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Real Vector Space and Related Notions[1]

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Summary. In this paper, we discuss the properties that hold in finite dimensional vector spaces and related spaces. In the Mizar language [1], [2], variables are strictly typed, and their type conversion requires a complicated process. Our purpose is to formalize that some properties of finite dimensional vector spaces are preserved in type transformations, and to contain the complexity of type transformations into this paper. Specifically, we show that properties such as algebraic structure, subsets, finite sequences and their sums, linear combination, linear independence, and affine independence are preserved in type conversions among `TOP-REAL(n)`, `REAL-NS(n)`, and `n-VectSp_over F_Real`. We referred to [4], [9], and [8] in the formalization.

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1. Common Properties Between Norm and Topology in Finite Dimensional Linear Spaces

From now on X denotes a set, n, m, k denote natural numbers, K denotes a field, f denotes an n-element, real-valued finite sequence, and M denotes a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$.

Now we state the propositions:

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(1) The RLS structure of $\mathcal{E}^n_{\mathrm{T}}$ = the RLS structure of $\langle \mathcal{E}^n, \|\cdot\| \rangle$.
PROOF: For every elements x, y of $\mathcal{R}^n$, $+_{\mathcal{E}^n}(x,y) = +_{\mathbb{R}^{\mathrm{Seg}\,n}}(x,y)$. For every element x of $\mathbb{R}$ and for every element y of $\mathcal{R}^n$, $\cdot_{\mathcal{E}^n}(x,y) = \cdot^{\mathbb{R}}_{\mathbb{R}^{\mathrm{Seg}\,n}}(x,y)$ by [3, (3)]. □

(2) $\mathcal{E}^n = \mathrm{MetricSpaceNorm}\langle \mathcal{E}^n, \|\cdot\| \rangle$.
PROOF: Set $X = \langle \mathcal{E}^n, \|\cdot\| \rangle$. For every elements x, y of $\mathcal{R}^n$, (the distance of $\mathcal{E}^n$)(x,y) = (the distance by norm of X)(x,y). □

(3) The topological structure of $\mathcal{E}^n_{\mathrm{T}}$ = $\mathrm{TopSpaceNorm}\langle \mathcal{E}^n, \|\cdot\| \rangle$. The theorem is a consequence of (2).

(4) The carrier of $\mathcal{E}^n_{\mathrm{T}}$ = the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. The theorem is a consequence of (1).

(5) The carrier of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$ = the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. The theorem is a consequence of (4).

(6) $0_{\mathcal{E}^n_{\mathrm{T}}} = 0_{\langle \mathcal{E}^n, \|\cdot\| \rangle}$. The theorem is a consequence of (1).

(7) Let us consider elements p, q of $\mathcal{E}^n_{\mathrm{T}}$, and elements f, g of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $p = f$ and $q = g$, then $p + q = f + g$. The theorem is a consequence of (1).

(8) Let us consider a real number r, an element q of $\mathcal{E}^n_{\mathrm{T}}$, and an element g of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $q = g$, then $r \cdot q = r \cdot g$. The theorem is a consequence of (1).

(9) Let us consider an element q of $\mathcal{E}^n_{\mathrm{T}}$, and an element g of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $q = g$, then $-q = -g$. The theorem is a consequence of (8).

(10) Let us consider elements p, q of $\mathcal{E}^n_{\mathrm{T}}$, and elements f, g of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $p = f$ and $q = g$, then $p - q = f - g$. The theorem is a consequence of (9) and (7).

Let us consider a set X and a natural number n.

(11) X is a linear combination of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ if and only if X is a linear combination of $\mathcal{E}^n_{\mathrm{T}}$. The theorem is a consequence of (4).

(12) X is a linear combination of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ if and only if X is a linear combination of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$. The theorem is a consequence of (11).

(13) Let us consider a linear combination L_5 of $\mathcal{E}^n_{\mathrm{T}}$, and a linear combination L_2 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $L_2 = L_5$. Then the support of L_2 = the support of L_5.

(14) Let us consider a linear combination L_5 of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$, and a linear combination L_2 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $L_2 = L_5$. Then the support of L_2 = the support of L_5. The theorem is a consequence of (11).

Let us consider a set F. Now we state the propositions:

(15) F is a subset of $\mathcal{E}^n_{\mathrm{T}}$ if and only if F is a subset of $\langle \mathcal{E}^n, \|\cdot\| \rangle$.

(16) F is a subset of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ if and only if F is a subset of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$.

(17) F is a finite sequence of elements of $\mathcal{E}^n_{\mathrm{T}}$ if and only if F is a finite sequence of elements of $\langle \mathcal{E}^n, \|\cdot\| \rangle$.

(18) F is a function from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathbb{R}$ if and only if F is a function from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ into $\mathbb{R}$. The theorem is a consequence of (4).

(19) Let us consider a finite sequence F_2 of elements of $\mathcal{E}^n_{\mathrm{T}}$, a function f_1 from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathbb{R}$, a finite sequence F_4 of elements of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and a function f_3 from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ into $\mathbb{R}$. If $f_1 = f_3$ and $F_2 = F_4$, then $f_1 \cdot F_2 = f_3 \cdot F_4$. The theorem is a consequence of (4) and (8).

(20) Let us consider a finite sequence F of elements of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, a function f_1 from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ into $\mathbb{R}$, a finite sequence F_4 of elements of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$, and a function f_3 from the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$ into $\mathbb{R}_{\mathrm{F}}$. If $f_1 = f_3$ and $F = F_4$, then $f_1 \cdot F = f_3 \cdot F_4$. The theorem is a consequence of (18), (4), and (19).

(21) Let us consider a finite sequence F_3 of elements of $\mathcal{E}^n_{\mathrm{T}}$, and a finite sequence F_2 of elements of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $F_3 = F_2$, then $\sum F_3 = \sum F_2$.
Proof: Set $T = \mathcal{E}^n_{\mathrm{T}}$. Set $V = \langle \mathcal{E}^n, \|\cdot\| \rangle$. Consider f being a sequence of the carrier of T such that $\sum F = f(\operatorname{len} F)$ and $f(0) = 0_T$ and for every natural number j and for every element v of T such that $j < \operatorname{len} F$ and $v = F(j+1)$ holds $f(j+1) = f(j) + v$.

Consider f_3 being a sequence of the carrier of V such that $\sum F_4 = f_3(\operatorname{len} F_4)$ and $f_3(0) = 0_V$ and for every natural number j and for every element v of V such that $j < \operatorname{len} F_4$ and $v = F_4(j+1)$ holds $f_3(j+1) = f_3(j) + v$. Define $\mathcal{S}[\text{natural number}] \equiv$ if $\$_1 \leqslant \operatorname{len} F$, then $f(\$_1) = f_3(\$_1)$. For every natural number i such that $\mathcal{S}[i]$ holds $\mathcal{S}[i+1]$. $\mathcal{S}[0]$. For every natural number n, $\mathcal{S}[n]$. $\square$

(22) Let us consider a finite sequence F of elements of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and a finite sequence F_4 of elements of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$. If $F_4 = F$, then $\sum F = \sum F_4$. The theorem is a consequence of (4) and (21).

(23) Let us consider a linear combination L_2 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and a linear combination L_4 of $\mathcal{E}^n_{\mathrm{T}}$. If $L_2 = L_4$, then $\sum L_2 = \sum L_4$. The theorem is a consequence of (4), (19), and (21).

(24) Let us consider a linear combination L_5 of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$, and a linear combination L_2 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $L_2 = L_5$, then $\sum L_2 = \sum L_5$. The theorem is a consequence of (11) and (23).

(25) Let us consider a subset A_3 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and a subset A_4 of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $A_3 = A_4$. Let us consider an object X. Then X is a linear combination

of A_3 if and only if X is a linear combination of A_4. The theorem is a consequence of (11).

(26) Let us consider a subset A_3 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and a subset A_4 of $\mathcal{E}^n_{\mathrm{T}}$. If $A_3 = A_4$, then $\Omega_{\mathrm{Lin}(A_3)} = \Omega_{\mathrm{Lin}(A_4)}$. The theorem is a consequence of (11) and (23).

(27) Let us consider a subset A_2 of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$, and a subset A_3 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $A_2 = A_3$, then $\Omega_{\mathrm{Lin}(A_3)} = \Omega_{\mathrm{Lin}(A_2)}$. The theorem is a consequence of (4) and (26).

(28) Let us consider a subset A_3 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and a subset A_4 of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $A_3 = A_4$. Then A_3 is linearly independent if and only if A_4 is linearly independent. The theorem is a consequence of (11), (6), and (23).

(29) Let us consider a subset A_2 of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$, and a subset A_3 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $A_2 = A_3$. Then A_2 is linearly independent if and only if A_3 is linearly independent. The theorem is a consequence of (4) and (28).

(30) Let us consider an object X. Then X is a subspace of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ if and only if X is a subspace of $\mathcal{E}^n_{\mathrm{T}}$. The theorem is a consequence of (1), (4), and (6).

(31) Let us consider a set X, a subspace U of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and a subspace W of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$. Suppose $\Omega_U = \Omega_W$. Then X is a linear combination of U if and only if X is a linear combination of W. The theorem is a consequence of (30).

(32) Let us consider a one-to-one finite sequence F of elements of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $\operatorname{rng} F$ is linearly independent. Then there exists a square matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension n such that

(i) M is invertible, and

(ii) $M \restriction \operatorname{len} F = F$.

The theorem is a consequence of (4) and (28).

(33) Let us consider a square matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension n, and a square matrix N over $\mathbb{R}$ of dimension n. Suppose $N = (\mathbb{R}_{\mathrm{F}} \to \mathbb{R})M$. Then M is invertible if and only if N is invertible.

(34) Let us consider a square matrix M over $\mathbb{R}$ of dimension n. Then M is invertible if and only if $(\mathbb{R} \to \mathbb{R}_{\mathrm{F}})M$ is invertible.

(35) Let us consider a one-to-one finite sequence F of elements of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $\operatorname{rng} F$ is linearly independent. Then there exists a square matrix M over $\mathbb{R}$ of dimension n such that

(i) M is invertible, and

(ii) $M \restriction \operatorname{len} F = F$.

The theorem is a consequence of (32) and (33).

(36) Let us consider a one-to-one finite sequence F of elements of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $\operatorname{rng} F$ is linearly independent. Let us consider an ordered basis B of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$. Suppose $B = \mathrm{MX2FinS}\, I_{\mathbb{R}_{\mathrm{F}}}^{n \times n}$. Let us consider a square matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension n. Suppose M is invertible and $M \restriction \operatorname{len} F = F$. Then $(\mathrm{Mx2Tran}(M))^{\circ}(\Omega_{\mathrm{Lin}(\operatorname{rng}(B \restriction \operatorname{len} F))}) = \Omega_{\mathrm{Lin}(\operatorname{rng} F)}$. The theorem is a consequence of (4), (28), and (26).

(37) Let us consider linearly independent subsets A, B of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $\overline{\overline{A}} = \overline{\overline{B}}$. Then there exists a square matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension n such that

(i) M is invertible, and

(ii) $(\mathrm{Mx2Tran}(M))^{\circ}(\Omega_{\mathrm{Lin}(A)}) = \Omega_{\mathrm{Lin}(B)}$.

