not collinear.

3. The Real Projective Plane and Pappus's Theorem

From now on o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 denote elements of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Now we state the propositions:

(17) Pappus theorem as "Pappos's Theorem: Nine proofs and three variations" [12] version:
Suppose $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear.
Then r_1, r_2 and r_3 are collinear.

(18) The projective space over $\mathcal{E}^3_{\mathrm{T}}$ is a Pappian, Desarguesian projective plane defined in terms of collinearity.

4. Proof: Special Case of Pascal's Theorem

In the sequel v_0, v_1, v_2, v_3, v_4, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, v_{100}, v_{101}, v_{102}, v_{103} denote elements of the projective space over $\mathcal{E}^3_{\mathrm{T}}$.
Now we state the propositions:

(19) Suppose $c_1 \neq c_2$ and $c_1 \neq c_3$ and $c_2 \neq c_3$ and $c_2 \neq c_4$ and $c_3 \neq c_4$ and $c_1 \neq c_5$ and $c_1 \neq c_6$ and $c_5 \neq c_6$ and $c_5 \neq c_7$ and $c_6 \neq c_7$ and c_1, c_4 and c_7 are not collinear and c_1, c_4 and c_2 are collinear and c_1, c_4 and c_3 are collinear and c_1, c_7 and c_5 are collinear and c_1, c_7 and c_6 are collinear and c_4, c_5 and c_8 are collinear and c_7, c_2 and c_8 are collinear and c_4, c_6 and c_9 are collinear and c_3, c_7 and c_9 are collinear and c_2, c_6 and c_{10} are collinear and c_3, c_5 and c_{10} are collinear.

Then it is not true that c_4, c_2 and c_7 are collinear or c_4, c_3 and c_7 are collinear or c_2, c_3 and c_7 are collinear or c_4, c_2 and c_5 are collinear or c_4, c_2 and c_6 are collinear or c_4, c_3 and c_5 are collinear or c_4, c_3 and c_6 are collinear or c_2, c_7 and c_5 are collinear or c_2, c_7 and c_6 are collinear or c_3, c_7 and c_5 are collinear or c_3, c_7 and c_6 are collinear or c_2, c_3 and c_5 are collinear or c_2, c_3 and c_6 are collinear or c_7, c_5 and c_4 are collinear or c_7, c_6.

And c_4 are collinear or c_5, c_6 and c_4 are collinear or c_5, c_6 and c_2 are collinear or c_4, c_5 and c_8 are not collinear or c_4, c_6 and c_9 are not collinear or c_2, c_7 and c_8 are not collinear or c_2, c_6 and c_{10} are not collinear or c_3, c_7 and c_9 are not collinear or c_3, c_5 and c_{10} are not collinear.

(20) conic$(0,0,0,0,0,0)$ = the carrier of the projective space over $\mathcal{E}^3_{\mathrm{T}}$.

(21) Suppose $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear.
Then p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 form the Pascal configuration.

(22) PAPPUS THEOREM AS A SPECIAL CASE OF PASCAL'S THEOREM: Suppose $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear.

And o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear.
Then r_1, r_2 and r_3 are collinear.

PROOF: p_1, p_2 and p_3 are collinear. Consider u_1, u_2, u_3 being elements of $\mathcal{E}^3_{\mathrm{T}}$ such that p_1 = the direction of u_1 and p_2 = the direction of u_2 and p_3 = the direction of u_3 and u_1 is not zero and u_2 is not zero and u_3 is not zero and u_1, u_2 and u_3 are lineary dependent. Set $x_1 = (u_2)_{\mathbf{2}} \cdot ((u_3)_{\mathbf{3}}) - (u_2)_{\mathbf{3}} \cdot ((u_3)_{\mathbf{2}})$. Set $x_2 = (u_2)_{\mathbf{3}} \cdot ((u_3)_{\mathbf{1}}) - (u_2)_{\mathbf{1}} \cdot ((u_3)_{\mathbf{3}})$. Set $x_3 = (u_2)_{\mathbf{1}} \cdot ((u_3)_{\mathbf{2}}) - (u_2)_{\mathbf{2}} \cdot ((u_3)_{\mathbf{1}})$. q_1, q_2 and q_3 are collinear.

Consider v_1, v_2, v_3 being elements of $\mathcal{E}^3_{\mathrm{T}}$ such that q_1 = the direction of v_1 and q_2 = the direction of v_2 and q_3 = the direction of v_3 and v_1 is not zero and v_2 is not zero and v_3 is not zero and v_1, v_2 and v_3 are lineary dependent. Set $y_1 = (v_2)_{\mathbf{2}} \cdot ((v_3)_{\mathbf{3}}) - (v_2)_{\mathbf{3}} \cdot ((v_3)_{\mathbf{2}})$. Set $y_2 = (v_2)_{\mathbf{3}} \cdot ((v_3)_{\mathbf{1}}) - (v_2)_{\mathbf{1}} \cdot ((v_3)_{\mathbf{3}})$. Set $y_3 = (v_2)_{\mathbf{1}} \cdot ((v_3)_{\mathbf{2}}) - (v_2)_{\mathbf{2}} \cdot ((v_3)_{\mathbf{1}})$. Set $x_4 = x_1 \cdot y_1$. Set $x_5 = x_2 \cdot y_2$. Set $x_6 = x_3 \cdot y_3$. Set $x_7 = x_1 \cdot y_2 + x_2 \cdot y_1$. Set $x_8 = x_1 \cdot y_3 + x_3 \cdot y_1$. Set $x_1 = x_2 \cdot y_3 + x_3 \cdot y_2$. For every point u of $\mathcal{E}^3_{\mathrm{T}}$, qfconic$(x_4, x_5, x_6, x_7, x_8, x_1, u) = |(u, u_2 \times u_3)| \cdot |(u, v_2 \times v_3)|$. □

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On Weakly Associative Lattices and Near Lattices

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Summary. The main aim of this article is to introduce formally two generalizations of lattices, namely weakly associative lattices and near lattices, which can be obtained from the former by certain weakening of the usual well-known axioms. We show selected propositions devoted to weakly associative lattices and near lattices from Chapter 6 of [15], dealing also with alternative versions of classical axiomatizations. Some of the results were proven in the Mizar [1], [2] system with the help of Prover9 [14] proof assistant.

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0. Introduction

Lattice theory is widely represented in the Mizar Mathematical Library, with Żukowski's first article [18], following Birkhoff [3] and Grätzer [11], [12]. In parallel, the theory of partially ordered sets was developed [4] treated generally as relational structures, even if informally the notions are quite similar [9], [7]. The review of the mechanization of lattice theory in Mizar, with the example of the solution of the Robbins problem, is contained in [6].

Our work can be seen as a step towards a Mizar support for [15] or [16], where original proof objects by OTTER/Prover9 were used. Some preliminary works in this direction were already done in [8] by present authors. We use the interface `ott2miz` [17] which allows for the automated translation of proofs;

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these automatically generated proofs are usually quite lengthy, even after native enhancements done by internal Mizar software for library revisions.

Weakly associative lattices were studied in [5]. In the present development, we deal with the parts of Chap. 6 “Lattice-like algebras” of [15], pp. 111–135, devoted to this class of lattices. In this sense, we continue the work started by Kulesza and Grabowski in [13], devoted to the formalization of quasi-lattices.

The class of weakly associative lattices (or WA-lattices, WAL) can be characterized from the standard set of axioms for lattices (with idempotence for the join and meet operations included), where the ordinary associative laws are replaced by the so-called part-preservation laws. The characteristic axiom is however W3 (or, dual W3' – compare Def. 1 and Def. 2). Section 2 contains also equivalent formulation of these axioms, using ordering relation on lattices. The earlier seems to be a bit more feasible taking into account the role of equality in the Mizar system [10] and the design of Prover9.

In Section 3 we show how described binary lattice operations can be associated with the corresponding ordering relation. Obviously, the associativity can only be shown under some conditions for elements (see theorems (15) and (16)). If we assume distributivity, the relation is transitive, as in usual lattices. Section 4 contains the proof that adding the distributivity condition to the set of usual WAL axioms, the associativity can be proven.

Then we deal with another generalization of lattices, i.e. near lattices (absorption law is weakened). Def. 6 and Def. 7 provide standard examples of these structures which are not necessarily lattices (see Def. 10 for the definition of the structure). The lattice operations are given by

$\sqcup$	0	1	2
0	0	1	0
1	1	1	2
2	0	2	2

$\sqcap$	0	1	2
0	0	0	2
1	0	1	1
2	2	1	2

Associativity laws do not hold here, so this is not a lattice.

The rest of the article is devoted to alternative axiomatizations of WAL. WAL-3 – equivalent set of axioms describing WAL is expressed in the form of five separate attributes to make Mizar registrations mechanism working (see Def. 11–Def. 15). It is shown that these adjectives imply the standard attributes for lattices.

In Section 8 WAL-4 is defined (the short sigle axiom system for WAL). We conclude with the proof if WAL-4 holds, then lattice operations are idempotent.

Some of the proofs were produced by means of Prover9, so they are definitely lengthy. The enhancement of the lemmas, including their shortening, reorganization and clustering, can be interesting and useful future work.

1. Preliminaries

From now on L denotes a non empty lattice structure and v_{100}, v_{102}, v_2, v_1, v_0, v_3, v_{101} denote elements of L.

Let us consider v_0, v_1, and v_2. Now we state the propositions:

(1) Suppose for every v_0, $v_0 \sqcap v_0 = v_0$ and for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_2, v_1, and v_0, $((v_0 \sqcup v_1) \sqcap (v_2 \sqcup v_1)) \sqcap v_1 = v_1$ and for every v_2, v_1, and v_0, $((v_0 \sqcap v_1) \sqcup (v_2 \sqcap v_1)) \sqcup v_1 = v_1$ and for every v_1, v_2, and v_0, $v_0 \sqcap (v_1 \sqcup (v_0 \sqcup v_2)) = v_0$. Then $(v_0 \sqcap v_1) \sqcap v_2 = v_0 \sqcap (v_1 \sqcap v_2)$.

(2) Suppose for every v_0, $v_0 \sqcap v_0 = v_0$ and for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_2, v_1, and v_0, $((v_0 \sqcup v_1) \sqcap (v_2 \sqcup v_1)) \sqcap v_1 = v_1$ and for every v_2, v_1, and v_0, $((v_0 \sqcap v_1) \sqcup (v_2 \sqcap v_1)) \sqcup v_1 = v_1$ and for every v_1, v_2, and v_0, $v_0 \sqcap (v_1 \sqcup (v_0 \sqcup v_2)) = v_0$. Then $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$.

Let us consider v_1 and v_2. Now we state the propositions:

(3) Suppose for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_1, v_2, and v_0, $v_0 \sqcap (v_1 \sqcup (v_0 \sqcup v_2)) = v_0$. Then $v_1 \sqcap (v_1 \sqcup v_2) = v_1$.

(4) Suppose for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_2, v_1, and v_0, $((v_0 \sqcap v_1) \sqcup (v_2 \sqcap v_1)) \sqcup v_1 = v_1$. Then $v_1 \sqcup (v_1 \sqcap v_2) = v_1$.

2. Definition of Attributes

Let L be a non empty lattice structure. We say that L is satisfying W3 if and only if

(Def. 1) for every elements v_2, v_1, v_0 of L, $((v_0 \sqcup v_1) \sqcap (v_2 \sqcup v_1)) \sqcap v_1 = v_1$.

We say that L is satisfying W3' if and only if

(Def. 2) for every elements v_2, v_1, v_0 of L, $((v_0 \sqcap v_1) \sqcup (v_2 \sqcap v_1)) \sqcup v_1 = v_1$.

Let L be a meet-absorbing, join-absorbing, meet-commutative, non empty lattice structure. Let us note that L is satisfying W3 if and only if the condition (Def. 3) is satisfied.

(Def. 3) for every elements v_2, v_1, v_0 of L, $v_1 \sqsubseteq (v_0 \sqcup v_1) \sqcap (v_2 \sqcup v_1)$.

Let us consider L. Observe that L is satisfying W3' if and only if the condition (Def. 4) is satisfied.

(Def. 4) for every v_2, v_1, and v_0, $(v_0 \sqcap v_1) \sqcup (v_2 \sqcap v_1) \sqsubseteq v_1$.

