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Intuitionistic Propositional Calculus in the Extended Framework with Modal Operator. Part II

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Summary. This paper is a continuation of Inoué [5]. As already mentioned in the paper, a number of intuitionistic provable formulas are given with a Hilbert-style proof. For that, we make use of a family of intuitionistic deduction theorems, which are also presented in this paper by means of Mizar system [2], [1]. Our axiom system of intuitionistic propositional logic IPC is based on the propositional subsystem of H_1-**IQC** in Troelstra and van Dalen [6, p. 68]. We also owe Heyting [4] and van Dalen [7]. Our treatment of a set-theoretic intuitionistic deduction theorem is due to Agata Darmochwał's Mizar article "Calculus of Quantifiers. Deduction Theorem" [3].

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1. The Notion of Proof in Intuitionistic Setting

From now on i, j, n, k, l denote natural numbers, T, S, X, Y, Z denote subsets of MC-w.f.f., p, q, r, t, F, H, G denote elements of MC-w.f.f., and s, U, V denote MC-formulas.

Let p, q be elements of MC-w.f.f.. The functor $p \Leftrightarrow q$ yielding an element of MC-w.f.f. is defined by the term

(Def. 1) $(p \Rightarrow q) \wedge (q \Rightarrow p)$.

The functor Proof-Step-Kinds-IPC yielding a set is defined by the term

(Def. 2) $\{k : k \leqslant 10\}$.

Now we state the proposition:

(1) (i) $0 \in$ Proof-Step-Kinds-IPC and ... and

(ii) $10 \in$ Proof-Step-Kinds-IPC.

One can verify that Proof-Step-Kinds-IPC is non empty and Proof-Step-Kinds-IPC is finite.

From now on f, g denote finite sequences of elements of MC-w.f.f. $\times$ Proof-Step-Kinds-IPC. Now we state the proposition:

(2) Let us consider a natural number n. If $1 \leqslant n \leqslant \operatorname{len} f$, then $(f(n))_{\mathbf{2}} = 0$ or ... or $(f(n))_{\mathbf{2}} = 10$.

Let P_1 be a finite sequence of elements of MC-w.f.f. $\times$ Proof-Step-Kinds-IPC and n be a natural number. Let us consider X. We say that P_1 is a correct n-th step w.r.t. IPC (X) if and only if

(Def. 3) (i) $(P_1(n))_{\mathbf{1}} \in X$, **if** $(P_1(n))_{\mathbf{2}} = 0$,

(ii) there exists p and there exists q such that $(P_1(n))_{\mathbf{1}} = p \Rightarrow (q \Rightarrow p)$, **if** $(P_1(n))_{\mathbf{2}} = 1$,

(iii) there exists p and there exists q and there exists r such that $(P_1(n))_{\mathbf{1}} = p \Rightarrow (q \Rightarrow r) \Rightarrow (p \Rightarrow q \Rightarrow (p \Rightarrow r))$, **if** $(P_1(n))_{\mathbf{2}} = 2$,

(iv) there exists p and there exists q such that $(P_1(n))_{\mathbf{1}} = p \wedge q \Rightarrow p$, **if** $(P_1(n))_{\mathbf{2}} = 3$,

(v) there exists p and there exists q such that $(P_1(n))_{\mathbf{1}} = p \wedge q \Rightarrow q$, **if** $(P_1(n))_{\mathbf{2}} = 4$,

(vi) there exists p and there exists q such that $(P_1(n))_{\mathbf{1}} = p \Rightarrow (q \Rightarrow p \wedge q)$, **if** $(P_1(n))_{\mathbf{2}} = 5$,

(vii) there exists p and there exists q such that $(P_1(n))_{\mathbf{1}} = p \Rightarrow p \vee q$, **if** $(P_1(n))_{\mathbf{2}} = 6$,

(viii) there exists p and there exists q such that $(P_1(n))_{\mathbf{1}} = q \Rightarrow p \vee q$, **if** $(P_1(n))_{\mathbf{2}} = 7$,

(ix) there exists p and there exists q and there exists r such that $(P_1(n))_{\mathbf{1}} = p \Rightarrow r \Rightarrow (q \Rightarrow r \Rightarrow (p \vee q \Rightarrow r))$, **if** $(P_1(n))_{\mathbf{2}} = 8$,

(x) there exists p such that $(P_1(n))_{\mathbf{1}} = \mathrm{FALSUM} \Rightarrow p$, **if** $(P_1(n))_{\mathbf{2}} = 9$,

(xi) there exists i and there exists j and there exists p and there exists q such that $1 \leqslant i < n$ and $1 \leqslant j < i$ and $p = (P_1(j))_{\mathbf{1}}$ and $q = (P_1(n))_{\mathbf{1}}$ and $(P_1(i))_{\mathbf{1}} = p \Rightarrow q$, **if** $(P_1(n))_{\mathbf{2}} = 10$.

Let us consider f. We say that f is a proof w.r.t. IPC (X) if and only if

(Def. 4) $f \neq \emptyset$ and for every n such that $1 \leqslant n \leqslant \operatorname{len} f$ holds f is a correct n-th step w.r.t. IPC (X).

Now we state the propositions:

(3) If f is a proof w.r.t. IPC (X), then $\operatorname{rng} f \neq \emptyset$.

(4) If f is a proof w.r.t. IPC (X), then $1 \leqslant \operatorname{len} f$.

(5) If f is a proof w.r.t. IPC (X), then $(f(1))_{\mathbf{2}} = 0$ or ... or $(f(1))_{\mathbf{2}} = 10$. The theorem is a consequence of (4) and (2).

(6) If $1 \leqslant n \leqslant \operatorname{len} f$, then f is a correct n-th step w.r.t. IPC (X) iff $f \frown g$ is a correct n-th step w.r.t. IPC (X).

Proof: If f is a correct n-th step w.r.t. IPC (X), then $f \frown g$ is a correct n-th step w.r.t. IPC (X). $(f(n))_{\mathbf{2}} = 0$ or ... or $(f(n))_{\mathbf{2}} = 10$. □

(7) If $1 \leqslant n \leqslant \operatorname{len} g$ and g is a correct n-th step w.r.t. IPC (X), then $f \frown g$ is a correct $n + \operatorname{len} f$-th step w.r.t. IPC (X). The theorem is a consequence of (2).

(8) If f is a proof w.r.t. IPC (X) and g is a proof w.r.t. IPC (X), then $f \frown g$ is a proof w.r.t. IPC (X). The theorem is a consequence of (6) and (7).

(9) If f is a proof w.r.t. IPC (X) and $X \subseteq Y$, then f is a proof w.r.t. IPC (Y). The theorem is a consequence of (2).

(10) If f is a proof w.r.t. IPC (X) and $1 \leqslant l \leqslant \operatorname{len} f$, then $(f(l))_{\mathbf{1}} \in \operatorname{CnIPC}(X)$.

Proof: For every n such that $1 \leqslant n \leqslant \operatorname{len} f$ holds $(f(n))_{\mathbf{1}} \in \operatorname{CnIPC}(X)$. □

Let us consider f. Assume $f \neq \emptyset$. The functor Effect-IPC(f) yielding an element of MC-w.f.f. is defined by the term

(Def. 5) $(f(\operatorname{len} f))_{\mathbf{1}}$.

Now we state the proposition:

(11) If f is a proof w.r.t. IPC (X), then Effect-IPC$(f) \in \operatorname{CnIPC}(X)$. The theorem is a consequence of (4) and (10).

2. A Consequence as a Set of All Intuitionistic Provable Formulas

Now we state the proposition:

(12) $X \subseteq \{F :$ there exists f such that f is a proof w.r.t. IPC (X) and Effect-IPC$(f) = F\}$. The theorem is a consequence of (1).

Let us consider X. Now we state the propositions:

(13) Suppose $Y = \{p :$ there exists f such that f is a proof w.r.t. IPC (X) and Effect-IPC$(f) = p\}$. Then Y is IPC theory.

(14) $\{p :$ there exists f such that f is a proof w.r.t. IPC (X) and Effect-IPC $(f) = p\} =$ CnIPC(X). The theorem is a consequence of (12) and (13).

(15) $p \in$ CnIPC(X) if and only if there exists f such that f is a proof w.r.t. IPC (X) and Effect-IPC$(f) = p$. The theorem is a consequence of (14).

(16) If $p \in$ CnIPC(X), then there exists Y such that $Y \subseteq X$ and Y is finite and $p \in$ CnIPC(Y).

Proof: Consider f such that f is a proof w.r.t. IPC (X) and Effect-IPC(f) $= p$. Consider A being a set such that A is finite and $A \subseteq$ MC-w.f.f. and $\operatorname{rng} f \subseteq A \times$ Proof-Step-Kinds-IPC. If $1 \leqslant n \leqslant \operatorname{len} f$, then f is a correct n-th step w.r.t. IPC (Y). $\square$

3. The Intuitionistic Provable Relation

Let us consider X and s. We say that $X \vdash_{IPC} (s)$ if and only if

(Def. 6) $s \in$ CnIPC(X).

We say that $\vdash_{IPC} s$ if and only if

(Def. 7) $\emptyset_{\text{MC-w.f.f.}} \vdash_{IPC} s$.

Now we state the propositions:

(17) $X \vdash_{IPC} (p \Rightarrow (q \Rightarrow p))$.

(18) $X \vdash_{IPC} (p \Rightarrow (q \Rightarrow r) \Rightarrow (p \Rightarrow q \Rightarrow (p \Rightarrow r)))$.

(19) $X \vdash_{IPC} (p \wedge q \Rightarrow p)$.

(20) $X \vdash_{IPC} (p \wedge q \Rightarrow q)$.

(21) $X \vdash_{IPC} (p \Rightarrow (q \Rightarrow p \wedge q))$.

(22) $X \vdash_{IPC} (p \Rightarrow p \vee q)$.

(23) $X \vdash_{IPC} (q \Rightarrow p \vee q)$.

(24) $X \vdash_{IPC} (p \Rightarrow r \Rightarrow (q \Rightarrow r \Rightarrow (p \vee q \Rightarrow r)))$.

(25) $X \vdash_{IPC} (\text{FALSUM} \Rightarrow p)$.

(26) If $X \vdash_{IPC} p$ and $X \vdash_{IPC} (p \Rightarrow q)$, then $X \vdash_{IPC} (q)$.

(27) $\vdash_{IPC} p \Rightarrow (q \Rightarrow p)$.

(28) $\vdash_{IPC} p \Rightarrow (q \Rightarrow r) \Rightarrow (p \Rightarrow q \Rightarrow (p \Rightarrow r))$.

(29) $\vdash_{IPC} p \wedge q \Rightarrow p$.

(30) $\vdash_{IPC} p \wedge q \Rightarrow q$.

(31) $\vdash_{IPC} p \Rightarrow (q \Rightarrow p \wedge q)$.

(32) $\vdash_{IPC} p \Rightarrow p \vee q$.

(33) $\vdash_{IPC} q \Rightarrow p \vee q$.

(34) $\vdash_{IPC} p \Rightarrow r \Rightarrow (q \Rightarrow r \Rightarrow (p \vee q \Rightarrow r))$.

(35) $\vdash_{IPC} \mathrm{FALSUM} \Rightarrow p$.

(36) If $\vdash_{IPC} p$ and $\vdash_{IPC} p \Rightarrow q$, then $\vdash_{IPC} q$.

Let us consider s. We say that s is IPC-valid if and only if

(Def. 8) $\emptyset_{\text{MC-w.f.f.}} \vdash_{IPC} (s)$.

One can verify that s is IPC-valid if and only if the condition (Def. 9) is satisfied.

(Def. 9) $s \in$ IPC-Taut.

Now we state the propositions:

(37) If p is IPC-valid, then $X \vdash_{IPC} (p)$.

(38) $p \Rightarrow (q \Rightarrow p)$ is IPC-valid.

(39) $p \Rightarrow (q \Rightarrow r) \Rightarrow (p \Rightarrow q \Rightarrow (p \Rightarrow r))$ is IPC-valid.

(40) $p \wedge q \Rightarrow p$ is IPC-valid.

(41) $p \wedge q \Rightarrow q$ is IPC-valid.

(42) $p \Rightarrow (q \Rightarrow p \wedge q)$ is IPC-valid.

(43) $p \Rightarrow p \vee q$ is IPC-valid.

(44) $q \Rightarrow p \vee q$ is IPC-valid.

(45) $p \Rightarrow r \Rightarrow (q \Rightarrow r \Rightarrow (p \vee q \Rightarrow r))$ is IPC-valid.

(46) FALSUM $\Rightarrow p$ is IPC-valid.

(47) If p is IPC-valid and $p \Rightarrow q$ is IPC-valid, then q is IPC-valid.

In the sequel X, T denote subsets of MC-w.f.f., F, G, H, p, q, r, t denote elements of MC-w.f.f., s, h denote MC-formulas, f denotes a finite sequence of elements of MC-w.f.f. $\times$ Proof-Step-Kinds-IPC, and i, j denote elements of $\mathbb{N}$.

4. The First Deduction Theorem for IPC

Now we state the propositions:

(48) $X \vdash_{IPC} (p \Rightarrow p)$. The theorem is a consequence of (26).

(49) $X \vdash_{IPC} (\mathrm{IVERUM})$.

(50) If $X \vdash_{IPC} (p)$, then $X \vdash_{IPC} (q \Rightarrow p)$.

(51) If p is IPC-valid, then $X \vdash_{IPC} (p)$.

(52) If $X \cup \{F\} \vdash_{IPC} (G)$, then $X \vdash_{IPC} (F \Rightarrow G)$.

PROOF: Consider f such that f is a proof w.r.t. IPC $(X \cup \{F\})$ and Effect-IPC$(f) = G$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $1 \leqslant \$_1 \leqslant \operatorname{len} f$, then for every H such that $H = (f(\$_1))_{\mathbf{1}}$ holds $X \vdash_{IPC} (F \Rightarrow H)$. For every natural number n such that for every natural number k such that $k < n$ holds $\mathcal{P}[k]$ holds $\mathcal{P}[n]$. For every natural number n, $\mathcal{P}[n]$. $1 \leqslant \operatorname{len} f$. $\square$

5. A Family of Deduction Theorems for IPC

From now on F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, F_{10}, G denote MC-formulas and x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x denote elements of MC-w.f.f..

Let x_1, x_2, x_3 be elements of MC-w.f.f.. Let us observe that the functor $\{x_1, x_2, x_3\}$ yields a subset of MC-w.f.f.. Let x_1, x_2, x_3, x_4 be elements of MC-w.f.f.. One can check that the functor $\{x_1, x_2, x_3, x_4\}$ yields a subset of MC-w.f.f.. Let x_1, x_2, x_3, x_4, x_5 be elements of MC-w.f.f.. One can verify that the functor $\{x_1, x_2, x_3, x_4, x_5\}$ yields a subset of MC-w.f.f.. Let x_1, x_2, x_3, x_4, x_5, x_6 be elements of MC-w.f.f.. One can verify that the functor $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ yields a subset of MC-w.f.f.. Let x_1, x_2, x_3, x_4, x_5, x_6, x_7 be elements of MC-w.f.f..

One can check that the functor $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ yields a subset of MC-w.f.f.. Let x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 be elements of MC-w.f.f.. Let us note that the functor $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ yields a subset of MC-w.f.f.. Let x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 be elements of MC-w.f.f.. One can verify that the functor $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ yields a subset of MC-w.f.f.. Let x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} be elements of MC-w.f.f.. Observe that the functor $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$ yields a subset of MC-w.f.f.. Now we state the propositions:

(53) If $\{F\} \vdash_{IPC} (G)$, then $\vdash_{IPC} F \Rightarrow G$. The theorem is a consequence of (52).

(54) If $\{F_1, F_2\} \vdash_{IPC} (G)$, then $\{F_2\} \vdash_{IPC} (F_1 \Rightarrow G)$. The theorem is a consequence of (52).

(55) If $\{F_1, F_2, F_3\} \vdash_{IPC} (G)$, then $\{F_2, F_3\} \vdash_{IPC} (F_1 \Rightarrow G)$. The theorem is a consequence of (52).

(56) If $\{F_1, F_2, F_3, F_4\} \vdash_{IPC} (G)$, then $\{F_2, F_3, F_4\} \vdash_{IPC} (F_1 \Rightarrow G)$. The theorem is a consequence of (52).

(57) If $\{F_1, F_2, F_3, F_4, F_5\} \vdash_{IPC} (G)$, then $\{F_2, F_3, F_4, F_5\} \vdash_{IPC} (F_1 \Rightarrow G)$. The theorem is a consequence of (52).

(58) If $\{F_1, F_2, F_3, F_4, F_5, F_6\} \vdash_{IPC} (G)$, then $\{F_2, F_3, F_4, F_5, F_6\} \vdash_{IPC} (F_1 \Rightarrow G)$. The theorem is a consequence of (52).

(59) Suppose $\{F_1, F_2, F_3, F_4, F_5, F_6, F_7\} \vdash_{IPC} (G)$. Then $\{F_2, F_3, F_4, F_5, F_6, F_7\} \vdash_{IPC} (F_1 \Rightarrow G)$. The theorem is a consequence of (52).

(60) Suppose $\{F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8\} \vdash_{IPC} (G)$. Then $\{F_2, F_3, F_4, F_5, F_6, F_7, F_8\} \vdash_{IPC} (F_1 \Rightarrow G)$. The theorem is a consequence of (52).

(61) Suppose $\{F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9\} \vdash_{IPC} (G)$. Then $\{F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9\} \vdash_{IPC} (F_1 \Rightarrow G)$. The theorem is a consequence of (52).

From now on x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} denote objects.

Now we state the propositions:

(62) $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\} = \{x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\} \cup \{x_1\}$.

(63) Suppose $\{F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, F_{10}\} \vdash_{IPC} (G)$. Then $\{F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, F_{10}\} \vdash_{IPC} (F_1 \Rightarrow G)$. The theorem is a consequence of (62) and (52).

6. Intuitionistic Provable Formulas and Theorems

Now we state the propositions:

(64) $\{p\} \vdash_{IPC} (p)$.

(65) If $X \vdash_{IPC} (p)$ and $X \subseteq Y$, then $Y \vdash_{IPC} (p)$. The theorem is a consequence of (15) and (9).

(66) If $p \in X$, then $X \vdash_{IPC} (p)$. The theorem is a consequence of (64) and (65).

(67) If $p \in X$, then $p \in \mathrm{CnIPC}(X)$. The theorem is a consequence of (66).

(68) If $p \in$ IPC-Taut, then $\vdash_{IPC} p$.

(69) If $\vdash_{IPC} p$, then $p \in$ IPC-Taut.

(70) $p \in$ IPC-Taut if and only if $\vdash_{IPC} p$.

(71) $\vdash_{IPC} p \Rightarrow (p \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM})$. The theorem is a consequence of (66), (26), (54), and (53).

(72) $\{p \wedge q\} \vdash_{IPC} (p)$. The theorem is a consequence of (19), (64), and (26).

(73) $\{p \wedge q\} \vdash_{IPC} (q)$. The theorem is a consequence of (20), (64), and (26).

(74) $\vdash_{IPC} (p \Rightarrow q) \wedge (p \Rightarrow (q \Rightarrow \mathrm{FALSUM})) \Rightarrow (p \Rightarrow \mathrm{FALSUM})$. The theorem is a consequence of (66), (19), (26), (20), (54), and (53).

(75) $\vdash_{IPC} p \Rightarrow \mathrm{FALSUM} \Rightarrow (p \Rightarrow q)$. The theorem is a consequence of (68).

(76) $\vdash_{IPC} (p \Rightarrow r) \wedge (q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)$. The theorem is a consequence of (72), (73), (24), (26), and (53).

(77) $\vdash_{IPC} p \wedge (p \Rightarrow q) \Rightarrow q$. The theorem is a consequence of (72), (73), (26), and (53).

(78) $\vdash_{IPC} p \Rightarrow (p \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM})$. The theorem is a consequence of (69), (71), and (68).

(79) $\vdash_{IPC} (p \Rightarrow \mathrm{FALSUM}) \vee q \Rightarrow (p \Rightarrow q)$. The theorem is a consequence of (69), (75), (76), and (68).

(80) $\vdash_{IPC} p \Rightarrow q \Rightarrow (q \Rightarrow \text{FALSUM} \Rightarrow (p \Rightarrow \text{FALSUM}))$.

(81) $\vdash_{IPC} (p \Rightarrow \text{FALSUM}) \vee (q \Rightarrow \text{FALSUM}) \Rightarrow (p \wedge q \Rightarrow \text{FALSUM})$. The theorem is a consequence of (69), (76), (80), and (68).

(82) Let us consider MC-formulas p, q. If $\vdash_{IPC} p$ and $\vdash_{IPC} q$, then $\vdash_{IPC} p \wedge q$. The theorem is a consequence of (31) and (36).

(83) If $\vdash_{IPC} p \Rightarrow q$ and $\vdash_{IPC} q \Rightarrow p$, then $\vdash_{IPC} p \Leftrightarrow q$.

(84) $\vdash_{IPC} p \Rightarrow p$. The theorem is a consequence of (27), (28), and (26).

(85) $\vdash_{IPC} p \Leftrightarrow p$. The theorem is a consequence of (84) and (82).

(86) $\vdash_{IPC} p \wedge q \Rightarrow \text{FALSUM} \Rightarrow (p \Rightarrow (q \Rightarrow \text{FALSUM}))$. The theorem is a consequence of (66), (21), (26), (55), (54), and (53).

(87) $\vdash_{IPC} p \Rightarrow (q \Rightarrow \text{FALSUM}) \Rightarrow (p \wedge q \Rightarrow \text{FALSUM})$. The theorem is a consequence of (66), (19), (26), (20), (54), and (53).

(88) $\vdash_{IPC} (p \wedge q \Rightarrow \text{FALSUM}) \Leftrightarrow (p \Rightarrow (q \Rightarrow \text{FALSUM}))$. The theorem is a consequence of (86), (87), and (83).

(89) $\vdash_{IPC} p \wedge q \Rightarrow \text{FALSUM} \Rightarrow (q \Rightarrow (p \Rightarrow \text{FALSUM}))$. The theorem is a consequence of (66), (21), (26), (55), (54), and (53).

(90) $\vdash_{IPC} q \Rightarrow (p \Rightarrow \text{FALSUM}) \Rightarrow (p \wedge q \Rightarrow \text{FALSUM})$. The theorem is a consequence of (66), (19), (26), (20), (54), and (53).

(91) $\vdash_{IPC} (q \Rightarrow (p \Rightarrow \text{FALSUM})) \Leftrightarrow (p \wedge q \Rightarrow \text{FALSUM})$. The theorem is a consequence of (89), (90), and (83).