The theorem is a consequence of (4), (28), and (26).

(38) Let us consider natural numbers n, m, a matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$, and a linearly independent subset A of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $\operatorname{rk}(M) = n$. Then $(\mathrm{Mx2Tran}(M))^{\circ} A$ is linearly independent. The theorem is a consequence of (4) and (28).

(39) Let us consider an element p of $\mathcal{E}^n_{\mathrm{T}}$, an element f of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, a subset H of $\mathcal{E}^n_{\mathrm{T}}$, and a subset I of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $p = f$ and $H = I$, then $p + H = f + I$. The theorem is a consequence of (4) and (7).

(40) Let us consider a subset A_3 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and a subset A_4 of $\mathcal{E}^n_{\mathrm{T}}$. If $A_3 = A_4$, then A_3 is affine iff A_4 is affine. The theorem is a consequence of (4), (8), and (7).

(41) Let us consider a set X. Then X is an affinely independent subset of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ if and only if X is an affinely independent subset of $\mathcal{E}^n_{\mathrm{T}}$. The theorem is a consequence of (4), (6), (9), (39), and (28).

(42) Let us consider natural numbers n, m, a matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$, and an affinely independent subset A of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $\operatorname{rk}(M) = n$. Then $(\mathrm{Mx2Tran}(M))^{\circ} A$ is affinely independent. The theorem is a consequence of (41).

(43) Let us consider a subset A_3 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and a subset A_4 of $\mathcal{E}^n_{\mathrm{T}}$. If $A_3 = A_4$, then $\operatorname{Affin} A_3 = \operatorname{Affin} A_4$. The theorem is a consequence of (4) and (40).

(44) Let us consider a linear combination L of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and a linear combination S of $\mathcal{E}^n_{\mathrm{T}}$. If $L = S$, then $\operatorname{sum} L = \operatorname{sum} S$. The theorem is a consequence of (4).

(45) Let us consider a subset A_3 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, a subset A_4 of $\mathcal{E}^n_{\mathrm{T}}$, an element v of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and an element w of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $A_3 = A_4$ and $v = w$ and $v \in \operatorname{Affin} A_3$ and A_3 is affinely independent. Then $v \to A_3 = w \to A_4$. The theorem is a consequence of (41), (25), (23), (44), and (43).

(46) Let us consider natural numbers n, m, a matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$, and an affinely independent subset A of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $\mathrm{rk}(M) = n$. Let us consider an element v of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $v \in \operatorname{Affin} A$. Then

(i) $(\mathrm{Mx2Tran}(M))(v) \in \operatorname{Affin}((\mathrm{Mx2Tran}(M))^{\circ} A)$, and

(ii) for every n-element, real-valued finite sequence f, $(v \to A)(f) = ((\mathrm{Mx2Tran}(M))(v) \to (\mathrm{Mx2Tran}(M))^{\circ} A)((\mathrm{Mx2Tran}(M))(f))$.

The theorem is a consequence of (41), (4), (43), and (45).

(47) Let us consider natural numbers n, m, a matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$, and a linearly independent subset A of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose $\mathrm{rk}(M) = n$. Then $(\mathrm{Mx2Tran}(M))^{-1}(A)$ is linearly independent. The theorem is a consequence of (4) and (28).

(48) Let us consider natural numbers n, m, a matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$, and an affinely independent subset A of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose $\mathrm{rk}(M) = n$. Then $(\mathrm{Mx2Tran}(M))^{-1}(A)$ is affinely independent. The theorem is a consequence of (41).

(49) Let us consider a real linear space V. Then every strict subspace of V is a strict subspace of Ω_V.

(50) Let us consider a set X. Then X is a basis of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$ if and only if X is a basis of $\mathcal{E}^n_{\mathrm{T}}$.

Let us consider a non empty natural number n.

(51) $+_{\mathbb{R}^{\mathrm{Seg}\, n}} = \pi^n(\text{the addition of } \mathbb{R}_{\mathrm{F}})$.

PROOF: Set $O_1 = +_{\mathbb{R}^{\mathrm{Seg}\, n}}$. Set $O_2 = \pi^n(\text{the addition of } \mathbb{R}_{\mathrm{F}})$. For every elements x, y of $\mathcal{R}^n$, $O_1(x, y) = O_2(x, y)$. □

(52) $\cdot^{\mathbb{R}}_{\mathbb{R}^{\mathrm{Seg}\, n}} = \cdot^{n}_{\mathbb{R}_{\mathrm{F}}}$.

PROOF: Set $O_1 = \cdot^{\mathbb{R}}_{\mathbb{R}^{\mathrm{Seg}\, n}}$. Set $O_2 = \cdot^{n}_{\mathbb{R}_{\mathrm{F}}}$. For every element x of $\mathbb{R}$ and for every element y of $\mathcal{R}^n$, $O_1(x, y) = O_2(x, y)$. □

(53) (i) $\mathcal{E}^n_{\mathrm{T}}$ is finite dimensional, and

(ii) $\dim(\mathcal{E}^n_{\mathrm{T}}) = n$.

The theorem is a consequence of (50).

(54) Let us consider a non empty natural number n. Then

(i) the carrier of $\mathcal{E}^n_{\mathrm{T}}$ = the carrier of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$, and

(ii) $0_{\mathcal{E}^n_{\mathrm{T}}} = 0_{\alpha}$, and

(iii) the addition of $\mathcal{E}^n_{\mathrm{T}}$ = the addition of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$, and

(iv) the external multiplication of $\mathcal{E}^n_{\mathrm{T}}$ = the left multiplication of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$,

where α is the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$. The theorem is a consequence of (51) and (52).

(55) Let us consider a non empty natural number n, elements x_2, y_2 of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$, and elements x_1, y_1 of $\mathcal{E}^n_{\mathrm{T}}$. If $x_2 = x_1$ and $y_2 = y_1$, then $x_2 + y_2 = x_1 + y_1$.

(56) Let us consider a non empty natural number n, an element a_1 of $\mathbb{R}_{\mathrm{F}}$, a real number a_2, an element x_2 of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$, and an element x_1 of $\mathcal{E}^n_{\mathrm{T}}$. If $a_1 = a_2$ and $x_2 = x_1$, then $a_1 \cdot x_2 = a_2 \cdot x_1$.

(57) Let us consider a non empty natural number n, an element x_2 of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$, and an element x_1 of $\mathcal{E}^n_{\mathrm{T}}$. If $x_2 = x_1$, then $-x_2 = -x_1$. The theorem is a consequence of (54).

(58) Let us consider a non empty natural number n, elements x_2, y_2 of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$, and elements x_1, y_1 of $\mathcal{E}^n_{\mathrm{T}}$. If $x_2 = x_1$ and $y_2 = y_1$, then $x_2 - y_2 = x_1 - y_1$. The theorem is a consequence of (57) and (54).

(59) Let us consider a non empty natural number n, a subset A_4 of $\mathcal{E}^n_{\mathrm{T}}$, and a subset A_3 of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$. Suppose $A_4 = A_3$. Then

(i) the carrier of $\mathrm{Lin}(A_4)$ = the carrier of $\mathrm{Lin}(A_3)$, and

(ii) $0_{\mathrm{Lin}(A_4)} = 0_{\mathrm{Lin}(A_3)}$, and

(iii) the addition of $\mathrm{Lin}(A_4)$ = the addition of $\mathrm{Lin}(A_3)$, and

(iv) the external multiplication of $\mathrm{Lin}(A_4)$ = the left multiplication of $\mathrm{Lin}(A_3)$.

The theorem is a consequence of (54).

(60) Let us consider a subset A_4 of $\mathcal{E}^n_{\mathrm{T}}$, and a subset A_3 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $A_4 = A_3$, then $\mathrm{Lin}(A_4) = \mathrm{Lin}(A_3)$. The theorem is a consequence of (26) and (1).

(61) Let us consider a set X. Then X is a basis of $\mathcal{E}^n_{\mathrm{T}}$ if and only if X is a basis of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. The theorem is a consequence of (4), (28), (49), and (26).

(62) (i) $\langle \mathcal{E}^n, \|\cdot\| \rangle$ is finite dimensional, and

(ii) $\dim(\langle \mathcal{E}^n, \|\cdot\| \rangle) = n$.

The theorem is a consequence of (53), (4), and (61).

2. Finite Dimensional Vector Spaces over Real Field

Note that there exists a real normed space which is finite dimensional.

Now we state the propositions:

(63) Let us consider a field K, a finite dimensional vector space V over K, and an ordered basis b of V. Then there exists a linear transformation T from V to the $\dim(V)$-dimension vector space over K such that

(i) T is bijective, and

(ii) for every element x of V, $T(x) = x \rightarrow b$.

Proof: Set W = the $\dim(V)$-dimension vector space over K. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an element x of V such that $\$_1 = x$ and $\$_2 = x \rightarrow b$.

For every element x of the carrier of V, there exists an element y of the carrier of W such that $\mathcal{P}[x, y]$. Consider f being a function from the carrier of V into the carrier of W such that for every element x of the carrier of V, $\mathcal{P}[x, f(x)]$. For every element x of V, $f(x) = x \rightarrow b$. For every elements x, y of V, $f(x + y) = f(x) + f(y)$. For every scalar a of K and for every vector x of V, $f(a \cdot x) = a \cdot f(x)$. For every objects x, y such that $x, y \in \operatorname{dom} f$ and $f(x) = f(y)$ holds $x = y$.

For every object y such that $y \in$ the carrier of W there exists an object x such that $x \in$ the carrier of V and $y = f(x)$ by [6, (102)], [7, (21)], [5, (36)]. $\square$

(64) Let us consider a field K, and a finite dimensional vector space V over K. Then there exists a linear transformation T from V to the $\dim(V)$-dimension vector space over K such that T is bijective. The theorem is a consequence of (63).

(65) Let us consider a field K, and finite dimensional vector spaces V, W over K. Then $\dim(V) = \dim(W)$ if and only if there exists a linear transformation T from V to W such that T is bijective. The theorem is a consequence of (64).

(66) Let us consider a real linear space X. Then

(i) the carrier of X = the carrier of $\mathrm{RLSp2RVSp}(X)$, and

(ii) the zero of X = the zero of $\mathrm{RLSp2RVSp}(X)$, and

(iii) the addition of X = the addition of $\mathrm{RLSp2RVSp}(X)$, and

(iv) the external multiplication of X =
the left multiplication of $\mathrm{RLSp2RVSp}(X)$.

(67) Let us consider a strict real linear space X.
Then $\mathrm{RVSp2RLSp}\,\mathrm{RLSp2RVSp}(X) = X$.

(68) Let us consider a strict vector space X over $\mathbb{R}_{\mathrm{F}}$.
Then $\mathrm{RLSp2RVSp}(\mathrm{RVSp2RLSp}\, X) = X$.

Let us consider a real linear space V and a set F.

(69) F is a subset of V if and only if F is a subset of $\mathrm{RLSp2RVSp}(V)$.

(70) F is a finite sequence of elements of V if and only if F is a finite sequence of elements of $\mathrm{RLSp2RVSp}(V)$.

(71) F is a function from V into $\mathbb{R}$ if and only if F is a function from $\mathrm{RLSp2RVSp}(V)$ into $\mathbb{R}$.