Let us note that every non empty lattice structure which is meet-commutative, join-idempotent, join-commutative, and satisfying W3' is also quasi-meet-absorbing and every non empty lattice structure which is meet-commutative, meet-idempotent, join-commutative, and satisfying W3 is also join-absorbing and every non empty lattice structure which is trivial is also satisfying W3' and there exists a non empty lattice structure which is satisfying W3, satisfying W3', join-idempotent, meet-idempotent, join-commutative, and meet-commutative.

A weakly associative lattice is a join-idempotent, meet-idempotent, join-commutative, meet-commutative, satisfying W3, satisfying W3', non empty lattice structure.

A WA-lattice is a weakly associative lattice. Note that every join-associative, meet-absorbing lattice is satisfying W3'.

Let L be a non empty lattice structure. We say that L is satisfying WA if and only if

(Def. 5) for every elements x, y, z of L, $x \sqcap (y \sqcup (x \sqcup z)) = x$.

3. On the Ordering Relation Generated by Weakly Associated Lattices

Let us note that every non empty lattice structure which is quasi-meet-absorbing, meet-commutative, and join-commutative is also meet-absorbing and every WA-lattice is meet-absorbing.

From now on L denotes a WA-lattice and x, y, z, u denote elements of L.

Now we state the propositions:

(5) $x \sqcup y = y$ if and only if $x \sqsubseteq y$.

(6) $x \sqcap y = x$ if and only if $x \sqsubseteq y$.

(7) $x \sqsubseteq x$.

(8) If $x \sqsubseteq y$ and $y \sqsubseteq x$, then $x = y$.

(9) $x \sqsubseteq x \sqcup y$.

(10) $x \sqcap y \sqsubseteq x$.

(11) If $x \sqsubseteq z$ and $y \sqsubseteq z$, then $x \sqcup y \sqsubseteq z$.

(12) There exists z such that

(i) $x \sqsubseteq z$, and

(ii) $y \sqsubseteq z$, and

(iii) for every u such that $x \sqsubseteq u$ and $y \sqsubseteq u$ holds $z \sqsubseteq u$.

The theorem is a consequence of (11) and (9).

(13) If $z \sqsubseteq x$ and $z \sqsubseteq y$, then $z \sqsubseteq x \sqcap y$.

(14) There exists z such that

(i) $z \sqsubseteq x$, and

(ii) $z \sqsubseteq y$, and

(iii) for every u such that $u \sqsubseteq x$ and $u \sqsubseteq y$ holds $u \sqsubseteq z$.

The theorem is a consequence of (13) and (10).

(15) If $x \sqsubseteq z$ and $y \sqsubseteq z$, then $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$.

(16) If $z \sqsubseteq x$ and $z \sqsubseteq y$, then $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$.

(17) If L is distributive and $x \sqsubseteq y \sqsubseteq z$, then $x \sqsubseteq z$.

4. Distributivity Implies Associativity

From now on L denotes a non empty lattice structure and v_0, v_1, v_2 denote elements of L.

Now we state the proposition:

(18) Suppose for every v_0, $v_0 \sqcap v_0 = v_0$ and for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_2, v_1, and v_0, $((v_0 \sqcup v_1) \sqcap (v_2 \sqcup v_1)) \sqcap v_1 = v_1$ and for every v_2, v_1, and v_0, $((v_0 \sqcap v_1) \sqcup (v_2 \sqcap v_1)) \sqcup v_1 = v_1$ and for every v_1 and v_0, $v_0 \sqcap (v_0 \sqcup v_1) = v_0$ and for every v_0, v_2, and v_1, $v_0 \sqcup (v_1 \sqcap v_2) = (v_0 \sqcup v_1) \sqcap (v_0 \sqcup v_2)$. $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$.

Observe that every WA-lattice which is distributive' is also join-associative.

5. Near Lattices

Let x, y be elements of $\{0, 1, 2\}$. The functors: $x \sqcap_{\mathrm{NL}} y$ and $x \sqcup_{\mathrm{NL}} y$ yielding elements of $\{0, 1, 2\}$ are defined by terms

(Def. 6) $\begin{cases} 2, & \textbf{if } x = 0 \text{ and } y = 2 \text{ or } x = 2 \text{ and } y = 0, \\ \min(x, y), & \textbf{otherwise}, \end{cases}$

(Def. 7) $\begin{cases} 0, & \textbf{if } x = 0 \text{ and } y = 2 \text{ or } x = 2 \text{ and } y = 0, \\ \max(x, y), & \textbf{otherwise}, \end{cases}$

respectively. The functors: $\sqcup_{\mathrm{NL}}$ and $\sqcap_{\mathrm{NL}}$ yielding binary operations on $\{0, 1, 2\}$ are defined by conditions

(Def. 8) for every elements x, y of $\{0, 1, 2\}$, $\sqcup_{\mathrm{NL}}(x, y) = x \sqcup_{\mathrm{NL}} y$,

(Def. 9) for every elements x, y of $\{0, 1, 2\}$, $\sqcap_{\mathrm{NL}}(x, y) = x \sqcap_{\mathrm{NL}} y$,

respectively.

6. Examples of Near Lattices

The functor ExNearLattice yielding a non empty lattice structure is defined by the term

(Def. 10) $\langle \{0,1,2\}, \sqcup_{\mathrm{N}L}, \sqcap_{\mathrm{N}L} \rangle$.

One can check that ExNearLattice is non join-associative and non meet-associative and every non empty lattice structure which is trivial is also meet-idempotent, join-commutative, quasi-meet-absorbing, and join-absorbing.

A near lattice is a join-idempotent, meet-idempotent, join-commutative, meet-commutative, quasi-meet-absorbing, join-absorbing, non empty lattice structure.

One can check that ExNearLattice is join-commutative, meet-commutative, join-idempotent, meet-idempotent, join-absorbing, and meet-absorbing and every join-commutative, meet-commutative, non empty lattice structure which is meet-absorbing is also quasi-meet-absorbing and every join-commutative, meet-commutative, non empty lattice structure which is quasi-meet-absorbing is also meet-absorbing.

Now we state the proposition:

(19) (i) ExNearLattice is a near lattice, and

(ii) ExNearLattice is not a lattice.

7. Alternative Axioms for WAL

From now on L denotes a non empty lattice structure and v_{101}, v_{100}, v_2, v_1, v_0, v_{102}, v_{103}, v_3 denote elements of L.

Now we state the proposition:

(20) Suppose for every v_1 and v_0, $(v_0 \sqcap v_1) \sqcup (v_0 \sqcap (v_0 \sqcup v_1)) = v_0$ and for every v_0 and v_1, $(v_0 \sqcap v_0) \sqcup (v_1 \sqcap (v_0 \sqcup v_0)) = v_0$ and for every v_1 and v_0, $(v_0 \sqcap v_1) \sqcup (v_1 \sqcap (v_0 \sqcup v_1)) = v_1$ and for every v_2, v_1, and v_0, $((v_0 \sqcup v_1) \sqcap (v_2 \sqcup v_0)) \sqcap v_0 = v_0$ and for every v_2, v_1, and v_0, $((v_0 \sqcap v_1) \sqcup (v_2 \sqcap v_0)) \sqcup v_0 = v_0$. $v_0 \sqcup v_0 = v_0$.

Let us consider v_0 and v_1. Now we state the propositions:

(21) Suppose for every v_1 and v_0, $(v_0 \sqcap v_1) \sqcup (v_0 \sqcap (v_0 \sqcup v_1)) = v_0$ and for every v_0 and v_1, $(v_0 \sqcap v_0) \sqcup (v_1 \sqcap (v_0 \sqcup v_0)) = v_0$ and for every v_1 and v_0, $(v_0 \sqcap v_1) \sqcup (v_1 \sqcap (v_0 \sqcup v_1)) = v_1$ and for every v_2, v_1, and v_0, $((v_0 \sqcup v_1) \sqcap (v_2 \sqcup v_0)) \sqcap v_0 = v_0$ and for every v_2, v_1, and v_0, $((v_0 \sqcap v_1) \sqcup (v_2 \sqcap v_0)) \sqcup v_0 = v_0$. Then $v_0 \sqcap v_1 = v_1 \sqcap v_0$. The theorem is a consequence of (24).

(22) Suppose for every v_1 and v_0, $(v_0 \sqcap v_1) \sqcup (v_0 \sqcap (v_0 \sqcup v_1)) = v_0$ and for every v_0 and v_1, $(v_0 \sqcap v_0) \sqcup (v_1 \sqcap (v_0 \sqcup v_0)) = v_0$ and for every v_1 and v_0, $(v_0 \sqcap v_1) \sqcup (v_1 \sqcap (v_0 \sqcup v_1)) = v_1$ and for every v_2, v_1, and v_0, $((v_0 \sqcup v_1) \sqcap (v_2 \sqcup v_0)) \sqcap v_0 = v_0$ and for every v_2, v_1, and v_0, $((v_0 \sqcap v_1) \sqcup (v_2 \sqcap v_0)) \sqcup v_0 = v_0$. Then $v_0 \sqcup v_1 = v_1 \sqcup v_0$. The theorem is a consequence of (24) and (21).

Let L be a non empty lattice structure. We say that L is satisfying WAL-3_1 if and only if

(Def. 11) for every elements v_1, v_0 of L, $(v_0 \sqcap v_1) \sqcup (v_0 \sqcap (v_0 \sqcup v_1)) = v_0$.

We say that L is satisfying WAL-3_2 if and only if

(Def. 12) for every elements v_0, v_1 of L, $(v_0 \sqcap v_0) \sqcup (v_1 \sqcap (v_0 \sqcup v_0)) = v_0$.

We say that L is satisfying WAL-3_3 if and only if

(Def. 13) for every elements v_1, v_0 of L, $(v_0 \sqcap v_1) \sqcup (v_1 \sqcap (v_0 \sqcup v_1)) = v_1$.

We say that L is satisfying WAL-3_4 if and only if

(Def. 14) for every elements v_2, v_1, v_0 of L, $((v_0 \sqcup v_1) \sqcap (v_2 \sqcup v_0)) \sqcap v_0 = v_0$.

We say that L is satisfying WAL-3_5 if and only if

(Def. 15) for every elements v_2, v_1, v_0 of L, $((v_0 \sqcap v_1) \sqcup (v_2 \sqcap v_0)) \sqcup v_0 = v_0$.

Let us note that every non empty lattice structure which is trivial is also satisfying WAL-3_1, satisfying WAL-3_2, satisfying WAL-3_3, satisfying WAL-3_4, and satisfying WAL-3_5 and every non empty lattice structure which is satisfying WAL-3_1, satisfying WAL-3_2, satisfying WAL-3_3, satisfying WAL-3_4, and satisfying WAL-3_5 is also join-idempotent, meet-idempotent, join-commutative, and meet-commutative.

8. Short Single Axiom for WAL

Let L be a non empty lattice structure. We say that L is satisfying WAL-4 if and only if

(Def. 16) for every elements v_2, v_0, v_5, v_4, v_3, v_1 of L, $(((v_0 \sqcap v_1) \sqcup (v_1 \sqcap (v_0 \sqcup v_1))) \sqcap v_2) \sqcup (((v_0 \sqcap (((v_1 \sqcap v_3) \sqcup (v_4 \sqcap v_1)) \sqcup v_1)) \sqcup (((v_1 \sqcap (((v_1 \sqcup v_3) \sqcap (v_4 \sqcup v_1)) \sqcap v_1)) \sqcup (v_5 \sqcap (v_1 \sqcup (((v_1 \sqcup v_3) \sqcap (v_4 \sqcup v_1)) \sqcap v_1)))) \sqcap (v_0 \sqcup (((v_1 \sqcap v_3) \sqcup (v_4 \sqcap v_1)) \sqcup v_1)))) \sqcap (((v_0 \sqcap v_1) \sqcup (v_1 \sqcap (v_0 \sqcup v_1))) \sqcup v_2)) = v_1$.

From now on L denotes a non empty lattice structure and v_{108}, v_{107}, v_{106}, v_{101}, v_{10}, v_9, v_8, v_7, v_6, v_{105}, v_{102}, v_{100}, v_{104}, v_{103}, v_5, v_4, v_3, v_2, v_1, v_0 denote elements of L.