(92) $\vdash_{IPC} p \Rightarrow (q \Rightarrow (p \wedge q \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM}))$. The theorem is a consequence of (66), (21), (65), (26), (55), (54), and (53).

(93) $\vdash_{IPC} q \Rightarrow (p \Rightarrow (p \wedge q \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM}))$. The theorem is a consequence of (66), (21), (65), (26), (55), (54), and (53).

(94) $\vdash_{IPC} p \Rightarrow (p \wedge q \Rightarrow \text{FALSUM} \Rightarrow (q \Rightarrow \text{FALSUM}))$. The theorem is a consequence of (66), (21), (65), (26), (55), (54), and (53).

(95) $\vdash_{IPC} q \Rightarrow (p \wedge q \Rightarrow \text{FALSUM} \Rightarrow (p \Rightarrow \text{FALSUM}))$. The theorem is a consequence of (66), (21), (65), (26), (55), (54), and (53).

(96) $\vdash_{IPC} p \vee q \Rightarrow \text{FALSUM} \Rightarrow (p \Rightarrow \text{FALSUM}) \wedge (q \Rightarrow \text{FALSUM})$. The theorem is a consequence of (68).

(97) $\vdash_{IPC} (p \Rightarrow \text{FALSUM}) \wedge (q \Rightarrow \text{FALSUM}) \Rightarrow (p \vee q \Rightarrow \text{FALSUM})$.

(98) $\vdash_{IPC} (p \vee q \Rightarrow \text{FALSUM}) \Leftrightarrow (p \Rightarrow \text{FALSUM}) \wedge (q \Rightarrow \text{FALSUM})$. The theorem is a consequence of (96), (97), and (83).

(99) $\vdash_{IPC} p \wedge (p \Rightarrow \text{FALSUM}) \Rightarrow \text{FALSUM}$.

(100) $\vdash_{IPC} \text{FALSUM} \Leftrightarrow p \wedge (p \Rightarrow \text{FALSUM})$. The theorem is a consequence of (35), (99), and (83).

(101) $\vdash_{IPC} p \Rightarrow \text{FALSUM} \Rightarrow (p \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM})$.

(102) $\vdash_{IPC} p \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow (p \Rightarrow \mathrm{FALSUM})$. The theorem is a consequence of (69), (71), and (68).

(103) $\vdash_{IPC} (p \Rightarrow \mathrm{FALSUM}) \Leftrightarrow (p \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM})$. The theorem is a consequence of (101), (102), and (83).

(104) $\vdash_{IPC} p \Rightarrow \mathrm{FALSUM} \Rightarrow q \Rightarrow (p \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow q)$. The theorem is a consequence of (66), (102), (65), (26), (54), and (53).

(105) $\vdash_{IPC} p \Rightarrow q \Rightarrow (p \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow (q \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM}))$. The theorem is a consequence of (69), (80), and (68).

(106) $\vdash_{IPC} p \wedge (q \Rightarrow \mathrm{FALSUM}) \Rightarrow (p \Rightarrow q \Rightarrow \mathrm{FALSUM})$. The theorem is a consequence of (66), (19), (26), (20), (54), and (53).

(107) $\vdash_{IPC} p \Rightarrow q \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow (p \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow (q \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM}))$. The theorem is a consequence of (66), (21), (26), (106), (80), (36), (65), (56), (55), (54), and (53).

(108) $\vdash_{IPC} p \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow (q \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM}) \Rightarrow (p \Rightarrow q \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM})$. The theorem is a consequence of (66), (79), (80), (36), (65), (26), (96), (19), (20), (54), and (53).

(109) $\vdash_{IPC} (p \Rightarrow q \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM}) \Leftrightarrow (p \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow (q \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM}))$. The theorem is a consequence of (107), (108), and (83).

(110) $\vdash_{IPC} p \wedge q \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow (p \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM}) \wedge (q \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM})$. The theorem is a consequence of (29), (30), (80), (36), and (68).

(111) $\vdash_{IPC} (p \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM}) \wedge (q \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM}) \Rightarrow (p \wedge q \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM})$. The theorem is a consequence of (66), (21), (26), (56), (19), (55), (20), (54), and (53).

(112) $\vdash_{IPC} (p \wedge q \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM}) \Leftrightarrow (p \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM}) \wedge (q \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM})$. The theorem is a consequence of (110), (111), and (83).

(113) $\vdash_{IPC} p \Rightarrow q \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow (p \Rightarrow (q \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM}))$. The theorem is a consequence of (66), (107), (65), (26), (71), (54), and (53).

(114) If $\vdash_{IPC} r$ and $\{r\} \vdash_{IPC} (q)$, then $\vdash_{IPC} q$. The theorem is a consequence of (53) and (36).

(115) If $X \vdash_{IPC} (r)$ and $X \cup \{r\} \vdash_{IPC} (q)$, then $X \vdash_{IPC} (q)$. The theorem is a consequence of (52) and (26).

(116) If $X \vdash_{IPC} (r)$ and $Y \cup \{r\} \vdash_{IPC} (q)$, then $X \cup Y \vdash_{IPC} (q)$. The theorem is a consequence of (52), (65), and (26).

(117) If $\vdash_{IPC} p$ and $\{r\} \vdash_{IPC} (q)$, then $\{p \Rightarrow r\} \vdash_{IPC} (q)$. The theorem is a consequence of (65), (64), (26), and (115).

(118) If $X \vdash_{IPC} (p)$ and $X \cup \{r\} \vdash_{IPC} (q)$, then $X \cup \{p \Rightarrow r\} \vdash_{IPC} (q)$. The theorem is a consequence of (65), (66), (26), and (115).

(119) $\{q\} \vdash_{IPC} (q \vee r)$. The theorem is a consequence of (64), (22), and (26).

(120) $\{r\} \vdash_{IPC} (q \vee r)$. The theorem is a consequence of (64), (23), and (26).

(121) If $\{p\} \vdash_{IPC} (r)$ and $\{q\} \vdash_{IPC} (r)$, then $\{p \vee q\} \vdash_{IPC} (r)$. The theorem is a consequence of (34), (53), (36), (65), (26), and (64).

(122) If $X \cup \{p\} \vdash_{IPC} (r)$ and $X \cup \{q\} \vdash_{IPC} (r)$, then $X \cup \{p \vee q\} \vdash_{IPC} (r)$. The theorem is a consequence of (52), (24), (26), (64), and (65).

(123) If $X \cup \{p\} \vdash_{IPC} (r)$ and $Y \cup \{q\} \vdash_{IPC} (r)$, then $(X \cup Y) \cup \{p \vee q\} \vdash_{IPC} (r)$. The theorem is a consequence of (52), (65), (24), (26), and (64).

(124) $\vdash_{IPC} p \Rightarrow q \vee (p \Rightarrow r) \Rightarrow (p \Rightarrow q \vee r)$. The theorem is a consequence of (120), (65), (64), (118), (119), (122), (52), and (53).

(125) $\vdash_{IPC} p \Rightarrow (p \Rightarrow \mathrm{FALSUM} \Rightarrow q)$. The theorem is a consequence of (66), (26), (25), (54), and (53).

(126) $\vdash_{IPC} p \Rightarrow q \Rightarrow (q \wedge r \Rightarrow \mathrm{FALSUM} \Rightarrow (p \wedge r \Rightarrow \mathrm{FALSUM}))$. The theorem is a consequence of (66), (20), (26), (19), (21), (55), (54), and (53).

(127) $\vdash_{IPC} p \Rightarrow q \Rightarrow (q \vee r \Rightarrow \mathrm{FALSUM} \Rightarrow (p \vee r \Rightarrow \mathrm{FALSUM}))$. The theorem is a consequence of (66), (68), (65), (26), (55), (54), and (53).

Let p be an element of MC-w.f.f.. Note that the functor neg(p) yields an element of MC-w.f.f. and is defined by the term

(Def. 10) $p \Rightarrow \mathrm{FALSUM}$.

The functor $\mathrm{neg}^2(p)$ yielding an element of MC-w.f.f. is defined by the term

(Def. 11) $p \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM}$.

The functor $\mathrm{neg}^3(p)$ yielding an element of MC-w.f.f. is defined by the term

(Def. 12) $p \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM}$.

The functor $\mathrm{neg}^4(p)$ yielding an element of MC-w.f.f. is defined by the term

(Def. 13) $p \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM}$.

The functor $\mathrm{neg}^5(p)$ yielding an element of MC-w.f.f. is defined by the term

(Def. 14) $p \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM}$.

Now we state the propositions:

(128) $\vdash_{IPC} p \Rightarrow \mathrm{neg}(\mathrm{neg}(p))$.

(129) $\vdash_{IPC} p \Rightarrow \mathrm{neg}^2(p)$.

(130) $\vdash_{IPC} (p \Rightarrow q) \wedge (p \Rightarrow \mathrm{neg}(q)) \Rightarrow \mathrm{neg}(p)$.

(131) $\vdash_{IPC} \mathrm{neg}(p) \Rightarrow (p \Rightarrow q)$.

(132) $\vdash_{IPC} p \Rightarrow \mathrm{neg}(\mathrm{neg}(\mathrm{neg}(\mathrm{neg}(p))))$.

(133) $\vdash_{IPC} \mathrm{neg}(p) \vee q \Rightarrow (p \Rightarrow q)$.

(134) $\vdash_{IPC} p \Rightarrow q \Rightarrow (\mathrm{neg}(q) \Rightarrow \mathrm{neg}(p))$.

(135) $\vdash_{IPC} \mathrm{neg}(p) \vee \mathrm{neg}(q) \Rightarrow \mathrm{neg}(p \wedge q)$.

(136) $\vdash_{IPC} \mathrm{neg}(p \wedge q) \Rightarrow (p \Rightarrow \mathrm{neg}(q))$.

(137) $\vdash_{IPC} p \Rightarrow \mathrm{neg}(q) \Rightarrow \mathrm{neg}(p \wedge q)$.

(138) $\vdash_{IPC} \mathrm{neg}(p \wedge q) \Leftrightarrow (p \Rightarrow \mathrm{neg}(q))$.

(139) $\vdash_{IPC} \mathrm{neg}(p \wedge q) \Rightarrow (q \Rightarrow \mathrm{neg}(p))$.

(140) $\vdash_{IPC} q \Rightarrow \mathrm{neg}(p) \Rightarrow \mathrm{neg}(p \wedge q)$.

(141) $\vdash_{IPC} (q \Rightarrow \mathrm{neg}(p)) \Leftrightarrow \mathrm{neg}(p \wedge q)$.

(142) $\vdash_{IPC} p \Rightarrow (q \Rightarrow \mathrm{neg}(\mathrm{neg}(p \wedge q)))$.

(143) $\vdash_{IPC} q \Rightarrow (p \Rightarrow \mathrm{neg}(\mathrm{neg}(p \wedge q)))$.

(144) $\vdash_{IPC} p \Rightarrow (\mathrm{neg}(p \wedge q) \Rightarrow \mathrm{neg}(q))$.

(145) $\vdash_{IPC} q \Rightarrow (\mathrm{neg}(p \wedge q) \Rightarrow \mathrm{neg}(p))$.

(146) $\vdash_{IPC} \mathrm{neg}(p \vee q) \Rightarrow \mathrm{neg}(p) \wedge \mathrm{neg}(q)$.

(147) $\vdash_{IPC} \mathrm{neg}(p) \wedge \mathrm{neg}(q) \Rightarrow \mathrm{neg}(p \vee q)$.

(148) $\vdash_{IPC} \mathrm{neg}(p \vee q) \Leftrightarrow \mathrm{neg}(p) \wedge \mathrm{neg}(q)$.

(149) $\vdash_{IPC} p \wedge \mathrm{neg}(p) \Rightarrow \mathrm{FALSUM}$.

(150) $\vdash_{IPC} \mathrm{FALSUM} \Leftrightarrow p \wedge \mathrm{neg}(p)$.

(151) $\vdash_{IPC} \mathrm{neg}(p) \Rightarrow \mathrm{neg}(\mathrm{neg}(\mathrm{neg}(p)))$.

(152) $\vdash_{IPC} \mathrm{neg}(\mathrm{neg}(\mathrm{neg}(p))) \Rightarrow \mathrm{neg}(p)$.

(153) $\vdash_{IPC} \mathrm{neg}(p) \Leftrightarrow \mathrm{neg}(\mathrm{neg}(\mathrm{neg}(p)))$.

(154) $\vdash_{IPC} \mathrm{neg}(p) \Rightarrow q \Rightarrow (\mathrm{neg}(\mathrm{neg}(\mathrm{neg}(p))) \Rightarrow q)$.

(155) $\vdash_{IPC} p \Rightarrow q \Rightarrow (\mathrm{neg}(\mathrm{neg}(p)) \Rightarrow \mathrm{neg}(\mathrm{neg}(q)))$.

(156) $\vdash_{IPC} p \wedge \mathrm{neg}(q) \Rightarrow \mathrm{neg}(p \Rightarrow q)$.

(157) $\vdash_{IPC} \mathrm{neg}(\mathrm{neg}(p \Rightarrow q)) \Rightarrow (\mathrm{neg}(\mathrm{neg}(p)) \Rightarrow \mathrm{neg}(\mathrm{neg}(q)))$.

(158) $\vdash_{IPC} \mathrm{neg}(\mathrm{neg}(p)) \Rightarrow \mathrm{neg}(\mathrm{neg}(q)) \Rightarrow \mathrm{neg}(\mathrm{neg}(p \Rightarrow q))$.

(159) $\vdash_{IPC} \mathrm{neg}(\mathrm{neg}(p \Rightarrow q)) \Leftrightarrow (\mathrm{neg}(\mathrm{neg}(p)) \Rightarrow \mathrm{neg}(\mathrm{neg}(q)))$.

(160) $\vdash_{IPC} \mathrm{neg}(\mathrm{neg}(p \wedge q)) \Rightarrow \mathrm{neg}(\mathrm{neg}(p)) \wedge \mathrm{neg}(\mathrm{neg}(q))$.

(161) $\vdash_{IPC} \mathrm{neg}(\mathrm{neg}(p)) \wedge \mathrm{neg}(\mathrm{neg}(q)) \Rightarrow \mathrm{neg}(\mathrm{neg}(p \wedge q))$.

(162) $\vdash_{IPC} \mathrm{neg}(\mathrm{neg}(p \wedge q)) \Leftrightarrow \mathrm{neg}(\mathrm{neg}(p)) \wedge \mathrm{neg}(\mathrm{neg}(q))$.

(163) $\vdash_{IPC} \mathrm{neg}(\mathrm{neg}(p \Rightarrow q)) \Rightarrow (p \Rightarrow \mathrm{neg}(\mathrm{neg}(q)))$.

(164) $\vdash_{IPC} p \Rightarrow (\mathrm{neg}(p) \Rightarrow q)$.

(165) $\vdash_{IPC} p \Rightarrow q \Rightarrow (\mathrm{neg}(q \wedge r) \Rightarrow \mathrm{neg}(p \wedge r))$.

(166) $\vdash_{IPC} p \Rightarrow q \Rightarrow (\mathrm{neg}(q \vee r) \Rightarrow \mathrm{neg}(p \vee r))$.

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Compactness of Neural Networks[1]

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Summary. In this article, Feed-forward Neural Network is formalized in the Mizar system [1], [2]. First, the multilayer perceptron [6], [7], [8] is formalized using functional sequences. Next, we show that a set of functions generated by these neural networks satisfies equicontinuousness and equiboundedness property [10], [5]. At last, we formalized the compactness of the function set of these neural networks by using the Ascoli-Arzela's theorem according to [4] and [3].

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Keywords: neural network; compactness; Ascoli-Arzela's theorem; equicontinuousness of continuous functions; equiboundedness of continuous functions

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1. Preliminaries

From now on R_1, R_2 denote real linear spaces.

Now we state the propositions:

(1) Suppose the RLS structure of R_1 = the RLS structure of R_2. Then the carrier of R_1 = the carrier of R_2.

(2) Suppose the RLS structure of R_1 = the RLS structure of R_2. Then $0_{R_1} = 0_{R_2}$.

(3) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider elements p, q of R_1, and elements f, g of R_2. If $p = f$ and $q = g$, then $p + q = f + g$.

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(4) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider a real number r, an element q of R_1, and an element g of R_2. If $q = g$, then $r \cdot q = r \cdot g$.

(5) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider an element q of R_1, and an element g of R_2. If $q = g$, then $-q = -g$.

(6) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider elements p, q of R_1, and elements f, g of R_2. If $p = f$ and $q = g$, then $p - q = f - g$.

(7) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider a set X, and a natural number n. Then X is a linear combination of R_2 if and only if X is a linear combination of R_1.

(8) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider a linear combination L_5 of R_1, and a linear combination L_3 of R_2. Suppose $L_3 = L_5$. Then the support of L_3 = the support of L_5.

Let us consider a set F. Now we state the propositions:

(9) Suppose the RLS structure of R_1 = the RLS structure of R_2. Then F is a subset of R_1 if and only if F is a subset of R_2.

(10) Suppose the RLS structure of R_1 = the RLS structure of R_2. Then F is a finite sequence of elements of R_1 if and only if F is a finite sequence of elements of R_2.

(11) Suppose the RLS structure of R_1 = the RLS structure of R_2. Then F is a function from R_1 into $\mathbb{R}$ if and only if F is a function from R_2 into $\mathbb{R}$.

(12) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider a finite sequence F_1 of elements of R_1, a function f_1 from R_1 into $\mathbb{R}$, a finite sequence F_3 of elements of R_2, and a function f_2 from R_2 into $\mathbb{R}$. If $f_1 = f_2$ and $F_1 = F_3$, then $f_1 \cdot F_1 = f_2 \cdot F_3$.

(13) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider a finite sequence F_2 of elements of R_1, and a finite sequence F_1 of elements of R_2. If $F_2 = F_1$, then $\sum F_2 = \sum F_1$.

Proof: Set $T = R_1$. Set $V = R_2$. Consider f being a sequence of the carrier of T such that $\sum F = f(\operatorname{len} F)$ and $f(0) = 0_T$ and for every natural number j and for every element v of T such that $j < \operatorname{len} F$ and $v = F(j+1)$ holds $f(j+1) = f(j)+v$. Consider f_2 being a sequence of the carrier of V such that $\sum F_3 = f_2(\operatorname{len} F_3)$ and $f_2(0) = 0_V$ and for every natural number j and for every element v of V such that $j < \operatorname{len} F_3$ and $v = F_3(j + 1)$ holds $f_2(j+1) = f_2(j)+v$. Define $\mathcal{S}$[natural number] $\equiv$ if $\$_1 \leqslant \operatorname{len} F$, then $f(\$_1) = f_2(\$_1)$. For every natural number i such that $\mathcal{S}[i]$ holds $\mathcal{S}[i + 1]$. For every natural number n, $\mathcal{S}[n]$. $\square$

(14) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider a linear combination L_3 of R_2, and a linear combination L_4 of R_1. If $L_3 = L_4$, then $\sum L_3 = \sum L_4$. The theorem is a consequence of (12) and (13).

(15) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider a subset A_1 of R_2, and a subset A_2 of R_1. Suppose $A_1 = A_2$. Let us consider an object X. Then X is a linear combination of A_1 if and only if X is a linear combination of A_2. The theorem is a consequence of (7).

Let us consider a subset A_1 of R_2 and a subset A_2 of R_1. Now we state the propositions:

(16) Suppose the RLS structure of R_1 = the RLS structure of R_2. Then if $A_1 = A_2$, then $\Omega_{\mathrm{Lin}(A_1)} = \Omega_{\mathrm{Lin}(A_2)}$. The theorem is a consequence of (7) and (14).

(17) Suppose the RLS structure of R_1 = the RLS structure of R_2. Then if $A_1 = A_2$, then A_1 is linearly independent iff A_2 is linearly independent. The theorem is a consequence of (7) and (14).

(18) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider an object X. Then X is a subspace of R_2 if and only if X is a subspace of R_1.

(19) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider a linear combination L of R_2, and a linear combination S of R_1. If $L = S$, then $\sum L = \sum S$. The theorem is a consequence of (12) and (13).

(20) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider a set X. Then X is a basis of R_1 if and only if X is a basis of R_2. The theorem is a consequence of (17) and (16).

(21) Let us consider real linear spaces R_1, R_2. Suppose the RLS structure of R_1 = the RLS structure of R_2 and R_1 is finite dimensional. Then
   (i) R_2 is finite dimensional, and
   (ii) $\dim(R_2) = \dim(R_1)$.

   The theorem is a consequence of (20).

Let us consider a real normed space R_3. Now we state the propositions:

(22) The normed structure of R_3 is a strict real normed space.

(23) There exists a normed linear topological space T such that the normed structure of R_3 = the normed structure of T.

   PROOF: Reconsider R_3 = the normed structure of $RNS0$ as a strict real normed space. Set L_2 = LinearTopSpaceNorm R_3. Reconsider N = the norm of R_3 as a function from the carrier of L_2 into $\mathbb{R}$. Set W =

$\langle$the carrier of L_2, the zero of L_2, the addition of L_2, the external multiplication of L_2, the topology of $L_2, N\rangle$. W is topological space-like, right complementable, Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, scalar unital, add-continuous, and mult-continuous. □

(24) Suppose R_3 is finite dimensional. Then there exists a normed linear topological space T such that

(i) the normed structure of $R_3 =$ the normed structure of T, and

(ii) T is finite dimensional.

The theorem is a consequence of (23) and (21).

(25) Let us consider a normed linear topological space T, and a real normed space R_3. Suppose T is finite dimensional and $R_3 =$ the normed structure of T. Then

(i) R_3 is finite dimensional, and

(ii) $\dim(R_3) = \dim(T)$.

The theorem is a consequence of (21).

2. The Ascoli-Arzela Theorem on Finite Dimensional Normed Linear Spaces

Let us consider a non empty metric space M, a non empty, compact topological space S, a normed linear topological space T, a subset G of (the carrier of $T)^{(\text{the carrier of } M)}$, and a non empty subset H of MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and T).

Now we state the propositions:

(26) Suppose $S = M_{\text{top}}$ and T is complete and finite dimensional and $\dim(T) \neq 0$. Then suppose $G = H$. Then MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and $T) \restriction H$ is totally bounded if and only if G is equibounded and equicontinuous.

Proof: For every point x of S and for every non empty subset H_1 of MetricSpaceNorm T such that $H_1 = \{f(x)$, where f is a function from S into $T : f \in H\}$ holds MetricSpaceNorm $T \restriction \overline{H_1}$ is compact by [9, (1)], (25). □

(27) Suppose $S = M_{\text{top}}$ and T is complete and finite dimensional and $\dim(T) \neq 0$. Then if $G = H$, then $\overline{H}$ is sequentially compact iff G is equibounded and equicontinuous. The theorem is a consequence of (26).