(72) Let us consider a real linear space T, and a set X. Then X is a linear combination of $\mathrm{RLSp2RVSp}(T)$ if and only if X is a linear combination of T.

(73) Let us consider a real linear space T, a linear combination L_5 of $\mathrm{RLSp2RVSp}(T)$, and a linear combination L_2 of T. Suppose $L_2 = L_5$. Then the support of $L_2 =$ the support of L_5.
Proof: The support of $L_2 \subseteq$ the support of L_5. Consider u being an element of $\mathrm{RLSp2RVSp}(T)$ such that $x = u$ and $L_5(u) \neq 0_{\mathbb{R}_{\mathrm{F}}}$. $\square$

(74) Let us consider a real linear space V, a finite sequence F_2 of elements of V, a function f_1 from V into $\mathbb{R}$, a finite sequence F_4 of elements of $\mathrm{RLSp2RVSp}(V)$, and a function f_3 from $\mathrm{RLSp2RVSp}(V)$ into $\mathbb{R}_{\mathrm{F}}$. If $f_1 = f_3$ and $F_2 = F_4$, then $f_1 \cdot F_2 = f_3 \cdot F_4$.

(75) Let us consider a real linear space T, a finite sequence F_3 of elements of T, and a finite sequence F_2 of elements of $\mathrm{RLSp2RVSp}(T)$. If $F_3 = F_2$, then $\sum F_3 = \sum F_2$.

(76) Let us consider a real linear space T, a linear combination L_5 of $\mathrm{RLSp2RVSp}(T)$, and a linear combination L_2 of T. If $L_2 = L_5$, then $\sum L_2 = \sum L_5$. The theorem is a consequence of (73) and (74).

Let us consider a real linear space T, a subset A_2 of $\mathrm{RLSp2RVSp}(T)$, and a subset A_3 of T. Now we state the propositions:

(77) If $A_2 = A_3$, then $\Omega_{\mathrm{Lin}(A_3)} = \Omega_{\mathrm{Lin}(A_2)}$. The theorem is a consequence of (72), (73), and (76).

(78) If $A_2 = A_3$, then A_2 is linearly independent iff A_3 is linearly independent. The theorem is a consequence of (72), (73), and (76).

(79) Let us consider a real linear space T, a set X, a subspace U of $\mathrm{RLSp2RVSp}(T)$, and a subspace W of T. Suppose $\Omega_U = \Omega_W$. Then X is a linear combination of U if and only if X is a linear combination of W.

(80) Let us consider a real linear space W, and a set X. Then X is a basis of $\mathrm{RLSp2RVSp}(W)$ if and only if X is a basis of W. The theorem is a consequence of (78) and (77).

Let us consider a real linear space W. Now we state the propositions:

(81) If W is finite dimensional, then RLSp2RVSp(W) is finite dimensional and dim(RLSp2RVSp(W)) = dim(W). The theorem is a consequence of (80).

(82) W is finite dimensional if and only if RLSp2RVSp(W) is finite dimensional. The theorem is a consequence of (80).

(83) Let us consider a non empty natural number n. Then RLSp2RVSp($\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\, n}$) = the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$. The theorem is a consequence of (51) and (52).

(84) Let us consider real linear spaces V, W, and a set X. Then X is a linear operator from V into W if and only if X is a linear transformation from RLSp2RVSp(V) to RLSp2RVSp(W).

(85) Let us consider real linear spaces X, Y, and a linear operator L from X into Y. Suppose L is bijective. Then there exists a linear operator K from Y into X such that

(i) $K = L^{-1}$, and

(ii) K is one-to-one and onto.

PROOF: Reconsider $K = L^{-1}$ as a function from the carrier of Y into the carrier of X. K is additive. For every vector x of Y and for every real number r, $K(r \cdot x) = r \cdot K(x)$. □

(86) Let us consider real linear spaces X, Y, Z, a linear operator L from X into Y, and a linear operator K from Y into Z. Then $K \cdot L$ is a linear operator from X into Z.

PROOF: Reconsider $T = K \cdot L$ as a function from X into Z. For every elements x, y of X, $T(x+y) = T(x) + T(y)$. For every real number a and for every vector x of X, $T(a \cdot x) = a \cdot T(x)$. □

(87) Let us consider real linear spaces V, W, a subset A of V, and a linear operator T from V into W. Suppose T is bijective. Then A is a basis of V if and only if $T^{\circ}A$ is a basis of W. The theorem is a consequence of (84) and (80).

(88) Let us consider a finite dimensional real linear space V, and a real linear space W. Suppose there exists a linear operator T from V into W such that T is bijective. Then

(i) W is finite dimensional, and

(ii) $\dim(W) = \dim(V)$.

The theorem is a consequence of (87).

(89) Let us consider a finite dimensional real linear space V. Suppose dim(V)

$\neq 0$. Then there exists a linear operator T from V into $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,\dim(V)}$ such that T is bijective. The theorem is a consequence of (81), (64), (83), and (84).

(90) Let us consider finite dimensional real linear spaces V, W. Suppose $\dim(V) \neq 0$. Then $\dim(V) = \dim(W)$ if and only if there exists a linear operator T from V into W such that T is bijective. The theorem is a consequence of (89), (85), (86), and (88).

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Splitting Fields

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Summary. In this article we further develop field theory in Mizar [1], [2]: we prove existence and uniqueness of splitting fields. We define the splitting field of a polynomial $p \in F[X]$ as the smallest field extension of F, in which p splits into linear factors. From this follows, that for a splitting field E of p we have $E = F(A)$ where A is the set of p's roots. Splitting fields are unique, however, only up to isomorphisms; to be more precise up to F-isomorphims i.e. isomorphisms i with $i|_F = \mathrm{Id}_F$. We prove that two splitting fields of $p \in F[X]$ are F-isomorphic using the well-known technique [4], [3] of extending isomorphisms from $F_1 \longrightarrow F_2$ to $F_1(a) \longrightarrow F_2(b)$ for a and b being algebraic over F_1 and F_2, respectively.

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1. Preliminaries

Now we state the propositions:

(1) Let us consider a ring R, a polynomial p over R, and an element q of the carrier of PolyRing(R). If $p = q$, then $-p = -q$.

(2) Let us consider a ring R, a polynomial p over R, and an element a of R. Then $a \cdot p = (a \restriction R) * p$.

(3) Let us consider a ring R, and an element a of R. Then $\mathrm{LC}(a \restriction R) = a$.

(4) Let us consider a ring R, a subring S of R, a finite sequence F of elements of R, and a finite sequence G of elements of S. If $F = G$, then $\prod F = \prod G$.

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Let F be a field. Let us observe that there exists

a field which is F-homomorphic, F-monomorphic, and F-isomorphic.

Let R be a ring. Observe that every R-isomorphic ring is R-homomorphic and R-monomorphic.

Let S be an R-homomorphic ring.

Observe that $\mathrm{PolyRing}(S)$ is $(\mathrm{PolyRing}(R))$-homomorphic.

Let F_1 be a field and F_2 be an F_1-isomorphic, F_1-homomorphic field. Observe that $\mathrm{PolyRing}(F_2)$ is $(\mathrm{PolyRing}(F_1))$-isomorphic.

2. More on Polynomials

Now we state the propositions:

(5) Let us consider a non degenerated ring R, a ring extension S of R, a polynomial p over R, and a polynomial q over S. If $p = q$, then $\mathrm{LC}\, p = \mathrm{LC}\, q$.

(6) Let us consider a field F, an element p of the carrier of $\mathrm{PolyRing}(F)$, an extension E of F, and an element q of the carrier of $\mathrm{PolyRing}(E)$. Suppose $p = q$. Let us consider an E-extending extension U of F, and an element a of U. Then $\mathrm{ExtEval}(q, a) = \mathrm{ExtEval}(p, a)$.

(7) Let us consider a ring R, a ring extension S of R, an element p of the carrier of $\mathrm{PolyRing}(R)$, and an element q of the carrier of $\mathrm{PolyRing}(S)$. Suppose $p = q$. Let us consider a ring extension T_1 of S, and a ring extension T_2 of R. If $T_1 = T_2$, then $\mathrm{Roots}(T_2, p) = \mathrm{Roots}(T_1, q)$.

(8) Let us consider an integral domain R, a non empty finite sequence F of elements of $\mathrm{PolyRing}(R)$, and a polynomial p over R. Suppose $p = \prod F$ and for every natural number i such that $i \in \mathrm{dom}\, F$ there exists an element a of R such that $F(i) = \mathrm{rpoly}(1, a)$. Then $\deg p = \mathrm{len}\, F$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every non empty finite sequence F of elements of $\mathrm{PolyRing}(R)$ for every polynomial p over R such that $\mathrm{len}\, F = \$_1$ and $p = \prod F$ and for every natural number i such that $i \in \mathrm{dom}\, F$ there exists an element a of R such that $F(i) = \mathrm{rpoly}(1, a)$ holds $\deg p = \mathrm{len}\, F$. For every natural number k, $\mathcal{P}[k]$. $\square$

(9) Let us consider a field F, a polynomial p over F, and a non zero element a of F. Then $a \cdot p$ splits in F if and only if p splits in F.

(10) Let us consider a field F, a non constant, monic polynomial p over F, and a non zero polynomial q over F. Suppose $p * q$ is a product of linear polynomials of F. Then p is a product of linear polynomials of F.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every non constant, monic polynomial p over F for every non zero polynomial q over F such that

$\deg(p * q) = \$_1$ and $p * q$ is a product of linear polynomials of F holds p is a product of linear polynomials of F. For every natural number i such that $1 \leqslant i$ holds $\mathcal{P}[i]$. $\square$

(11) Let us consider a field F, a non constant polynomial p over F, and a non zero polynomial q over F. If $p * q$ splits in F, then p splits in F. The theorem is a consequence of (10) and (9).

(12) Let us consider a field F, and polynomials p, q over F. If p splits in F and q splits in F, then $p * q$ splits in F.

(13) Let us consider a ring R, an R-homomorphic ring S, a homomorphism h from R to S, and an element a of R. Then $(\mathrm{PolyHom}(h))(a \restriction R) = h(a) \restriction S$.

(14) Let us consider a field F_1, an F_1-isomorphic, F_1-homomorphic field F_2, an isomorphism h between F_1 and F_2, and elements p, q of the carrier of $\mathrm{PolyRing}(F_1)$. Then $p \mid q$ if and only if $(\mathrm{PolyHom}(h))(p) \mid (\mathrm{PolyHom}(h))(q)$.

(15) Let us consider a field F, an extension E of F, an F-algebraic element a of E, and an irreducible element p of the carrier of $\mathrm{PolyRing}(F)$. Suppose $\mathrm{ExtEval}(p, a) = 0_E$. Then $\mathrm{MinPoly}(a, F) = \mathrm{NormPoly}\, p$.

(16) Let us consider a field F_1, an F_1-monomorphic, F_1-homomorphic field F_2, a monomorphism h of F_1 and F_2, and an element p of the carrier of $\mathrm{PolyRing}(F_1)$. Then $\mathrm{NormPoly}(\mathrm{PolyHom}(h))(p) = (\mathrm{PolyHom}(h))(\mathrm{NormPoly}\, p)$.