Let us consider v_0. Now we state the propositions:

(23) Suppose for every v_2, v_0, v_5, v_4, v_3, and v_1, $(((v_0 \sqcap v_1) \sqcup (v_1 \sqcap (v_0 \sqcup v_1))) \sqcap v_2) \sqcup (((v_0 \sqcap (((v_1 \sqcap v_3) \sqcup (v_4 \sqcap v_1)) \sqcup v_1)) \sqcup (((v_1 \sqcap (((v_1 \sqcup v_3) \sqcap (v_4 \sqcup v_1)) \sqcap$

$v_1)) \sqcup (v_5 \sqcap (v_1 \sqcup (((v_1 \sqcup v_3) \sqcap (v_4 \sqcup v_1)) \sqcap v_1)))) \sqcap (v_0 \sqcup (((v_1 \sqcap v_3) \sqcup (v_4 \sqcap v_1)) \sqcup v_1)))) \sqcap (((v_0 \sqcap v_1) \sqcup (v_1 \sqcap (v_0 \sqcup v_1))) \sqcup v_2)) = v_1$. Then $v_0 \sqcap v_0 = v_0$.

(24) Suppose for every v_2, v_0, v_5, v_4, v_3, and v_1, $(((v_0 \sqcap v_1) \sqcup (v_1 \sqcap (v_0 \sqcup v_1))) \sqcap v_2) \sqcup (((v_0 \sqcap (((v_1 \sqcap v_3) \sqcup (v_4 \sqcap v_1)) \sqcup v_1)) \sqcup (((v_1 \sqcap (((v_1 \sqcup v_3) \sqcap (v_4 \sqcup v_1)) \sqcap v_1)) \sqcup (v_5 \sqcap (v_1 \sqcup (((v_1 \sqcup v_3) \sqcap (v_4 \sqcup v_1)) \sqcap v_1)))) \sqcap (v_0 \sqcup (((v_1 \sqcap v_3) \sqcup (v_4 \sqcap v_1)) \sqcup v_1)))) \sqcap (((v_0 \sqcap v_1) \sqcup (v_1 \sqcap (v_0 \sqcup v_1))) \sqcup v_2)) = v_1$. Then $v_0 \sqcup v_0 = v_0$. The theorem is a consequence of (23).

One can check that every non empty lattice structure which is trivial is also satisfying WAL-4 and every non empty lattice structure which is satisfying WAL-4 is also join-idempotent and meet-idempotent.

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Ascoli-Arzelà Theorem[1]

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Summary. In this article we formalize the Ascoli-Arzelà theorem [5], [6], [8] in Mizar [1], [2]. First, we gave definitions of equicontinuousness and equiboundedness of a set of continuous functions [12], [7], [3], [9]. Next, we formalized the Ascoli-Arzelà theorem using those definitions, and proved this theorem.

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1. Equicontinuousness and Equiboundedness of Continuous Functions

From now on S, T denote real normed spaces and F denotes a subset of (the carrier of T)$^{(\text{the carrier of } S)}$.

Let X be a non empty metric space and Y be a subset of X. The functor $\overline{Y}$ yielding a subset of X is defined by

(Def. 1) there exists a subset Z of $X_{\rm top}$ such that $Z = Y$ and $it = \overline{Z}$.

Now we state the proposition:

(1) Let us consider a real normed space X, a subset Y of X, and a subset Z of MetricSpaceNorm X. If $Y = Z$, then $\overline{Y} = \overline{Z}$.

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Let X be a non empty metric space and H be a non empty subset of X. Observe that $\overline{H}$ is non empty.

Now we state the propositions:

(2) Let us consider a topological space S, and a finite sequence F of elements of 2^{α}. Suppose for every natural number i such that $i \in \operatorname{Seg} \operatorname{len} F$ holds $F_{/i}$ is compact. Then $\bigcup \operatorname{rng} F$ is compact, where α is the carrier of S.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence F of elements of $2^{(\text{the carrier of } S)}$ such that $\operatorname{len} F = \$_1$ and for every natural number i such that $i \in \operatorname{Seg} \operatorname{len} F$ holds $F_{/i}$ is compact holds $\bigcup \operatorname{rng} F$ is compact. $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number n, $\mathcal{P}[n]$. $\square$

(3) Let us consider a non empty topological space S, a normed linear topological space T, a function f from S into T, and a point x of S. Then f is continuous at x if and only if for every real number e such that $0 < e$ there exists a subset H of S such that H is open and $x \in H$ and for every point y of S such that $y \in H$ holds $\|f(x) - f(y)\| < e$.
Proof: For every subset G of T such that G is open and $f(x) \in G$ there exists a subset H of S such that H is open and $x \in H$ and $f^{\circ}H \subseteq G$. $\square$

(4) Let us consider a non empty metric space S, a non empty, compact topological space V, a normed linear topological space T, and a function f from V into T. Suppose $V = S_{\text{top}}$. Then f is continuous if and only if for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every points x_1, x_2 of S such that $\rho(x_1, x_2) < d$ holds $\|f_{/x_1} - f_{/x_2}\| < e$.
Proof: For every point x of V, f is continuous at x. $\square$

Let S be a non empty metric space, T be a real normed space, and F be a subset of $(\text{the carrier of } T)^{(\text{the carrier of } S)}$. We say that F is equibounded if and only if

(Def. 2) there exists a real number K such that for every function f from the carrier of S into the carrier of T such that $f \in F$ for every element x of S, $\|f(x)\| \leqslant K$.

Let x_0 be a point of S. We say that F is equicontinuous at x_0 if and only if

(Def. 3) for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every function f from the carrier of S into the carrier of T such that $f \in F$ for every point x of S such that $\rho(x, x_0) < d$ holds $\|f(x) - f(x_0)\| < e$.

We say that F is equicontinuous if and only if

(Def. 4) for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every function f from the carrier of S into the carrier of T such that $f \in F$ for every points x_1, x_2 of S such that $\rho(x_1, x_2) < d$ holds $\|f(x_1) - f(x_2)\| < e$.

Now we state the proposition:

(5) Let us consider a non empty metric space S, a real normed space T, and a subset F of (the carrier of $T)^\alpha$. Suppose $S_{\rm top}$ is compact. Then F is equicontinuous if and only if for every point x of S, F is equicontinuous at x, where α is the carrier of S.

Proof: Define $\mathcal{P}[\text{element of } S, \text{real number}] \equiv 0 < \$_2$ and for every function f from the carrier of S into the carrier of T such that $f \in F$ for every point x of S such that $\rho(x, \$_1) < \$_2$ holds $\|f(x) - f(\$_1)\| < \frac{e}{2}$. For every element x_0 of the carrier of S, there exists an element d of $\mathbb{R}$ such that $\mathcal{P}[x_0, d]$.

Consider D being a function from the carrier of S into $\mathbb{R}$ such that for every element x_0 of the carrier of S, $\mathcal{P}[x_0, D(x_0)]$. Set $C_1 =$ the set of all $\mathrm{Ball}(x_0, \frac{D(x_0)}{2})$ where x_0 is an element of S. $C_1 \subseteq 2^\alpha$, where α is the carrier of $S_{\rm top}$. For every subset P of $S_{\rm top}$ such that $P \in C_1$ holds P is open. The carrier of $S_{\rm top} \subseteq \bigcup C_1$. Consider G being a family of subsets of $S_{\rm top}$ such that $G \subseteq C_1$ and G is cover of $\Omega_{S_{\rm top}}$ and finite. Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exists a point x_0 of S such that $\$_2 = x_0$ and $\$_1 = \mathrm{Ball}(x_0, \frac{D(x_0)}{2})$. For every object Z such that $Z \in G$ there exists an object x_0 such that $x_0 \in$ the carrier of S and $\mathcal{Q}[Z, x_0]$.

Consider H being a function from G into the carrier of S such that for every object Z such that $Z \in G$ holds $\mathcal{Q}[Z, H(Z)]$. For every object Z such that $Z \in G$ holds $Z = \mathrm{Ball}(H_{/Z}, \frac{D(H(Z))}{2})$. Reconsider $D_0 = D^\circ(\mathrm{rng}\, H)$ as a finite subset of $\mathbb{R}$. $G \neq \emptyset$. Consider x_3 being an object such that $x_3 \in G$. Consider x_3 being an object such that $x_3 \in \mathrm{rng}\, H$. Set $d_0 = \inf D_0$. Consider x_3 being an object such that $x_3 \in \mathrm{dom}\, D$ and $x_3 \in \mathrm{rng}\, H$ and $d_0 = D(x_3)$. For every function f from S into T such that $f \in F$ for every points x_1, x_2 of S such that $\rho(x_1, x_2) < d$ holds $\|f(x_1) - f(x_2)\| < e$. $\square$

2. Ascoli-Arzelà Theorem

From now on S, Z denote real normed spaces, T denotes a real Banach space, and F denotes a subset of (the carrier of $T)^{(\text{the carrier of } S)}$.

Now we state the proposition:

(6) Let us consider a real normed space Z. Then Z is complete if and only if MetricSpaceNorm Z is complete.

Proof: For every sequence s of Z such that s is Cauchy sequence by norm holds s is convergent by [10, (8)], [4, (5)]. $\square$

Let us consider a real normed space Z and a non empty subset H of MetricSpaceNorm Z. Now we state the propositions:

(7) If Z is complete, then MetricSpaceNorm $Z \restriction \overline{H}$ is complete.

PROOF: Reconsider $F = H$ as a non empty subset of Z. $\overline{F} = \overline{H}$. Set $N =$ MetricSpaceNorm $Z \restriction \overline{H}$. For every sequence S_2 of N such that S_2 is Cauchy holds S_2 is convergent. □

(8) MetricSpaceNorm $Z \restriction H$ is totally bounded if and only if MetricSpaceNorm $Z \restriction \overline{H}$ is totally bounded.

PROOF: Reconsider $F = H$ as a non empty subset of Z. Consider D being a subset of (MetricSpaceNorm $Z)_{\rm top}$ such that $D = H$ and $\overline{H} = \overline{D}$. $\overline{F} = \overline{H}$. MetricSpaceNorm $Z \restriction H$ is totally bounded. □

(9) Let us consider a real normed space Z, a non empty subset F of Z, and a non empty subset H of MetricSpaceNorm Z. Suppose Z is complete and $H = F$ and MetricSpaceNorm $Z \restriction H$ is totally bounded. Then

(i) $\overline{H}$ is sequentially compact, and

(ii) MetricSpaceNorm $Z \restriction \overline{H}$ is compact, and

(iii) $\overline{F}$ is compact.

The theorem is a consequence of (1), (7), and (8).

(10) Let us consider a real normed space Z, a non empty subset F of Z, a non empty subset H of MetricSpaceNorm Z, and a subset T of TopSpaceNorm Z. Suppose Z is complete and $H = F$ and $H = T$. Then

(i) MetricSpaceNorm $Z \restriction H$ is totally bounded iff $\overline{H}$ is sequentially compact, and

(ii) MetricSpaceNorm $Z \restriction H$ is totally bounded iff MetricSpaceNorm $Z \restriction \overline{H}$ is compact, and

(iii) MetricSpaceNorm $Z \restriction H$ is totally bounded iff $\overline{F}$ is compact, and

(iv) MetricSpaceNorm $Z \restriction H$ is totally bounded iff $\overline{T}$ is compact.

The theorem is a consequence of (1), (7), and (8).

(11) Let us consider a non empty, compact topological space S, and a normed linear topological space T. Suppose T is complete. Let us consider a non empty subset H of MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and T).

Then $\overline{H}$ is sequentially compact if and only if MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and $T) \restriction H$ is totally bounded. The theorem is a consequence of (7) and (8).

(12) Let us consider a non empty, compact topological space S, and a normed linear topological space T. Suppose T is complete. Let us consider a non

empty subset F of the $\mathbb{R}$-norm space of continuous functions of S and T, and a non empty subset H of MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and T). Suppose $H = F$. Then $\overline{F}$ is compact if and only if MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and T)$\restriction H$ is totally bounded. The theorem is a consequence of (1) and (11).

Let us consider a non empty metric space M, a non empty, compact topological space S, a normed linear topological space T, a subset G of (the carrier of $T)^{(\text{the carrier of } M)}$, and a non empty subset H of MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and T). Now we state the propositions:

(13) Suppose $S = M_{\text{top}}$ and T is complete. Then suppose $G = H$ and MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and T)$\restriction H$ is totally bounded. Then G is equibounded and equicontinuous.

Proof: Set $Z =$ the $\mathbb{R}$-norm space of continuous functions of S and T. Set $M_1 = \text{MetricSpaceNorm}\, Z\restriction H$. Consider L being a family of subsets of M_1 such that L is finite and the carrier of $M_1 = \bigcup L$ and for every subset C of M_1 such that $C \in L$ there exists an element w of M_1 such that $C = \text{Ball}(w, 1)$.

Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exists a point w of M_1 such that $\$_2 = w$ and $\$_1 = \text{Ball}(w, 1)$. For every object D such that $D \in L$ there exists an object w such that $w \in$ the carrier of M_1 and $\mathcal{Q}[D, w]$. Consider U being a function from L into the carrier of M_1 such that for every object D such that $D \in L$ holds $\mathcal{Q}[D, U(D)]$. For every object D such that $D \in L$ holds $D = \text{Ball}(U_{/D}, 1)$. Set $N_1 =$ the norm of Z. Reconsider $N_2 = N_1{}^\circ(\operatorname{rng} U)$ as a finite subset of $\mathbb{R}$. Consider x_3 being an object such that $x_3 \in L$. Consider x_3 being an object such that $x_3 \in \operatorname{rng} U$. Set $d_0 = \sup N_2$. Set $K = d_0 + 1$.

For every function f from the carrier of M into the carrier of T such that $f \in G$ for every element x of M, $\|f(x)\| \leqslant K$. For every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every function f from the carrier of M into the carrier of T such that $f \in G$ for every points x_1, x_2 of M such that $\rho(x_1, x_2) < d$ holds $\|f(x_1) - f(x_2)\| < e$. $\square$

(14) Suppose $S = M_{\text{top}}$ and T is complete. Then suppose $G = H$ and MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and T)$\restriction H$ is totally bounded. Then

(i) for every point x of S and for every non empty subset H_2 of MetricSpa–ceNorm T such that $H_2 = \{f(x)$, where f is a function from S into $T : f \in H\}$ holds MetricSpaceNorm $T\restriction H_2$ is totally bounded, and

(ii) G is equicontinuous.

PROOF: For every point x of S and for every non empty subset H_2 of MetricSpaceNorm T such that $H_2 = \{f(x)$, where f is a function from S into $T : f \in H\}$ holds MetricSpaceNorm $T \restriction H_2$ is totally bounded. □

(15) Let us consider a normed linear topological space T, and a real normed space R. Suppose $R =$ the normed structure of T and the topology of $T =$ the topology of TopSpaceNorm R. Then

(i) the distance by norm of $R =$ the distance by norm of T, and

(ii) MetricSpaceNorm $R =$ MetricSpaceNorm T, and

(iii) TopSpaceNorm $T =$ TopSpaceNorm R.

PROOF: For every points x, y of R, (the distance by norm of $T)(x, y) = \|x - y\|$ by [11, (19)]. □

Let us consider a non empty metric space M, a non empty, compact topological space S, a normed linear topological space T, a subset G of (the carrier of $T)^{(\text{the carrier of } M)}$, and a non empty subset H of MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and T). Now we state the propositions:

(16) Suppose $S = M_{\rm top}$ and T is complete and $G = H$. Then MetricSpaceNo–rm(the $\mathbb{R}$-norm space of continuous functions of S and $T) \restriction H$ is totally bounded if and only if G is equicontinuous and for every point x of S and for every non empty subset H_2 of MetricSpaceNorm T such that $H_2 = \{f(x)$, where f is a function from S into $T : f \in H\}$ holds MetricSpaceNorm $T \restriction \overline{H_2}$ is compact.

PROOF: Set $Z =$ the $\mathbb{R}$-norm space of continuous functions of S and T. Set $M_1 =$ MetricSpaceNorm $Z \restriction H$. For every real number e such that $e > 0$ there exists a family L of subsets of M_1 such that L is finite and the carrier of $M_1 = \bigcup L$ and for every subset C of M_1 such that $C \in L$ there exists an element w of M_1 such that $C = \text{Ball}(w, e)$. □

(17) Suppose $S = M_{\rm top}$ and T is complete and $G = H$. Then $\overline{H}$ is sequentially compact if and only if G is equicontinuous and for every point x of S and for every non empty subset H_2 of MetricSpaceNorm T such that $H_2 = \{f(x)$, where f is a function from S into $T : f \in H\}$ holds MetricSpaceNorm $T \restriction \overline{H_2}$ is compact. The theorem is a consequence of (11) and (16).

Let us consider a non empty metric space M, a non empty, compact topological space S, a normed linear topological space T, a non empty subset F of the $\mathbb{R}$-norm space of continuous functions of S and T, and a subset G of (the carrier of $T)^{(\text{the carrier of } M)}$. Now we state the propositions:

(18) Suppose $S = M_{\mathrm{top}}$ and T is complete and $G = F$. Then $\overline{F}$ is compact if and only if G is equicontinuous and for every point x of S and for every non empty subset F_1 of MetricSpaceNorm T such that $F_1 = \{f(x)$, where f is a function from S into $T : f \in F\}$ holds MetricSpaceNorm $T \restriction \overline{F_1}$ is compact. The theorem is a consequence of (12) and (16).

(19) Suppose $S = M_{\mathrm{top}}$ and T is complete and $G = F$. Then $\overline{F}$ is compact if and only if for every point x of M, G is equicontinuous at x and for every point x of S and for every non empty subset F_1 of MetricSpaceNorm T such that $F_1 = \{f(x)$, where f is a function from S into $T : f \in F\}$ holds MetricSpaceNorm $T \restriction \overline{F_1}$ is compact. The theorem is a consequence of (18) and (5).

(20) Let us consider a normed linear topological space T. Then T is compact if and only if TopSpaceNorm T is compact. The theorem is a consequence of (15).

(21) Let us consider a normed linear topological space T, and a set X. Then X is a compact subset of T if and only if X is a compact subset of TopSpaceNorm T. The theorem is a consequence of (15).

(22) Let us consider a normed linear topological space T. If T is compact, then T is complete. The theorem is a consequence of (20) and (6).

Let us observe that every normed linear topological space which is compact is also complete.

Now we state the proposition:

(23) Let us consider a non empty metric space M, a non empty, compact topological space S, a normed linear topological space T, a compact subset U of T, a non empty subset F of the $\mathbb{R}$-norm space of continuous functions of S and T, and a subset G of (the carrier of $T)^{\alpha}$. Suppose $S = M_{\mathrm{top}}$ and T is complete and $G = F$ and for every function f such that $f \in F$ holds $\mathrm{rng}\, f \subseteq U$. Then

(i) if $\overline{F}$ is compact, then G is equibounded and equicontinuous, and

(ii) if G is equicontinuous, then $\overline{F}$ is compact,

where α is the carrier of M.

Proof: Reconsider $H = F$ as a non empty subset of MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and T). Set $Z =$ the $\mathbb{R}$-norm space of continuous functions of S and T. MetricSpaceNorm $Z \restriction H$ is totally bounded iff $\overline{F}$ is compact. For every point x of S and for every non empty subset F_1 of MetricSpaceNorm T such that $F_1 = \{f(x)$, where f is a function from S into $T : f \in F\}$ holds MetricSpaceNorm $T \restriction \overline{F_1}$ is compact. $\square$

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On Primary Ideals. Part I

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Summary. We formalize in the Mizar System [3], [4], definitions and basic propositions about primary ideals of a commutative ring along with Chapter 4 of [1] and Chapter III of [8]. Additionally other necessary basic ideal operations such as compatibilities taking radical and intersection of finite number of ideals are formalized as well in order to prove theorems relating primary ideals. These basic operations are mainly quoted from Chapter 1 of [1] and compiled as preliminaries in the first half of the article.

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Keywords: primary ideal; radical ideal; prime ideal

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From now on R denotes a commutative ring, A denotes a non degenerated, commutative ring, I, J, $\mathfrak{p}$ denote ideals of A, $\mathfrak{q}$ denotes a prime ideal of A, and M, M_1, M_2 denote ideals of ${}^{A}/_{\mathfrak{p}}$.

Let us consider A and $\mathfrak{p}$. We introduce the notation $\pi_{A \to A/\mathfrak{p}}$ as a synonym of the canonical homomorphism of $\mathfrak{p}$ into quotient field.

Now we state the proposition:

(1) Let us consider ideals a, b of A, and a prime ideal $\mathfrak{q}$ of A. If $a \cap b \subseteq \mathfrak{q}$, then $a \subseteq \mathfrak{q}$ or $b \subseteq \mathfrak{q}$.

Let us consider A. Let a be a non empty finite sequence of elements of Ideals A and i be an element of dom a. Let us observe that the functor $a(i)$ yields a non empty subset of A. One can check that $a(i)$ is closed under addition, left and right ideal as a subset of A and $\bigcap \operatorname{rng} a$ is closed under addition, left and right ideal as a subset of A.

Now we state the proposition:

(2) [1, p.8, Prop. 1.11 ii)]:

Let us consider a non empty finite sequence a of elements of Ideals A, and

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a prime ideal $\mathfrak{q}$ of A. Suppose $\bigcap \operatorname{rng} a \subseteq \mathfrak{q}$. Then there exists an object i such that

(i) $i \in \operatorname{dom} a$, and

(ii) $a(i) \subseteq \mathfrak{q}$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every non empty finite sequence a of elements of Ideals A for every prime ideal $\mathfrak{q}$ of A such that $\operatorname{len} a = \$_1$ holds if $\bigcap \operatorname{rng} a \subseteq \mathfrak{q}$, then there exists an object i such that $i \in \operatorname{dom} a$ and $a(i) \subseteq \mathfrak{q}$. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every non zero natural number i, $\mathcal{P}[i]$. □

Let us consider A. Let I be an ideal of A. The functor $\%I$ yielding a function from $2^{(\text{the carrier of } A)}$ into $2^{(\text{the carrier of } A)}$ is defined by

(Def. 1) for every subset x of A, $it(x) = x \,\%\, I$.

Now we state the propositions:

(3) Let us consider a proper ideal I of A, and a non empty finite sequence F of elements of Ideals A. Then

(i) $\operatorname{rng}(\%I) \cdot F \neq \emptyset$, and

(ii) $\operatorname{rng} F \neq \emptyset$, and

(iii) $\bigcap \operatorname{rng}(\%I) \cdot F \subseteq$ the carrier of A.

(4) [1, P.8, EX.1.12. IV)]:
Let us consider a proper ideal I of A, and a non empty finite sequence F of elements of Ideals A. Then $(\%I)(\bigcap \operatorname{rng} F) = \bigcap \operatorname{rng}(\%I) \cdot F$.
PROOF: $\operatorname{rng}(\%I) \cdot F \neq \emptyset$. For every object x such that $x \in (\%I)(\bigcap \operatorname{rng} F)$ holds $x \in \bigcap \operatorname{rng}(\%I) \cdot F$. $\bigcap \operatorname{rng}(\%I) \cdot F \subseteq (\%I)(\bigcap \operatorname{rng} F)$. □

(5) $I * \Omega_A = I$.

(6) Let us consider finite sequences f, g of elements of 2^{α}. Suppose $\operatorname{len} f \geqslant \operatorname{len} g > 0$ and $I^{\operatorname{len} f} = f(\operatorname{len} f)$ and $f(1) = I$ and for every natural number i such that i, $i+1 \in \operatorname{dom} f$ holds $f(i+1) = I * f_{/i}$ and $I^{\operatorname{len} g} = g(\operatorname{len} g)$ and $g(1) = I$ and for every natural number i such that i, $i+1 \in \operatorname{dom} g$ holds $g(i+1) = I * g_{/i}$. Then $f \restriction \operatorname{dom} g = g$, where α is the carrier of A.
PROOF: Set $f_1 = f \restriction \operatorname{dom} g$. For every natural number i such that i, $i+1 \in \operatorname{dom} f_1$ holds $f_1(i+1) = I * f_{1/i}$. $f_1 = g$. □

(7) Let us consider a natural number n. If $n > 0$, then $I^{n+1} = I * I^n$. The theorem is a consequence of (6).

(8) [1, P.9, EX.1.13 II)]:
$\sqrt{I} = \sqrt{\sqrt{I}}$.