(28) Let us consider a non empty metric space M, a non empty, compact topological space S, and a normed linear topological space T. Suppose $S = M_{\mathrm{top}}$ and T is complete and finite dimensional and $\dim(T) \neq 0$. Let us consider a subset G of (the carrier of $T)^{\alpha}$, and a non empty subset F of the $\mathbb{R}$-norm space of continuous functions of S and T. Suppose $G = F$. Then $\overline{F}$ is compact if and only if G is equibounded and equicontinuous, where α is the carrier of M. The theorem is a consequence of (27).

(29) Let us consider a non empty real normed space R_3, a normed linear topological space T, a non empty subset X of R_3, a non empty, compact, strict topological space S, and a non empty subset G of the $\mathbb{R}$-norm space of continuous functions of S and T.

Suppose S is a subspace of TopSpaceNorm R_3 and the carrier of $S = X$ and X is compact and T is complete and finite dimensional and $\dim(T) \neq 0$ and there exist real numbers K, D such that $0 < K$ and $0 < D$ and for every function F from X into T such that $F \in G$ holds for every points x, y of R_3 such that x, $y \in X$ holds $\|F_{/x} - F_{/y}\| \leqslant D \cdot \|x - y\|$ and for every point x of R_3 such that $x \in X$ holds $\|F_{/x}\| \leqslant K$. Then $\overline{G}$ is compact.

PROOF: Reconsider $Y = X$ as a non empty subset of MetricSpaceNorm R_3. Reconsider $M =$ MetricSpaceNorm $R_3 \restriction Y$ as a non empty metric space. For every object z, $z \in$ the topology of S iff $z \in$ the open set family of M. For every object z such that $z \in$ the continuous functions of S and T holds $z \in$ (the carrier of $T)^{\alpha}$, where α is the carrier of M. Reconsider $H = G$ as a subset of (the carrier of $T)^{(\text{the carrier of } M)}$. $\overline{G}$ is compact iff H is equibounded and equicontinuous.

Consider K, D being real numbers such that $0 < K$ and $0 < D$ and for every function F from X into T such that $F \in G$ holds for every points x, y of R_3 such that x, $y \in X$ holds $\|F_{/x} - F_{/y}\| \leqslant D \cdot \|x - y\|$ and for every point x of R_3 such that $x \in X$ holds $\|F_{/x}\| \leqslant K$. For every function f from the carrier of M into the carrier of T such that $f \in H$ for every element x of M, $\|f(x)\| \leqslant K$. For every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every function f from the carrier of M into the carrier of T such that $f \in H$ for every points x_1, x_2 of M such that $\rho(x_1, x_2) < d$ holds $\|f(x_1) - f(x_2)\| < e$. $\square$

3. High-Order and Multilayer Perceptron

Let n be a natural number, k be a finite sequence of elements of $\mathbb{N}$, and N be a finite sequence. We say that N is a multilayer perceptron with k and n if and only if

(Def. 1) $\operatorname{len} N = n$ and $\operatorname{len} N+1 = \operatorname{len} k$ and for every natural number i such that $1 \leqslant i < \operatorname{len} k$ holds $N(i)$ is a function from $\langle \mathcal{E}^{k(i)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(i+1)}, \|\cdot\| \rangle$.

We say that N is a multilayer perceptron-like if and only if

(Def. 2) there exists a finite sequence k of elements of $\mathbb{N}$ such that $\operatorname{len} N+1 = \operatorname{len} k$ and for every natural number i such that $1 \leqslant i < \operatorname{len} k$ holds $N(i)$ is a function from $\langle \mathcal{E}^{k(i)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(i+1)}, \|\cdot\| \rangle$.

Observe that there exists a finite sequence which is a multilayer perceptron-like. A multilayer perceptron is multilayer perceptron-like finite sequence. Now we state the proposition:

(30) Let us consider a multilayer perceptron N. Then there exists a finite sequence k of elements of $\mathbb{N}$ such that

(i) $\operatorname{len} N + 1 = \operatorname{len} k$, and

(ii) for every natural number i such that $1 \leqslant i < \operatorname{len} k$ holds $N(i)$ is a function from $\langle \mathcal{E}^{k(i)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(i+1)}, \|\cdot\| \rangle$.

Let n be a natural number, k be a finite sequence of elements of $\mathbb{N}$, and N be a finite sequence. Assume N is a multilayer perceptron with k and n. Assume $\operatorname{len} N \neq 0$. The functor OutputFunc$(N, k, n)$ yielding a function from $\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(n+1)}, \|\cdot\| \rangle$ is defined by

(Def. 3) there exists a finite sequence p such that $\operatorname{len} p = \operatorname{len} N$ and $p(1) = N(1)$ and for every natural number i such that $1 \leqslant i < \operatorname{len} N$ there exists a function N_2 from $\langle \mathcal{E}^{k(i+1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(i+2)}, \|\cdot\| \rangle$ and there exists a function p_2 from $\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(i+1)}, \|\cdot\| \rangle$ such that $N_2 = N(i+1)$ and $p_2 = p(i)$ and $p(i+1) = N_2 \cdot p_2$ and $it = p(\operatorname{len} N)$.

Now we state the proposition:

(31) Let us consider a natural number n, a finite sequence k of elements of $\mathbb{N}$, and a non empty finite sequence N. Suppose $n \neq 0$ and N is a multilayer perceptron with k and $n+1$. Then there exists a finite sequence k_1 of elements of $\mathbb{N}$ and there exists a non empty finite sequence N_1 and there exists a function N_2 from $\langle \mathcal{E}^{k(n+1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(n+2)}, \|\cdot\| \rangle$ such that $N_1 = N{\restriction}n$ and $k_1 = k{\restriction}(n+1)$ and $N_2 = N(n+1)$ and N_1 is a multilayer perceptron with k_1 and n and OutputFunc$(N, k, n+1) = N_2 \cdot (\text{OutputFunc}(N_1, k_1, n))$.

PROOF: Reconsider $N_1 = N{\restriction}n$ as a non empty finite sequence. Reconsider $k_1 = k{\restriction}(n+1)$ as a finite sequence of elements of $\mathbb{N}$. For every natural number i such that $1 \leqslant i < \operatorname{len} k_1$ holds $N_1(i)$ is a function from $\langle \mathcal{E}^{k_1(i)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k_1(i+1)}, \|\cdot\| \rangle$. Consider p being a finite sequence such that $\operatorname{len} p = \operatorname{len} N$ and $p(1) = N(1)$ and for every natural number i such that $1 \leqslant i < \operatorname{len} N$ there exists a function N_2 from $\langle \mathcal{E}^{k(i+1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(i+2)}, \|\cdot\| \rangle$ and there exists a function p_2 from $\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(i+1)}, \|\cdot\| \rangle$ such that $N_2 = N(i+1)$ and $p_2 = p(i)$ and $p(i+1) = N_2 \cdot p_2$ and $\operatorname{OutputFunc}(N, k, n+1) = p(\operatorname{len} N)$. Consider N_2 being a function from $\langle \mathcal{E}^{k(n+1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(n+2)}, \|\cdot\| \rangle$, p_2 being a function from $\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(n+1)}, \|\cdot\| \rangle$ such that $N_2 = N(n+1)$ and $p_2 = p(n)$ and $p(n+1) = N_2 \cdot p_2$. $\square$

Let n be a natural number and k be a finite sequence of elements of $\mathbb{N}$. The functor $\operatorname{Neurons}(n, k)$ yielding a subset of

(the carrier of $\langle \mathcal{E}^{k(n+1)}, \|\cdot\| \rangle)^{(\text{the carrier of } \langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle)}$ is defined by the term

(Def. 4) $\{F$, where F is a function from $\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(n+1)}, \|\cdot\| \rangle$: there exists a finite sequence N such that N is a multilayer perceptron with k and n and $F = \operatorname{OutputFunc}(N, k, n)\}$.

Now we state the propositions:

(32) Let us consider a natural number n, a finite sequence k of elements of $\mathbb{N}$, a non empty, compact, strict topological space S, a non empty subspace M of $\operatorname{MetricSpaceNorm}\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$, a non empty subset X of $\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$, and a normed linear topological space T. Suppose $S = M_{\mathrm{top}}$ and the carrier of $M = X$ and X is compact and T is complete and finite dimensional and $\dim(T) \neq 0$ and $\langle \mathcal{E}^{k(n+1)}, \|\cdot\| \rangle =$ the normed structure of T.

Let us consider a subset G of (the carrier of $T)^{\alpha}$, and a non empty subset F of the $\mathbb{R}$-norm space of continuous functions of S and T. Suppose $G = F$ and $G \subseteq \{f{\restriction}X$, where f is a function from $\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(n+1)}, \|\cdot\| \rangle : f \in \operatorname{Neurons}(n, k)\}$. Then $\overline{F}$ is compact if and only if G is equibounded and equicontinuous, where α is the carrier of M.

(33) Let us consider a natural number n, a finite sequence k of elements of $\mathbb{N}$, a non empty, compact, strict topological space S, a non empty subset X of $\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$, and a normed linear topological space T. Suppose S is a subspace of $\operatorname{TopSpaceNorm}\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$ and the carrier of $S = X$ and X is compact and T is complete and finite dimensional and $\dim(T) \neq 0$ and $\langle \mathcal{E}^{k(n+1)}, \|\cdot\| \rangle =$ the normed structure of T. Let us consider a non empty subset G of the $\mathbb{R}$-norm space of continuous functions of S and T.

Suppose $G \subseteq \{f{\restriction}X$, where f is a function from $\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(n+1)}, \|\cdot\| \rangle : f \in \operatorname{Neurons}(n, k)\}$ and there exist real numbers K, D

such that $0 < K$ and $0 < D$ and for every function F from X into T such that $F \in G$ holds for every points x, y of $\langle \mathcal{E}^{k(1)}, \| \cdot \| \rangle$ such that $x, y \in X$ holds $\|F_{/x} - F_{/y}\| \leqslant D \cdot \|x - y\|$ and for every point x of $\langle \mathcal{E}^{k(1)}, \| \cdot \| \rangle$ such that $x \in X$ holds $\|F_{/x}\| \leqslant K$. Then $\overline{G}$ is compact.

Let X, Y be real normed spaces, F be a function from X into Y, and D, K be real numbers. We say that F is a layer function of D and K if and only if

(Def. 5) for every points x, y of X, $\|F(x) - F(y)\| \leqslant D \cdot \|x - y\|$ and for every point x of X, $\|F(x)\| \leqslant K$.

Let n be a natural number, k be a finite sequence of elements of $\mathbb{N}$, and N be a finite sequence. We say that N is a layer sequence of D, K, k and n if and only if

(Def. 6) $\operatorname{len} N = n$ and N is a multilayer perceptron with k and n and for every natural number i such that $1 \leqslant i < \operatorname{len} k$ there exists a function N_3 from $\langle \mathcal{E}^{k(i)}, \| \cdot \| \rangle$ into $\langle \mathcal{E}^{k(i+1)}, \| \cdot \| \rangle$ such that $N(i) = N_3$ and N_3 is a layer function of D and K.

Now we state the propositions:

(34) Let us consider real numbers D, K. Suppose $0 \leqslant D$ and $0 \leqslant K$. Let us consider a natural number n, a finite sequence k of elements of $\mathbb{N}$, and a non empty finite sequence N. Suppose N is a layer sequence of D, K, k and n. Then $\mathrm{OutputFunc}(N, k, n)$ is a layer function of D^n and K.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence k of elements of $\mathbb{N}$ for every non empty finite sequence N such that $\operatorname{len} N = \$_1$ and N is a layer sequence of D, K, k and $\$_1$ holds $\mathrm{OutputFunc}(N, k, \$_1)$ is a layer function of $D^{\$_1}$ and K. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. □

(35) Let us consider a natural number n, a finite sequence k of elements of $\mathbb{N}$, a non empty, compact, strict topological space S, a non empty subset X of $\langle \mathcal{E}^{k(1)}, \| \cdot \| \rangle$, and a normed linear topological space T. Suppose S is a subspace of $\mathrm{TopSpaceNorm}\langle \mathcal{E}^{k(1)}, \| \cdot \| \rangle$ and the carrier of $S = X$ and X is compact and T is complete and finite dimensional and $\dim(T) \neq 0$ and $\langle \mathcal{E}^{k(n+1)}, \| \cdot \| \rangle =$ the normed structure of T.

Let us consider a non empty subset G of the $\mathbb{R}$-norm space of continuous functions of S and T, and real numbers D, K. Suppose $0 < D$ and $0 < K$ and $G \subseteq \{F \restriction X$, where F is a function from $\langle \mathcal{E}^{k(1)}, \| \cdot \| \rangle$ into $\langle \mathcal{E}^{k(n+1)}, \| \cdot \| \rangle$: there exists a non empty finite sequence N such that N is a layer sequence of D, K, k and n and $F = \mathrm{OutputFunc}(N, k, n)\}$. Then $\overline{G}$ is compact.

Proof: Set $K_1 = K + 1$. Set $D_1 = D^n + 1$. For every function F from X into T such that $F \in G$ holds for every points x, y of $\langle \mathcal{E}^{k(1)}, \| \cdot \| \rangle$ such that $x, y \in X$ holds $\|F_{/x} - F_{/y}\| \leqslant D_1 \cdot \|x - y\|$ and for every point x of $\langle \mathcal{E}^{k(1)}, \|\cdot\|\rangle$ such that $x \in X$ holds $\|F_{/x}\| \leqslant K_1$. $\square$

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Splitting Fields for the Rational Polynomials X^2-2, X^2+X+1, X^3-1, and X^3-2

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Summary. In [11] the existence (and uniqueness) of splitting fields has been formalized. In this article we apply this result by providing splitting fields for the polynomials X^2-2, X^3-1, X^2+X+1 and X^3-2 over $\mathcal{Q}$ using the Mizar [2], [1] formalism. We also compute the degrees and bases for these splitting fields, which requires some additional registrations to adopt types properly.

The main result, however, is that the polynomial X^3-2 does not split over $\mathcal{Q}(\sqrt[3]{2})$. Because X^3-2 obviously has a root over $\mathcal{Q}(\sqrt[3]{2})$, this shows that the field extension $\mathcal{Q}(\sqrt[3]{2})$ is not normal over $\mathcal{Q}$ [3], [4], [5] and [7].

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1. Preliminaries

Let L be a non empty double loop structure and a, b, c be elements of L. Note that the functor $\{a,b,c\}$ yields a subset of L. Let i be an integer. Let us observe that i^3 is integer.

Let i be an even integer. Let us observe that i^3 is even.

Let i be an odd integer. Let us observe that i^3 is odd.

Now we state the propositions:

(1) Let us consider complex numbers r, s. Then $(r \cdot s)^3 = r^3 \cdot s^3$.

(2) Let us consider a rational number r. Then $r^3 \geqslant 0$ if and only if $r \geqslant 0$.

(3) There exists no rational number r such that $r^3 = 2$. The theorem is a consequence of (2) and (1).

Note that $\mathrm{root}_3(2)$ is non rational. Now we state the proposition:

(4) Let us consider finite sets X_1, X_2. Suppose $X_1 \subseteq X_2$ and $\overline{\overline{X_1}} = \overline{\overline{X_2}}$. Then $X_1 = X_2$.

Let F be a field. Observe that there exists an element of the carrier of PolyRing(F) which is linear and there exists an element of the carrier of PolyRing (F) which is non linear and non constant.

Let us consider a field F and an element p of the carrier of PolyRing(F). Now we state the propositions:

(5) If $\deg(p) = 2$, then p is reducible iff p has roots.

(6) If $\deg(p) = 3$, then p is reducible iff p has roots.

2. More on Field Extensions

One can check that $\mathbb{C}_{\mathrm{F}}$ is ($\mathbb{F}_{\mathbb{Q}}$)-extending and there exists an element of $\mathbb{R}_{\mathrm{F}}$ which is ($\mathbb{F}_{\mathbb{Q}}$)-membered and there exists an element of $\mathbb{R}_{\mathrm{F}}$ which is non ($\mathbb{F}_{\mathbb{Q}}$)-membered and there exists an element of $\mathbb{C}_{\mathrm{F}}$ which is ($\mathbb{R}_{\mathrm{F}}$)-membered and there exists an element of $\mathbb{C}_{\mathrm{F}}$ which is non ($\mathbb{R}_{\mathrm{F}}$)-membered and there exists an element of $\mathbb{C}_{\mathrm{F}}$ which is ($\mathbb{F}_{\mathbb{Q}}$)-membered and there exists an element of $\mathbb{C}_{\mathrm{F}}$ which is non ($\mathbb{F}_{\mathbb{Q}}$)-membered.

Now we state the propositions:

(7) Let us consider a field F, an extension E of F, an E-extending extension K of F, an element p of the carrier of PolyRing(F), and an element q of the carrier of PolyRing(E). If $p = q$, then $\mathrm{Roots}(K, p) = \mathrm{Roots}(K, q)$.

(8) Let us consider a field F, an extension E of F, an F-extending extension K of E, an element a of E, and an element b of K. Suppose $b = a$. Then $\mathrm{RAdj}(F, \{a\}) = \mathrm{RAdj}(F, \{b\})$.

(9) Let us consider a field F, an extension E of F, an F-extending extension K of E, an F-algebraic element a of E, and an F-algebraic element b of K. Suppose $b = a$. Then $\mathrm{FAdj}(F, \{a\}) = \mathrm{FAdj}(F, \{b\})$.

(10) Let us consider a field F, an extension E of F, an E-extending extension K of F, an F-algebraic element a of E, and an F-algebraic element b of K. If $a = b$, then $\mathrm{MinPoly}(a, F) = \mathrm{MinPoly}(b, F)$.

(11) Let us consider a field F, an F-finite extension E of F, and an element a of E. Then $\deg(\mathrm{MinPoly}(a, F)) \mid \deg(E, F)$.

Let F be a field, E be an extension of F, and T_1, T_2 be subsets of E. One can check that $\mathrm{FAdj}(F, T_1 \cup T_2)$ is $(\mathrm{FAdj}(F, T_1))$-extending and $(\mathrm{FAdj}(F, T_2))$-extending.

Let a, b be elements of E. Observe that $\mathrm{FAdj}(F, \{a, b\})$ is $(\mathrm{FAdj}(F, \{a\}))$-extending and $(\mathrm{FAdj}(F, \{b\}))$-extending. Let a, b, c be elements of E. Let us observe that $\mathrm{FAdj}(F, \{a, b, c\})$ is $(\mathrm{FAdj}(F, \{a, b\}))$-extending, $(\mathrm{FAdj}(F, \{a, c\}))$-extending, and $(\mathrm{FAdj}(F, \{b, c\}))$-extending.

3. The Rational Polynomials $X^2 - 2$, $X^3 - 1$, $X^2 + X + 1$ and $X^3 - 2$

The functors: $\mathrm{X}^2{-}2$, $\mathrm{X}^3{-}1$, $\mathrm{X}^3{-}2$, and $\mathrm{X}^2 + \mathrm{X} + 1$ yielding elements of the carrier of $\mathrm{PolyRing}(\mathbb{F}_\mathbb{Q})$ are defined by terms

(Def. 1) $\langle -(1_{\mathbb{F}_\mathbb{Q}} + 1_{\mathbb{F}_\mathbb{Q}}), 0_{\mathbb{F}_\mathbb{Q}}, 1_{\mathbb{F}_\mathbb{Q}} \rangle$,

(Def. 2) $(\mathbf{0}.\mathbb{F}_\mathbb{Q} +\cdot (0, -1)) +\cdot (3, 1)$,

(Def. 3) $(\mathbf{0}.\mathbb{F}_\mathbb{Q} +\cdot (0, -2)) +\cdot (3, 1)$,

(Def. 4) $\langle 1_{\mathbb{F}_\mathbb{Q}}, 1_{\mathbb{F}_\mathbb{Q}}, 1_{\mathbb{F}_\mathbb{Q}} \rangle$,

respectively. The functors: $\sqrt{2}$ and $\sqrt[3]{2}$ yielding non zero elements of $\mathbb{R}_\mathrm{F}$ are defined by terms

(Def. 5) $\sqrt{2}$,

(Def. 6) $\mathrm{root}_3(2)$,

respectively. The functors: $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{-3}$ yielding non zero elements of $\mathbb{C}_\mathrm{F}$ are defined by terms

(Def. 7) $\sqrt{2}$,

(Def. 8) $\mathrm{root}_3(2)$,

(Def. 9) $(i) \cdot \sqrt{3}$,

respectively. The functor ζ yielding a non zero element of $\mathbb{C}_\mathrm{F}$ is defined by the term

(Def. 10) $\frac{-1+(i)\cdot\sqrt{3}}{2}$.

Observe that $\mathrm{X}^2{-}2$ is monic, purely quadratic, and irreducible and $\mathrm{X}^3{-}2$ is monic, non constant, and irreducible and $\mathrm{X}^3{-}1$ is monic, non constant, and reducible and $\mathrm{X}^2 + \mathrm{X} + 1$ is monic, quadratic, and irreducible and $\sqrt{2}$ is non $(\mathbb{F}_\mathbb{Q})$-membered and $(\mathbb{F}_\mathbb{Q})$-algebraic and $\sqrt{2}$ is non $(\mathbb{F}_\mathbb{Q})$-membered and $(\mathbb{F}_\mathbb{Q})$-algebraic and $\sqrt[3]{2}$ is non $(\mathbb{F}_\mathbb{Q})$-membered and $(\mathbb{F}_\mathbb{Q})$-algebraic and $\sqrt[3]{2}$ is non $(\mathbb{F}_\mathbb{Q})$-membered and $(\mathbb{F}_\mathbb{Q})$-algebraic and ζ is non $(\mathbb{R}_\mathrm{F})$-membered and $(\mathbb{F}_\mathbb{Q})$-algebraic.

$(\zeta)^{\mathbf{2}}$ is non $(\mathbb{R}_\mathrm{F})$-membered and $(\mathbb{F}_\mathbb{Q})$-algebraic and $\mathrm{FAdj}(\mathbb{F}_\mathbb{Q}, \{\sqrt[3]{2}\})$ is $(\mathbb{F}_\mathbb{Q})$-finite and $\mathrm{FAdj}(\mathbb{F}_\mathbb{Q}, \{\sqrt[3]{2}, \zeta\})$ is $(\mathbb{F}_\mathbb{Q})$-finite and $\mathbb{R}_\mathrm{F}$ is $(\mathrm{FAdj}(\mathbb{F}_\mathbb{Q}, \{\sqrt{2}\}))$-extending and $\mathbb{R}_\mathrm{F}$ is $(\mathrm{FAdj}(\mathbb{F}_\mathbb{Q}, \{\sqrt[3]{2}\}))$-extending and $\mathbb{C}_\mathrm{F}$ is $(\mathrm{FAdj}(\mathbb{F}_\mathbb{Q}, \{\sqrt{2}\}))$-extending and $\mathbb{C}_\mathrm{F}$ is $(\mathrm{FAdj}(\mathbb{F}_\mathbb{Q}, \{\sqrt[3]{2}\}))$-extending and $\mathbb{C}_\mathrm{F}$ is $(\mathrm{FAdj}(\mathbb{F}_\mathbb{Q}, \{\sqrt[3]{2}, \zeta\}))$-extending.