Let F_1 be a field, F_2 be an F_1-isomorphic, F_1-homomorphic field, h be an isomorphism between F_1 and F_2, and p be a constant element of the carrier of $\mathrm{PolyRing}(F_1)$. One can check that $(\mathrm{PolyHom}(h))(p)$ is constant as an element of the carrier of $\mathrm{PolyRing}(F_2)$.

Let p be a non constant element of the carrier of $\mathrm{PolyRing}(F_1)$. Note that $(\mathrm{PolyHom}(h))(p)$ is non constant as an element of the carrier of $\mathrm{PolyRing}(F_2)$.

Let p be an irreducible element of the carrier of $\mathrm{PolyRing}(F_1)$. Let us note that $(\mathrm{PolyHom}(h))(p)$ is irreducible as an element of the carrier of $\mathrm{PolyRing}(F_2)$.

Now we state the propositions:

(17) Let us consider a field F_1, a non constant element p of the carrier of $\mathrm{PolyRing}(F_1)$, an F_1-isomorphic field F_2, and an isomorphism h between F_1 and F_2. Then p splits in F_1 if and only if $(\mathrm{PolyHom}(h))(p)$ splits in F_2.

(18) Let us consider a field F, an element p of the carrier of $\mathrm{PolyRing}(F)$, an extension E of F, and an E-extending extension U of F.
Then $\mathrm{Roots}(E, p) \subseteq \mathrm{Roots}(U, p)$.

(19) Let us consider a field F, a non constant element p of the carrier of $\mathrm{PolyRing}(F)$, an extension E of F, and an extension U of E. If p splits in E, then p splits in U. The theorem is a consequence of (2).

3. More on Products of Linear Polynomials

Now we state the propositions:

(20) Let us consider a field F, and a non empty finite sequence G of elements of the carrier of PolyRing(F). Suppose for every natural number i such that $i \in \operatorname{dom} G$ there exists an element a of F such that $G(i) = \operatorname{rpoly}(1, a)$. Then G is a factorization of $\prod G$.

(21) Let us consider a field F, and non empty finite sequences G_1, G_2 of elements of PolyRing(F). Suppose for every natural number i such that $i \in \operatorname{dom} G_1$ there exists an element a of F such that $G_1(i) = \operatorname{rpoly}(1, a)$ and for every natural number i such that $i \in \operatorname{dom} G_2$ there exists an element a of F such that $G_2(i) = \operatorname{rpoly}(1, a)$ and $\prod G_1 = \prod G_2$. Let us consider an element a of F. Then there exists a natural number i such that $i \in \operatorname{dom} G_1$ and $G_1(i) = \operatorname{rpoly}(1, a)$ if and only if there exists a natural number i such that $i \in \operatorname{dom} G_2$ and $G_2(i) = \operatorname{rpoly}(1, a)$. The theorem is a consequence of (20).

(22) Let us consider a field F, an extension E of F, and a non empty finite sequence G_1 of elements of PolyRing(F). Suppose for every natural number i such that $i \in \operatorname{dom} G_1$ there exists an element a of F such that $G_1(i) = \operatorname{rpoly}(1, a)$.

Let us consider a non empty finite sequence G_2 of elements of PolyRing (E). Suppose for every natural number i such that $i \in \operatorname{dom} G_2$ there exists an element a of E such that $G_2(i) = \operatorname{rpoly}(1, a)$. Suppose $\prod G_1 = \prod G_2$.

Let us consider an element a of E. Then there exists a natural number i such that $i \in \operatorname{dom} G_1$ and $G_1(i) = \operatorname{rpoly}(1, a)$ if and only if there exists a natural number i such that $i \in \operatorname{dom} G_2$ and $G_2(i) = \operatorname{rpoly}(1, a)$. The theorem is a consequence of (4) and (21).

(23) Let us consider a field F, a product of linear polynomials p of F, and an element a of F. Then $\operatorname{LC} a \cdot p = a$.

(24) Let us consider a field F, and an extension E of F. Then every product of linear polynomials of F is a product of linear polynomials of E.

(25) Let us consider a field F, an extension E of F, a non zero element a of F, a non zero element b of E, a product of linear polynomials p of F, and a product of linear polynomials q of E. If $a \cdot p = b \cdot q$, then $a = b$ and $p = q$. The theorem is a consequence of (5) and (2).

(26) Let us consider a field F, an extension E of F, and a non empty finite sequence G of elements of the carrier of PolyRing(E). Suppose for every natural number i such that $i \in \operatorname{dom} G$ there exists an element a of F such that $G(i) = \operatorname{rpoly}(1, a)$. Then $\prod G$ is a product of linear polynomials of F.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every non empty finite sequence G of elements of PolyRing(E) such that $\operatorname{len} G = \$_1$ and for every natural number i such that $i \in \operatorname{dom} G$ there exists an element a of F such that $G(i) = \operatorname{rpoly}(1, a)$ holds $\prod G$ is a product of linear polynomials of F. For every natural number k, $\mathcal{P}[k]$. Consider n being a natural number such that $\operatorname{len} G = n$. $\square$

4. Existence of Splitting Fields

Let us consider a field F, a non constant element p of the carrier of PolyRing (F), an extension U of F, and a U-extending extension E of F. Now we state the propositions:

(27) If p splits in E, then p splits in U iff $\operatorname{Roots}(E, p) \subseteq$ the carrier of U.

(28) If p splits in E, then p splits in U iff $\operatorname{Roots}(E, p) \subseteq \operatorname{Roots}(U, p)$. The theorem is a consequence of (27).

(29) If p splits in E, then p splits in U iff $\operatorname{Roots}(E, p) = \operatorname{Roots}(U, p)$. The theorem is a consequence of (28) and (18).

(30) Let us consider a field F, a non constant element p of the carrier of PolyRing(F), and an extension E of F. If p splits in E, then p splits in $\operatorname{FAdj}(F, \operatorname{Roots}(E, p))$. The theorem is a consequence of (27).

Let F be a field and p be a non constant element of the carrier of PolyRing(F). A splitting field of p is an extension of F defined by

(Def. 1) p splits in *it* and for every extension E of F such that p splits in E and E is a subfield of *it* holds $E \approx$ *it*.

Let us consider a field F and a non constant element p of the carrier of PolyRing(F). Now we state the propositions:

(31) There exists an extension E of F such that E is a splitting field of p.

(32) There exists an extension E of F such that $\operatorname{FAdj}(F, \operatorname{Roots}(E, p))$ is a splitting field of p. The theorem is a consequence of (30), (18), and (28).

(33) Let us consider a field F, a non constant element p of the carrier of PolyRing(F), and an extension E of F. Suppose p splits in E. Then $\operatorname{FAdj}(F, \operatorname{Roots}(E, p))$ is a splitting field of p. The theorem is a consequence of (30), (18), and (28).

(34) Let us consider a field F, a non constant element p of the carrier of PolyRing(F), and a splitting field E of p. Then $E \approx \operatorname{FAdj}(F, \operatorname{Roots}(E, p))$. The theorem is a consequence of (33).

Let F be a field and p be a non constant element of the carrier of PolyRing(F). Let us observe that there exists a splitting field of p which is strict and every splitting field of p is F-finite.

5. Fixing and Extending Automorphisms

Let R be a ring. Let us observe that there exists a function from R into R which is isomorphism.

A homomorphism of R is an additive, multiplicative, unity-preserving function from R into R.

A monomorphism of R is a monomorphic function from R into R.

An automorphism of R is an isomorphism function from R into R. Let R, S_2 be rings, S_1 be a ring extension of R, and h be a function from S_1 into S_2. We say that h is R-fixing if and only if

(Def. 2) for every element a of R, $h(a) = a$.

Now we state the propositions:

(35) Let us consider rings R, S_2, a ring extension S_1 of R, and a function h from S_1 into S_2. Then h is R-fixing if and only if $h \restriction R = \mathrm{id}_R$.

(36) Let us consider a field F, an extension E_1 of F, an E_1-homomorphic extension E_2 of F, and a homomorphism h from E_1 to E_2. Then h is F-fixing if and only if h is a linear transformation from $\mathrm{VecSp}(E_1, F)$ to $\mathrm{VecSp}(E_2, F)$.

(37) Let us consider a field F, an extension E of F, an E-extending extension E_1 of F, an E-extending extension E_2 of F, and a function h from E_1 into E_2. If h is E-fixing, then h is F-fixing.

Let R be a ring, S_1, S_2 be ring extensions of R, and h be a function from S_1 into S_2. We say that h is R-homomorphism if and only if

(Def. 3) h is R-fixing, additive, multiplicative, and unity-preserving.

We say that h is R-monomorphism if and only if

(Def. 4) h is R-fixing and monomorphic.

We say that h is R-isomorphism if and only if

(Def. 5) h is R-fixing and isomorphism.

Let S be a ring extension of R. Observe that there exists an automorphism of S which is R-fixing.

Now we state the propositions:

(38) Let us consider a ring R, a ring extension S of R, an element p of the carrier of PolyRing(R), an R-fixing monomorphism h of S, and an element a of S. Then $a \in \mathrm{Roots}(S, p)$ if and only if $h(a) \in \mathrm{Roots}(S, p)$.

(39) Let us consider an integral domain R, a domain ring extension S of R, a non zero element p of the carrier of PolyRing(R), and an R-fixing monomorphism h of S. Then $h \restriction \mathrm{Roots}(S, p)$ is a permutation of $\mathrm{Roots}(S, p)$. The theorem is a consequence of (38).

Let R_1, R_2, S_2 be rings, S_1 be a ring extension of R_1, h_1 be a function from R_1 into R_2, and h_2 be a function from S_1 into S_2. We say that h_2 is h_1-extending if and only if

(Def. 6) for every element a of R_1, $h_2(a) = h_1(a)$.

Now we state the proposition:

(40) Let us consider rings R_1, R_2, S_2, a ring extension S_1 of R_1, a function h_1 from R_1 into R_2, and a function h_2 from S_1 into S_2. Then h_2 is h_1-extending if and only if $h_2 \restriction R_1 = h_1$.

Let R be a ring and S be a ring extension of R. Let us note that every automorphism of S which is R-fixing is also (id_R)-extending and every automorphism of S which is (id_R)-extending is also R-fixing.

Now we state the proposition:

(41) Let us consider fields F_1, F_2, an extension E_1 of F_1, an extension E_2 of F_2, an E_1-extending extension K_1 of F_1, an E_2-extending extension K_2 of F_2, a function h_1 from F_1 into F_2, a function h_2 from E_1 into E_2, and a function h_3 from K_1 into K_2. Suppose h_2 is h_1-extending and h_3 is h_2-extending. Then h_3 is h_1-extending.

Let F be a field and E_1, E_2 be extensions of F. We say that E_1 and E_2 are isomorphic over F if and only if

(Def. 7) there exists a function i from E_1 into E_2 such that i is F-isomorphism.

Now we state the propositions:

(42) Let us consider a field F, and an extension E of F. Then E and E are isomorphic over F.

(43) Let us consider a field F, and extensions E_1, E_2 of F. If E_1 and E_2 are isomorphic over F, then E_2 and E_1 are isomorphic over F.

PROOF: Consider f being a function from E_1 into E_2 such that f is F-isomorphism. Reconsider $g = f^{-1}$ as a function from E_2 into E_1. g is additive. g is multiplicative. $\square$

(44) Let us consider a field F, and extensions E_1, E_2, E_3 of F. Suppose E_1 and E_2 are isomorphic over F and E_2 and E_3 are isomorphic over F. Then E_1 and E_3 are isomorphic over F.