PROOF: For every object o such that $o \in \sqrt{\sqrt{I}}$ holds $o \in \sqrt{I}$. □

(9) [1, P.9, EX.1.13 III)]:
$\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.
PROOF: For every object o such that $o \in \sqrt{I \cap J}$ holds $o \in \sqrt{I} \cap \sqrt{J}$. $\sqrt{I} \cap \sqrt{J} \subseteq \sqrt{I \cap J}$. $\square$

(10) [1, P.9, EX.1.13 IV)]:
$\sqrt{I} = \Omega_A$ if and only if $I = \Omega_A$.
PROOF: If $\sqrt{I} = \Omega_A$, then $I = \Omega_A$ by [7, (2)], [2, (19)]. $\square$

(11) [1, P.9, EX.1.13 V)]:
$\sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}$.
PROOF: For every object o such that $o \in \sqrt{I + J}$ holds $o \in \sqrt{\sqrt{I} + \sqrt{J}}$. $\sqrt{\sqrt{I} + \sqrt{J}} \subseteq \sqrt{I + J}$. $\square$

(12) [1, P.9, EX.1.13 VI)]:
Let us consider a prime ideal $\mathfrak{q}$ of A, and a non zero natural number n. Then $\sqrt{\mathfrak{q}^n} = \mathfrak{q}$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv \sqrt{\mathfrak{q}^{\$_1}} = \mathfrak{q}$. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every non zero natural number i, $\mathcal{P}[i]$. $\square$

(13) [1, P.9, PROP1.16]:
If $\sqrt{I}$ and $\sqrt{J}$ are co-prime, then I and J are co-prime. The theorem is a consequence of (11) and (10).

(14) Let us consider elements x, y of the carrier of $A/\mathfrak{p}$. Suppose $x, y \in (\pi_{A \to A/\mathfrak{p}})^\circ I$. Then $x + y \in (\pi_{A \to A/\mathfrak{p}})^\circ I$.

(15) Let us consider elements a, x of the carrier of $A/\mathfrak{p}$. Suppose $x \in (\pi_{A \to A/\mathfrak{p}})^\circ I$. Then $a \cdot x \in (\pi_{A \to A/\mathfrak{p}})^\circ I$.

(16) $(\pi_{A \to A/\mathfrak{p}})^\circ I$ is an ideal of $A/\mathfrak{p}$. The theorem is a consequence of (14) and (15).

(17) Let us consider elements x, y of the carrier of A. Suppose $x, y \in (\pi_{A \to A/\mathfrak{p}})^{-1}(M_1)$. Then $x + y \in (\pi_{A \to A/\mathfrak{p}})^{-1}(M_1)$.

(18) Let us consider elements r, x of the carrier of A.
Suppose $x \in (\pi_{A \to A/\mathfrak{p}})^{-1}(M_1)$. Then $r \cdot x \in (\pi_{A \to A/\mathfrak{p}})^{-1}(M_1)$.

(19) $(\pi_{A \to A/\mathfrak{p}})^{-1}(M_1)$ is an ideal of A. The theorem is a consequence of (17) and (18).

(20) $\mathfrak{p} \subseteq (\pi_{A \to A/\mathfrak{p}})^{-1}(M_1)$.
PROOF: For every object x such that $x \in \mathfrak{p}$ holds $x \in (\pi_{A \to A/\mathfrak{p}})^{-1}(M_1)$ by [5, (13)]. $\square$

(21) $(\pi_{A \to A/\mathfrak{p}})^\circ((\pi_{A \to A/\mathfrak{p}})^{-1}(M_1)) = M_1$.

(22) If $\mathfrak{p} \subseteq I$, then $(\pi_{A \to A/\mathfrak{p}})^{-1}((\pi_{A \to A/\mathfrak{p}})^{\circ} I) = I$.
PROOF: $(\pi_{A \to A/\mathfrak{p}})^{-1}((\pi_{A \to A/\mathfrak{p}})^{\circ} I) \subseteq I$. $\square$

(23) If $I \subseteq J$, then $(\pi_{A \to A/\mathfrak{p}})^{\circ} I \subseteq (\pi_{A \to A/\mathfrak{p}})^{\circ} J$.

(24) If $M_1 \subseteq M_2$, then $(\pi_{A \to A/\mathfrak{p}})^{-1}(M_1) \subseteq (\pi_{A \to A/\mathfrak{p}})^{-1}(M_2)$.

(25) $(\pi_{A \to A/\mathfrak{p}})^{-1}(\Omega_{A/\mathfrak{p}}) = \Omega_A$.

(26) If M is proper, then $(\pi_{A \to A/\mathfrak{p}})^{-1}(M)$ is proper. The theorem is a consequence of (21).

(27) If $\mathfrak{p} \subseteq I$ and I is maximal, then $(\pi_{A \to A/\mathfrak{p}})^{\circ} I$ is maximal. The theorem is a consequence of (16), (25), (22), (26), (19), and (24).

Let us consider A and $\mathfrak{p}$. The functor $\Psi_{\mathfrak{p}}$ yielding a function from Ideals $A/\mathfrak{p}$ into Ideals$(A, \mathfrak{p})$ is defined by

(Def. 2) for every element x of Ideals $A/\mathfrak{p}$, $it(x) = (\pi_{A \to A/\mathfrak{p}})^{-1}(x)$.

Let J be a proper ideal of A. Observe that A/J is non degenerated and commutative.

[1, p.2, Prop. 1.1]:

Let us consider A. Let $\mathfrak{p}$ be an ideal of A. Let us note that $\Psi_{\mathfrak{p}}$ is one-to-one and $\subseteq$-monotone.

[1, p.50, Chapter 4]:

Let A be a well unital, non empty double loop structure and S be a subset of A. We say that S is quasi-primary if and only if

(Def. 3) for every elements x, y of A such that $x \cdot y \in S$ holds $x \in S$ or $y \in \sqrt{S}$.

We say that S is primary if and only if

(Def. 4) S is proper and quasi-primary.

Let K be a well unital, non empty double loop structure. Let us note that every subset of K which is primary is also proper and quasi-primary and every subset of K which is proper and quasi-primary is also primary.

Now we state the proposition:

(28) Let us consider an ideal $\mathfrak{q}$ of A. If $\mathfrak{q}$ is prime, then $\mathfrak{q}$ is primary.
PROOF: For every elements x, y of A such that $x \cdot y \in \mathfrak{q}$ holds $x \in \mathfrak{q}$ or $y \in \sqrt{\mathfrak{q}}$. $\square$

Let us consider A. One can verify that every ideal of A which is prime is also primary.

Let A be a non degenerated, commutative ring. Let us observe that there exists a proper ideal of A which is primary.

Now we state the propositions:

(29) I is primary if and only if $I \neq \Omega_A$ and for every elements x, y of A such that $x \cdot y \in I$ and $x \notin I$ holds $y \in \sqrt{I}$.

(30) $I \neq \Omega_A$ and for every elements x, y of A such that $x \cdot y \in I$ and $x \notin I$ holds $y \in \sqrt{I}$ if and only if ${}^{A}\!/_{I}$ is not degenerated and for every element z of ${}^{A}\!/_{I}$ such that z is zero-divisible holds z is nilpotent.

Proof: If $I \neq \Omega_A$ and for every elements x, y of A such that $x \cdot y \in I$ and $x \notin I$ holds $y \in \sqrt{I}$, then ${}^{A}\!/_{I}$ is not degenerated and for every element z of ${}^{A}\!/_{I}$ such that z is zero-divisible holds z is nilpotent. If ${}^{A}\!/_{I}$ is not degenerated and for every element z of ${}^{A}\!/_{I}$ such that z is zero-divisible holds z is nilpotent, then $I \neq \Omega_A$ and for every elements x_1, y_1 of A such that $x_1 \cdot y_1 \in I$ and $x_1 \notin I$ holds $y_1 \in \sqrt{I}$ by [6, (2)]. □

(31) I is primary if and only if ${}^{A}\!/_{I}$ is not degenerated and for every element x of ${}^{A}\!/_{I}$ such that x is zero-divisible holds x is nilpotent. The theorem is a consequence of (29) and (30).

[1, p.50, Prop. 4.1]:

Let us consider A. Let Q be a primary ideal of A. Note that $\sqrt{Q}$ is prime.

Let I be a primary ideal of A. One can verify that every element of ${}^{A}\!/_{I}$ which is zero-divisible is also nilpotent.

Let P, Q be ideals of A. We say that Q is P-primary if and only if

(Def. 5) $\sqrt{Q} = P$.

The functor PrimaryIdeals(A) yielding a family of subsets of the carrier of A is defined by the term

(Def. 6) the set of all I where I is a primary ideal of A.

Note that PrimaryIdeals(A) is non empty.

Let us consider $\mathfrak{q}$. The functor PrimaryIdeals$(A, \mathfrak{q})$ yielding a non empty subset of PrimaryIdeals(A) is defined by the term

(Def. 7) $\{I$, where I is a primary ideal of $A : I$ is $\mathfrak{q}$-primary$\}$.

Let us consider a proper ideal $\mathfrak{p}$ of A. Now we state the propositions:

(32) $(\pi_{A \to A/\mathfrak{p}})^{\circ}\sqrt{\mathfrak{p}} = \mathrm{nilrad}({}^{A}\!/_{\mathfrak{p}})$.

Proof: For every object x such that $x \in \mathrm{nilrad}({}^{A}\!/_{\mathfrak{p}})$ holds $x \in (\pi_{A \to A/\mathfrak{p}})^{\circ}\sqrt{\mathfrak{p}}$. $(\pi_{A \to A/\mathfrak{p}})^{\circ}\sqrt{\mathfrak{p}} \subseteq \mathrm{nilrad}({}^{A}\!/_{\mathfrak{p}})$. □

(33) If $\sqrt{\mathfrak{p}}$ is maximal, then ${}^{A}\!/_{\mathfrak{p}}$ is local.

Proof: Set $m = \sqrt{\mathfrak{p}}$. $(\pi_{A \to A/\mathfrak{p}})^{\circ}m = \mathrm{nilrad}({}^{A}\!/_{\mathfrak{p}})$. For every objects m_1, m_2 such that m_1, $m_2 \in$ m-Spectrum$({}^{A}\!/_{\mathfrak{p}})$ holds $m_1 = m_2$. □

(34) [1, p.51, Prop. 4.2]:

Let us consider a proper ideal $\mathfrak{p}$ of A. If $\sqrt{\mathfrak{p}}$ is maximal, then $\mathfrak{p}$ is primary.

Proof: Set $m = \sqrt{\mathfrak{p}}$. $(\pi_{A \to A/\mathfrak{p}})^{\circ}m$ is maximal. ${}^{A}\!/_{\mathfrak{p}}$ is local. For every element x of ${}^{A}\!/_{\mathfrak{p}}$ such that x is zero-divisible holds x is nilpotent. □

(35) [1, P.51, PROP. 4.2] CASE OF M IS MAXIMAL IDEAL:
Let us consider a maximal ideal M of A, and a non zero natural number n. Then $M^n \in \mathrm{PrimaryIdeals}(A, M)$. The theorem is a consequence of (12) and (34).

(36) Let us consider ideals q_1, q_2 of A. Suppose $q_1, q_2 \in \mathrm{PrimaryIdeals}(A, \mathfrak{q})$. Then $q_1 \cap q_2 \in \mathrm{PrimaryIdeals}(A, \mathfrak{q})$. The theorem is a consequence of (9) and (29).

(37) [1, P.51, LEMMA 4.3]:
Let us consider a prime ideal $\mathfrak{q}$ of A, and a non empty finite sequence F of elements of $\mathrm{PrimaryIdeals}(A, \mathfrak{q})$. Then $\bigcap \mathrm{rng}\, F \in \mathrm{PrimaryIdeals}(A, \mathfrak{q})$.
PROOF: $\bigcap \mathrm{rng}\, F \in \mathrm{PrimaryIdeals}(A, \mathfrak{q})$. □

(38) Let us consider a proper ideal I of A, and an element x of $\sqrt{I}$. Then there exists a natural number m such that

(i) $m \in \{n$, where n is an element of $\mathbb{N} : x^n \notin I\}$, and

(ii) $x^{m+1} \in I$.

PROOF: Consider x_1 being an element of A such that $x_1 = x$ and there exists an element n of $\mathbb{N}$ such that ${x_1}^n \in I$. Consider n_1 being an element of $\mathbb{N}$ such that ${x_1}^{n_1} \in I$. $n_1 \notin \{n$, where n is an element of $\mathbb{N} : x^n \notin I\}$. $0 \in \{n$, where n is an element of $\mathbb{N} : x^n \notin I\}$. $\{n$, where n is an element of $\mathbb{N} : x^n \notin I\} = \mathbb{N}$. □

(39) Let us consider proper ideals I, J of A. Suppose $I \subseteq J \subseteq \sqrt{I}$ and for every elements x, y of A such that $x \cdot y \in I$ and $x \notin I$ holds $y \in J$. Then

(i) I is primary, and

(ii) $\sqrt{I} = J$, and

(iii) J is prime.