Now we state the propositions:

(12) $\zeta = -\frac{1}{2} + (i) \cdot \frac{\sqrt{3}}{2}$.

(13) $(\zeta)^{\mathbf{2}} = -\frac{1}{2} - \frac{(i)\cdot\sqrt{3}}{2}$.

(14) (i) $\zeta^2 \neq 1$, and

(ii) $\zeta^3 = 1$, and

(iii) $\zeta^2 = -\zeta - 1$.

(15) (i) ζ is a complex root of 3, 1, and

(ii) $(\zeta)^{\mathbf{2}}$ is a complex root of 3, 1.

(16) $\sqrt[3]{2}^3 = 2$.

(17) $\mathrm{X}^3{-}1 = (\mathrm{X}{-}\,1_{\mathbb{F}_{\mathbb{Q}}}) \cdot (\mathrm{X}^2 + \mathrm{X} + 1)$.

(18) (i) $\deg(\mathrm{X}^2{-}2) = 2$, and

(ii) $\deg(\mathrm{X}^3{-}2) = 3$, and

(iii) $\deg(\mathrm{X}^3{-}1) = 3$, and

(iv) $\deg(\mathrm{X}^2 + \mathrm{X} + 1) = 2$.

Let us consider an element x of $\mathbb{F}_{\mathbb{Q}}$. Now we state the propositions:

(19) $\mathrm{eval}(\mathrm{X}^2{-}2, x) = x^2 - 2$.

(20) $\mathrm{eval}(\mathrm{X}^3{-}1, x) = x^3 - 1$.

(21) $\mathrm{eval}(\mathrm{X}^2 + \mathrm{X} + 1, x) = x^2 + x + 1$.

(22) $\mathrm{eval}(\mathrm{X}^3{-}2, x) = x^3 - 2$.

(23) Let us consider an element r of $\mathbb{R}_{\mathrm{F}}$. Then $\mathrm{ExtEval}(\mathrm{X}^2{-}2, r) = r^2 - 2$.

Let us consider an element z of $\mathbb{C}_{\mathrm{F}}$. Now we state the propositions:

(24) $\mathrm{ExtEval}(\mathrm{X}^3{-}1, z) = z^3 - 1$.

(25) $\mathrm{ExtEval}(\mathrm{X}^2 + \mathrm{X} + 1, z) = z^2 + z + 1$.

(26) $\mathrm{ExtEval}(\mathrm{X}^3{-}2, z) = z^3 - 2$.

(27) Let us consider an element z of the carrier of $\mathbb{C}_{\mathrm{F}}$.
Then $\mathrm{ExtEval}(\mathrm{X}^3{-}1, z) = 0_{\mathbb{C}_{\mathrm{F}}}$ if and only if z is a complex root of 3, 1.

(28) $\mathrm{Discriminant}(\mathrm{X}^2 + \mathrm{X} + 1) = -3$.

(29) $\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\zeta\}) = \mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt{-3}\})$.
PROOF: $\{\zeta\}$ is a subset of $\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt{-3}\})$ by [10, (35)], [9, (12)], [6, (2)]. $\{\sqrt{-3}\}$ is a subset of $\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\zeta\})$. $\square$

4. A Splitting Field of $X^2 - 2$

Now we state the propositions:

(30) $\mathrm{MinPoly}(\sqrt{2}, \mathbb{F}_{\mathbb{Q}}) = \mathrm{X}^2 - 2$.

(31) $\deg(\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt{2}\}), \mathbb{F}_{\mathbb{Q}}) = 2$.

(32) $\{1, \sqrt{2}\}$ is a basis of $\mathrm{VecSp}(\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt{2}\}), \mathbb{F}_{\mathbb{Q}})$. The theorem is a consequence of (30).

(33) $\mathrm{Roots}(\mathrm{X}^2 - 2) = \emptyset$.

(34) $\mathrm{X}^2 - 2$ does not split in $\mathbb{F}_{\mathbb{Q}}$.

(35) $\mathrm{Roots}(\mathbb{R}_{\mathrm{F}}, \mathrm{X}^2 - 2) = \{\sqrt{2}, -\sqrt{2}\}$.

Proof: $\overline{\overline{\mathrm{Roots}(\mathbb{R}_{\mathrm{F}}, \mathrm{X}^2 - 2)}} = 2$ by [12, (22)], [13, (13)]. $\square$

(36) $\mathrm{X}^2 - 2 = (\mathrm{X} - \sqrt{2}) \cdot (\mathrm{X} + \sqrt{2})$.

(37) $\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt{2}\})$ is a splitting field of $\mathrm{X}^2 - 2$.

Proof: Set $F = \mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt{2}\})$. $\mathrm{X}^2 - 2 = 1_{\mathbb{R}_{\mathrm{F}}} \cdot (\mathrm{rpoly}(1, \sqrt{2}) * \mathrm{rpoly}(1, -\sqrt{2}))$. $\{\sqrt{2}, -\sqrt{2}\} \subseteq$ the carrier of F. $\mathrm{X}^2 - 2$ splits in F. $\square$

(38) $\sqrt[3]{2}$ is not an element of $\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt{2}\})$. The theorem is a consequence of (10), (30), and (11).

(39) $\mathbb{R}_{\mathrm{F}}$ is not a splitting field of $\mathrm{X}^2 - 2$. The theorem is a consequence of (37) and (38).

(40) $\mathbb{C}_{\mathrm{F}}$ is not a splitting field of $\mathrm{X}^2 - 2$. The theorem is a consequence of (37) and (38).

5. A Splitting Field of $X^3 - 1$ and $X^2 + X + 1$

Now we state the propositions:

(41) $\mathrm{Roots}(\mathrm{X}^3 - 1) = \{1\}$.

(42) $\mathrm{Roots}(\mathrm{X}^2 + \mathrm{X} + 1) = \emptyset$.

(43) $\mathrm{MinPoly}(\zeta, \mathbb{F}_{\mathbb{Q}}) = \mathrm{X}^2 + \mathrm{X} + 1$.

(44) $\mathrm{Roots}(\mathbb{C}_{\mathrm{F}}, \mathrm{X}^3 - 1) = \{1, \zeta, (\zeta)^{\mathbf{2}}\}$.

(45) $\mathrm{Roots}(\mathbb{C}_{\mathrm{F}}, \mathrm{X}^2 + \mathrm{X} + 1) = \{\zeta, (\zeta)^{\mathbf{2}}\}$.

(46) $\mathrm{X}^3 - 1$ does not split in $\mathbb{F}_{\mathbb{Q}}$.

(47) $\mathrm{X}^3 - 1$ does not split in $\mathbb{R}_{\mathrm{F}}$.

(48) $\mathrm{X}^2 + \mathrm{X} + 1$ does not split in $\mathbb{F}_{\mathbb{Q}}$.

(49) $\mathrm{X}^2 + \mathrm{X} + 1$ does not split in $\mathbb{R}_{\mathrm{F}}$.

(50) $\mathrm{X}^2 + \mathrm{X} + 1 = (\mathrm{X} - \zeta) \cdot (\mathrm{X} - (\zeta)^{\mathbf{2}})$.

(51) $\mathrm{X}^3-1 = (\mathrm{X}- 1_{\mathbb{C}_\mathrm{F}}) \cdot (\mathrm{X}- \zeta) \cdot (\mathrm{X}-(\zeta)^{\mathbf{2}})$. The theorem is a consequence of (50).

(52) $\mathrm{FAdj}(\mathbb{F}_\mathbb{Q}, \{\zeta\})$ is a splitting field of $\mathrm{X}^2 + \mathrm{X} + 1$.
PROOF: Set $F = \mathrm{FAdj}(\mathbb{F}_\mathbb{Q}, \{\zeta\})$. $\mathrm{Roots}(\mathbb{C}_\mathrm{F}, \mathrm{X}^2 + \mathrm{X} + 1) \subseteq$ the carrier of F. □

(53) $\mathrm{FAdj}(\mathbb{F}_\mathbb{Q}, \{\zeta\})$ is a splitting field of X^3-1.
PROOF: Set $F = \mathrm{FAdj}(\mathbb{F}_\mathbb{Q}, \{\zeta\})$. $\mathrm{Roots}(\mathbb{C}_\mathrm{F}, \mathrm{X}^3-1) \subseteq$ the carrier of F. □

(54) $\deg(\mathrm{FAdj}(\mathbb{F}_\mathbb{Q}, \{\zeta\}), \mathbb{F}_\mathbb{Q}) = 2$.

(55) $\{1, \zeta\}$ is a basis of $\mathrm{VecSp}(\mathrm{FAdj}(\mathbb{F}_\mathbb{Q}, \{\zeta\}), \mathbb{F}_\mathbb{Q})$. The theorem is a consequence of (43).

(56) $\sqrt{2}$ is not an element of $\mathrm{FAdj}(\mathbb{F}_\mathbb{Q}, \{\zeta\})$. The theorem is a consequence of (55).

(57) $\mathbb{C}_\mathrm{F}$ is not a splitting field of $\mathrm{X}^2 + \mathrm{X} + 1$. The theorem is a consequence of (52) and (56).

(58) $\mathbb{C}_\mathrm{F}$ is not a splitting field of X^3-1. The theorem is a consequence of (53) and (56).

6. A Splitting Field of $X^3 - 2$

Now we state the propositions:

(59) $\mathrm{MinPoly}(\sqrt[3]{2}, \mathbb{F}_\mathbb{Q}) = \mathrm{X}^3-2$.

(60) $\deg(\mathrm{FAdj}(\mathbb{F}_\mathbb{Q}, \{\sqrt[3]{2}\}), \mathbb{F}_\mathbb{Q}) = 3$.

(61) $\{1, \sqrt[3]{2}, \sqrt[3]{2}^{\mathbf{2}}\}$ is a basis of $\mathrm{VecSp}(\mathrm{FAdj}(\mathbb{F}_\mathbb{Q}, \{\sqrt[3]{2}\}), \mathbb{F}_\mathbb{Q})$. The theorem is a consequence of (59).

(62) $\mathrm{Roots}(\mathrm{X}^3-2) = \emptyset$. The theorem is a consequence of (6).

(63) X^3-2 does not split in $\mathbb{F}_\mathbb{Q}$. The theorem is a consequence of (6).

(64) $\mathrm{Roots}(\mathrm{FAdj}(\mathbb{F}_\mathbb{Q}, \{\sqrt[3]{2}\}), \mathrm{X}^3-2) = \{\sqrt[3]{2}\}$.

(65) X^3-2 does not split in $\mathrm{FAdj}(\mathbb{F}_\mathbb{Q}, \{\sqrt[3]{2}\})$.

(66) $\mathrm{Roots}(\mathbb{R}_\mathrm{F}, \mathrm{X}^3-2) = \{\sqrt[3]{2}\}$.

(67) X^3-2 does not split in $\mathbb{R}_\mathrm{F}$.

(68) $\mathrm{Roots}(\mathbb{C}_\mathrm{F}, \mathrm{X}^3-2) = \{\sqrt[3]{2}, \sqrt[3]{2} \cdot \zeta, \sqrt[3]{2} \cdot (\zeta)^{\mathbf{2}}\}$.

(69) $\mathrm{X}^3-2 = (\mathrm{X}- \sqrt[3]{2}) \cdot (\mathrm{X}- \sqrt[3]{2} \cdot \zeta) \cdot (\mathrm{X}- \sqrt[3]{2} \cdot (\zeta)^{\mathbf{2}})$.
PROOF: Set $F = \mathbb{C}_\mathrm{F}$. Set $a = \sqrt[3]{2} \cdot \zeta$. Set $b = \sqrt[3]{2} \cdot (\zeta)^{\mathbf{2}}$. Set $c = \sqrt[3]{2}$. Reconsider $p_1 = \mathrm{X}- c$ as a polynomial over F. $p_1 * \langle a \cdot b, -b + -a, 1_F\rangle = \mathrm{X}^3-2$ by [8, (10)]. □

(70) $\mathrm{FAdj}(\mathbb{F}_\mathbb{Q}, \{\sqrt[3]{2}, \zeta\})$ is a splitting field of X^3-2.

Proof: Set $F = \mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt[3]{2}, \zeta\})$. $\mathrm{Roots}(\mathbb{C}_{\mathrm{F}}, \mathrm{X}^3{-}2) \subseteq$ the carrier of F. $\square$

Let us observe that $\mathbb{C}_{\mathrm{F}}$ is $(\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt[3]{2}\}))$-extending and $\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt[3]{2}, \zeta\})$ is $(\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt[3]{2}\}))$-extending and ζ is $(\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt[3]{2}\}))$-algebraic.

Now we state the propositions:

(71) $\mathrm{MinPoly}(\zeta, \mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt[3]{2}\})) = \mathrm{X}^2 + \mathrm{X} + 1$. The theorem is a consequence of (9), (5), and (7).

(72) $\deg(\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt[3]{2}, \zeta\}), \mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt[3]{2}\})) = 2$. The theorem is a consequence of (71).

(73) $\{1, \zeta\}$ is a basis of $\mathrm{VecSp}(\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt[3]{2}, \zeta\}), \mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt[3]{2}\}))$. The theorem is a consequence of (71).

(74) $\deg(\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt[3]{2}, \zeta\}), \mathbb{F}_{\mathbb{Q}}) = 6$. The theorem is a consequence of (59), (9), and (72).

(75) $\{1, \sqrt[3]{2}, \sqrt[3]{2}^{\mathbf{2}}, \zeta, \sqrt[3]{2}\cdot\zeta, \sqrt[3]{2}^{\mathbf{2}}\cdot\zeta\}$ is a basis of $\mathrm{VecSp}(\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt[3]{2}, \zeta\}), \mathbb{F}_{\mathbb{Q}})$.
Proof: Set $F = \mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt[3]{2}, \zeta\})$. Set $K = \mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt[3]{2}\})$. $K = \mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt[3]{2}\})$. Set $M = \{1, \sqrt[3]{2}, \sqrt[3]{2}^{\mathbf{2}}, \zeta, \sqrt[3]{2} \cdot \zeta, \sqrt[3]{2}^{\mathbf{2}} \cdot \zeta\}$. Reconsider $B_1 = \{1, \sqrt[3]{2}, \sqrt[3]{2}^{\mathbf{2}}\}$ as a basis of $\mathrm{VecSp}(K, \mathbb{F}_{\mathbb{Q}})$. Reconsider $B_2 = \{1, \zeta\}$ as a basis of $\mathrm{VecSp}(F, K)$. $\mathrm{Base}(B_1, B_2) = M$. $\square$

One can verify that $\mathbb{C}_{\mathrm{F}}$ is $(\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt{2}\}))$-extending and $\mathbb{C}_{\mathrm{F}}$ is $(\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt{2}, \zeta\}))$-extending and $\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt{2}, \zeta\})$ is $(\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt{2}\}))$-extending and $\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt[3]{2}, \zeta, \sqrt{2}\})$ is $(\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt{2}, \zeta\}))$-extending and ζ is $(\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt{2}\}))$-algebraic and $\sqrt[3]{2}$ is $(\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt{2}, \zeta\}))$-algebraic and $\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt[3]{2}, \zeta, \sqrt{2}\})$ is $(\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt{2}, \zeta\}))$-finite.

Now we state the propositions:

(76) $\mathrm{MinPoly}(\zeta, \mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt{2}\})) = \mathrm{X}^2 + \mathrm{X} + 1$. The theorem is a consequence of (9), (5), and (7).

(77) $\deg(\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt{2}, \zeta\}), \mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt{2}\})) = 2$. The theorem is a consequence of (76).

(78) $\deg(\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt{2}, \zeta\}), \mathbb{F}_{\mathbb{Q}}) = 4$. The theorem is a consequence of (30), (10), and (77).

(79) $\sqrt{2}$ is not an element of $\mathrm{FAdj}(\mathbb{F}_{\mathbb{Q}}, \{\sqrt[3]{2}, \zeta\})$. The theorem is a consequence of (78) and (74).

(80) $\mathbb{C}_{\mathrm{F}}$ is not a splitting field of $\mathrm{X}^3{-}2$. The theorem is a consequence of (70) and (79).

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Absolutely Integrable Functions

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Summary. The goal of this article is to clarify the relationship between Riemann's improper integrals and Lebesgue integrals. In previous articles [6], [7], we treated Riemann's improper integrals [1], [11] and [4] on arbitrary intervals. Therefore, in this article, we will continue to clarify the relationship between improper integrals and Lebesgue integrals [8], using the Mizar [3], [2] formalism.

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1. Preliminaries

Let s be a without $-\infty$ sequence of extended reals. One can check that $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$ is without $-\infty$.

Let s be a without $+\infty$ sequence of extended reals. One can verify that $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$ is without $+\infty$.

Now we state the propositions:

(1) Let us consider a without $-\infty$ sequence f_1 of extended reals, and a without $+\infty$ sequence f_2 of extended reals. Then

(i) $(\sum_{\alpha=0}^{\kappa}(f_1 - f_2)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa\in\mathbb{N}} - (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa\in\mathbb{N}}$, and

(ii) $(\sum_{\alpha=0}^{\kappa}(f_2 - f_1)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa\in\mathbb{N}} - (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa\in\mathbb{N}}$.

PROOF: Set $P_1 = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa\in\mathbb{N}}$. Set $P_2 = (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa\in\mathbb{N}}$. Set $P_{12} = (\sum_{\alpha=0}^{\kappa}(f_1-f_2)(\alpha))_{\kappa\in\mathbb{N}}$. Set $P_{21} = (\sum_{\alpha=0}^{\kappa}(f_2-f_1)(\alpha))_{\kappa\in\mathbb{N}}$. Define $\mathcal{C}$[natural number] $\equiv P_{12}(\$_1) = P_1(\$_1) - P_2(\$_1)$. For every natural number k such that $\mathcal{C}[k]$ holds $\mathcal{C}[k+1]$. For every natural number k, $\mathcal{C}[k]$. For every element k of $\mathbb{N}$, $P_{12}(k) = (P_1 - P_2)(k)$. Define $\mathcal{C}$[natural number] $\equiv P_{21}(\$_1) = P_2(\$_1) - P_1(\$_1)$. For every natural number k such that $\mathcal{C}[k]$ holds $\mathcal{C}[k+1]$.

For every natural number k, $\mathcal{C}[k]$. For every element k of $\mathbb{N}$, $P_{21}(k) = (P_2 - P_1)(k)$ by [5, (7)]. $\square$

(2) Let us consider sets X, A, and a partial function f from X to $\mathbb{R}$. If f is non-positive, then $f \restriction A$ is non-positive.

(3) Let us consider a set X, and a partial function f from X to $\mathbb{R}$. If f is non-positive, then $-f$ is non-negative.

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number a, and a real number x. Now we state the propositions:

(4) If f is left convergent in a and non-decreasing, then if $x \in \operatorname{dom} f$ and $x < a$, then $f(x) \leqslant \lim_{a^-} f$.

(5) If f is left convergent in a and non-increasing, then if $x \in \operatorname{dom} f$ and $x < a$, then $f(x) \geqslant \lim_{a^-} f$.

(6) If f is right convergent in a and non-decreasing, then if $x \in \operatorname{dom} f$ and $a < x$, then $f(x) \geqslant \lim_{a^+} f$.

(7) If f is right convergent in a and non-increasing, then if $x \in \operatorname{dom} f$ and $a < x$, then $f(x) \leqslant \lim_{a^+} f$.

(8) If f is convergent in $-\infty$ and non-increasing, then if $x \in \operatorname{dom} f$, then $f(x) \leqslant \lim_{-\infty} f$.

(9) If f is convergent in $+\infty$ and non-decreasing, then if $x \in \operatorname{dom} f$, then $f(x) \leqslant \lim_{+\infty} f$.

Let us consider real numbers a, b and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(10) Suppose $a \leqslant b$ and $[a, b] \subseteq \operatorname{dom} f$ and $f \restriction [a, b]$ is bounded and non-negative. Then $\int\limits_a^b f(x)dx \geqslant 0$.

(11) Suppose $a \leqslant b$ and $[a, b] \subseteq \operatorname{dom} f$ and $f \restriction [a, b]$ is bounded and f is integrable on $[a, b]$ and $f \restriction [a, b]$ is non-positive. Then $\int\limits_a^b f(x)dx \leqslant 0$. The theorem is a consequence of (3) and (10).

Let us consider real numbers a, b, c, d and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(12) Suppose $c \leqslant d$ and $[c, d] \subseteq [a, b] \subseteq \operatorname{dom} f$ and $f \restriction [a, b]$ is bounded and f is integrable on $[a, b]$ and $f \restriction [a, b]$ is non-negative. Then $\int\limits_c^d f(x)dx \leqslant$

$\int_a^b f(x)dx$. The theorem is a consequence of (10).

(13) Suppose $c \leqslant d$ and $[c,d] \subseteq [a,b] \subseteq \operatorname{dom} f$ and $f{\restriction}[a,b]$ is bounded and f is integrable on $[a,b]$ and $f{\restriction}[a,b]$ is non-positive. Then $\int_c^d f(x)dx \geqslant \int_a^b f(x)dx$. The theorem is a consequence of (2) and (11).

2. Fundamental Properties of Measure and Integral

Now we state the propositions:

(14) Let us consider a non empty set X, a partial function f from X to $\mathbb{R}$, and a set E. Then $\overline{\mathbb{R}}(f){\restriction}E = \overline{\mathbb{R}}(f{\restriction}E)$.

(15) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, an element A of S, and a sequence E of subsets of S. Suppose f is A-measurable and $A = \operatorname{dom} f$ and E is disjoint valued and $A = \bigcup E$ and ($\int^+ \max_+(f)\,\mathrm{d}M < +\infty$ or $\int^+ \max_-(f)\,\mathrm{d}M < +\infty$). Then there exists a sequence I of extended reals such that

(i) for every natural number n, $I(n) = \int f{\restriction}E(n)\,\mathrm{d}M$, and

(ii) I is summable, and

(iii) $\int f\,\mathrm{d}M = \sum I$.