PROOF: Consider f being a function from E_1 into E_2 such that f is F-isomorphism. Consider g being a function from E_2 into E_3 such that g is F-isomorphism. $\mathrm{dom}(g \cdot f) =$ the carrier of E_1. Reconsider $h = g \cdot f$ as

a function from E_1 into E_3. h is F-fixing. $\square$

(45) Let us consider a field F, an F-finite extension E_1 of F, and an extension E_2 of F. Suppose E_1 and E_2 are isomorphic over F. Then

(i) E_2 is F-finite, and

(ii) $\deg(E_1, F) = \deg(E_2, F)$.

The theorem is a consequence of (36).

6. Some More Preliminaries

Let R be a ring, S_1, S_2 be ring extensions of R, and h be a relation between the carrier of S_1 and the carrier of S_2. We say that h is R-isomorphism if and only if

(Def. 8) there exists a function g from S_1 into S_2 such that $g = h$ and g is R-isomorphism.

Now we state the propositions:

(46) Let us consider a field F, an extension E of F, and an F-algebraic element a of E. Then

(i) $0_{\mathrm{FAdj}(F,\{a\})} = \mathrm{ExtEval}(\mathbf{0}.F, a)$, and

(ii) $1_{\mathrm{FAdj}(F,\{a\})} = \mathrm{ExtEval}(\mathbf{1}.F, a)$.

(47) Let us consider a field F, an extension E of F, an F-algebraic element a of E, elements x, y of $\mathrm{FAdj}(F, \{a\})$, and polynomials p, q over F. Suppose $x = \mathrm{ExtEval}(p, a)$ and $y = \mathrm{ExtEval}(q, a)$. Then

(i) $x + y = \mathrm{ExtEval}(p + q, a)$, and

(ii) $x \cdot y = \mathrm{ExtEval}(p * q, a)$.

(48) Let us consider a field F, an extension E of F, an F-algebraic element a of E, and an element x of F. Then $x = \mathrm{ExtEval}(x \restriction F, a)$.

Let us consider a field F, an extension E of F, and an element a of E. Now we state the propositions:

(49) $\mathrm{HomExtEval}(a, F)$ is a function from $\mathrm{PolyRing}(F)$ into $\mathrm{RAdj}(F, \{a\})$.

(50) $\mathrm{HomExtEval}(a, F)$ is a function from $\mathrm{PolyRing}(F)$ into $\mathrm{FAdj}(F, \{a\})$. The theorem is a consequence of (49).

(51) Let us consider a field F_1, an F_1-isomorphic, F_1-homomorphic field F_2, an isomorphism h between F_1 and F_2, an extension E_1 of F_1, an extension E_2 of F_2, an F_1-algebraic element a of E_1, an F_2-algebraic element b of E_2, and an irreducible element p of the carrier of $\mathrm{PolyRing}(F_1)$. Suppose $\mathrm{ExtEval}(p, a) = 0_{E_1}$ and $\mathrm{ExtEval}((\mathrm{PolyHom}(h))(p), b) = 0_{E_2}$. Then

$(\mathrm{PolyHom}(h))(\mathrm{MinPoly}(a, F_1)) = \mathrm{MinPoly}(b, F_2)$. The theorem is a consequence of (15) and (16).

(52) Let us consider a field F_1, an F_1-isomorphic, F_1-homomorphic field F_2, an isomorphism h between F_1 and F_2, an extension E_1 of F_1, an extension E_2 of F_2, an F_1-algebraic element a of E_1, and an F_2-algebraic element b of E_2. Suppose $\mathrm{ExtEval}((\mathrm{PolyHom}(h))(\mathrm{MinPoly}(a, F_1)), b) = 0_{E_2}$. Then $(\mathrm{PolyHom}(h))(\mathrm{MinPoly}(a, F_1)) = \mathrm{MinPoly}(b, F_2)$. The theorem is a consequence of (15) and (16).

(53) Let us consider a field F_1, a non constant element p_1 of the carrier of $\mathrm{PolyRing}(F_1)$, an extension F_2 of F_1, a non constant element p_2 of the carrier of $\mathrm{PolyRing}(F_2)$, and a splitting field E of p_1. Suppose $p_2 = p_1$ and E is F_2-extending. Then E is a splitting field of p_2.

7. Uniqueness of Splitting Fields

Let F be a field, E be an extension of F, and a, b be F-algebraic elements of E. The functor $\Phi(a, b)$ yielding a relation between the carrier of $\mathrm{FAdj}(F, \{a\})$ and the carrier of $\mathrm{FAdj}(F, \{b\})$ is defined by the term

(Def. 9) the set of all $\langle \mathrm{ExtEval}(p, a), \mathrm{ExtEval}(p, b) \rangle$ where p is a polynomial over F.

Note that $\Phi(a, b)$ is quasi-total. Now we state the proposition:

(54) Let us consider a field F, an extension E of F, and F-algebraic elements a, b of E. Then $\Phi(a, b)$ is F-isomorphism if and only if $\mathrm{MinPoly}(a, F) = \mathrm{MinPoly}(b, F)$. The theorem is a consequence of (46), (47), and (48).

Let F_1 be a field, F_2 be an F_1-isomorphic, F_1-homomorphic field, h be an isomorphism between F_1 and F_2, E_1 be an extension of F_1, E_2 be an extension of F_2, a be an element of E_1, b be an element of E_2, and p be an irreducible element of the carrier of $\mathrm{PolyRing}(F_1)$.

Assume $\mathrm{ExtEval}(p, a) = 0_{E_1}$ and $\mathrm{ExtEval}((\mathrm{PolyHom}(h))(p), b) = 0_{E_2}$. The functor $\Psi(a, b, h, p)$ yielding a function from $\mathrm{FAdj}(F_1, \{a\})$ into $\mathrm{FAdj}(F_2, \{b\})$ is defined by

(Def. 10) for every element r of the carrier of $\mathrm{PolyRing}(F_1)$, $it(\mathrm{ExtEval}(r, a)) = \mathrm{ExtEval}((\mathrm{PolyHom}(h))(r), b)$.

Now we state the propositions:

(55) Let us consider a field F_1, an F_1-isomorphic, F_1-homomorphic field F_2, an isomorphism h between F_1 and F_2, an extension E_1 of F_1, an extension E_2 of F_2, an element a of E_1, an element b of E_2, and an irreducible element p of the carrier of $\mathrm{PolyRing}(F_1)$. Suppose $\mathrm{ExtEval}(p, a) = 0_{E_1}$

and $\mathrm{ExtEval}((\mathrm{PolyHom}(h))(p), b) = 0_{E_2}$. Then $\Psi(a, b, h, p)$ is h-extending and isomorphism.

PROOF: Set $f = \Psi(a, b, h, p)$. Set $F_3 = \mathrm{FAdj}(F_1, \{a\})$. Set $F_5 = \mathrm{FAdj}(F_2, \{b\})$. $f(1_{F_3}) = 1_{F_5}$ by [6, (36)], [5, (14)], [7, (14)], (13). f is onto by [6, (56), (45)]. □

(56) Let us consider a field F, an extension E of F, an irreducible element p of the carrier of $\mathrm{PolyRing}(F)$, and elements a, b of E. Suppose a is a root of p in E and b is a root of p in E. Then $\mathrm{FAdj}(F, \{a\})$ and $\mathrm{FAdj}(F, \{b\})$ are isomorphic. The theorem is a consequence of (55).

(57) Let us consider a field F_1, an F_1-homomorphic, F_1-isomorphic field F_2, an isomorphism h between F_1 and F_2, a non constant element p of the carrier of $\mathrm{PolyRing}(F_1)$, a splitting field E_1 of p, and a splitting field E_2 of $(\mathrm{PolyHom}(h))(p)$. Then there exists a function f from E_1 into E_2 such that f is h-extending and isomorphism.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every field F_1 for every F_1-homomorphic, F_1-isomorphic field F_2 for every isomorphism h between F_1 and F_2 for every non constant element p of the carrier of $\mathrm{PolyRing}(F_1)$ for every splitting field E_1 of p for every splitting field E_2 of $(\mathrm{PolyHom}(h))(p)$ such that $\overline{\overline{(\mathrm{Roots}(E_1, p)) \setminus (\text{the carrier of } F_1)}} = \$_1$ there exists a function f from E_1 into E_2 such that f is h-extending and isomorphism.

For every natural number k, $\mathcal{P}[k]$. Consider n being a natural number such that $\overline{\overline{(\mathrm{Roots}(E_1, p)) \setminus \alpha}} = n$, where α is the carrier of F_1. □

(58) Let us consider a field F, a non constant element p of the carrier of $\mathrm{PolyRing}(F)$, and splitting fields E_1, E_2 of p. Then E_1 and E_2 are isomorphic over F.

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Algorithm NextFit for the Bin Packing Problem[1]

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Summary. The bin packing problem is a fundamental and important optimization problem in theoretical computer science [4], [6]. An instance is a sequence of items, each being of positive size at most one. The task is to place all the items into bins so that the total size of items in each bin is at most one and the number of bins that contain at least one item is minimum.

Approximation algorithms have been intensively studied. Algorithm NextFit would be the simplest one. The algorithm repeatedly does the following: If the first unprocessed item in the sequence can be placed, in terms of size, additionally to the bin into which the algorithm has placed an item the last time, place the item into that bin; otherwise place the item into an empty bin. Johnson [5] proved that the number of the resulting bins by algorithm NextFit is less than twice the number of the fewest bins that are needed to contain all items.

In this article, we formalize in Mizar [1], [2] the bin packing problem as follows: An instance is a sequence of positive real numbers that are each at most one. The task is to find a function that maps the indices of the sequence to positive integers such that the sum of the subsequence for each of the inverse images is at most one and the size of the image is minimum. We then formalize algorithm NextFit, its feasibility, its approximation guarantee, and the tightness of the approximation guarantee.

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1. Preliminaries

Let a be a non empty finite sequence of elements of $\mathbb{R}$ and i be an element of $\operatorname{dom} a$. Let us observe that the functor $a(i)$ yields an element of $\mathbb{R}$. Let h be a non empty finite sequence of elements of $\mathbb{N}^*$ and i be an element of $\operatorname{dom} h$. Let us observe that the functor $h(i)$ yields a finite sequence of elements of $\mathbb{N}$. Now we state the propositions:

(1) Let us consider a natural number n. If n is odd, then $1 \leqslant n$ and $n + 1 \operatorname{div} 2 = \frac{n+1}{2}$.

(2) Let us consider a set D, and a finite sequence p. Suppose for every natural number i such that $i \in \operatorname{dom} p$ holds $p(i) \in D$. Then p is a finite sequence of elements of D.

(3) Let us consider objects x, y. Then $\{\langle x, y\rangle\}^{-1}(\{y\}) = \{x\}$.
Proof: For every object v, $v \in \{x\}$ iff $v \in \operatorname{dom}\{\langle x, y\rangle\}$ and $\{\langle x, y\rangle\}(v) \in \{y\}$. $\square$

(4) Let us consider natural numbers a, b, and a set s. If $\operatorname{Seg} a \cup \{s\} = \operatorname{Seg} b$, then $a = b$ or $a + 1 = b$. Proof: $b - a \leqslant 1$. $\square$

Let D be a non empty set, f be a D-valued finite sequence, and I be a set. The functor $\operatorname{Seq}(f, I)$ yielding a D-valued finite sequence is defined by the term

(Def. 1) $\operatorname{Seq}(f \restriction I)$.