PROOF: $\sqrt{I} \subseteq J$. □

(40) Let us consider a proper ideal Q of A. Suppose for every elements x, y of A such that $x \cdot y \in Q$ and $y \notin \sqrt{Q}$ holds $x \in Q$. Then

(i) Q is primary, and

(ii) $\sqrt{Q}$ is prime.

The theorem is a consequence of (39).

(41) [1, P.51, LEMMA 4.4 I)]:
Let us consider an ideal $\mathfrak{p}$ of A, and an element x of A. Suppose $x \in \mathfrak{p}$. Then $\mathfrak{p} \,\%\, \{x\}\text{–ideal} = \Omega_A$.
PROOF: Set $I = \{x\}\text{–ideal}$. If $x \in \mathfrak{p}$, then $\mathfrak{p} \,\%\, I = \Omega_A$. □

(42) [1, p.51, Lemma 4.4 ii)]:
Let us consider an ideal $\mathfrak{p}$ of A, and an element x of A. Suppose $\mathfrak{p} \in \text{PrimaryIdeals}(A, \mathfrak{q})$. If $x \notin \mathfrak{p}$, then $\mathfrak{p} \,\%\, \{x\}$–ideal $\in \text{PrimaryIdeals}(A, \mathfrak{q})$.
Proof: Set $I = \{x\}$–ideal. Consider q_1 being a primary ideal of A such that $q_1 = \mathfrak{p}$ and q_1 is $\mathfrak{q}$-primary. If $x \notin \mathfrak{p}$, then $\mathfrak{p} \,\%\, I \in \text{PrimaryIdeals}(A, \mathfrak{q})$. □

(43) [1, p.51, Lemma 4.4 iii)]:
Let us consider an ideal $\mathfrak{p}$ of A, and an element x of A. Suppose $\mathfrak{p} \in \text{PrimaryIdeals}(A, \mathfrak{q})$. If $x \notin \mathfrak{q}$, then $\mathfrak{p} \,\%\, \{x\}$–ideal $= \mathfrak{p}$.
Proof: Set $I = \{x\}$–ideal. Consider Q being a primary ideal of A such that $Q = \mathfrak{p}$ and Q is $\mathfrak{q}$-primary. If $x \notin \mathfrak{q}$, then $\mathfrak{p} \,\%\, I = \mathfrak{p}$. □

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Some Properties of Membership Functions Composed of Triangle Functions and Piecewise Linear Functions[1]

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Summary. IF-THEN rules in fuzzy inference is composed of multiple fuzzy sets (membership functions). IF-THEN rules can therefore be considered as a pair of membership functions [7]. The evaluation function of fuzzy control is composite function with fuzzy approximate reasoning and is functional on the set of membership functions. We obtained continuity of the evaluation function and compactness of the set of membership functions [12]. Therefore, we proved the existence of pair of membership functions, which maximizes (minimizes) evaluation function and is considered IF-THEN rules, in the set of membership functions by using extreme value theorem. The set of membership functions (fuzzy sets) is defined in this article to verifier our proofs before by Mizar [9], [10], [4]. Membership functions composed of triangle function, piecewise linear function and Gaussian function used in practice are formalized using existing functions.

On the other hand, not only curve membership functions mentioned above but also membership functions composed of straight lines (piecewise linear function) like triangular and trapezoidal functions are formalized. Moreover, different from the definition in [3] formalizations of triangular and trapezoidal function composed of two straight lines, minimum function and maximum functions are proposed. We prove, using the Mizar [2], [1] formalism, some properties of membership functions such as continuity and periodicity [13], [8].

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1. Preliminaries

Now we state the propositions:

(1) Let us consider real numbers a, b, c, d. Then $|\max(c, \min(d, a)) - \max(c, \min(d, b))| \leqslant |a - b|$.

(2) Let us consider a real number x. Then $|\sin x| \leqslant |x|$.

(3) Let us consider real numbers x, y. Then $|\sin x - \sin y| \leqslant |x - y|$. The theorem is a consequence of (2).

(4) Let us consider a real number x. If $\exp x = 1$, then $x = 0$.

(5) Let us consider real numbers a, b, p, q. Suppose $a > 0$ and $p > 0$ and $\frac{-b}{a} < \frac{q}{p}$. Then

(i) $\frac{-b}{a} < \frac{q-b}{a+p} < \frac{q}{p}$, and

(ii) $\frac{a \cdot q + b \cdot p}{a+p} > 0$.

(6) Let us consider real numbers a, b, p, q, s. Suppose $a > 0$ and $p > 0$ and $\frac{s-b}{a} = \frac{s-q}{-p}$. Then

(i) $\frac{s-b}{a} = \frac{q-b}{a+p}$, and

(ii) $\frac{s-q}{-p} = \frac{q-b}{a+p}$.

(7) Let us consider real numbers a, b, p, q, s. Suppose $a > 0$ and $p > 0$ and $\frac{s-b}{a} < \frac{s-q}{-p}$. Then $\frac{s-b}{a} < \frac{q-b}{a+p} < \frac{s-q}{-p}$.

2. The Set of Membership Functions

Let X be a non empty set. The functor Membership-Funcs(X) yielding a set is defined by

(Def. 1) for every object f, $f \in$ *it* iff f is a membership function of X.

Now we state the propositions:

(8) Let us consider a non empty set X, and an object x. Suppose $x \in$ Membership-Funcs(X). Then there exists a membership function f of X such that

(i) $x = f$, and

(ii) $\operatorname{dom} f = X$.

(9) Membership-Funcs($\mathbb{R}$) $= \{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R} : f$ is a fuzzy set of $\mathbb{R}\}$. The theorem is a consequence of (8).

(10) Let us consider non empty sets A, X.
Then $\{\chi_{A,X}\} \subseteq$ Membership-Funcs(X).

(11) $\{\chi_{[a,b],\mathbb{R}}$, where a, b are real numbers : $a \leqslant b\} \subseteq$ Membership-Funcs($\mathbb{R}$).

(12) $\{\chi_{A,\mathbb{R}}$, where A is a subset of $\mathbb{R}$: $A \subseteq \mathbb{R}\} \subseteq$ Membership-Funcs($\mathbb{R}$).

(13) $\{f$, where f is a fuzzy set of $\mathbb{R}$: there exists a subset A of $\mathbb{R}$ such that $f = \chi_{A,\mathbb{R}}\} \subseteq$ Membership-Funcs($\mathbb{R}$).

(14) Let us consider functions f, g from $\mathbb{R}$ into $\mathbb{R}$, and a real number a. Suppose g is continuous and for every real number x, $f(x) = \min(a, g(x))$. Then f is continuous.

Proof: For every real number x_0 such that $x_0 \in \operatorname{dom} f$ holds f is continuous in x_0. $\square$

Let us consider functions F, f, g from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(15) If f is continuous and g is continuous and for every real number x, $F(x) = \min(f(x), g(x))$, then F is continuous.

Proof: For every real number x_0 such that $x_0 \in \operatorname{dom} F$ holds F is continuous in x_0. $\square$

(16) If f is continuous and g is continuous and for every real number x, $F(x) = \max(f(x), g(x))$, then F is continuous.

Proof: For every real number x_0 such that $x_0 \in \operatorname{dom} F$ holds F is continuous in x_0. $\square$

(17) Let us consider functions f, g from $\mathbb{R}$ into $\mathbb{R}$, and a real number a. Suppose g is continuous and for every real number x, $f(x) = \max(a, g(x))$. Then f is continuous. The theorem is a consequence of (16).

(18) Let us consider functions f, g from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b. Suppose g is continuous and for every real number x, $f(x) = \max(a, \min(b, g(x)))$. Then f is continuous.

Proof: Define $\mathcal{H}(\text{element of } \mathbb{R}) = (\min(b, g(\$_1)))(\in \mathbb{R})$. Consider h being a function from $\mathbb{R}$ into $\mathbb{R}$ such that for every element x of $\mathbb{R}$, $h(x) = \mathcal{H}(x)$. For every real number x, $h(x) = \min(b, g(x))$. h is continuous. For every real number x, $f(x) = \max(a, h(x))$. $\square$

(19) Let us consider functions f, g from $\mathbb{R}$ into $\mathbb{R}$. Suppose g is continuous and for every real number x, $f(x) = \max(0, \min(1, g(x)))$. Then f is continuous.

Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$ and real numbers a, b. Now we state the propositions:

(20) If for every real number t_1, $f(t_1) = \frac{1}{2} \cdot (\sin(a \cdot t_1 + b)) + \frac{1}{2}$, then f is continuous.

Proof: For every real number x_0 such that $x_0 \in \operatorname{dom} f$ holds f is continuous in x_0. $\square$

(21) If for every real number x, $f(x) = \frac{1}{2} \cdot (\sin(a \cdot x + b)) + \frac{1}{2}$, then f is continuous.

(22) Let us consider real numbers r, s, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose for every real number x, $f(x) = \max(r, \min(s, x))$. Then f is Lipschitzian. The theorem is a consequence of (1).

(23) Let us consider a function g from $\mathbb{R}$ into $\mathbb{R}$. Then $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}$: for every real number $x, f(x) = \min(1, \max(0, g(x)))\} \subseteq$ Membership-Funcs($\mathbb{R}$).

Proof: Consider f being a function from $\mathbb{R}$ into $\mathbb{R}$ such that $f_0 = f$ and for every real number x, $f(x) = \min(1, \max(0, g(x)))$. rng $f \subseteq [0, 1]$. □

(24) $\{f$, where f, g are functions from $\mathbb{R}$ into $\mathbb{R}$: for every real number $x, f(x) = \max(0, \min(1, g(x)))\} \subseteq$ Membership-Funcs($\mathbb{R}$).

Let us consider functions f, g from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(25) If for every real number x, $f(x) = \max(0, \min(1, g(x)))$, then f is a fuzzy set of $\mathbb{R}$.

(26) If for every real number x, $f(x) = \min(1, \max(0, g(x)))$, then f is a fuzzy set of $\mathbb{R}$. The theorem is a consequence of (23).

(27) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}$: there exist real numbers a, b such that for every real number $t_1, f(t_1) = \frac{1}{2} \cdot (\sin(a \cdot t_1 + b)) + \frac{1}{2}\} \subseteq$ Membership-Funcs($\mathbb{R}$).

Proof: Consider f being a function from $\mathbb{R}$ into $\mathbb{R}$ such that $x = f$ and there exist real numbers a, b such that for every real number t_1, $f(t_1) = \frac{1}{2} \cdot (\sin(a \cdot t_1 + b)) + \frac{1}{2}$. rng $f \subseteq [0, 1]$. □

(28) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}$, a, b are real numbers : for every real number $t_1, f(t_1) = \frac{1}{2} \cdot (\sin(a \cdot t_1 + b)) + \frac{1}{2}\} \subseteq$ Membership-Funcs($\mathbb{R}$).

Proof: Consider f being a function from $\mathbb{R}$ into $\mathbb{R}$, a, b being real numbers such that $x = f$ and for every real number t_1, $f(t_1) = \frac{1}{2} \cdot (\sin(a \cdot t_1 + b)) + \frac{1}{2}$. rng $f \subseteq [0, 1]$. □

(29) Let us consider real numbers a, b, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose for every real number t_1, $f(t_1) = \frac{1}{2} \cdot (\sin(a \cdot t_1 + b)) + \frac{1}{2}$. Then f is a fuzzy set of $\mathbb{R}$. The theorem is a consequence of (28).

(30) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}$: there exist real numbers a, b such that for every real number $t_1, f(t_1) = \frac{1}{2} \cdot (\cos(a \cdot t_1 + b)) + \frac{1}{2}\} \subseteq$ Membership-Funcs($\mathbb{R}$).

Proof: Consider f being a function from $\mathbb{R}$ into $\mathbb{R}$ such that $x = f$ and there exist real numbers a, b such that for every real number t_1, $f(t_1) = \frac{1}{2} \cdot (\cos(a \cdot t_1 + b)) + \frac{1}{2}$. rng $f \subseteq [0, 1]$. □

(31) Let us consider real numbers a, b, and a function f from $\mathbb{R}$ into $\mathbb{R}$.

Suppose for every real number t_1, $f(t_1) = \frac{1}{2} \cdot (\cos(a \cdot t_1 + b)) + \frac{1}{2}$. Then f is a fuzzy set of $\mathbb{R}$. The theorem is a consequence of (30).