Proof: Consider I_1 being a non-negative sequence of extended reals such that for every natural number n, $I_1(n) = \int \max_+(f){\restriction}E(n)\,\mathrm{d}M$ and I_1 is summable and $\int \max_+(f)\,\mathrm{d}M = \sum I_1$. Consider I_2 being a non-negative sequence of extended reals such that for every natural number n, $I_2(n) = \int \max_-(f){\restriction}E(n)\,\mathrm{d}M$ and I_2 is summable and $\int \max_-(f)\,\mathrm{d}M = \sum I_2$. For every natural number n, $E(n)$ is an element of S and $E(n) \subseteq \operatorname{dom} f$. For every natural number n, $I_1(n) = \int^+ \max_+(f){\restriction}E(n)\,\mathrm{d}M$. For every natural number n, $I_2(n) = \int^+ \max_-(f){\restriction}E(n)\,\mathrm{d}M$. □

(16) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, and elements A, B of S. Suppose $A \cup B \subseteq \operatorname{dom} f$ and f is $(A \cup B)$-measurable and A misses B and ($\int^+ \max_+(f{\restriction}(A \cup B))\,\mathrm{d}M < +\infty$ or $\int^+ \max_-(f{\restriction}(A \cup B))\,\mathrm{d}M < +\infty$). Then $\int f{\restriction}(A \cup B)\,\mathrm{d}M = \int f{\restriction}A\,\mathrm{d}M + \int f{\restriction}B\,\mathrm{d}M$.

(17) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, an element A of S, and

a sequence E of subsets of S. Suppose f is A-measurable and $A = \operatorname{dom} f$ and E is non descending and $\lim E \subseteq A$ and $M(A \setminus (\lim E)) = 0$ and $(\int^+ \max_+(f)\,\mathrm{d}M < +\infty$ or $\int^+ \max_-(f)\,\mathrm{d}M < +\infty)$. Then there exists a sequence I of extended reals such that

(i) for every natural number n, $I(n) =$ $\int f \restriction(\text{the partial unions of } E)(n)\,\mathrm{d}M$, and

(ii) I is convergent, and

(iii) $\int f\,\mathrm{d}M = \lim I$.

Proof: Reconsider $L_2 = \lim E$ as an element of S. Reconsider $F =$ the partial diff-unions of E as a sequence of subsets of S. Set $g = f \restriction L_2$. Consider J being a sequence of extended reals such that for every natural number n, $J(n) = \int g \restriction F(n)\,\mathrm{d}M$ and J is summable and $\int g\,\mathrm{d}M = \sum J$. Reconsider $I = (\sum_{\alpha=0}^{\kappa} J(\alpha))_{\kappa \in \mathbb{N}}$ as a sequence of extended reals.

For every natural number n, $g \restriction(\text{the partial unions of } F)(n) =$ $f \restriction(\text{the partial unions of } E)(n)$. For every natural number n, (the partial unions of $E)(n) \subseteq \bigcup E$. Define $\mathcal{P}[\text{natural number}] \equiv I(\$_1) = \int g \restriction(\text{the partial unions of } F)(\$_1)\,\mathrm{d}M$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. For every natural number n, $I(n) = \int f \restriction(\text{the partial unions of } E)(n)\,\mathrm{d}M$. $\square$

(18) Let us consider non empty sets X, Y, a set A, a sequence F of X, and a sequence G of Y. Suppose for every element n of $\mathbb{N}$, $G(n) = A \cap F(n)$. Then $\bigcup \operatorname{rng} G = A \cap \bigcup \operatorname{rng} F$.

(19) Let us consider a non empty set X, a σ-field S of subsets of X, a sequence E of S, and a partial function f from X to $\overline{\mathbb{R}}$. Suppose for every natural number n, f is $(E(n))$-measurable. Then f is $(\bigcup E)$-measurable.
Proof: For every real number r, $\bigcup E \cap \text{LE-dom}(f, r) \in S$. $\square$

(20) Let us consider real numbers a, b, and a natural number n. If $a < b$, then $a \leqslant b - \frac{b-a}{n+1} < b$ and $a < a + \frac{b-a}{n+1} \leqslant b$.

Let us consider real numbers a, b. Now we state the propositions:

(21) Suppose $a < b$. Then there exists a sequence E of subsets of L-Field such that

(i) for every natural number n, $E(n) = [a, b - \frac{b-a}{n+1}]$ and $E(n) \subseteq [a, b[$ and $E(n)$ is a non empty, closed interval subset of $\mathbb{R}$, and

(ii) E is non descending and convergent, and

(iii) $\bigcup E = [a, b[$.

Proof: Define $\mathcal{F}(\text{element of } \mathbb{N}) = [a, b - \frac{b-a}{\$_1+1}]$. Consider E being a function from $\mathbb{N}$ into $2^{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $E(n) = \mathcal{F}(n)$. For every natural number n, $E(n) = [a, b - \frac{b-a}{n+1}]$. For every natural number n, $E(n) = [a, b - \frac{b-a}{n+1}]$ and $E(n) \subseteq [a, b[$ and $E(n)$ is a non empty, closed interval subset of $\mathbb{R}$. $\square$

(22) Suppose $a < b$. Then there exists a sequence E of subsets of L-Field such that

(i) for every natural number n, $E(n) = [a + \frac{b-a}{n+1}, b]$ and $E(n) \subseteq]a, b]$ and $E(n)$ is a non empty, closed interval subset of $\mathbb{R}$, and

(ii) E is non descending and convergent, and

(iii) $\bigcup E =]a, b]$.

PROOF: Define $\mathcal{F}(\text{element of } \mathbb{N}) = [a + \frac{b-a}{\$_1+1}, b]$. Consider E being a function from $\mathbb{N}$ into $2^{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $E(n) = \mathcal{F}(n)$. For every natural number n, $E(n) = [a + \frac{b-a}{n+1}, b]$ and $E(n) \subseteq]a, b]$ and $E(n)$ is a non empty, closed interval subset of $\mathbb{R}$. $\square$

Let us consider a real number a. Now we state the propositions:

(23) There exists a sequence E of subsets of L-Field such that

(i) for every natural number n, $E(n) = [a, a + n]$, and

(ii) E is non descending and convergent, and

(iii) $\bigcup E = [a, +\infty[$.

PROOF: Define $\mathcal{F}(\text{element of } \mathbb{N}) = [a, a + \$_1]$. Consider E being a function from $\mathbb{N}$ into $2^{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $E(n) = \mathcal{F}(n)$. For every natural number n, $E(n) = [a, a + n]$. $\square$

(24) There exists a sequence E of subsets of L-Field such that

(i) for every natural number n, $E(n) = [a - n, a]$, and

(ii) E is non descending and convergent, and

(iii) $\bigcup E =]-\infty, a]$.

PROOF: Define $\mathcal{F}(\text{element of } \mathbb{N}) = [a - \$_1, a]$. Consider E being a function from $\mathbb{N}$ into $2^{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $E(n) = \mathcal{F}(n)$. For every natural number n, $E(n) = [a - n, a]$. $\square$

(25) Let us consider a set X, a σ-field S of subsets of X, a σ-measure M on S, and a set A with measure zero w.r.t. M. Then $A \in \text{COM}(S, M)$.

(26) Let us consider a real number r. Then $\{r\} \in$ L-Field. The theorem is a consequence of (25).

(27) Let us consider a non empty set X, a σ-field S of subsets of X, an element E of S, and a partial function f from X to $\overline{\mathbb{R}}$. If $E = \emptyset$, then f is E-measurable.

(28) Let us consider a non empty set X, a σ-field S of subsets of X, an element E of S, and a partial function f from X to $\mathbb{R}$. If $E = \emptyset$, then f is E-measurable. The theorem is a consequence of (27).

(29) Let us consider a real number r, an element E of L-Field, and a partial function f from $\mathbb{R}$ to $\overline{\mathbb{R}}$. If $E = \{r\}$, then f is E-measurable.
PROOF: For every real number a, $E \cap \text{LE-dom}(f, a) \in$ L-Field. □

(30) Let us consider a real number r, an element E of L-Field, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. If $E = \{r\}$, then f is E-measurable. The theorem is a consequence of (29).

Let us consider real numbers a, b, a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and an element E of L-Field. Now we state the propositions:

(31) Suppose $[a, b[\subseteq \operatorname{dom} f$ and f is right improper integrable on a and b. Then if $E \subseteq [a, b[$, then f is E-measurable. The theorem is a consequence of (21), (19), and (28).

(32) Suppose $]a, b] \subseteq \operatorname{dom} f$ and f is left improper integrable on a and b. Then if $E \subseteq]a, b]$, then f is E-measurable. The theorem is a consequence of (22), (20), (19), and (28).

(33) Suppose $]a, b[\subseteq \operatorname{dom} f$ and f is improper integrable on a and b. Then if $E \subseteq]a, b[$, then f is E-measurable. The theorem is a consequence of (32) and (31).

Let us consider a real number a, a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and an element E of L-Field. Now we state the propositions:

(34) Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and f is improper integrable on $[a, +\infty[$. Then if $E \subseteq [a, +\infty[$, then f is E-measurable.
PROOF: Set $A = [a, +\infty[$. Consider K being a sequence of subsets of L-Field such that for every natural number n, $K(n) = [a, a + n]$ and K is non descending and convergent and $\bigcup K = [a, +\infty[$. Reconsider $K_1 = K$ as a sequence of L-Field. For every natural number n, $\overline{\mathbb{R}}(f)$ is $(K_1(n))$-measurable by [8, (49)]. $\overline{\mathbb{R}}(f)$ is A-measurable. □

(35) Suppose $]-\infty, a] \subseteq \operatorname{dom} f$ and f is improper integrable on $]-\infty, a]$. Then if $E \subseteq]-\infty, a]$, then f is E-measurable.
PROOF: Consider K being a sequence of subsets of L-Field such that for every natural number n, $K(n) = [a - n, a]$ and K is non descending and convergent and $\bigcup K =]-\infty, a]$. For every element n of $\mathbb{N}$, $K(n)$ is a non empty, closed interval subset of $\mathbb{R}$. Reconsider $K_1 = K$ as a sequence of L-Field. For every natural number n, $\overline{\mathbb{R}}(f)$ is $(K_1(n))$-measurable by [8, (49)]. $\overline{\mathbb{R}}(f)$ is $(\bigcup K_1)$-measurable. □

(36) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $\operatorname{dom} f = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$. Let us consider an element E of L-Field.

Then f is E-measurable. The theorem is a consequence of (34) and (35).

3. Relation between Improper Integral and Lebesgue Integral

Now we state the propositions:

(37) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\mathbb{R}$, and an element A of S. Suppose $A = \operatorname{dom} f$ and f is A-measurable. Then $\int -f \, \mathrm{d}M = -\int f \, \mathrm{d}M$.

(38) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\mathbb{R}$, and elements A, B, E of S. Suppose $E = \operatorname{dom} f$ and f is E-measurable and non-positive and $A \subseteq B$. Then $\int f{\restriction}A \, \mathrm{d}M \geqslant \int f{\restriction}B \, \mathrm{d}M$.

Proof: For every set x such that $x \in \operatorname{dom}(\overline{\mathbb{R}}(f))$ holds $(\overline{\mathbb{R}}(f))(x) \leqslant 0$. $\int \overline{\mathbb{R}}(f{\restriction}A) \, \mathrm{d}M \geqslant \int \overline{\mathbb{R}}(f){\restriction}B \, \mathrm{d}M$. $\int \overline{\mathbb{R}}(f{\restriction}A) \, \mathrm{d}M \geqslant \int \overline{\mathbb{R}}(f{\restriction}B) \, \mathrm{d}M$. $\square$

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, real numbers a, b, and a non empty subset A of $\mathbb{R}$. Now we state the propositions:

(39) Suppose $[a, b[\subseteq \operatorname{dom} f$ and $A = [a, b[$ and f is right improper integrable on a and b and $f{\restriction}A$ is non-negative. Then

(i) right-improper-integral$(f, a, b) = \int f{\restriction}A \, \mathrm{d}$ L-Meas, and

(ii) if f is right extended Riemann integrable on a, b, then $f{\restriction}A$ is integrable on L-Meas, and

(iii) if f is not right extended Riemann integrable on a, b, then $\int f{\restriction}A \, \mathrm{d}$ L-Meas $= +\infty$.

The theorem is a consequence of (12), (21), (31), (14), (17), (20), and (4).

(40) Suppose $[a, b[\subseteq \operatorname{dom} f$ and $A = [a, b[$ and f is right improper integrable on a and b and $f{\restriction}A$ is non-positive. Then

(i) right-improper-integral$(f, a, b) = \int f{\restriction}A \, \mathrm{d}$ L-Meas, and

(ii) if f is right extended Riemann integrable on a, b, then $f{\restriction}A$ is integrable on L-Meas, and

(iii) if f is not right extended Riemann integrable on a, b, then $\int f{\restriction}A \, \mathrm{d}$ L-Meas $= -\infty$.

The theorem is a consequence of (3), (39), and (31).

(41) Suppose $]a, b] \subseteq \operatorname{dom} f$ and $A =]a, b]$ and f is left improper integrable on a and b and $f{\restriction}A$ is non-negative. Then

(i) left-improper-integral$(f, a, b) = \int f{\restriction}A \, \mathrm{d}$ L-Meas, and

(ii) if f is left extended Riemann integrable on a, b, then $f{\restriction}A$ is integrable on L-Meas, and

(iii) if f is not left extended Riemann integrable on a, b, then $\int f{\restriction}A \,\mathrm{d}\,$L-Meas $= +\infty$.

The theorem is a consequence of (12), (22), (32), (14), (17), (20), and (7).

(42) Suppose $]a, b] \subseteq \mathrm{dom}\, f$ and $A =]a, b]$ and f is left improper integrable on a and b and $f{\restriction}A$ is non-positive. Then

(i) left-improper-integral$(f, a, b) = \int f{\restriction}A \,\mathrm{d}\,$L-Meas, and

(ii) if f is left extended Riemann integrable on a, b, then $f{\restriction}A$ is integrable on L-Meas, and

(iii) if f is not left extended Riemann integrable on a, b, then $\int f{\restriction}A \,\mathrm{d}\,$L-Meas $= -\infty$.

The theorem is a consequence of (3), (41), and (32).

(43) Suppose $]a, b[\subseteq \mathrm{dom}\, f$ and $A =]a, b[$ and f is improper integrable on a and b and $f{\restriction}A$ is non-negative. Then

(i) improper-integral$(f, a, b) = \int f{\restriction}A \,\mathrm{d}\,$L-Meas, and

(ii) if there exists a real number c such that $a < c < b$ and f is left extended Riemann integrable on a, c and right extended Riemann integrable on c, b, then $f{\restriction}A$ is integrable on L-Meas, and

(iii) if for every real number c such that $a < c < b$ holds f is not left extended Riemann integrable on a, c or f is not right extended Riemann integrable on c, b, then $\int f{\restriction}A \,\mathrm{d}\,$L-Meas $= +\infty$.

The theorem is a consequence of (31), (32), (41), (39), (26), and (33).

(44) Suppose $]a, b[\subseteq \mathrm{dom}\, f$ and $A =]a, b[$ and f is improper integrable on a and b and $f{\restriction}A$ is non-positive. Then

(i) improper-integral$(f, a, b) = \int f{\restriction}A \,\mathrm{d}\,$L-Meas, and

(ii) if there exists a real number c such that $a < c < b$ and f is left extended Riemann integrable on a, c and right extended Riemann integrable on c, b, then $f{\restriction}A$ is integrable on L-Meas, and

(iii) if for every real number c such that $a < c < b$ holds f is not left extended Riemann integrable on a, c or f is not right extended Riemann integrable on c, b, then $\int f{\restriction}A \,\mathrm{d}\,$L-Meas $= -\infty$.

The theorem is a consequence of (3), (43), (33), and (37).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number b, and a non empty subset A of $\mathbb{R}$. Now we state the propositions:

(45) Suppose $]-\infty, b] \subseteq \mathrm{dom}\, f$ and $A =]-\infty, b]$ and f is improper integrable on $]-\infty, b]$ and f is non-negative. Then

(i) $\int\limits_{-\infty}^{b} f(x)dx = \int f\restriction A \, \mathrm{d}\,\text{L-Meas}$, and

(ii) if f is extended Riemann integrable on $-\infty$, b, then $f\restriction A$ is integrable on L-Meas, and

(iii) if f is not extended Riemann integrable on $-\infty$, b, then $\int f\restriction A \, \mathrm{d}\,\text{L-Meas} = +\infty$.

The theorem is a consequence of (12), (24), (35), (14), (17), and (8).

(46) Suppose $]-\infty, b] \subseteq \operatorname{dom} f$ and $A =]-\infty, b]$ and f is improper integrable on $]-\infty, b]$ and f is non-positive. Then

(i) $\int\limits_{-\infty}^{b} f(x)dx = \int f\restriction A \, \mathrm{d}\,\text{L-Meas}$, and

(ii) if f is extended Riemann integrable on $-\infty$, b, then $f\restriction A$ is integrable on L-Meas, and

(iii) if f is not extended Riemann integrable on $-\infty$, b, then $\int f\restriction A \, \mathrm{d}\,\text{L-Meas} = -\infty$.

Proof: Reconsider $A_1 = A$ as an element of L-Field. For every object x such that $x \in \operatorname{dom}(-f)$ holds $0 \leqslant (-f)(x)$. $\int\limits_{-\infty}^{b} (-f)(x)dx = \int (-f)\restriction A \, \mathrm{d}\,\text{L-Meas}$. $f\restriction A$ is A_1-measurable. $\int -f\restriction A \, \mathrm{d}\,\text{L-Meas} = -\int f\restriction A \, \mathrm{d}\,\text{L-Meas}$. $\square$

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number a, and a non empty subset A of $\mathbb{R}$. Now we state the propositions:

(47) Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and $A = [a, +\infty[$ and f is improper integrable on $[a, +\infty[$ and f is non-negative. Then

(i) $\int\limits_{a}^{+\infty} f(x)dx = \int f\restriction A \, \mathrm{d}\,\text{L-Meas}$, and

(ii) if f is extended Riemann integrable on a, $+\infty$, then $f\restriction A$ is integrable on L-Meas, and

(iii) if f is not extended Riemann integrable on a, $+\infty$, then $\int f\restriction A \, \mathrm{d}\,\text{L-Meas} = +\infty$.

The theorem is a consequence of (12), (23), (34), (14), (17), and (9).

(48) Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and $A = [a, +\infty[$ and f is improper integrable on $[a, +\infty[$ and f is non-positive. Then

(i) $\int\limits_{a}^{+\infty} f(x)dx = \int f \restriction A \, \mathrm{d}\,$L-Meas, and

(ii) if f is extended Riemann integrable on a, $+\infty$, then $f \restriction A$ is integrable on L-Meas, and

(iii) if f is not extended Riemann integrable on a, $+\infty$, then $\int f \restriction A \, \mathrm{d}\,$L-Meas $= -\infty$.

PROOF: Reconsider $A_1 = A$ as an element of L-Field. For every object x such that $x \in \mathrm{dom}(-f)$ holds $0 \leqslant (-f)(x)$. $\int\limits_{a}^{+\infty} (-f)(x)dx = \int (-f) \restriction A \, \mathrm{d}\,$L-Meas. $f \restriction A$ is A_1-measurable. $\int -f \restriction A \, \mathrm{d}\,$L-Meas $= -\int f \restriction A \, \mathrm{d}\,$L-Meas. □

(49) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, and elements A, B of S. Suppose there exists an element E of S such that $E = \mathrm{dom}\, f$ and f is E-measurable and f is non-negative. Then $\int^{+} f \restriction (A \cup B) \, \mathrm{d}M \leqslant \int^{+} f \restriction A \, \mathrm{d}M + \int^{+} f \restriction B \, \mathrm{d}M$.

(50) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, and sets A, B. Suppose $A \subseteq \mathrm{dom}\, f$ and $B \subseteq \mathrm{dom}\, f$ and $f \restriction A$ is integrable on M and $f \restriction B$ is integrable on M. Then $f \restriction (A \cup B)$ is integrable on M. The theorem is a consequence of (49).

(51) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\mathbb{R}$, and sets A, B. Suppose $A \subseteq \mathrm{dom}\, f$ and $B \subseteq \mathrm{dom}\, f$ and $f \restriction A$ is integrable on M and $f \restriction B$ is integrable on M. Then $f \restriction (A \cup B)$ is integrable on M. The theorem is a consequence of (14) and (50).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number a, and a non empty subset A of $\mathbb{R}$. Now we state the propositions:

(52) Suppose $\mathrm{dom}\, f = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$ and f is non-negative. Then

(i) $\int\limits_{-\infty}^{+\infty} f(x)dx = \int f \, \mathrm{d}\,$L-Meas, and

(ii) if f is ∞-extended Riemann integrable, then f is integrable on L-Meas, and

(iii) if f is not ∞-extended Riemann integrable, then $\int f \, \mathrm{d}\,$L-Meas $= +\infty$.

The theorem is a consequence of (45), (36), (26), (47), and (51).

(53) Suppose $\operatorname{dom} f = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$ and f is non-positive. Then

(i) $\int\limits_{-\infty}^{+\infty} f(x)dx = \int f \, \mathrm{d}\,\text{L-Meas}$, and

(ii) if f is ∞-extended Riemann integrable, then f is integrable on L-Meas, and

(iii) if f is not ∞-extended Riemann integrable, then $\int f \, \mathrm{d}\,\text{L-Meas} = -\infty$.

Proof: For every object x such that $x \in \operatorname{dom}(-f)$ holds $0 \leqslant (-f)(x)$. Reconsider $E = \mathbb{R}$ as an element of L-Field. f is E-measurable. $-\int\limits_{-\infty}^{+\infty} f(x)dx = \int -f \, \mathrm{d}\,\text{L-Meas}$. $-\int\limits_{-\infty}^{+\infty} f(x)dx = -\int f \, \mathrm{d}\,\text{L-Meas}$. □

4. Absolutely Integrable Function

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b. Now we state the propositions:

(54) Suppose $[a, b[= \operatorname{dom} f$. Then there exists a sequence F of partial functions from $\mathbb{R}$ into $\mathbb{R}$ such that

(i) for every natural number n, $\operatorname{dom}(F(n)) = \operatorname{dom} f$ and for every real number x such that $x \in [a, b - \frac{1}{n+1}]$ holds $F(n)(x) = f(x)$ and for every real number x such that $x \notin [a, b - \frac{1}{n+1}]$ holds $F(n)(x) = 0$, and

(ii) $\lim \overline{\mathbb{R}}(F) = f$.