Let a be a non empty finite sequence of elements of $\mathbb{R}$, f be a function, and s be a set. The functor $\operatorname{SumBin}(a, f, s)$ yielding a real number is defined by the term

(Def. 2) $\sum \operatorname{Seq}(a, f^{-1}(s))$.

Let us observe that there exists a non empty finite sequence of elements of $\mathbb{R}$ which is positive. Let a be a finite sequence of elements of $\mathbb{R}$. We say that a is at most one if and only if

(Def. 3) for every natural number i such that $1 \leqslant i \leqslant \operatorname{len} a$ holds $a(i) \leqslant 1$.

Note that there exists a non empty, positive finite sequence of elements of $\mathbb{R}$ which is at most one. Let us consider a finite sequence f of elements of $\mathbb{N}$ and natural numbers j, b. Now we state the propositions:

(5) If $b = j$, then $(f \frown \langle b\rangle)^{-1}(\{j\}) = f^{-1}(\{j\}) \cup \{\operatorname{len} f + 1\}$.
Proof: For every object z, $z \in (f \frown \langle b\rangle)^{-1}(\{j\})$ iff $z \in f^{-1}(\{j\}) \cup \{\operatorname{len} f + 1\}$. $\square$

(6) If $b \neq j$, then $(f \frown \langle b\rangle)^{-1}(\{j\}) = f^{-1}(\{j\})$.
Proof: For every object z, $z \in (f \frown \langle b\rangle)^{-1}(\{j\})$ iff $z \in f^{-1}(\{j\})$. $\square$

(7) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, a set p, and a natural number i. Suppose $p \cup \{i\} \subseteq \operatorname{dom} a$ and for every natural

number m such that $m \in p$ holds $m < i$. Then $\mathrm{Seq}(a{\restriction}(p \cup \{i\})) = \mathrm{Seq}(a{\restriction}p)^\frown \langle a(i) \rangle$.

Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, a finite sequence f of elements of $\mathbb{N}$, and natural numbers j, b. Now we state the propositions:

(8) Suppose $\mathrm{len}\, f + 1 \leqslant \mathrm{len}\, a$. Then if $b = j$, then $\mathrm{SumBin}(a, f^\frown \langle b \rangle, \{j\}) = \mathrm{SumBin}(a, f, \{j\}) + a(\mathrm{len}\, f + 1)$.
PROOF: $(f^\frown \langle b \rangle)^{-1}(\{j\}) = f^{-1}(\{j\}) \cup \{\mathrm{len}\, f + 1\}$. For every natural number m such that $m \in f^{-1}(\{j\})$ holds $m < \mathrm{len}\, f + 1$. $\square$

(9) Suppose $\mathrm{len}\, f + 1 \leqslant \mathrm{len}\, a$. Then if $b \neq j$, then $\mathrm{SumBin}(a, f^\frown \langle b \rangle, \{j\}) = \mathrm{SumBin}(a, f, \{j\})$.

(10) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, and a finite sequence f of elements of $\mathbb{N}$. Suppose $\mathrm{dom}\, f = \mathrm{dom}\, a$. Then $\mathrm{SumBin}(a, f, \mathrm{rng}\, f) = \sum a$.

(11) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, a finite sequence f of elements of $\mathbb{N}$, and sets s, t. Suppose $\mathrm{dom}\, f \subseteq \mathrm{dom}\, a$ and s misses t. Then $\mathrm{SumBin}(a, f, s \cup t) = \mathrm{SumBin}(a, f, s) + \mathrm{SumBin}(a, f, t)$.
PROOF: Reconsider $F = a$ as a partial function from $\mathbb{N}$ to $\mathbb{R}$. For every set W such that $W \subseteq \mathrm{dom}\, a$ holds $\sum_{\kappa=0}^{W} F(\kappa) = \sum \mathrm{Seq}(a, W)$ by [3, (51)]. $\square$

(12) Let us consider a non empty, positive finite sequence a of elements of $\mathbb{R}$, a finite sequence f of elements of $\mathbb{N}$, and a set s. If $\mathrm{dom}\, f \subseteq \mathrm{dom}\, a$, then $0 \leqslant \mathrm{SumBin}(a, f, s)$.
PROOF: Reconsider $s_1 = \mathrm{Seq}(a, f^{-1}(s))$ as a real-valued finite sequence. For every natural number i such that $i \in \mathrm{dom}\, s_1$ holds $0 \leqslant s_1(i)$. $\square$

(13) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, a finite sequence f of elements of $\mathbb{N}$, and a set s. If s misses $\mathrm{rng}\, f$, then $\mathrm{SumBin}(a, f, s) = 0$.

2. OPTIMAL PACKING

Now we state the propositions:

(14) Let us consider a non empty, at most one finite sequence a of elements of $\mathbb{R}$. Then there exists a natural number k and there exists a non empty finite sequence f of elements of $\mathbb{N}$ such that $\mathrm{dom}\, f = \mathrm{dom}\, a$ and for every natural number j such that $j \in \mathrm{rng}\, f$ holds $\mathrm{SumBin}(a, f, \{j\}) \leqslant 1$ and $k = \overline{\overline{\mathrm{rng}\, f}}$.
PROOF: Set $k_1 = \mathrm{len}\, a$. Set $f_1 = \mathrm{idseq}(k_1)$. For every natural number j such that $j \in \mathrm{rng}\, f_1$ holds $\mathrm{SumBin}(a, f_1, \{j\}) \leqslant 1$. There exists a non

empty finite sequence f of elements of $\mathbb{N}$ such that $\operatorname{dom} f = \operatorname{dom} a$ and for every natural number j such that $j \in \operatorname{rng} f$ holds $\operatorname{SumBin}(a, f, \{j\}) \leqslant 1$ and $k_1 = \overline{\overline{\operatorname{rng} f}}$. $\square$

(15) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, and a finite sequence f of elements of $\mathbb{N}$. Suppose $\operatorname{dom} f = \operatorname{dom} a$ and for every natural number j such that $j \in \operatorname{rng} f$ holds $\operatorname{SumBin}(a, f, \{j\}) \leqslant 1$. Then there exists a finite sequence f_2 of elements of $\mathbb{N}$ such that

(i) $\operatorname{dom} f_2 = \operatorname{dom} a$, and

(ii) for every natural number j such that $j \in \operatorname{rng} f_2$ holds $\operatorname{SumBin}(a, f_2, \{j\}) \leqslant 1$, and

(iii) there exists a natural number k such that $\operatorname{rng} f_2 = \operatorname{Seg} k$, and

(iv) $\overline{\overline{\operatorname{rng} f}} = \overline{\overline{\operatorname{rng} f_2}}$.

PROOF: Reconsider $g_3 = \operatorname{Sgm}_0 \operatorname{rng} f$ as a finite 0-sequence of $\mathbb{N}$. Reconsider $g_2 = \operatorname{XFS2FS}(g_3)$ as a one-to-one function. Reconsider $g = {g_2}^{-1}$ as a one-to-one function. Reconsider $f_3 = g \cdot f$ as a finite sequence. Consider k_0 being a natural number such that $\operatorname{dom} g_2 = \operatorname{Seg} k_0$. For every natural number j such that $j \in \operatorname{rng} f_3$ holds $\operatorname{SumBin}(a, f_3, \{j\}) \leqslant 1$. $\square$

Let a be a non empty, at most one finite sequence of elements of $\mathbb{R}$. The functor $\operatorname{Opt}(a)$ yielding an element of $\mathbb{N}$ is defined by

(Def. 4) there exists a non empty finite sequence g of elements of $\mathbb{N}$ such that $\operatorname{dom} g = \operatorname{dom} a$ and for every natural number j such that $j \in \operatorname{rng} g$ holds $\operatorname{SumBin}(a, g, \{j\}) \leqslant 1$ and $it = \overline{\overline{\operatorname{rng} g}}$ and for every non empty finite sequence f of elements of $\mathbb{N}$ such that $\operatorname{dom} f = \operatorname{dom} a$ and for every natural number j such that $j \in \operatorname{rng} f$ holds $\operatorname{SumBin}(a, f, \{j\}) \leqslant 1$ holds $it \leqslant \overline{\overline{\operatorname{rng} f}}$.

Now we state the propositions:

(16) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, a finite sequence f of elements of $\mathbb{N}$, a natural number k, and a real-valued finite sequence R_1. Suppose $\operatorname{dom} f = \operatorname{dom} a$ and $\operatorname{rng} f = \operatorname{Seg} k$ and $\operatorname{len} R_1 = k$ and for every natural number j such that $j \in \operatorname{dom} R_1$ holds $R_1(j) = \operatorname{SumBin}(a, f, \{j\})$. Then $\sum R_1 = \operatorname{SumBin}(a, f, \operatorname{rng} f)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every real-valued finite sequence r_1 such that $r_1 = R_1 \restriction \operatorname{Seg} \$_1$ holds $\sum r_1 = \operatorname{SumBin}(a, f, \operatorname{Seg} \$_1)$. For every real-valued finite sequence r_1 such that $r_1 = R_1 \restriction \operatorname{Seg} 1$ holds $\sum r_1 = \operatorname{SumBin}(a, f, \operatorname{Seg} 1)$. For every element i of $\mathbb{N}$ such that $1 \leqslant i < k$ and $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant k$ holds $\mathcal{P}[i]$. $\square$

(17) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, and a finite sequence f of elements of $\mathbb{N}$. Suppose $\operatorname{dom} f = \operatorname{dom} a$ and for every natural number j such that $j \in \operatorname{rng} f$ holds $\operatorname{SumBin}(a, f, \{j\}) \leqslant 1$. Then $\lceil \sum a \rceil \leqslant \overline{\overline{\operatorname{rng} f}}$.

PROOF: Consider f_2 being a finite sequence of elements of $\mathbb{N}$ such that $\operatorname{dom} f_2 = \operatorname{dom} a$ and for every natural number j such that $j \in \operatorname{rng} f_2$ holds $\operatorname{SumBin}(a, f_2, \{j\}) \leqslant 1$ and there exists a natural number k such that $\operatorname{rng} f_2 = \operatorname{Seg} k$ and $\overline{\overline{\operatorname{rng} f}} = \overline{\overline{\operatorname{rng} f_2}}$. Consider i being a natural number such that $\operatorname{rng} f_2 = \operatorname{Seg} i$. Define $\mathcal{N}(\text{natural number}) = \operatorname{SumBin}(a, f_2, \{\$_1\})$.

There exists a finite sequence p such that $\operatorname{len} p = i$ and for every natural number j such that $j \in \operatorname{dom} p$ holds $p(j) = \mathcal{N}(j)$. Consider R_1 being a finite sequence such that $\operatorname{len} R_1 = i$ and for every natural number j such that $j \in \operatorname{dom} R_1$ holds $R_1(j) = \operatorname{SumBin}(a, f_2, \{j\})$. For every natural number j such that $j \in \operatorname{dom} R_1$ holds $R_1(j) \in \mathbb{R}$. R_1 is a finite sequence of elements of $\mathbb{R}$.