(32) Let us consider real numbers a, b, and a fuzzy set f of $\mathbb{R}$. Suppose $a \neq 0$ and for every real number t_1, $f(t_1) = \frac{1}{2} \cdot (\sin(a \cdot t_1 + b)) + \frac{1}{2}$. Then f is normalized.

Proof: There exists an element x of $\mathbb{R}$ such that $f(x) = 1$. □

(33) Let us consider a fuzzy set f of $\mathbb{R}$. Suppose $f \in \{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}$: there exist real numbers a, b such that $a \neq 0$ and for every real number $t_1, f(t_1) = \frac{1}{2} \cdot (\sin(a \cdot t_1 + b)) + \frac{1}{2}\}$. Then f is normalized.

Proof: Consider f_2 being a function from $\mathbb{R}$ into $\mathbb{R}$ such that $f = f_2$ and there exist real numbers a, b such that $a \neq 0$ and for every real number t_1, $f_2(t_1) = \frac{1}{2} \cdot (\sin(a \cdot t_1 + b)) + \frac{1}{2}$. Consider a, b being real numbers such that $a \neq 0$ and for every real number t_1, $f_2(t_1) = \frac{1}{2} \cdot (\sin(a \cdot t_1 + b)) + \frac{1}{2}$. There exists an element x of $\mathbb{R}$ such that $f(x) = 1$. □

(34) Let us consider a fuzzy set f of $\mathbb{R}$, and real numbers a, b. Suppose $a \neq 0$ and for every real number t_1, $f(t_1) = \frac{1}{2} \cdot (\cos(a \cdot t_1 + b)) + \frac{1}{2}$. Then f is normalized.

Proof: There exists an element x of $\mathbb{R}$ such that $f(x) = 1$. □

(35) Let us consider a fuzzy set f of $\mathbb{R}$. Suppose $f \in \{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}$: there exist real numbers a, b such that $a \neq 0$ and for every real number $t_1, f(t_1) = \frac{1}{2} \cdot (\cos(a \cdot t_1 + b)) + \frac{1}{2}\}$. Then f is normalized. The theorem is a consequence of (34).

(36) Let us consider a function F from $\mathbb{R}$ into $\mathbb{R}$, real numbers a, b, c, d, and an integer i. Suppose $a \neq 0$ and $i \neq 0$ and for every real number x, $F(x) = \max(0, \min(1, c \cdot (\sin(a \cdot x + b)) + d))$. Then F is $(\frac{2 \cdot \pi}{a} \cdot i)$-periodic.

Proof: For every real number x such that $x \in \operatorname{dom} F$ holds $x + \frac{2 \cdot \pi}{a} \cdot i$, $x - \frac{2 \cdot \pi}{a} \cdot i \in \operatorname{dom} F$ and $F(x) = F(x + \frac{2 \cdot \pi}{a} \cdot i)$. □

(37) Let us consider a function F from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, c, d. Suppose for every real number x, $F(x) = \max(0, \min(1, c \cdot (\sin(a \cdot x + b)) + d))$. Then F is periodic.

Proof: There exists a real number t such that F is t-periodic by (36), [6, (1)]. □

(38) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}$: there exist real numbers a, b such that for every real number $t_1, f(t_1) = \max(0, \sin(a \cdot t_1 + b))\} \subseteq$ Membership-Funcs($\mathbb{R}$).

Proof: Consider f being a function from $\mathbb{R}$ into $\mathbb{R}$ such that $x = f$ and there exist real numbers a, b such that for every real number t_1, $f(t_1) = \max(0, \sin(a \cdot t_1 + b))$. $\operatorname{rng} f \subseteq [0, 1]$ by [5, (4)]. □

(39) Let us consider real numbers a, b, and a function f from $\mathbb{R}$ into $\mathbb{R}$.

Suppose for every real number x, $f(x) = \max(0, \sin(a \cdot x + b))$. Then f is a fuzzy set of $\mathbb{R}$. The theorem is a consequence of (38).

(40) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}$: there exist real numbers a, b such that for every real number $x, f(x) = \max(0, \cos(a \cdot x + b))\} \subseteq$ Membership-Funcs($\mathbb{R}$).

PROOF: Consider f being a function from $\mathbb{R}$ into $\mathbb{R}$ such that $x = f$ and there exist real numbers a, b such that for every real number t_1, $f(t_1) = \max(0, \cos(a \cdot t_1 + b))$. $\operatorname{rng} f \subseteq [0, 1]$. □

(41) Let us consider real numbers a, b, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose for every real number x, $f(x) = \max(0, \cos(a \cdot x + b))$. Then f is a fuzzy set of $\mathbb{R}$. The theorem is a consequence of (40).

(42) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}, a, b, c, d$ are real numbers : for every real number $x, f(x) = \max(0, \min(1, c \cdot (\sin(a \cdot x + b)) + d))\} \subseteq \{f$, where f, g are functions from $\mathbb{R}$ into $\mathbb{R}$: for every real number $x, f(x) = \max(0, \min(1, g(x)))\}$.

(43) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}, a, b, c, d$ are real numbers : for every real number $x, f(x) = \max(0, \min(1, c \cdot (\sin(a \cdot x + b)) + d))\} \subseteq$ Membership-Funcs($\mathbb{R}$).

PROOF: Consider f being a function from $\mathbb{R}$ into $\mathbb{R}$, a, b, c, d being real numbers such that $f = g$ and for every real number x, $f(x) = \max(0, \min(1, c \cdot (\sin(a \cdot x + b)) + d))$. f is a fuzzy set of $\mathbb{R}$. □

(44) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, c, d. Suppose for every real number x, $f(x) = \max(0, \min(1, c \cdot (\sin(a \cdot x + b)) + d))$. Then f is a fuzzy set of $\mathbb{R}$. The theorem is a consequence of (43).

(45) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}, a, b, c, d$ are real numbers : for every real number $x, f(x) = \max(0, \min(1, c \cdot (\arctan(a \cdot x + b)) + d))\} \subseteq \{f$, where f, g are functions from $\mathbb{R}$ into $\mathbb{R}$: for every real number $x, f(x) = \max(0, \min(1, g(x)))\}$.

(46) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}, a, b, c, d$ are real numbers : for every real number $x, f(x) = \max(0, \min(1, c \cdot (\arctan(a \cdot x + b)) + d))\} \subseteq$ Membership-Funcs($\mathbb{R}$).

(47) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, c, d. Suppose for every real number x, $f(x) = \max(0, \min(1, c \cdot (\arctan(a \cdot x + b)) + d))$. Then f is a fuzzy set of $\mathbb{R}$. The theorem is a consequence of (68) and (24).

(48) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, c, d, r, s. Suppose for every real number x, $f(x) = \max(r, \min(s, c \cdot (\sin(a \cdot x + b)) + d))$. Then f is Lipschitzian.

PROOF: There exists a real number r such that $0 < r$ and for every real numbers x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} f$ holds $|f(x_1)-f(x_2)| \leqslant r\cdot|x_1-x_2|$. $\square$

(49) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, c, d. Suppose for every real number x, $f(x) = \max(0, \min(1, c\cdot(\sin(a\cdot x+b))+d))$. Then f is Lipschitzian.

Let us consider real numbers a, b and a function f from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(50) If $b \neq 0$ and for every real number x, $f(x) = \exp(-\frac{(x-a)^2}{2\cdot b^2})$, then f is a fuzzy set of $\mathbb{R}$.

Proof: $\operatorname{rng} f \subseteq [0,1]$. $\square$

(51) If $b \neq 0$ and for every real number x, $f(x) = \exp(-\frac{(x-a)^2}{2\cdot b^2})$, then f is a fuzzy set of $\mathbb{R}$.

Proof: For every real number x, $f(x) = \exp(-\frac{(x-a)^2}{2\cdot b^2})$. $\square$

(52) Let us consider real numbers a, b. Suppose $b \neq 0$. Then $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}$: for every real number $x, f(x) = \exp(-\frac{(x-a)^2}{2\cdot b^2})\}$ $\subseteq$ Membership-Funcs($\mathbb{R}$). The theorem is a consequence of (51).

Let us consider real numbers a, b and a fuzzy set f of $\mathbb{R}$. Now we state the propositions:

(53) If for every real number x, $f(x) = \exp(-\frac{(x-a)^2}{2\cdot b^2})$, then f is normalized.

Proof: There exists an element x of $\mathbb{R}$ such that $f(x) = 1$. $\square$

(54) If $b \neq 0$ and for every real number x, $f(x) = \exp(-\frac{(x-a)^2}{2\cdot b^2})$, then f is strictly normalized.

Proof: There exists an element x of $\mathbb{R}$ such that $f(x) = 1$ and for every element y of $\mathbb{R}$ such that $f(y) = 1$ holds $y = x$ by [11, (20)], (4). $\square$

(55) Let us consider real numbers a, b, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose $b \neq 0$ and for every real number x, $f(x) = \exp(-\frac{(x-a)^2}{2\cdot b^2})$. Then f is continuous.

Proof: Set $h = \text{AffineMap}(1, -a)$. $f = (\text{the function exp}) \cdot ((\frac{-1}{2\cdot b^2}\cdot h)\cdot h)$. $\square$

(56) Let us consider real numbers a, b, c, r, s, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose $b \neq 0$ and for every real number x, $f(x) = \max(r, \min(s, \exp(-\frac{(x-a)^2}{2\cdot b^2})+c))$. Then f is continuous.

Proof: Define $\mathcal{H}(\text{element of } \mathbb{R}) = (\exp(-\frac{(\$_1-a)^2}{2\cdot b^2}))(\in \mathbb{R})$. Consider h being a function from $\mathbb{R}$ into $\mathbb{R}$ such that for every element x of $\mathbb{R}$, $h(x) = \mathcal{H}(x)$. For every real number x_0 such that $x_0 \in \operatorname{dom} f$ holds f is continuous in x_0. $\square$

Let us consider real numbers a, b, c and a function f from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(57) Suppose $b \neq 0$ and for every real number x, $f(x) = \max(0, \min(1, \exp(-\frac{(x-a)^2}{2 \cdot b^2}) + c))$. Then f is continuous.

(58) Suppose $b \neq 0$ and for every real number x, $f(x) = \max(0, \min(1, \exp(-\frac{(x-a)^2}{2 \cdot b^2}) + c))$. Then f is a fuzzy set of $\mathbb{R}$. The theorem is a consequence of (25).

(59) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}, a, b, c$ are real numbers : $b \neq 0$ and for every real number $x, f(x) = \max(0, \min(1, \exp(-\frac{(x-a)^2}{2 \cdot b^2}) + c))\}$ $\subseteq$ Membership-Funcs($\mathbb{R}$). The theorem is a consequence of (58).

(60) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, r, s. Suppose for every real number x, $f(x) = \max(r, \min(s, (\text{AffineMap}(a, b))(x)))$. Then f is Lipschitzian.

Proof: There exists a real number r such that $0 < r$ and for every real numbers x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} f$ holds $|f(x_1) - f(x_2)| \leqslant r \cdot |x_1 - x_2|$. □

Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$ and real numbers a, b. Now we state the propositions:

(61) If for every real number x, $f(x) = \max(0, \min(1, (\text{AffineMap}(a, b))(x)))$, then f is Lipschitzian.

(62) If for every real number x, $f(x) = \max(0, \min(1, (\text{AffineMap}(a, b))(x)))$, then f is a fuzzy set of $\mathbb{R}$.

(63) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}, a, b$ are real numbers : for every real number $x, f(x) = \max(0, \min(1, (\text{AffineMap}(a, b))(x)))\} \subseteq$ Membership-Funcs($\mathbb{R}$). The theorem is a consequence of (25).

(64) Let us consider real numbers a, b, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose for every real number x, $f(x) = \max(0, 1 - |\frac{x-a}{b}|)$. Then f is a fuzzy set of $\mathbb{R}$.

Proof: $\operatorname{rng} f \subseteq [0, 1]$. □

(65) Let us consider real numbers a, b. Suppose $b > 0$. Let us consider a real number x. Then $(\text{TriangularFS}((a-b), a, (a+b)))(x) = \max(0, 1 - |\frac{x-a}{b}|)$.