Proof: For every element n of $\mathbb{N}$, $[a, b - \frac{1}{n+1}] \subseteq \operatorname{dom} f$. Define $\mathcal{P}[\text{element of } \mathbb{N}, \text{object}] \equiv \$_2 = \chi_{[a, b - \frac{1}{\$_1+1}], \operatorname{dom} f}$. For every element n of $\mathbb{N}$, there exists an element $\langle$ of $\mathbb{R} \dot{\to} \mathbb{R}$ such that $P[n, \langle]$. Consider C_2 being a sequence of $\mathbb{R} \dot{\to} \mathbb{R}$ such that for every element n of $\mathbb{N}$, $P[n, C_2(n)]$. Define $\mathcal{Q}[\text{element of } \mathbb{N}, \text{object}] \equiv \$_2 = f \cdot C_2(\$_1)$. For every element n of $\mathbb{N}$, there exists an element F of $\mathbb{R} \dot{\to} \mathbb{R}$ such that $Q[n, F]$. Consider F being a sequence of $\mathbb{R} \dot{\to} \mathbb{R}$ such that for every element n of $\mathbb{N}$, $Q[n, F(n)]$. For every natural number n, $\operatorname{dom}(F(n)) = \operatorname{dom} f$ and for every real number x such that $x \in [a, b - \frac{1}{n+1}]$ holds $F(n)(x) = f(x)$ and for every real number x such that $x \notin [a, b - \frac{1}{n+1}]$ holds $F(n)(x) = 0$. For every element x of $\mathbb{R}$ such that $x \in \operatorname{dom}(\lim \overline{\mathbb{R}}(F))$ holds $(\lim \overline{\mathbb{R}}(F))(x) = (\overline{\mathbb{R}}(f))(x)$ by [9, (16)]. □

(55) Suppose $a < b$ and $[a, b[\subseteq \operatorname{dom} f$ and f is right improper integrable on a and b and $|f|$ is right extended Riemann integrable on a, b. Then

(i) f is right extended Riemann integrable on a, b, and

(ii) right-improper-integral$(f, a, b) \leqslant$ right-improper-integral$(|f|, a, b) < +\infty$.

PROOF: Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int\limits_a^x f(x)dx$ and I is left convergent in b or left divergent to $+\infty$ in b or left divergent to $-\infty$ in b. Consider A_I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} A_I = [a, b[$ and for every real number x such that $x \in \operatorname{dom} A_I$ holds $A_I(x) = \int\limits_a^x |f|(x)dx$ and A_I is left convergent in b. For every real numbers r_1, r_2 such that r_1, $r_2 \in \operatorname{dom} A_I$ and $r_1 < r_2$ holds $A_I(r_1) \leqslant A_I(r_2)$. Consider r being a real number such that $0 < r < b - a$. For every real number g such that $g \in \operatorname{dom} I \cap]b - r, b[$ holds $I(g) \leqslant A_I(g)$ by [10, (8)]. □

(56) Suppose $a < b$ and $]a, b] \subseteq \operatorname{dom} f$ and f is left improper integrable on a and b and $|f|$ is left extended Riemann integrable on a, b. Then

(i) f is left extended Riemann integrable on a, b, and

(ii) left-improper-integral$(f, a, b) \leqslant$ left-improper-integral$(|f|, a, b) < +\infty$.

PROOF: Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int\limits_x^b f(x)dx$ and I is right convergent in a or right divergent to $+\infty$ in a or right divergent to $-\infty$ in a. Consider A_I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} A_I =]a, b]$ and for every real number x such that $x \in \operatorname{dom} A_I$ holds $A_I(x) = \int\limits_x^b |f|(x)dx$ and A_I is right convergent in a. For every real numbers r_1, r_2 such that r_1, $r_2 \in \operatorname{dom} A_I$ and $r_1 < r_2$ holds $A_I(r_1) \geqslant A_I(r_2)$. Consider r being a real number such that $0 < r < b - a$. For every real number g such that $g \in \operatorname{dom} I \cap]a, a + r[$ holds $I(g) \leqslant A_I(g)$. □

(57) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a non empty, closed interval subset A of $\mathbb{R}$. Suppose $A \subseteq \operatorname{dom} f$. Then

(i) $\max_+(f \restriction A) = \max_+(f \restriction A)$, and

(ii) $\max_-(f \restriction A) = \max_-(f\restriction A)$.

(58) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose $]-\infty, b] \subseteq \operatorname{dom} f$ and f is improper integrable on $]-\infty, b]$ and $|f|$ is extended Riemann integrable on $-\infty$, b. Then

(i) f is extended Riemann integrable on $-\infty$, b, and

(ii) $\int\limits_{-\infty}^{b} f(x)dx \leqslant \int\limits_{-\infty}^{b} |f|(x)dx < +\infty$.

Proof: Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int\limits_{x}^{b} f(x)dx$ and I is convergent in $-\infty$ or divergent in $-\infty$ to $+\infty$ or divergent in $-\infty$ to $-\infty$. Consider A_I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} A_I =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} A_I$ holds $A_I(x) = \int\limits_{x}^{b} |f|(x)dx$ and A_I is convergent in $-\infty$. For every real numbers r_1, r_2 such that r_1, $r_2 \in \operatorname{dom} A_I$ and $r_1 < r_2$ holds $A_I(r_1) \geqslant A_I(r_2)$. For every real number g such that $g \in \operatorname{dom} I \cap]-\infty, 1[$ holds $I(g) \leqslant A_I(g)$. □

(59) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number a. Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and f is improper integrable on $[a, +\infty[$ and $|f|$ is extended Riemann integrable on a, $+\infty$. Then

(i) f is extended Riemann integrable on a, $+\infty$, and

(ii) $\int\limits_{a}^{+\infty} f(x)dx \leqslant \int\limits_{a}^{+\infty} |f|(x)dx < +\infty$.

Proof: Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int\limits_{a}^{x} f(x)dx$ and I is convergent in $+\infty$ or divergent in $+\infty$ to $+\infty$ or divergent in $+\infty$ to $-\infty$. Consider A_I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} A_I = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom} A_I$ holds $A_I(x) = \int\limits_{a}^{x} |f|(x)dx$ and A_I is convergent in $+\infty$. For every real numbers r_1, r_2 such that r_1, $r_2 \in \operatorname{dom} A_I$ and $r_1 < r_2$ holds $A_I(r_1) \leqslant A_I(r_2)$. For every real number g such that $g \in \operatorname{dom} I \cap]1, +\infty[$ holds $I(g) \leqslant A_I(g)$. □

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b. Now we state the propositions:

(60) Suppose $a \leqslant b$ and $[a, b] \subseteq \operatorname{dom} f$ and f is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded. Then

(i) $\max_+(f)$ is integrable on $[a, b]$, and

(ii) $\max_-(f)$ is integrable on $[a, b]$, and

(iii) $2 \cdot (\int\limits_a^b \max\limits_+(f)(x)dx) = \int\limits_a^b f(x)dx + \int\limits_a^b |f|(x)dx$, and

(iv) $2 \cdot (\int\limits_a^b \max\limits_-(f)(x)dx) = -\int\limits_a^b f(x)dx + \int\limits_a^b |f|(x)dx$, and

(v) $\int\limits_a^b f(x)dx = \int\limits_a^b \max\limits_+(f)(x)dx - \int\limits_a^b \max\limits_-(f)(x)dx$.

(61) Suppose $a < b$ and $]a, b] \subseteq \operatorname{dom} f$ and f is left extended Riemann integrable on a, b and $|f|$ is left extended Riemann integrable on a, b. Then $\max_+(f)$ is left extended Riemann integrable on a, b.

PROOF: Set $G = (R^<)\int\limits_a^b f(x)dx$. Set $A_G = (R^<)\int\limits_a^b |f|(x)dx$. Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int\limits_x^b f(x)dx$ and I is right convergent in a and $G = \lim_{a^+} I$.

Consider A_I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} A_I =]a, b]$ and for every real number x such that $x \in \operatorname{dom} A_I$ holds $A_I(x) = \int\limits_x^b |f|(x)dx$ and A_I is right convergent in a and $A_G = \lim_{a^+} A_I$. For every real number d such that $a < d \leqslant b$ holds $\max_+(f)$ is integrable on $[d, b]$ and $\max_+(f) \restriction [d, b]$ is bounded. There exists a partial function I_3 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_3 =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I_3$ holds $I_3(x) = \int\limits_x^b \max\limits_+(f)(x)dx$ and I_3 is right convergent in a. $\square$

(62) Suppose $a < b$ and $[a, b[\subseteq \operatorname{dom} f$ and f is right extended Riemann integrable on a, b and $|f|$ is right extended Riemann integrable on a, b. Then $\max_+(f)$ is right extended Riemann integrable on a, b.

PROOF: Set $G = (R^{>})\int\limits_{a}^{b} f(x)dx$. Set $A_G = (R^{>})\int\limits_{a}^{b} |f|(x)dx$. Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int\limits_{a}^{x} f(x)dx$ and I is left convergent in b and $G = \lim_{b^-} I$.

Consider A_I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} A_I = [a, b[$ and for every real number x such that $x \in \operatorname{dom} A_I$ holds $A_I(x) = \int\limits_{a}^{x} |f|(x)dx$ and A_I is left convergent in b and $A_G = \lim_{b^-} A_I$. For every real number d such that $a \leqslant d < b$ holds $\max_+(f)$ is integrable on $[a, d]$ and $\max_+(f)\restriction[a, d]$ is bounded. There exists a partial function I_3 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_3 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_3$ holds $I_3(x) = \int\limits_{a}^{x} \max\limits_{+}(f)(x)dx$ and I_3 is left convergent in b. $\square$

(63) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose $]-\infty, b] \subseteq \operatorname{dom} f$ and f is extended Riemann integrable on $-\infty$, b and $|f|$ is extended Riemann integrable on $-\infty$, b. Then $\max_+(f)$ is extended Riemann integrable on $-\infty$, b.

PROOF: Set $G = (R^{<})\int\limits_{-\infty}^{b} f(x)dx$. Set $A_G = (R^{<})\int\limits_{-\infty}^{b} |f|(x)dx$. Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int\limits_{x}^{b} f(x)dx$ and I is convergent in $-\infty$ and $G = \lim_{-\infty} I$.

Consider A_I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} A_I =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} A_I$ holds $A_I(x) = \int\limits_{x}^{b} |f|(x)dx$ and A_I is convergent in $-\infty$ and $A_G = \lim_{-\infty} A_I$. For every real number d such that $d \leqslant b$ holds $\max_+(f)$ is integrable on $[d, b]$ and $\max_+(f)\restriction[d, b]$ is bounded. There exists a partial function I_3 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_3 =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} I_3$ holds $I_3(x) = \int\limits_{x}^{b} \max\limits_{+}(f)(x)dx$ and I_3 is convergent in $-\infty$. $\square$

(64) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number

a. Suppose $[a, +\infty[\subseteq \mathrm{dom}\, f$ and f is extended Riemann integrable on a, $+\infty$ and $|f|$ is extended Riemann integrable on a, $+\infty$. Then $\max_+(f)$ is extended Riemann integrable on a, $+\infty$.

PROOF: Set $G = (R^{>}) \int\limits_a^{+\infty} f(x)dx$. Set $A_G = (R^{>}) \int\limits_a^{+\infty} |f|(x)dx$. Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\mathrm{dom}\, I = [a, +\infty[$ and for every real number x such that $x \in \mathrm{dom}\, I$ holds $I(x) = \int\limits_a^x f(x)dx$ and I is convergent in $+\infty$ and $G = \lim_{+\infty} I$.

Consider A_I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\mathrm{dom}\, A_I = [a, +\infty[$ and for every real number x such that $x \in \mathrm{dom}\, A_I$ holds $A_I(x) = \int\limits_a^x |f|(x)dx$ and A_I is convergent in $+\infty$ and $A_G = \lim_{+\infty} A_I$. For every real number d such that $a \leqslant d$ holds $\max_+(f)$ is integrable on $[a, d]$ and $\max_+(f){\restriction}[a, d]$ is bounded. There exists a partial function I_3 from $\mathbb{R}$ to $\mathbb{R}$ such that $\mathrm{dom}\, I_3 = [a, +\infty[$ and for every real number x such that $x \in \mathrm{dom}\, I_3$ holds $I_3(x) = \int\limits_a^x \max_+(f)(x)dx$ and I_3 is convergent in $+\infty$. □

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b. Now we state the propositions:

(65) Suppose $a < b$ and $]a, b] \subseteq \mathrm{dom}\, f$ and f is left extended Riemann integrable on a, b and $|f|$ is left extended Riemann integrable on a, b. Then $\max_-(f)$ is left extended Riemann integrable on a, b. The theorem is a consequence of (61).

(66) Suppose $a < b$ and $[a, b[\subseteq \mathrm{dom}\, f$ and f is right extended Riemann integrable on a, b and $|f|$ is right extended Riemann integrable on a, b. Then $\max_-(f)$ is right extended Riemann integrable on a, b. The theorem is a consequence of (62).

(67) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose $]-\infty, b] \subseteq \mathrm{dom}\, f$ and f is extended Riemann integrable on $-\infty$, b and $|f|$ is extended Riemann integrable on $-\infty$, b. Then $\max_-(f)$ is extended Riemann integrable on $-\infty$, b. The theorem is a consequence of (63).

(68) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number a. Suppose $[a, +\infty[\subseteq \mathrm{dom}\, f$ and f is extended Riemann integrable on a, $+\infty$ and $|f|$ is extended Riemann integrable on a, $+\infty$. Then $\max_-(f)$ is extended Riemann integrable on a, $+\infty$. The theorem is a consequence of

(64).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b. Now we state the propositions:

(69) Suppose $]a, b] \subseteq \operatorname{dom} f$ and $\max_+(f)$ is left extended Riemann integrable on a, b and $\max_-(f)$ is left extended Riemann integrable on a, b. Then

(i) f is left extended Riemann integrable on a, b, and

(ii) left-improper-integral$(f, a, b) =$ left-improper-integral$(\max_+(f), a, b) -$ left-improper-integral$(\max_-(f), a, b)$.

PROOF: Consider I_1 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{x}^{b} \max\limits_{+}(f)(x)dx$ and I_1 is right convergent in a. Consider I_2 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_2 =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I_2$ holds $I_2(x) = \int\limits_{x}^{b} \max\limits_{-}(f)(x)dx$ and I_2 is right convergent in a. For every real number d such that $a < d \leqslant b$ holds f is integrable on $[d, b]$ and $f{\restriction}[d, b]$ is bounded. For every real number x such that $x \in \operatorname{dom}(I_1 - I_2)$ holds $(I_1 - I_2)(x) = \int\limits_{x}^{b} f(x)dx$. $\square$

(70) Suppose $[a, b[\subseteq \operatorname{dom} f$ and $\max_+(f)$ is right extended Riemann integrable on a, b and $\max_-(f)$ is right extended Riemann integrable on a, b. Then

(i) f is right extended Riemann integrable on a, b, and

(ii) right-improper-integral$(f, a, b) =$ right-improper-integral$(\max_+(f), a, b) -$ right-improper-integral$(\max_-(f), a, b)$.

PROOF: Consider I_1 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{a}^{x} \max\limits_{+}(f)(x)dx$ and I_1 is left convergent in b. Consider I_2 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_2 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_2$ holds $I_2(x) = \int\limits_{a}^{x} \max\limits_{-}(f)(x)dx$ and I_2 is left convergent in b. For every real number d such that $a \leqslant d < b$ holds f is integrable on $[a, d]$ and $f{\restriction}[a, d]$ is bounded. For every real number x

such that $x \in \operatorname{dom}(I_1 - I_2)$ holds $(I_1 - I_2)(x) = \int\limits_a^x f(x)dx$. $\square$

(71) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose $]-\infty, b] \subseteq \operatorname{dom} f$ and $\max_+(f)$ is extended Riemann integrable on $-\infty$, b and $\max_-(f)$ is extended Riemann integrable on $-\infty$, b. Then

(i) f is extended Riemann integrable on $-\infty$, b, and

(ii) $\int\limits_{-\infty}^b f(x)dx = \int\limits_{-\infty}^b \max_+(f)(x)dx - \int\limits_{-\infty}^b \max_-(f)(x)dx$.

Proof: Consider I_1 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_x^b \max_+(f)(x)dx$ and I_1 is convergent in $-\infty$. Consider I_2 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_2 =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} I_2$ holds $I_2(x) = \int\limits_x^b \max_-(f)(x)dx$ and I_2 is convergent in $-\infty$. For every real number d such that $d \leqslant b$ holds f is integrable on $[d, b]$ and $f{\restriction}[d, b]$ is bounded. For every real number x such that $x \in \operatorname{dom}(I_1 - I_2)$ holds $(I_1 - I_2)(x) = \int\limits_x^b f(x)dx$. $\square$

(72) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number a. Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and $\max_+(f)$ is extended Riemann integrable on a, $+\infty$ and $\max_-(f)$ is extended Riemann integrable on a, $+\infty$. Then

(i) f is extended Riemann integrable on a, $+\infty$, and

(ii) $\int\limits_a^{+\infty} f(x)dx = \int\limits_a^{+\infty} \max_+(f)(x)dx - \int\limits_a^{+\infty} \max_-(f)(x)dx$.

Proof: Consider I_1 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_a^x \max_+(f)(x)dx$ and I_1 is convergent in $+\infty$. Consider I_2 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_2 = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom} I_2$ holds $I_2(x) = \int\limits_a^x \max_-(f)(x)dx$ and I_2 is convergent in $+\infty$. For every real number d such that $a \leqslant d$ holds

f is integrable on $[a, d]$ and $f \upharpoonright [a, d]$ is bounded. For every real number x such that $x \in \operatorname{dom}(I_1 - I_2)$ holds $(I_1 - I_2)(x) = \int\limits_a^x f(x)dx$. □

5. Improper Integral of Absolutely Integrable Functions

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, real numbers a, b, and a non empty subset A of $\mathbb{R}$. Now we state the propositions:

(73) Suppose $]a, b] \subseteq \operatorname{dom} f$ and $A =]a, b]$ and f is left improper integrable on a and b and $|f|$ is left extended Riemann integrable on a, b and $f \upharpoonright A$ is non-negative. Then

(i) $f \upharpoonright A$ is integrable on L-Meas, and

(ii) left-improper-integral$(f, a, b) = \int f \upharpoonright A \, \mathrm{d}\, \text{L-Meas}$.

The theorem is a consequence of (56) and (41).

(74) Suppose $[a, b[\subseteq \operatorname{dom} f$ and $A = [a, b[$ and f is right improper integrable on a and b and $|f|$ is right extended Riemann integrable on a, b and $f \upharpoonright A$ is non-negative. Then

(i) $f \upharpoonright A$ is integrable on L-Meas, and

(ii) right-improper-integral$(f, a, b) = \int f \upharpoonright A \, \mathrm{d}\, \text{L-Meas}$.

The theorem is a consequence of (55) and (39).

(75) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number b, and a non empty subset A of $\mathbb{R}$. Suppose $]-\infty, b] \subseteq \operatorname{dom} f$ and $A =]-\infty, b]$ and f is improper integrable on $]-\infty, b]$ and $|f|$ is extended Riemann integrable on $-\infty$, b and f is non-negative. Then

(i) $f \upharpoonright A$ is integrable on L-Meas, and

(ii) $\int\limits_{-\infty}^{b} f(x)dx = \int f \upharpoonright A \, \mathrm{d}\, \text{L-Meas}$.

The theorem is a consequence of (58) and (45).

(76) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number a, and a non empty subset A of $\mathbb{R}$. Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and $A = [a, +\infty[$ and f is improper integrable on $[a, +\infty[$ and $|f|$ is extended Riemann integrable on a, $+\infty$ and f is non-negative. Then

(i) $f \upharpoonright A$ is integrable on L-Meas, and

(ii) $\int\limits_{a}^{+\infty} f(x)dx = \int f \upharpoonright A \, \mathrm{d}\, \text{L-Meas}$.

The theorem is a consequence of (59) and (47).

(77) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b. Suppose $a < b$ and $[a, b[\subseteq \operatorname{dom} f$ and f is right improper integrable on a and b and $|f|$ is right extended Riemann integrable on a, b. Then $\max_+(f)$ is right extended Riemann integrable on a, b. The theorem is a consequence of (55) and (62).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, real numbers a, b, and a non empty subset A of $\mathbb{R}$. Now we state the propositions:

(78) Suppose $[a, b[\subseteq \operatorname{dom} f$ and $A = [a, b[$ and f is right improper integrable on a and b and $|f|$ is right extended Riemann integrable on a, b. Then

(i) $f \restriction A$ is integrable on L-Meas, and

(ii) right-improper-integral$(f, a, b) = \int f \restriction A \, \mathrm{d}\,$L-Meas.

The theorem is a consequence of (55), (62), (74), (66), and (70).

(79) Suppose $]a, b] \subseteq \operatorname{dom} f$ and $A =]a, b]$ and f is left improper integrable on a and b and $|f|$ is left extended Riemann integrable on a, b. Then

(i) $f \restriction A$ is integrable on L-Meas, and

(ii) left-improper-integral$(f, a, b) = \int f \restriction A \, \mathrm{d}\,$L-Meas.

The theorem is a consequence of (56), (61), (73), (65), and (69).

(80) Suppose $]a, b[\subseteq \operatorname{dom} f$ and $A =]a, b[$ and f is improper integrable on a and b and there exists a real number c such that $a < c < b$ and $|f|$ is left extended Riemann integrable on a, c and right extended Riemann integrable on c, b. Then

(i) $f \restriction A$ is integrable on L-Meas, and

(ii) improper-integral$(f, a, b) = \int f \restriction A \, \mathrm{d}\,$L-Meas.

The theorem is a consequence of (79), (78), (51), and (26).

(81) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number b, and a non empty subset A of $\mathbb{R}$. Suppose $]-\infty, b] \subseteq \operatorname{dom} f$ and $A =]-\infty, b]$ and f is improper integrable on $]-\infty$, $b]$ and $|f|$ is extended Riemann integrable on $-\infty$, b. Then

(i) $f \restriction A$ is integrable on L-Meas, and

(ii) $\int\limits_{-\infty}^{b} f(x)dx = \int f \restriction A \, \mathrm{d}\,$L-Meas.

The theorem is a consequence of (58), (63), (75), (67), and (71).

(82) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number a, and a non empty subset A of $\mathbb{R}$. Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and $A = [a, +\infty[$

and f is improper integrable on $[a, +\infty[$ and $|f|$ is extended Riemann integrable on a, $+\infty$. Then

(i) $f \restriction A$ is integrable on L-Meas, and

(ii) $\int\limits_{a}^{+\infty} f(x)dx = \int f \restriction A \,\mathrm{d}\,\text{L-Meas}$.

The theorem is a consequence of (59), (64), (76), (68), and (72).

(83) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $\operatorname{dom} f = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$ and $|f|$ is ∞-extended Riemann integrable. Then

(i) f is integrable on L-Meas, and

(ii) $\int\limits_{-\infty}^{+\infty} f(x)dx = \int f \,\mathrm{d}\,\text{L-Meas}$.

The theorem is a consequence of (81), (82), (51), and (36).