Reconsider $R_2 = i \mapsto 1$ as a real-valued, i-element finite sequence. For every natural number j such that $j \in \operatorname{Seg} i$ holds $R_1(j) \leqslant R_2(j)$. $\sum R_1 = \operatorname{SumBin}(a, f_2, \operatorname{rng} f_2)$. $\sum a \leqslant \overline{\overline{\operatorname{rng} f}}$. $\square$

(18) Let us consider a non empty, at most one finite sequence a of elements of $\mathbb{R}$. Then $\lceil \sum a \rceil \leqslant \operatorname{Opt}(a)$. The theorem is a consequence of (17).

3. ONLINE ALGORITHMS

Let a be a non empty finite sequence of elements of $\mathbb{R}$ and A be a function from $\mathbb{R} \times \mathbb{N}^*$ into $\mathbb{N}$. The functor $\operatorname{OnlinePackingHistory}(a, A)$ yielding a non empty finite sequence of elements of $\mathbb{N}^*$ is defined by

(Def. 5) $\operatorname{len} it = \operatorname{len} a$ and $it(1) = \langle 1 \rangle$ and for every natural number i such that $1 \leqslant i < \operatorname{len} a$ there exists an element d_1 of $\mathbb{R}$ and there exists a finite sequence d_2 of elements of $\mathbb{N}$ such that $d_1 = a(i+1)$ and $d_2 = it(i)$ and $it(i+1) = d_2 \frown \langle A(d_1, d_2) \rangle$.

Now we state the propositions:

(19) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, and a function A from $\mathbb{R} \times \mathbb{N}^*$ into $\mathbb{N}$. Then $(\operatorname{OnlinePackingHistory}(a, A))(1) = \{\langle 1, 1 \rangle\}$.

(20) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, a function A from $\mathbb{R} \times \mathbb{N}^*$ into $\mathbb{N}$, and a non empty finite sequence h of elements of $\mathbb{N}^*$. Suppose $h = \operatorname{OnlinePackingHistory}(a, A)$.

Then $\operatorname{SumBin}(a, h(1), \{h(1)(1)\}) = a(1)$. The theorem is a consequence of (3).

Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, a function A from $\mathbb{R} \times \mathbb{N}^*$ into $\mathbb{N}$, a non empty finite sequence h of elements of $\mathbb{N}^*$, and a natural number i. Now we state the propositions:

(21) If $h = \mathrm{OnlinePackingHistory}(a, A)$, then if $1 \leqslant i \leqslant \operatorname{len} a$, then $h(i)$ is a finite sequence of elements of $\mathbb{N}$.

(22) If $h = \mathrm{OnlinePackingHistory}(a, A)$, then if $1 \leqslant i \leqslant \operatorname{len} a$, then $\operatorname{len} h(i) = i$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \operatorname{len} h(\$_1) = \$_1$. For every element i of $\mathbb{N}$ such that $1 \leqslant i < \operatorname{len} a$ and $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \operatorname{len} a$ holds $\mathcal{P}[i]$. For every natural number i such that $1 \leqslant i \leqslant \operatorname{len} a$ holds $\mathcal{P}[i]$. $\square$

(23) If $h = \mathrm{OnlinePackingHistory}(a, A)$, then if $1 \leqslant i < \operatorname{len} a$, then $h(i+1) = h(i) \frown \langle A(a(i+1), h(i)) \rangle$ and $h(i+1)(i+1) = A(a(i+1), h(i))$. The theorem is a consequence of (22).

(24) If $h = \mathrm{OnlinePackingHistory}(a, A)$, then if $1 \leqslant i < \operatorname{len} a$, then $\operatorname{rng} h(i+1) = \operatorname{rng} h(i) \cup \{h(i+1)(i+1)\}$. The theorem is a consequence of (23).

(25) Let us consider a non empty, positive finite sequence a of elements of $\mathbb{R}$, a function A from $\mathbb{R} \times \mathbb{N}^*$ into $\mathbb{N}$, and a non empty finite sequence h of elements of $\mathbb{N}^*$. Suppose $h = \mathrm{OnlinePackingHistory}(a, A)$. Let us consider natural numbers i, l. Suppose $1 \leqslant i < \operatorname{len} a$. Then $\mathrm{SumBin}(a, h(i), \{l\}) \leqslant \mathrm{SumBin}(a, h(i+1), \{l\})$. The theorem is a consequence of (21), (22), (23), (8), and (6).

Let a be a non empty finite sequence of elements of $\mathbb{R}$ and A be a function from $\mathbb{R} \times \mathbb{N}^*$ into $\mathbb{N}$. The functor $\mathrm{OnlinePacking}(a, A)$ yielding a non empty finite sequence of elements of $\mathbb{N}$ is defined by the term

(Def. 6) $(\mathrm{OnlinePackingHistory}(a, A))(\operatorname{len} \mathrm{OnlinePackingHistory}(a, A))$.

Now we state the proposition:

(26) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, a function A from $\mathbb{R} \times \mathbb{N}^*$ into $\mathbb{N}$, a non empty finite sequence h of elements of $\mathbb{N}^*$, and a non empty finite sequence f of elements of $\mathbb{N}$. Then $\operatorname{dom}(\mathrm{OnlinePacking}(a, A)) = \operatorname{dom} a$. The theorem is a consequence of (22).

4. Feasibility of Algorithm NextFit

Let a be a non empty finite sequence of elements of $\mathbb{R}$. The functor $\mathrm{NextFit}(a)$ yielding a function from $\mathbb{R} \times \mathbb{N}^*$ into $\mathbb{N}$ is defined by

(Def. 7) for every real number s and for every finite sequence f of elements of $\mathbb{N}$, if $s + \mathrm{SumBin}(a, f, \{f(\operatorname{len} f)\}) \leqslant 1$, then $it(s, f) = f(\operatorname{len} f)$ and if $s + \mathrm{SumBin}(a, f, \{f(\operatorname{len} f)\}) > 1$, then $it(s, f) = f(\operatorname{len} f) + 1$.

Now we state the propositions:

(27) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, and a non empty finite sequence h of elements of $\mathbb{N}^*$.
Suppose $h = \mathrm{OnlinePackingHistory}(a, \mathrm{NextFit}(a))$. Let us consider a natural number i. Suppose $1 \leqslant i \leqslant \operatorname{len} a$. Then there exists a natural number k such that

(i) $\operatorname{rng} h(i) = \operatorname{Seg} k$, and

(ii) $h(i)(i) = k$.

PROOF: Define $\mathcal{R}[\text{natural number}] \equiv$ there exists a natural number k such that $\operatorname{rng} h(\$_1) = \operatorname{Seg} k$ and $h(\$_1)(\$_1) = k$. For every element i of $\mathbb{N}$ such that $1 \leqslant i < \operatorname{len} a$ and $\mathcal{R}[i]$ holds $\mathcal{R}[i+1]$. For every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \operatorname{len} a$ holds $\mathcal{R}[i]$. For every natural number i such that $1 \leqslant i \leqslant \operatorname{len} a$ holds $\mathcal{R}[i]$. $\square$

(28) Let us consider a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$, and a non empty finite sequence h of elements of $\mathbb{N}^*$. Suppose $h = \mathrm{OnlinePackingHistory}(a, \mathrm{NextFit}(a))$. Let us consider a natural number i. Suppose $1 \leqslant i \leqslant \operatorname{len} a$. Then $\mathrm{SumBin}(a, h(i), \{h(i)(i)\}) \leqslant 1$.
PROOF: Define $\mathcal{T}[\text{natural number}] \equiv \mathrm{SumBin}(a, h(\$_1), \{h(\$_1)(\$_1)\}) \leqslant 1$. $\mathrm{SumBin}(a, h(1), \{h(1)(1)\}) \leqslant 1$. For every element i of $\mathbb{N}$ such that $1 \leqslant i < \operatorname{len} a$ and $\mathcal{T}[i]$ holds $\mathcal{T}[i+1]$. For every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \operatorname{len} a$ holds $\mathcal{T}[i]$. For every natural number i such that $1 \leqslant i \leqslant \operatorname{len} a$ holds $\mathcal{T}[i]$. $\square$

(29) Let us consider a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$, and a non empty finite sequence h of elements of $\mathbb{N}^*$. Suppose $h = \mathrm{OnlinePackingHistory}(a, \mathrm{NextFit}(a))$. Let us consider natural numbers i, j. Suppose $1 \leqslant i \leqslant \operatorname{len} a$ and $j \in \operatorname{rng} h(i)$. Then $\mathrm{SumBin}(a, h(i), \{j\}) \leqslant 1$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every natural number j such that $j \in \operatorname{rng} h(\$_1)$ holds $\mathrm{SumBin}(a, h(\$_1), \{j\}) \leqslant 1$. For every natural number j such that $j \in \operatorname{rng} h(1)$ holds $\mathrm{SumBin}(a, h(1), \{j\}) \leqslant 1$. For every element i_0 of $\mathbb{N}$ such that $1 \leqslant i_0 < \operatorname{len} a$ and $\mathcal{P}[i_0]$ holds $\mathcal{P}[i_0 + 1]$.

For every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \operatorname{len} a$ holds $\mathcal{P}[i]$. For every natural numbers i, j such that $1 \leqslant i \leqslant \operatorname{len} a$ and $j \in \operatorname{rng} h(i)$ holds $\mathrm{SumBin}(a, h(i), \{j\}) \leqslant 1$. $\square$

(30) Let us consider a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$, and a non empty finite sequence f of elements of $\mathbb{N}$. Suppose $f = \mathrm{OnlinePacking}(a, \mathrm{NextFit}(a))$. Let us consider a natural number j. If $j \in \operatorname{rng} f$, then $\mathrm{SumBin}(a, f, \{j\}) \leqslant 1$. The theorem is a consequence of (29).

5. Approximation Guarantee of Algorithm NextFit

Let us consider a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$, a non empty finite sequence h of elements of $\mathbb{N}^*$, and natural numbers i, k. Now we state the propositions:

(31) If $h = \mathrm{OnlinePackingHistory}(a, \mathrm{NextFit}(a))$, then if $1 \leqslant i \leqslant \mathrm{len}\, a$ and $\mathrm{rng}\, h(i) = \mathrm{Seg}\, k$, then $h(i)(i) = k$. The theorem is a consequence of (27).

(32) Suppose $h = \mathrm{OnlinePackingHistory}(a, \mathrm{NextFit}(a))$. Then suppose $1 \leqslant i < \mathrm{len}\, a$ and $\mathrm{rng}\, h(i) = \mathrm{Seg}\, k$ and $\mathrm{rng}\, h(i+1) = \mathrm{Seg}(k+1)$. Then $\mathrm{SumBin}(a, h(i+1), \{k\}) + \mathrm{SumBin}(a, h(i+1), \{k+1\}) > 1$. The theorem is a consequence of (21), (22), (23), (31), (24), (6), (8), and (12).

(33) Let us consider a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$, and a non empty finite sequence h of elements of $\mathbb{N}^*$. Suppose $h = \mathrm{OnlinePackingHistory}(a, \mathrm{NextFit}(a))$. Let us consider natural numbers i, l, k. Suppose $1 \leqslant i \leqslant \mathrm{len}\, a$ and $\mathrm{rng}\, h(i) = \mathrm{Seg}\, k$ and $2 \leqslant k$ and $1 \leqslant l < k$. Then $\mathrm{SumBin}(a, h(i), \{l\}) + \mathrm{SumBin}(a, h(i), \{l+1\}) > 1$.