Proof: Set $f_1 = (\text{AffineMap}(0, 0)) \restriction \mathbb{R} \setminus]a-b, a+b[$.
Set $f_2 = (\text{AffineMap}(\frac{1}{a-(a-b)}, -\frac{a-b}{a-(a-b)})) \restriction [a-b, a]$.
Set $f_3 = (\text{AffineMap}(-\frac{1}{a+b-a}, \frac{a+b}{a+b-a})) \restriction [a, a+b]$. Set $F = (f_1 +\cdot f_2) +\cdot f_3$. $F(x) = \max(0, 1 - |\frac{x-a}{b}|)$. □

Let us consider real numbers a, b and a fuzzy set f of $\mathbb{R}$. Now we state the propositions:

(66) If $b > 0$ and for every real number x, $f(x) = \max(0, 1 - |\frac{x-a}{b}|)$, then f is triangular. The theorem is a consequence of (65).

(67) If $b > 0$ and for every real number x, $f(x) = \max(0, 1 - |\frac{x-a}{b}|)$, then f is strictly normalized.

PROOF: There exists an element x of $\mathbb{R}$ such that $f(x) = 1$ and for every element y of $\mathbb{R}$ such that $f(y) = 1$ holds $y = x$. $\square$

(68) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, c. Suppose for every real number x, $f(x) = \max(0, \min(1, c \cdot (1 - |\frac{x-a}{b}|)))$. Then f is a fuzzy set of $\mathbb{R}$. The theorem is a consequence of (25).

(69) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b. Suppose $b > 0$ and for every real number x, $f(x) = \max(0, 1 - |\frac{x-a}{b}|)$. Then f is continuous.

PROOF: $f = \mathrm{TriangularFS}((a - b), a, (a + b))$. $\square$

(70) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, c, r, s. Suppose $b \neq 0$ and for every real number x, $f(x) = \max(r, \min(s, c \cdot (1 - |\frac{x-a}{b}|)))$. Then f is Lipschitzian.

PROOF: There exists a real number r such that $0 < r$ and for every real numbers x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} f$ holds $|f(x_1) - f(x_2)| \leqslant r \cdot |x_1 - x_2|$. $\square$

(71) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, c. Suppose $b \neq 0$ and for every real number x, $f(x) = \max(0, \min(1, c \cdot (1 - |\frac{x-a}{b}|)))$. Then f is Lipschitzian.

(72) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}, a, b$ are real numbers : $b > 0$ and for every real number $x, f(x) = \max(0, 1 - |\frac{x-a}{b}|)\} \subseteq$ Membership-Funcs($\mathbb{R}$).

PROOF: $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}, a, b$ are real numbers : $b > 0$ and for every real number $x, f(x) = \max(0, 1 - |\frac{x-a}{b}|)\} \subseteq$ $\{\mathrm{TriangularFS}(a, b, c)$, where a, b, c are real numbers : $a < b < c\}$. $\square$

(73) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}, a, b, c, d$ are real numbers : $b \neq 0$ and for every real number $x, f(x) = \max(0, \min(1, c \cdot (1 - |\frac{x-a}{b}|)))\} \subseteq$ Membership-Funcs($\mathbb{R}$). The theorem is a consequence of (68).

(74) Let us consider real numbers a, b, p, q, s.
Then $(\mathrm{AffineMap}(a, b)) \restriction]-\infty, s[+\cdot (\mathrm{AffineMap}(p, q)) \restriction [s, +\infty[$ is a function from $\mathbb{R}$ into $\mathbb{R}$.

(75) Let us consider real numbers a, b, p, q, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose for every real number x, $f(x) =$
$\max(0, \min(1, ((\mathrm{AffineMap}(a, b)) \restriction]-\infty, \frac{q-b}{a-p}[+\cdot (\mathrm{AffineMap}(p, q)) \restriction [\frac{q-b}{a-p}, +\infty[)(x)))$. Then f is a fuzzy set of $\mathbb{R}$. The theorem is a consequence of (74) and (25).

(76) Let us consider real numbers a, b, c. Suppose $a < b < c$. Then

(i) (TriangularFS$(a, b, c))(a) = 0$, and

(ii) (TriangularFS$(a, b, c))(b) = 1$, and

(iii) (TriangularFS$(a, b, c))(c) = 0$.

(77) Let us consider real numbers a, b, c, d. Suppose $a < b < c < d$. Then

(i) (TrapezoidalFS$(a, b, c, d))(a) = 0$, and

(ii) (TrapezoidalFS$(a, b, c, d))(b) = 1$, and

(iii) (TrapezoidalFS$(a, b, c, d))(c) = 1$, and

(iv) (TrapezoidalFS$(a, b, c, d))(d) = 0$.

Let us consider real numbers a, b, p, q and a real number x. Now we state the propositions:

(78) Suppose $a > 0$ and $p > 0$ and $\frac{-b}{a} < \frac{q}{p}$ and $\frac{1-b}{a} = \frac{1-q}{-p}$. Then (TriangularFS $(\frac{-b}{a}, \frac{1-b}{a}, \frac{q}{p}))(x) = \max(0, \min(1, ((\text{AffineMap}(a, b))\restriction]-\infty, \frac{q-b}{a+p}[+\cdot(\text{Affine}$–$\text{Map}(-p, q))\restriction[\frac{q-b}{a+p}, +\infty[)(x)))$.

Proof: For every real number x, (TriangularFS$(\frac{-b}{a}, \frac{1-b}{a}, \frac{q}{p}))(x) = \max(0, \min(1, ((\text{AffineMap}(a, b))\restriction]-\infty, \frac{q-b}{a+p}[+\cdot(\text{AffineMap}(-p, q))\restriction[\frac{q-b}{a+p}, +\infty[)(x)))$. $\square$

(79) Suppose $a > 0$ and $p > 0$ and $\frac{1-b}{a} < \frac{1-q}{-p}$.
Then (TrapezoidalFS$(\frac{-b}{a}, \frac{1-b}{a}, \frac{1-q}{-p}, \frac{q}{p}))(x) = \max(0, \min(1, ((\text{AffineMap}(a, b))\restriction]-\infty, \frac{q-b}{a+p}[+\cdot(\text{AffineMap}(-p, q))\restriction[\frac{q-b}{a+p}, +\infty[)(x)))$.

Proof: Set $f_4 = (\text{AffineMap}(a, b))\restriction]-\infty, \frac{q-b}{a+p}[$.
Set $f_5 = (\text{AffineMap}(-p, q))\restriction[\frac{q-b}{a+p}, +\infty[$.
For every real number x, (TrapezoidalFS$(\frac{-b}{a}, \frac{1-b}{a}, \frac{1-q}{-p}, \frac{q}{p}))(x) = \max(0, \min(1, (f_4 +\cdot f_5)(x)))$. $\square$

(80) Let us consider real numbers a, b, p, q, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a > 0$ and $p > 0$ and $f = (\text{AffineMap}(a, b))\restriction]-\infty, \frac{q-b}{a+p}[+\cdot(\text{Affine}$–$\text{Map}(-p, q))\restriction[\frac{q-b}{a+p}, +\infty[$. Then f is Lipschitzian.

Proof: There exists a real number r such that $0 < r$ and for every real numbers x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} f$ holds $|f(x_1) - f(x_2)| \leqslant r \cdot |x_1 - x_2|$. $\square$

(81) Let us consider real numbers a, b, p, q. Suppose $a > 0$ and $p > 0$. Then there exists a real number r such that

(i) $0 < r$, and

(ii) for every real numbers x_1, x_2 such that $x_1, x_2 \in \mathrm{dom}((\mathrm{AffineMap}(a,b)){\restriction}]-\infty, \frac{q-b}{a+p}[+\cdot(\mathrm{AffineMap}(-p,q)){\restriction}[\frac{q-b}{a+p}, +\infty[)$ holds $|((\mathrm{AffineMap}(a,b)){\restriction}]-\infty, \frac{q-b}{a+p}[+\cdot(\mathrm{AffineMap}(-p,q)){\restriction}[\frac{q-b}{a+p}, +\infty[)$ $(x_1) - ((\mathrm{AffineMap}(a,b)){\restriction}]-\infty, \frac{q-b}{a+p}[+\cdot(\mathrm{AffineMap}(-p,q)){\restriction}[\frac{q-b}{a+p}, +\infty[)$ $(x_2)| \leqslant r \cdot |x_1 - x_2|$.

The theorem is a consequence of (74) and (80).

(82) Let us consider real numbers a, b, p, q, r, s, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a > 0$ and $p > 0$ and for every real number x, $f(x) = \max(r, \min(s, ((\mathrm{AffineMap}(a,b)){\restriction}]-\infty, \frac{q-b}{a+p}[+\cdot(\mathrm{AffineMap}(-p,q)){\restriction}[\frac{q-b}{a+p}, +\infty[)(x)))$. Then f is Lipschitzian. The theorem is a consequence of (74), (81), and (1).

(83) Let us consider real numbers a, b, c. Suppose $a < b < c$. Let us consider a real number x. Then $(\mathrm{TriangularFS}(a,b,c))(x) = \max(0, \min(1, ((\mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a})){\restriction}]-\infty, b[+\cdot(\mathrm{AffineMap}(-\frac{1}{c-b}, \frac{c}{c-b})){\restriction}[b, +\infty[)(x)))$. The theorem is a consequence of (78).

(84) Let us consider real numbers a, b, c, d. Suppose $a < b < c < d$. Let us consider a real number x. Then $(\mathrm{TrapezoidalFS}(a,b,c,d))(x) = \max(0, \min(1, ((\mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a})){\restriction}]-\infty, \frac{b \cdot d - a \cdot c}{d-c+b-a}[+\cdot(\mathrm{AffineMap}(-\frac{1}{d-c}, \frac{d}{d-c})){\restriction}[\frac{b \cdot d - a \cdot c}{d-c+b-a}, +\infty[)(x)))$. The theorem is a consequence of (79).

(85) Let us consider real numbers a, b, p, q, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a > 0$ and $p > 0$ and for every real number x, $f(x) = \max(0, \min(1, ((\mathrm{AffineMap}(a,b)){\restriction}]-\infty, \frac{q-b}{a+p}[+\cdot(\mathrm{AffineMap}(-p,q)){\restriction}[\frac{q-b}{a+p}, +\infty[)(x)))$. Then f is Lipschitzian.

(86) Let us consider real numbers a, b, c. If $a < b < c$, then $\mathrm{TriangularFS}(a,b,c)$ is Lipschitzian. The theorem is a consequence of (83) and (82).

(87) Let us consider real numbers a, b, c, d. If $a < b < c < d$, then Trapezoidal–FS(a,b,c,d) is Lipschitzian. The theorem is a consequence of (84) and (82).

Let us consider real numbers a, b, p, q and a fuzzy set f of $\mathbb{R}$. Now we state the propositions:

(88) Suppose $a > 0$ and $p > 0$ and $\frac{-b}{a} < \frac{q}{p}$ and $\frac{1-b}{a} = \frac{1-q}{-p}$ and for every real number x, $f(x) = \max(0, \min(1, ((\mathrm{AffineMap}(a,b)){\restriction}]-\infty, \frac{q-b}{a+p}[+\cdot(\mathrm{AffineMap}(-p,q)){\restriction}[\frac{q-b}{a+p}, +\infty[)(x)))$. Then f is triangular and strictly normalized. The theorem is a consequence of (78).

(89) Suppose $a > 0$ and $p > 0$ and $\frac{1-b}{a} < \frac{1-q}{-p}$ and for every real number x, $f(x) = \max(0, \min(1, ((\mathrm{AffineMap}(a,b)){\restriction}]-\infty, \frac{q-b}{a+p}[+\cdot(\mathrm{AffineMap}$

$(-p, q)) \restriction [\frac{q-b}{a+p}, +\infty[)(x)))$. Then f is trapezoidal and normalized. The theorem is a consequence of (79).

(90) $\{f$, where f is a fuzzy set of $\mathbb{R}$: f is triangular$\} \subseteq$ Membership-Funcs($\mathbb{R}$).

(91) $\{$TriangularFS(a, b, c), where a, b, c are real numbers : $a < b < c\} \subseteq$ Membership-Funcs($\mathbb{R}$).

(92) $\{f$, where f is a fuzzy set of $\mathbb{R}$: f is trapezoidal$\} \subseteq$ Membership-Funcs($\mathbb{R}$).

(93) $\{$TrapezoidalFS(a, b, c, d), where a, b, c, d are real numbers : $a < b < c < d\} \subseteq$ Membership-Funcs($\mathbb{R}$).

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