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Non-Trivial Universes and Sequences of Universes[1]

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Summary. Universe is a concept which is present from the beginning of the creation of the Mizar Mathematical Library (MML) in several forms (`Universe`, `Universe_closure`, `UNIVERSE`) [25], then later as `the_universe_of`, [33], and recently with the definition `GrothendieckUniverse` [26], [11], [11]. These definitions are useful in many articles [28, 33, 8, 35], [19, 32, 31, 15, 6], but also [34, 12, 20, 22, 21], [27, 2, 3, 23, 16, 7, 4, 5].

In this paper, using the Mizar system [9] [10], we trivially show that Grothendieck's definition of Universe as defined in [26], coincides with the original definition of Universe defined by Artin, Grothendieck, and Verdier (*Chapitre 0 Univers et Appendice "Univers" (par N. Bourbaki) de l'Exposé I. "PREFAISCEAUX"*) [1], and how the different definitions of MML concerning universes are related. We also show that the definition of Universe introduced by Mac Lane ([18]) is compatible with the MML's definition.

Although a universe may be empty, we consider the properties of non-empty universes, completing the properties proved in [25].

We introduce the notion of "trivial" and "non-trivial" Universes, depending on whether or not they contain the set ω (`NAT`), following the notion of Robert M. Solovay[2]. The following result links the universes $\mathbf{U}_0$ (`FinSETS`) and $\mathbf{U}_1$ (`SETS`):

$$\text{GrothendieckUniverse } \omega = \text{GrothendieckUniverse } \mathbf{U}_0 = \mathbf{U}_1$$

Before turning to the last section, we establish some trivial propositions allowing the construction of sets outside the considered universe.

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[2]`https://cs.nyu.edu/pipermail/fom/2008-March/012783.html`

The last section is devoted to the construction, in Tarski-Grothendieck, of a tower of universes indexed by the ordinal numbers (See 8. Examples, Grothendieck universe, ncatlab.org [24]).

Grothendieck's universe is referenced in current works: "Assuming the existence of a sufficient supply of (Grothendieck) univers", Jacob Lurie in "Higher Topos Theory" [17], "Annexe B – Some results on Grothendieck universes", Olivia Caramello and Riccardo Zanfa in "Relative topos theory via stacks" [13], "Remark 1.1.5 (quoting Michael Shulman [30])", Emily Riehl in "Category theory in Context" [29], and more specifically "Strict Universes for Grothendieck Topoi" [14].

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1. Preliminaries

Now we state the propositions:

(1) Let us consider a set X. Then $\pi_1(X), \pi_2(X) \in 2^{\bigcup \bigcup X}$.

(2) $\mathbb{R}^* =$ the set of all X where X is a finite sequence of elements of $\mathbb{R}$.

One can verify that there exists a Grothendieck which is empty and there exists a Grothendieck which is non empty.

Let X be a set. One can verify that every Grothendieck of X is non empty.

2. Original Definitions of Grothendieck's Universe

Let $\mathcal{G}$ be a set. We say that $\mathcal{G}$ satisfies axiom GU_1 if and only if

(Def. 1) for every sets x, y such that $x \in \mathcal{G}$ and $y \in x$ holds $y \in \mathcal{G}$.

We say that $\mathcal{G}$ satisfies axiom GU_2 if and only if

(Def. 2) for every sets x, y such that $x, y \in \mathcal{G}$ holds $\{x, y\} \in \mathcal{G}$.

We say that $\mathcal{G}$ satisfies axiom GU_3 if and only if

(Def. 3) for every set x such that $x \in \mathcal{G}$ holds $2^x \in \mathcal{G}$.

Let $\mathcal{G}$ be a non empty set. We say that $\mathcal{G}$ satisfies axiom GU_4 if and only if

(Def. 4) for every element I of $\mathcal{G}$ and for every $\mathcal{G}$-valued many sorted set x indexed by I, $\bigcup \operatorname{rng} x \in \mathcal{G}$.

3. Equivalences of Definitions

Now we state the propositions:

(3) Let us consider a set X. Then X satisfies axiom GU_1 if and only if X is transitive.

(4) Let us consider a non empty set X. Then X satisfies axiom GU_4 if and only if X is Family-Union-closed.

(5) Let us consider a Family-Union-closed set X, and a function f. Suppose $\operatorname{dom} f \in X$ and $\operatorname{rng} f \subseteq X$. Then $\bigcup \operatorname{rng} f \in X$.

One can check that every Grothendieck satisfies axiom GU_1, axiom GU_2, and axiom GU_3 and every non empty Grothendieck satisfies axiom GU_4.

Now we state the proposition:

(6) Let us consider a non empty set $\mathcal{G}$. Suppose $\mathcal{G}$ satisfies axiom GU_1, axiom GU_2, axiom GU_3, and axiom GU_4. Then $\mathcal{G}$ is a non empty Grothendieck.

Let us consider a set X. Now we state the propositions:

(7) X is a universal class if and only if X is a non empty Grothendieck.

(8) $\mathbf{T}(\{X\}^{*\in})$ is a Grothendieck of X.

(9) The universe of $\{X\}$ is a Grothendieck of X. The theorem is a consequence of (8).

(10) $\mathrm{Universe\_closure}(\{X\}) = \mathrm{GrothendieckUniverse}(X)$.

4. Equivalences of Mac Lane Definition

Now we state the propositions:

(11) Let us consider a Grothendieck U. Suppose $\omega \in U$. Then

(i) for every sets x, u such that $x \in u \in U$ holds $x \in U$, and

(ii) for every sets u, v such that $u, v \in U$ holds $\{u, v\}$, $\langle u, v \rangle$, $u \times v \in U$, and

(iii) for every set x such that $x \in U$ holds 2^x, $\bigcup x \in U$, and

(iv) $\omega \in U$, and

(v) for every sets a, b and for every function f from a into b such that $\operatorname{dom} f = a$ and f is onto and $a \in U$ and $b \subseteq U$ holds $b \in U$.

(12) Let us consider a set U. Suppose for every sets x, u such that $x \in u \in U$ holds $x \in U$ and for every set x such that $x \in U$ holds 2^x, $\bigcup x \in U$ and $\omega \in U$ and for every sets a, b and for every function f from a into b such that $\operatorname{dom} f = a$ and f is onto and $a \in U$ and $b \subseteq U$ holds $b \in U$. Then U is a Grothendieck. The theorem is a consequence of (4) and (3).

5. Properties of Universe, Following [25]

From now on X denotes a set and $\mathcal{U}$ denotes a universal class.

Now we state the proposition:

(13) Suppose X satisfies axiom GU_1 and axiom GU_3. Then

(i) for every set y and for every subset x of y such that $y \in X$ holds $x \in X$, and

(ii) for every sets x, y such that $x \subseteq y$ and $y \in X$ holds $x \in X$, and

(iii) if X is not empty, then $\emptyset \in X$.

Let $\mathcal{U}$ be a universal class. The functor $\emptyset_{\mathcal{U}}$ yielding an element of $\mathcal{U}$ is defined by the term

(Def. 5) $\emptyset$.

Now we state the propositions:

(14) $\mathcal{U}$ is a Grothendieck of $\emptyset$. The theorem is a consequence of (13).

(15) Let us consider elements u, v of $\mathcal{U}$. Then $v^u \subseteq$ the set of all f where f is a function from u into v.

Let $\mathcal{U}$ be a universal class and u be an element of $\mathcal{U}$. Note that the functor $\operatorname{succ} u$ yields an element of $\mathcal{U}$. Now we state the propositions:

(16) Let us consider a natural number n. Then $n \in \mathcal{U}$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv \$_1 \in \mathcal{U}$. $\mathcal{P}[0]$. For every natural number n, $\mathcal{P}[n]$. $\square$

(17) $\omega \subseteq \mathcal{U}$.

(18) (i) $\mathbb{N} \in \mathcal{U}$, or

(ii) $\mathbb{N} \approx \mathcal{U}$.

The theorem is a consequence of (16).

Let us note that every universal class is infinite. Now we state the proposition:

(19) $\mathbf{U}_0$ is denumerable.

Observe that there exists a universal class which is denumerable.

Now we state the proposition:

(20) $\mathcal{U}$ is not denumerable if and only if $\omega \in \mathcal{U}$.

Observe that there exists a universal class which is non denumerable.

Let $\mathcal{U}$ be a universal class. We say that $\mathcal{U}$ is trivial if and only if

(Def. 6) $\omega \notin \mathcal{U}$.

Now we state the proposition:

(21) (i) $\mathbf{U}_0$ is trivial, and

(ii) $\mathbf{U}_1$ is not trivial.

The theorem is a consequence of (16), (13), (19), and (20).

One can check that there exists a universal class which is trivial and there exists a universal class which is non trivial and every non trivial universal class is non denumerable. Now we state the proposition:

(22) Let us consider an element x of $\mathcal{U}$, and objects y, z. Suppose $x = \langle y, z \rangle$. Then

(i) y is an element of $\mathcal{U}$, and

(ii) z is an element of $\mathcal{U}$.

Let $\mathcal{U}$ be a universal class. Let us note that there exists an element of $\mathcal{U}$ which is pair. Now we state the proposition:

(23) Let us consider elements u, v of $\mathcal{U}$. Then the set of all f where f is a function from u into v is an element of $\mathcal{U}$. The theorem is a consequence of (13).

Let $\mathcal{U}$ be a universal class, I be an element of $\mathcal{U}$, and x be a $\mathcal{U}$-valued many sorted set indexed by I. Let us observe that the functor $\prod x$ yields an element of $\mathcal{U}$. Let x, y be elements of $\mathcal{U}$. The functor $x \uplus y$ yielding an element of $\mathcal{U}$ is defined by the term

(Def. 7) $[x \longmapsto \emptyset_{\mathcal{U}}, y \longmapsto \{\emptyset_{\mathcal{U}}\}]$.

Now we state the propositions:

(24) Let us consider elements x, y of $\mathcal{U}$. Then $x \uplus y$ is a subset of $\{x, y\} \times \{\emptyset, \{\emptyset\}\}$.

(25) Let us consider an element u of $\mathcal{U}$. Then $u \uplus u = \{\langle u, \{\emptyset\}\rangle\}$.

Let $\mathcal{U}$ be a universal class, I be an element of $\mathcal{U}$, and x be a $\mathcal{U}$-valued many sorted set indexed by I. Note that the functor $\operatorname{dom} x$ yields an element of $\mathcal{U}$. Note that the functor $\bigcup x$ yields an element of $\mathcal{U}$. Let us note that the functor $\operatorname{disjoint} x$ yields a $\mathcal{U}$-valued many sorted set indexed by I. The functor $\biguplus x$ yielding an element of $\mathcal{U}$ is defined by the term

(Def. 8) $\bigcup \operatorname{disjoint} x$.

Let us consider an element I of $\mathcal{U}$ and a $\mathcal{U}$-valued many sorted set x indexed by I. Now we state the propositions:

(26) $\bigcup \operatorname{coprod}(x)$ is an element of $\mathcal{U}$.

(27) $\biguplus x$ is a subset of $\bigcup \operatorname{rng} x \times I$.

(28) If X satisfies axiom GU_2, then for every set x such that $x \in X$ holds $\{x\} \in X$.

Let us consider an element u of $\mathcal{U}$. Now we state the propositions:

(29) $\overline{\overline{u}} \in \mathcal{U}$.

(30) (i) $u \not\approx \mathcal{U}$, and

(ii) $\overline{\overline{u}} \in \overline{\overline{\mathcal{U}}}$.

(31) Let us consider elements u, v of $\mathcal{U}$. Then $\{\langle u, \emptyset\rangle, \langle v, \{\emptyset\}\rangle\} = \{u\} \times \{\emptyset\} \cup \{v\} \times \{\{\emptyset\}\}$.

(32) Let us consider elements I, a, b, u, v of $\mathcal{U}$, and a $\mathcal{U}$-valued many sorted set x indexed by I. Suppose $I = \{a, b\}$ and $x(a) = u$ and $x(b) = v$. Then $\biguplus x = u \times \{a\} \cup v \times \{b\}$.

Let us consider elements I, u, v of $\mathcal{U}$ and a $\mathcal{U}$-valued many sorted set x indexed by I. Now we state the propositions:

(33) Suppose $I = \{\emptyset, \{\emptyset\}\}$ and $x(\emptyset) = u$ and $x(\{\emptyset\}) = v$. Then $\biguplus x = u \times \{\emptyset\} \cup v \times \{\{\emptyset\}\}$. The theorem is a consequence of (32).

(34) Suppose $I = \{\emptyset, \{\emptyset\}\}$ and $x(\emptyset) = \{u\}$ and $x(\{\emptyset\}) = \{v\}$ and $u \neq v$. Then $\biguplus x = u \uplus v$. The theorem is a consequence of (33) and (31).

(35) Let us consider an element x of $\mathcal{U}$, and objects y, z. Suppose $x = \langle y, z\rangle$. Then

(i) y is an element of $\mathcal{U}$, and

(ii) z is an element of $\mathcal{U}$.

Let $\mathcal{U}$ be a universal class. Observe that there exists an element of $\mathcal{U}$ which is pair.

Let u be a pair element of $\mathcal{U}$. The functors: $(u)_{\mathbf{1}}$ and $(u)_{\mathbf{2}}$ yield elements of $\mathcal{U}$. Now we state the proposition:

(36) Let us consider an element X of $\mathcal{U}$. Then

(i) $\pi_1(X)$ is an element of $\mathcal{U}$, and

(ii) $\pi_2(X)$ is an element of $\mathcal{U}$.

The theorem is a consequence of (1).

Let us consider a binary relation R. Now we state the propositions:

(37) If $R \in \mathcal{U}$, then $\operatorname{dom} R$, $\operatorname{rng} R \in \mathcal{U}$. The theorem is a consequence of (36).

(38) If $\operatorname{dom} R$ is an element of $\mathcal{U}$ and $\operatorname{rng} R$ is an element of $\mathcal{U}$, then R is an element of $\mathcal{U}$. The theorem is a consequence of (13).

(39) Let us consider a set X, a non empty set Y, and a function f from X into Y. If $f \in \mathcal{U}$, then $X \in \mathcal{U}$. The theorem is a consequence of (37).

(40) Let us consider non empty sets A, B. Suppose $A \times B$ is an element of $\mathcal{U}$. Then

(i) A is an element of $\mathcal{U}$, and

(ii) B is an element of $\mathcal{U}$.

The theorem is a consequence of (36).

(41) Let us consider a set X. Suppose id_X is an element of $\mathcal{U}$. Then X is an element of $\mathcal{U}$. The theorem is a consequence of (37).

(42) Let us consider elements x, y, z of $\mathcal{U}$. Then $\langle x, y \rangle \longmapsto z$ is an element of $\mathcal{U}$.

6. Properties of Universe Containing ω

Now we state the propositions:

(43) $\omega \subset \mathbf{U}_0$. The theorem is a consequence of (16).

(44) Let us consider a set X. Then $\mathbf{T}(\emptyset) \subseteq \mathbf{T}(X)$.

(45) Let us consider a Grothendieck $\mathcal{G}$ of X. Then $\mathbf{U}_0 \subseteq \mathcal{G}$. The theorem is a consequence of (44).

(46) (i) GrothendieckUniverse$(\emptyset) = \mathbf{U}_0$, and

(ii) GrothendieckUniverse$(\emptyset) = \mathbf{U}_\emptyset$.

(47) Let us consider a set X, and a Grothendieck $\mathcal{G}$ of X. Then Grothendieck Universe$(\emptyset) \subseteq$ GrothendieckUniverse$(X) \subseteq \mathcal{G}$.

(48) Let us consider an element n of $\mathbf{U}_0$. Then GrothendieckUniverse$(n) = \mathbf{U}_0$. The theorem is a consequence of (45).

(49) the empty Grothendieck $\subset \omega \subset$ GrothendieckUniverse$(\emptyset) \subset$ Grothendieck Universe(ω). The theorem is a consequence of (16), (46), (43), (19), and (20).

(50) Let us consider a non empty Grothendieck $\mathcal{G}$. Suppose $\mathcal{G} \neq$ Grothendieck Universe(ω). Then

(i) GrothendieckUniverse$(\omega) \in \mathcal{G}$, or

(ii) $\mathcal{G} \in$ GrothendieckUniverse(ω).

(51) $\mathbf{T}(\omega) =$ GrothendieckUniverse(ω).

(52) Let us consider sets N_1, N_2. Suppose $N_1 = \mathbb{N} \times \mathbb{N} \cup \mathbb{N}$ and $N_2 = N_1 \cup 2^{N_1}$. Then $\mathbb{R} \subseteq N_2 \cup \mathbb{N} \times N_2$.

Let us consider a non trivial universal class $\mathcal{U}$. Now we state the propositions:

(53) $\mathbb{R}$ is an element of $\mathcal{U}$. The theorem is a consequence of (52) and (13).

(54) $\overline{\mathbb{R}}$ is an element of $\mathcal{U}$. The theorem is a consequence of (53) and (13).

(55) $\mathbb{C} \in \mathcal{U}$. The theorem is a consequence of (16), (53), and (13).

(56) $\mathbb{H} \in \mathcal{U}$. The theorem is a consequence of (16), (53), (55), and (13).

(57) Let us consider a natural number n. Then $\mathrm{Seg}\, n \in \mathcal{U}$. The theorem is a consequence of (16) and (13).

(58) Let us consider a set D. If $D \in \mathcal{U}$, then for every natural number n, $D^n \in \mathcal{U}$. The theorem is a consequence of (57).

(59) Let us consider a non trivial universal class $\mathcal{U}$, and a natural number n. Then $\mathcal{R}^n \in \mathcal{U}$. The theorem is a consequence of (53) and (58).

Let us consider a set X and a natural number n. Now we state the propositions:

(60) If $X \in \mathcal{U}$, then $X^n \in \mathcal{U}$. The theorem is a consequence of (57).

(61) $X^n \subseteq X^*$.

(62) Let us consider a non empty set X, and an object x. If $x \in X^*$, then there exists a natural number n such that $x \in X^n$.

(63) Let us consider a non empty set X. Then there exists a function f such that

(i) $\operatorname{dom} f = \mathbb{N}$, and

(ii) for every natural number n, $f(n) = X^n$, and

(iii) $\bigcup \operatorname{rng} f = X^*$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a natural number n such that $\$_1 = n$ and $\$_2 = X^n$. For every object x such that $x \in \mathbb{N}$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = \mathbb{N}$ and for every object x such that $x \in \mathbb{N}$ holds $\mathcal{P}[x, f(x)]$. For every natural number n, $f(n) = X^n$. $\bigcup \operatorname{rng} f = X^*$. $\square$

(64) Let us consider a non trivial universal class $\mathcal{U}$, and a non empty set X. If $X \in \mathcal{U}$, then $X^* \in \mathcal{U}$. The theorem is a consequence of (63) and (58).

Let us consider a non trivial universal class $\mathcal{U}$. Now we state the propositions:

(65) $\mathbb{R}^* \in \mathcal{U}$. The theorem is a consequence of (53) and (64).

(66) $\overline{\mathbb{R}}^* \in \mathcal{U}$. The theorem is a consequence of (54) and (64).

(67) $\mathbb{C}^* \in \mathcal{U}$.

(68) $(\mathbb{H})^* \in \mathcal{U}$.

(69) Let us consider a universal class $\mathcal{U}$, and a set X. If $X \in \mathcal{U}$, then for every finite sequence s of elements of X, $s \in \mathcal{U}$. The theorem is a consequence of (57) and (13).

(70) Let us consider an empty set X, and a finite sequence f of elements of X^*. Then $f = \operatorname{len} f \mapsto 0$.

(71) Let us consider a non trivial universal class $\mathcal{U}$, and a non empty set D. If $D \in \mathcal{U}$, then for every matrix M over D, $M \in \mathcal{U}$.

(72) $\mathbf{U}_0, \mathbb{N}, \mathbb{R}, \overline{\mathbb{R}} \in \mathbf{U}_1$. The theorem is a consequence of (16), (13), (53), and (54).

(73) Let us consider a set X, and a universal class $\mathcal{U}$. If $\mathcal{U} \in \mathbf{T}(X)$, then $\mathbf{T}(\mathcal{U}) \subseteq \mathbf{T}(X)$.

(74) $\mathbf{U}_0 \in \mathbf{T}(\omega)$. The theorem is a consequence of (19) and (20).

(75) $\mathbf{U}_1 = \mathbf{T}(\omega)$. The theorem is a consequence of (72), (73), and (74).

(76) $\mathrm{GrothendieckUniverse}(\omega) = \mathbf{U}_1$.

(77) $\mathrm{GrothendieckUniverse}(\omega) = \mathrm{GrothendieckUniverse}(\mathbf{U}_0) = \mathbf{U}_1$.
Proof: $\mathrm{GrothendieckUniverse}(\omega) = \mathrm{GrothendieckUniverse}(\mathbf{U}_0)$. □

Let us consider a non empty set X, a Grothendieck $\mathcal{G}'$ of X, and a universal class $\mathcal{G}$. Now we state the propositions:

(78) If X misses $\mathcal{G}$, then $\mathcal{G}' \neq \mathcal{G}$.

(79) If X misses $\mathcal{G}$, then $\mathcal{G}' \in \mathcal{G}$ or $\mathcal{G} \in \mathcal{G}'$.

(80) Let us consider universal classes $\mathcal{U}, \mathcal{U}'$, and an element a of $\mathcal{U}$. If $a \notin \mathcal{U}'$, then $\mathcal{U}' \in \mathcal{U}$. The theorem is a consequence of (78).

(81) Let us consider a Grothendieck $\mathcal{G}$. Then $\bigcup \mathcal{G} = \mathcal{G}$.

One can verify that every Grothendieck is limit ordinal.

Now we state the proposition:

(82) Let us consider a universal class $\mathcal{U}$, and a non empty element V of $\mathcal{U}$. Then $\mathrm{Funcs}\, V$ is a subset of $\mathcal{U}$. The theorem is a consequence of (81).

7. How to Get Out of a Universe?

Now we state the propositions:

(83) There exists a set a such that $a \notin \mathcal{U}$.

(84) There exists a subset A of $\mathcal{U}$ such that $A \notin \mathcal{U}$.

(85) the set of all u where u is an element of $\mathcal{U}$ is not an element of $\mathcal{U}$.

(86) Let us consider an element X of $\mathcal{U}$. Then $\mathcal{U} \setminus X$ is not an element of $\mathcal{U}$.
Proof: $\mathcal{U} \setminus X \notin \mathcal{U}$. □

(87) $2^{\mathcal{U}} \notin \mathcal{U}$.

8. A Sequence of Universes

Now we state the proposition:

(88) Let us consider a set X. Then there exists a function f such that

(i) $\mathrm{dom}\, f = \mathbb{N}$, and

(ii) $f(0) = X$, and

(iii) for every natural number n, $f(n+1) = \mathrm{GrothendieckUniverse}(f(n))$.