Proof: Define $\mathcal{N}[\text{natural number}] \equiv$ for every natural number l for every natural number k such that $\mathrm{rng}\, h(\$_1) = \mathrm{Seg}\, k$ and $2 \leqslant k$ and $1 \leqslant l < k$ holds $\mathrm{SumBin}(a, h(\$_1), \{l\}) + \mathrm{SumBin}(a, h(\$_1), \{l+1\}) > 1$. For every natural number l and for every natural number k such that $\mathrm{rng}\, h(1) = \mathrm{Seg}\, k$ and $2 \leqslant k$ and $1 \leqslant l < k$ holds $\mathrm{SumBin}(a, h(1), \{l\}) + \mathrm{SumBin}(a, h(1), \{l+1\}) > 1$.

For every element i_0 of $\mathbb{N}$ such that $1 \leqslant i_0 < \mathrm{len}\, a$ and $\mathcal{N}[i_0]$ holds $\mathcal{N}[i_0+1]$. For every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \mathrm{len}\, a$ holds $\mathcal{N}[i]$. For every natural numbers i, l, k such that $1 \leqslant i \leqslant \mathrm{len}\, a$ and $\mathrm{rng}\, h(i) = \mathrm{Seg}\, k$ and $2 \leqslant k$ and $1 \leqslant l < k$ holds $\mathrm{SumBin}(a, h(i), \{l\}) + \mathrm{SumBin}(a, h(i), \{l+1\}) > 1$. $\square$

(34) Let us consider a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$, and a non empty finite sequence h of elements of $\mathbb{N}^*$. Suppose $h = \mathrm{OnlinePackingHistory}(a, \mathrm{NextFit}(a))$. Let us consider natural numbers i, j, k. Suppose $1 \leqslant i \leqslant \mathrm{len}\, a$ and $\mathrm{rng}\, h(i) = \mathrm{Seg}\, k$ and $2 \leqslant k$ and $1 \leqslant j \leqslant k \operatorname{div} 2$. Then $\mathrm{SumBin}(a, h(i), \{2 \cdot j - 1\}) + \mathrm{SumBin}(a, h(i), \{2 \cdot j\}) > 1$. The theorem is a consequence of (33).

(35) Let us consider a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$, a non empty finite sequence h of elements of $\mathbb{N}^*$, and a finite sequence f of elements of $\mathbb{N}$. Suppose $f = \mathrm{OnlinePacking}(a, \mathrm{NextFit}(a))$. Then there exists a natural number k such that $\mathrm{rng}\, f = \mathrm{Seg}\, k$. The theorem is a consequence of (27).

(36) Let us consider a non empty, positive, at most one finite sequence a of

elements of $\mathbb{R}$, a non empty finite sequence f of elements of $\mathbb{N}$, and a natural number k. Suppose $f = \mathrm{OnlinePacking}(a, \mathrm{NextFit}(a))$ and $\mathrm{rng}\, f = \mathrm{Seg}\, k$. Let us consider a natural number j. Suppose $1 \leqslant j \leqslant k \operatorname{div} 2$. Then $\mathrm{SumBin}(a, f, \{2 \cdot j - 1\}) + \mathrm{SumBin}(a, f, \{2 \cdot j\}) > 1$. The theorem is a consequence of (34).

Let us consider a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$, a non empty finite sequence f of elements of $\mathbb{N}$, and a natural number k. Now we state the propositions:

(37) If $f = \mathrm{OnlinePacking}(a, \mathrm{NextFit}(a))$ and $k = \overline{\overline{\mathrm{rng}\, f}}$, then $k \operatorname{div} 2 < \sum a$. The theorem is a consequence of (35), (26), (2), (36), (12), (16), and (10).

(38) Suppose $f = \mathrm{OnlinePacking}(a, \mathrm{NextFit}(a))$ and $k = \overline{\overline{\mathrm{rng}\, f}}$. Then $k \leqslant 2 \cdot \lceil \sum a \rceil - 1$.
PROOF: $k \operatorname{div} 2 < \lceil \sum a \rceil$. $\frac{k-1}{2} \leqslant k \operatorname{div} 2$ by [8, (4), (5)]. $\square$

(39) If $f = \mathrm{OnlinePacking}(a, \mathrm{NextFit}(a))$ and $k = \overline{\overline{\mathrm{rng}\, f}}$, then $k \leqslant 2 \cdot (\mathrm{Opt}(a)) - 1$. The theorem is a consequence of (38) and (18).

6. TIGHTNESS OF APPROXIMATION GUARANTEE OF ALGORITHM NEXTFIT

Now we state the propositions:

(40) Let us consider a natural number n, a real number ε, a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$, and a non empty finite sequence f of elements of $\mathbb{N}$. Suppose n is odd and $\mathrm{len}\, a = n$ and $\varepsilon = \frac{1}{n+1}$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds if i is odd, then $a(i) = 2 \cdot \varepsilon$ and if i is even, then $a(i) = 1 - \varepsilon$ and $f = \mathrm{OnlinePacking}(a, \mathrm{NextFit}(a))$. Then $n = \overline{\overline{\mathrm{rng}\, f}}$.
PROOF: $1 \leqslant n$. Set $h = \mathrm{OnlinePackingHistory}(a, \mathrm{NextFit}(a))$. Define $\mathcal{N}[\text{natural number}] \equiv$ if $\$_1$ is odd, then $\mathrm{SumBin}(a, h(\$_1), \{h(\$_1)(\$_1)\}) = 2 \cdot \varepsilon$ and if $\$_1$ is even, then $\mathrm{SumBin}(a, h(\$_1), \{h(\$_1)(\$_1)\}) = 1 - \varepsilon$ and $h(\$_1)(\$_1) = \$_1$ and $\mathrm{rng}\, h(\$_1) = \mathrm{Seg}\, \$_1$. $\mathcal{N}[1]$. For every element i of $\mathbb{N}$ such that $1 \leqslant i < \mathrm{len}\, a$ and $\mathcal{N}[i]$ holds $\mathcal{N}[i+1]$. For every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \mathrm{len}\, a$ holds $\mathcal{N}[i]$. $\square$

(41) Let us consider a natural number n, a real number ε, and a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$. Suppose n is odd and $\mathrm{len}\, a = n$ and $\varepsilon = \frac{1}{n+1}$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds if i is odd, then $a(i) = 2 \cdot \varepsilon$ and if i is even, then $a(i) = 1 - \varepsilon$. Then $\sum a = \frac{n+1}{2} + \frac{1}{n+1} - \frac{1}{2}$.
PROOF: $1 \leqslant n$. $n + 1 \operatorname{div} 2 = \frac{n+1}{2}$. Define $\mathcal{N}[\text{natural number}] \equiv$ if $\$_1$ is odd, then $\sum(a \restriction \$_1) = 2 \cdot \varepsilon \cdot (\$_1 + 1 \operatorname{div} 2) + (1 - \varepsilon) \cdot ((\$_1 + 1 \operatorname{div} 2) - 1)$ and

if $\$_1$ is even, then $\sum(a{\restriction}\$_1) = 2 \cdot \varepsilon \cdot (\$_1 \operatorname{div} 2) + (1-\varepsilon) \cdot (\$_1 \operatorname{div} 2)$. For every element i of $\mathbb{N}$ such that $1 \leqslant i < \operatorname{len} a$ and $\mathcal{N}[i]$ holds $\mathcal{N}[i+1]$. For every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \operatorname{len} a$ holds $\mathcal{N}[i]$. $\square$

(42) Let us consider a natural number n, a real number ε, a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$, and a non empty finite sequence f of elements of $\mathbb{N}$. Suppose n is odd and $\operatorname{len} a = n$ and $\varepsilon = \frac{1}{n+1}$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds if i is odd, then $a(i) = 2 \cdot \varepsilon$ and if i is even, then $a(i) = 1 - \varepsilon$ and $\operatorname{dom} f = \operatorname{dom} a$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds if i is odd, then $f(i) = 1$ and if i is even, then $f(i) = (i \operatorname{div} 2) + 1$. Let us consider a natural number j. If $j \in \operatorname{rng} f$, then $\operatorname{SumBin}(a, f, \{j\}) \leqslant 1$.

PROOF: $1 \leqslant n$. $n + 1 \operatorname{div} 2 = \frac{n+1}{2}$. Set $n_1 = n + 1 \operatorname{div} 2$. $1 + 1 \leqslant n + 1$. For every object y, $y \in \operatorname{Seg} n_1$ iff there exists an object x such that $x \in \operatorname{dom} f$ and $y = f(x)$. $\square$

(43) Let us consider a natural number n, a real number ε, and a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$. Suppose n is odd and $\operatorname{len} a = n$ and $\varepsilon = \frac{1}{n+1}$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds if i is odd, then $a(i) = 2 \cdot \varepsilon$ and if i is even, then $a(i) = 1 - \varepsilon$. Then $n = 2 \cdot (\operatorname{Opt}(a)) - 1$.

PROOF: $1 \leqslant n$. $n + 1 \operatorname{div} 2 = \frac{n+1}{2}$. There exists a non empty finite sequence g of elements of $\mathbb{N}$ such that $\operatorname{dom} g = \operatorname{dom} a$ and for every natural number j such that $j \in \operatorname{rng} g$ holds $\operatorname{SumBin}(a, g, \{j\}) \leqslant 1$ and $n + 1 \operatorname{div} 2 = \overline{\overline{\operatorname{rng} g}}$ and for every non empty finite sequence f of elements of $\mathbb{N}$ such that $\operatorname{dom} f = \operatorname{dom} a$ and for every natural number j such that $j \in \operatorname{rng} f$ holds $\operatorname{SumBin}(a, f, \{j\}) \leqslant 1$ holds $n + 1 \operatorname{div} 2 \leqslant \overline{\overline{\operatorname{rng} f}}$. $\square$

(44) Let us consider a natural number n. Suppose n is odd. Then there exists a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$ such that

(i) $\operatorname{len} a = n$, and

(ii) for every non empty finite sequence f of elements of $\mathbb{N}$ such that $f = \operatorname{OnlinePacking}(a, \operatorname{NextFit}(a))$ holds $n = \overline{\overline{\operatorname{rng} f}}$ and $n = 2 \cdot (\operatorname{Opt}(a)) - 1$.

PROOF: $1 \leqslant n$. Set $\varepsilon = \frac{1}{n+1}$. Define $\mathcal{P}[\text{natural number}, \text{object}] \equiv$ if $\$_1$ is odd, then $\$_2 = 2 \cdot \varepsilon$ and if $\$_1$ is even, then $\$_2 = 1 - \varepsilon$. For every natural number i such that $i \in \operatorname{Seg} n$ there exists an object x such that $\mathcal{P}[i, x]$. Consider a_0 being a finite sequence such that $\operatorname{dom} a_0 = \operatorname{Seg} n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, a_0(i)]$. For every natural number i such that $i \in \operatorname{dom} a_0$ holds $a_0(i) \in \mathbb{R}$. a_0 is positive by (1), [7,

(22)]. For every natural number i such that $1 \leqslant i \leqslant \operatorname{len} a_0$ holds $a_0(i) \leqslant 1$. $\square$

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