Proof: Define $\mathcal{G}(\text{set}, \text{set}) = \text{GrothendieckUniverse}(\$_2)$. There exists a function f such that $\operatorname{dom} f = \mathbb{N}$ and $f(0) = X$ and for every natural number n, $f(n+1) = \mathcal{G}(n, f(n))$. □

The Construction of X, GrothendieckUniverse(X), GrothendieckUniverse (GrothendieckUniverse(X)), ...

Let X be a set. The functor sequence-universe(X) yielding a function is defined by

(Def. 9) $\operatorname{dom} it = \mathbb{N}$ and $it(0) = X$ and for every natural number n, $it(n+1) =$ GrothendieckUniverse($it(n)$).

Now we state the propositions:

(89) Let us consider a set X. Then sequence-universe(X) is a transfinite sequence.

(90) Let us consider a set X, and a transfinite sequence S. If $\operatorname{dom} S = \mathbb{N}$, then $\operatorname{last} S = S(\mathbb{N})$.

(91) Let us consider a transfinite sequence S. Suppose $\operatorname{dom} S = \mathbb{N}$. Then

(i) $S(\mathbb{N}) = \emptyset$, and

(ii) $\operatorname{last} S = \emptyset$.

The theorem is a consequence of (90).

(92) Let us consider a set X, and a transfinite sequence S. Suppose $S =$ sequence-universe(X). Then

(i) $\operatorname{last} S = \emptyset$, and

(ii) $S(\mathbb{N}) = \emptyset$.

The theorem is a consequence of (91).

The Construction of $X \cup$GrothendieckUniverse(X)$\cup$GrothendieckUniverse(GrothendieckUniverse(X)) $\cup \ldots$.

Let X be a set. The functor union-sequence-universe(X) yielding a non empty set is defined by the term

(Def. 10) $\bigcup \operatorname{rng}$ sequence-universe(X).

Now we state the proposition:

(93) Let us consider a set X. Then rng sequence-universe(X) $\subseteq$ union-sequence-universe(X).

The Formal Counterpart of $\emptyset (= \mathcal{U}_0) \in \mathcal{U}_1 \in \mathcal{U}_2 \in \ldots$: Sequence of universes indexed by the ordinal numbers (see 8. Examples, Grothendieck Universe [24]).

The functor sequence-universe yielding a sequence of union-sequence-universe($\emptyset$) is defined by the term

(Def. 11) sequence-universe($\emptyset$).

Now we state the propositions:

(94) $\emptyset$, $\mathbf{U}_0$, $\mathbf{U}_1 \in \operatorname{rng}$ sequence-universe. The theorem is a consequence of (45) and (77).

(95) $\bigcup_{n<\omega} \mathcal{U}_n$ IS NOT A UNIVERSE:
$\bigcup \operatorname{rng}$ sequence-universe is not a Grothendieck. The theorem is a consequence of (72) and (94).

(96) (i) $\mathbf{T}(\mathbf{U}_0) =$ GrothendieckUniverse($\mathbf{U}_0$), and

(ii) $\mathbf{T}(\mathbf{U}_1) =$ GrothendieckUniverse($\mathbf{U}_1$).

(97) Let us consider a set X, and a natural number n. Then

(i) (sequence-universe(X))($n+1$) is transitive, and

(ii) $\mathbf{T}$((sequence-universe(X))($n+1$)) $=$
GrothendieckUniverse((sequence-universe(X))($n+1$)).

Let us consider a natural number n. Now we state the propositions:

(98) $\mathbf{T}$((sequence-universe($\mathbf{U}_0$))(n)) $=$
GrothendieckUniverse((sequence-universe($\mathbf{U}_0$))(n)). The theorem is a consequence of (77).

(99) $\mathbf{U}_n \in \mathbf{U}_{n+1}$.

(100) (sequence-universe($\mathbf{U}_0$))(n) $= \mathbf{U}_n$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ (sequence-universe($\mathbf{U}_0$))($\$_1$) $= \mathbf{U}_{\$_1}$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. $\square$

(101) GrothendieckUniverse((sequence-universe($\emptyset$))(n)) $=$
(sequence-universe(GrothendieckUniverse($\emptyset$)))(n).
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ GrothendieckUniverse((sequence-universe($\emptyset$))($\$_1$)) $=$ (sequence-universe(GrothendieckUniverse($\emptyset$)))($\$_1$). $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. $\square$

(102) (sequence-universe)($n+1$) $= \mathbf{U}_n$. The theorem is a consequence of (46), (100), and (101).

Let us note that there exists an element of $\bigcup \operatorname{rng}$ sequence-universe which is non empty.

Now we state the propositions:

(103) $\mathbf{U}_0$, $\mathbf{U}_1 \in$ GrothendieckUniverse(sequence-universe). The theorem is a consequence of (45) and (77).

(104) Let us consider a natural number n. Then (sequence-universe)($n+1$) $\in$ GrothendieckUniverse(sequence-universe). The theorem is a consequence of (45) and (102).

The Construction of $\mathcal{U}_\omega$: Tower of universes indexed by the ordinal numbers (see 8. Examples, Grothendieck Universe [24]).

The functor $\mathcal{U}_\omega$ yielding a non trivial universal class is defined by the term

(Def. 12) GrothendieckUniverse(sequence-universe).

Now we state the proposition:

(105) Let us consider a natural number n. Then (sequence-universe)$(n) \subseteq$ (sequence-universe)$(n+1)$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ (sequence-universe)$(\$_1) \subseteq$ (sequence-universe)$(\$_1 + 1)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number n, $\mathcal{P}[n]$. $\square$

Let X be an element of $\bigcup$ rng sequence-universe. The functor rank-universe(X) yielding a natural number is defined by

(Def. 13) $X \in$ (sequence-universe)(it) and for every natural number n such that $n < it$ holds $X \notin$ (sequence-universe)(n).

Now we state the propositions:

(106) Let us consider an element X of $\bigcup$ rng sequence-universe, and a natural number n. Suppose rank-universe$(X) \leqslant n$.
Then $X \in$ (sequence-universe)(n).

Proof: Define $\mathcal{P}$[natural number] $\equiv X \in$ (sequence-universe)$(\$_1)$. For every natural number j such that rank-universe$(X) \leqslant j$ and $\mathcal{P}[j]$ holds $\mathcal{P}[j+1]$. For every natural number i such that rank-universe$(X) \leqslant i$ holds $\mathcal{P}[i]$. $\square$

(107) Let us consider a natural number i. Then there exists a set x such that $x \in$ (sequence-universe)$(i + 1) \setminus$ (sequence-universe)(i). The theorem is a consequence of (105) and (102).

(108) Let us consider a natural number n. Then $\mathbf{U}_{n+1} \setminus (\mathbf{U}_n) \notin \mathbf{U}_{n+1}$. The theorem is a consequence of (99) and (86).

The functor ComplUniverse yielding a function from $\mathbb{N}$ into $\bigcup$ rng sequence-universe is defined by

(Def. 14) for every natural number n, $it(n) = \mathbf{U}_{n+1} \setminus (\mathbf{U}_n)$.

Let us consider a natural number n. Now we state the propositions:

(109) (ComplUniverse)(n) is not empty. The theorem is a consequence of (99).

(110) (ComplUniverse)$(n) \subseteq \mathbf{U}_{n+1}$.

(111) There exists a function f from $\mathbb{N}$ into $\bigcup\bigcup$ rng sequence-universe such that for every natural number i, $f(i) \in$ (ComplUniverse)(i).

Proof: Set $g =$ the choice of ComplUniverse. rng $g \subseteq \bigcup\bigcup$ rng sequence-universe. For every natural number i, $g(i) \in$ (ComplUniverse)(i). $\square$

(112) Let us consider a function f from $\mathbb{N}$ into $\bigcup \operatorname{rng}$ sequence-universe. Then $f \in \mathcal{U}_\omega$. The theorem is a consequence of (13) and (104).

(113) Let us consider a function f from $\mathbb{N}$ into $\bigcup \bigcup \operatorname{rng}$ sequence-universe. Then $f \in \mathcal{U}_\omega$. The theorem is a consequence of (13) and (104).

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Isomorphism between Spaces of Multilinear Maps and Nested Compositions over Real Normed Vector Spaces

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Summary. This paper formalizes in Mizar [1], [2], that the isometric isomorphisms between spaces formed by an $(n + 1)$-dimensional multilinear map and an n-fold composition of linear maps on real normed spaces. This result is used to describe the space of nth-order derivatives of the Frechet derivative as a multilinear space. In Section 1, we discuss the spaces of 1-dimensional multilinear maps and 0-fold compositions as a preparation, and in Section 2, we extend the discussion to the spaces of $(n + 1)$-dimensional multilinear map and an n-fold compositions. We referred to [4], [11], [8], [9] in this formalization.

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1. Preliminaries

Let X be a real linear space. The functor IsoCPRLSP(X) yielding a linear operator from X into $\prod\langle X\rangle$ is defined by

(Def. 1) for every point x of X, $it(x) = \langle x\rangle$.

Now we state the proposition:

(1) Let us consider a real linear space X.
Then $0_{\prod\langle X\rangle} = (\text{IsoCPRLSP}(X))(0_X)$.

Let X be a real linear space. Observe that IsoCPRLSP(X) is one-to-one and onto and there exists a linear operator from X into $\prod\langle X\rangle$ which is one-to-one and onto.

Let f be a bijective linear operator from X into $\prod\langle X\rangle$. Let us note that the functor f^{-1} yields a linear operator from $\prod\langle X\rangle$ into X. Let f be a one-to-one, onto linear operator from X into $\prod\langle X\rangle$. Let us note that f^{-1} is bijective as a linear operator from $\prod\langle X\rangle$ into X and there exists a linear operator from $\prod\langle X\rangle$ into X which is one-to-one and onto.

Now we state the propositions:

(2) Let us consider a real linear space X, and a point x of X. Then $((\mathrm{IsoCPRLSP}(X))^{-1})(\langle x\rangle) = x$.
PROOF: Set $I = \mathrm{IsoCPRLSP}(X)$. Set $J = I^{-1}$. For every point x of X, $J(\langle x\rangle) = x$. □

(3) Let us consider a real linear space X. Then $((\mathrm{IsoCPRLSP}(X))^{-1})(0_{\prod\langle X\rangle}) = 0_X$. The theorem is a consequence of (1).

(4) Let us consider a real linear space G. Then

(i) for every set x, x is a point of $\prod\langle G\rangle$ iff there exists a point x_1 of G such that $x = \langle x_1\rangle$, and

(ii) for every points x, y of $\prod\langle G\rangle$ and for every points x_1, y_1 of G such that $x = \langle x_1\rangle$ and $y = \langle y_1\rangle$ holds $x + y = \langle x_1 + y_1\rangle$, and

(iii) $0_{\prod\langle G\rangle} = \langle 0_G\rangle$, and

(iv) for every point x of $\prod\langle G\rangle$ and for every point x_1 of G such that $x = \langle x_1\rangle$ holds $-x = \langle -x_1\rangle$, and

(v) for every point x of $\prod\langle G\rangle$ and for every point x_1 of G and for every real number a such that $x = \langle x_1\rangle$ holds $a \cdot x = \langle a \cdot x_1\rangle$.

PROOF: Consider I being a function from G into $\prod\langle G\rangle$ such that I is one-to-one and onto and for every point x of G, $I(x) = \langle x\rangle$ and for every points v, w of G, $I(v+w) = I(v)+I(w)$ and for every point v of G and for every element r of $\mathbb{R}$, $I(r \cdot v) = r \cdot I(v)$ and $0_{\prod\langle G\rangle} = I(0_G)$. For every set x, x is a point of $\prod\langle G\rangle$ iff there exists a point x_1 of G such that $x = \langle x_1\rangle$.

For every points x, y of $\prod\langle G\rangle$ and for every points x_1, y_1 of G such that $x = \langle x_1\rangle$ and $y = \langle y_1\rangle$ holds $x + y = \langle x_1 + y_1\rangle$. For every point x of $\prod\langle G\rangle$ and for every point x_1 of G such that $x = \langle x_1\rangle$ holds $-x = \langle -x_1\rangle$. For every point x of $\prod\langle G\rangle$ and for every point x_1 of G and for every real number a such that $x = \langle x_1\rangle$ holds $a \cdot x = \langle a \cdot x_1\rangle$. □

(5) Let us consider real linear spaces X, Y, and a function f from X into Y. Then f is a linear operator from X into Y if and only if

$f \cdot ((\mathrm{IsoCPRLSP}(X))^{-1})$ is a linear operator from $\prod\langle X\rangle$ into Y.

(6) Let us consider real linear spaces X, Y, and a function f from $\prod\langle X\rangle$ into Y. Then f is a linear operator from $\prod\langle X\rangle$ into Y if and only if $f \cdot (\mathrm{IsoCPRLSP}(X))$ is a linear operator from X into Y. The theorem is a consequence of (5).

(7) Let us consider a real linear space X, a point s of $\prod\langle X\rangle$, and an element i of $\mathrm{dom}\langle X\rangle$. Then $\mathrm{reproj}(i, s) = \mathrm{IsoCPRLSP}(X)$.
Proof: For every element x of X, $(\mathrm{reproj}(i, s))(x) = (\mathrm{IsoCPRLSP}(X))(x)$. □

(8) Let us consider real linear spaces X, Y, and an object f. Then f is a linear operator from $\prod\langle X\rangle$ into Y if and only if f is a multilinear operator from $\langle X\rangle$ into Y. The theorem is a consequence of (6) and (7).

Let us consider real linear spaces X, Y. Now we state the propositions:

(9) $\mathrm{MultOpers}(\langle X\rangle, Y) = \mathrm{LinearOperators}(\prod\langle X\rangle, Y)$. The theorem is a consequence of (8).

(10) $\mathrm{VectorSpaceOfMultOpers}_{\mathbb{R}}(\langle X\rangle, Y) = \mathrm{VectorSpaceOfLinearOpers}_{\mathbb{R}}(\prod\langle X\rangle, Y)$. The theorem is a consequence of (9).

(11) Let us consider a real normed space G. Then

(i) for every set x, x is a point of $\prod\langle G\rangle$ iff there exists a point x_1 of G such that $x = \langle x_1\rangle$, and

(ii) for every points x, y of $\prod\langle G\rangle$ and for every points x_1, y_1 of G such that $x = \langle x_1\rangle$ and $y = \langle y_1\rangle$ holds $x + y = \langle x_1 + y_1\rangle$, and

(iii) $0_{\prod\langle G\rangle} = \langle 0_G\rangle$, and

(iv) for every point x of $\prod\langle G\rangle$ and for every point x_1 of G such that $x = \langle x_1\rangle$ holds $-x = \langle -x_1\rangle$, and

(v) for every point x of $\prod\langle G\rangle$ and for every point x_1 of G and for every real number a such that $x = \langle x_1\rangle$ holds $a \cdot x = \langle a \cdot x_1\rangle$, and

(vi) for every point x of $\prod\langle G\rangle$ and for every point x_1 of G such that $x = \langle x_1\rangle$ holds $\|x\| = \|x_1\|$.

Proof: Consider I being a function from G into $\prod\langle G\rangle$ such that I is one-to-one and onto and for every point x of G, $I(x) = \langle x\rangle$ and for every points v, w of G, $I(v + w) = I(v) + I(w)$ and for every point v of G and for every element r of $\mathbb{R}$, $I(r \cdot v) = r \cdot I(v)$ and $0_{\prod\langle G\rangle} = I(0_G)$ and for every point v of G, $\|I(v)\| = \|v\|$. For every set x, x is a point of $\prod\langle G\rangle$ iff there exists a point x_1 of G such that $x = \langle x_1\rangle$.

For every points x, y of $\prod\langle G\rangle$ and for every points x_1, y_1 of G such that $x = \langle x_1\rangle$ and $y = \langle y_1\rangle$ holds $x + y = \langle x_1 + y_1\rangle$. For every point x of $\prod\langle G\rangle$ and for every point x_1 of G such that $x = \langle x_1\rangle$ holds $-x = \langle -x_1\rangle$. For every point x of $\prod\langle G\rangle$ and for every point x_1 of G and for every real number a such that $x = \langle x_1\rangle$ holds $a \cdot x = \langle a \cdot x_1\rangle$. For every point x of $\prod\langle G\rangle$ and for every point x_1 of G such that $x = \langle x_1\rangle$ holds $\|x\| = \|x_1\|$. □

Let X be a real normed space. The functor IsoCPNrSP(X) yielding a linear operator from X into $\prod\langle X\rangle$ is defined by

(Def. 2) for every point x of X, $it(x) = \langle x\rangle$.

Now we state the proposition:

(12) Let us consider a real normed space X.
Then $0_{\prod\langle X\rangle} = (\text{IsoCPNrSP}(X))(0_X)$.

Let X be a real normed space. Let us note that IsoCPNrSP(X) is one-to-one, onto, and isometric and there exists a linear operator from X into $\prod\langle X\rangle$ which is one-to-one, onto, and isometric.

Let I be a one-to-one, onto, isometric linear operator from X into $\prod\langle X\rangle$. Let us observe that the functor I^{-1} yields a linear operator from $\prod\langle X\rangle$ into X. One can check that I^{-1} is one-to-one, onto, and isometric as a linear operator from $\prod\langle X\rangle$ into X and there exists a linear operator from $\prod\langle X\rangle$ into X which is one-to-one, onto, and isometric. Let us consider real normed spaces X, Y and a function f from X into Y. Now we state the propositions:

(13) f is a linear operator from X into Y if and only if $f \cdot ((\text{IsoCPNrSP}(X))^{-1})$ is a linear operator from $\prod\langle X\rangle$ into Y.

(14) f is a Lipschitzian linear operator from X into Y if and only if $f \cdot ((\text{IsoCPNrSP}(X))^{-1})$ is a Lipschitzian linear operator from $\prod\langle X\rangle$ into Y.

Let us consider real normed spaces X, Y and a function f from $\prod\langle X\rangle$ into Y. Now we state the propositions:

(15) f is a linear operator from $\prod\langle X\rangle$ into Y if and only if $f \cdot (\text{IsoCPNrSP}(X))$ is a linear operator from X into Y. The theorem is a consequence of (13).

(16) f is a Lipschitzian linear operator from $\prod\langle X\rangle$ into Y if and only if $f \cdot (\text{IsoCPNrSP}(X))$ is a Lipschitzian linear operator from X into Y. The theorem is a consequence of (14).

(17) Let us consider a real normed space X, a point s of $\prod\langle X\rangle$, and an element i of dom$\langle X\rangle$. Then reproj$(i, s) = \text{IsoCPNrSP}(X)$.
Proof: For every element x of X, $(\text{reproj}(i, s))(x) = (\text{IsoCPNrSP}(X))(x)$. □

(18) Let us consider a real normed space X, and a point x of $\prod\langle X\rangle$. Then NrProduct $x = \|x\|$. The theorem is a consequence of (11).

Let us consider real normed spaces X, Y and an object f. Now we state the propositions:

(19) f is a linear operator from $\prod\langle X\rangle$ into Y if and only if f is a multilinear operator from $\langle X\rangle$ into Y. The theorem is a consequence of (15) and (17).

(20) f is a Lipschitzian linear operator from $\prod\langle X\rangle$ into Y if and only if f is a Lipschitzian multilinear operator from $\langle X\rangle$ into Y. The theorem is a consequence of (16), (18), (17), and (11).

Let us consider real normed spaces X, Y. Now we state the propositions:

(21) $\mathrm{MultOpers}(\langle X\rangle, Y) = \mathrm{LinearOperators}(\prod\langle X\rangle, Y)$. The theorem is a consequence of (19).

(22) $\mathrm{BoundedMultOpers}(\langle X\rangle, Y) = \mathrm{BdLinOps}(\prod\langle X\rangle, Y)$. The theorem is a consequence of (20).

(23) $\mathrm{BoundedMultOpersNorm}(\langle X\rangle, Y) = \mathrm{BdLinOpsNorm}(\prod\langle X\rangle, Y)$.
Proof: Set $n_1 = \mathrm{BoundedMultOpersNorm}(\langle X\rangle, Y)$. Set $n_2 = \mathrm{BdLinOpsNorm}(\prod\langle X\rangle, Y)$. $\mathrm{BoundedMultOpers}(\langle X\rangle, Y) = \mathrm{BdLinOps}(\prod\langle X\rangle, Y)$. For every object f such that $f \in \mathrm{BoundedMultOpers}(\langle X\rangle, Y)$ holds $n_1(f) = n_2(f)$. $\square$

(24) $\mathrm{VectorSpaceOfMultOpers}_{\mathbb{R}}(\langle X\rangle, Y) = \mathrm{VectorSpaceOfLinearOpers}_{\mathbb{R}}(\prod\langle X\rangle, Y)$. The theorem is a consequence of (21).

(25) $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(\langle X\rangle, Y) =$ the real norm space of bounded linear operators from $\prod\langle X\rangle$ into Y. The theorem is a consequence of (24) and (23).

(26) Let us consider a real normed space X. If X is complete, then $\prod\langle X\rangle$ is complete.

2. Spaces of Multilinear Maps and Nested Compositions over Real Normed Vector Spaces

Now we state the propositions:

(27) Let us consider real norm space sequences X, Y, a real normed space Z, and a Lipschitzian bilinear operator f from $\prod X \times \prod Y$ into Z. Then $f \cdot ((\mathrm{IsoCPNrSP}(\prod X, \prod Y))^{-1})$ is a Lipschitzian multilinear operator from $\langle \prod X, \prod Y\rangle$ into Z.

(28) Let us consider real norm space sequences X, Y, a real normed space Z, and a point f of $\mathrm{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(\prod X, \prod Y, Z)$. Then $f \cdot ((\mathrm{IsoCPNrSP}(\prod X, \prod Y))^{-1})$ is a point of NormSpaceOfBoundedMult-$\mathrm{Opers}_{\mathbb{R}}(\langle \prod X, \prod Y\rangle, Z)$.

(29) Let us consider real linear space sequences X, Y. Then $\overline{X \frown Y} = \overline{X} \frown \overline{Y}$.
PROOF: Reconsider $C_1 = \overline{X}$, $C_2 = \overline{Y}$ as a finite sequence. For every natural number i such that $i \in \operatorname{dom} \overline{X \frown Y}$ holds $\overline{X \frown Y}(i) = (C_1 \frown C_2)(i)$. $\square$

(30) Let us consider a real linear space X. Then

(i) $\operatorname{len} \overline{\langle X \rangle} = \operatorname{len} \langle X \rangle$, and

(ii) $\operatorname{len} \overline{\langle X \rangle} = 1$, and

(iii) $\overline{\langle X \rangle} = \langle$the carrier of $X\rangle$.

(31) Let us consider a real norm space sequence X, an element x of $\prod X$, a real normed space Y, an element z of $\prod(X \frown \langle Y \rangle)$, an element i of $\operatorname{dom} X$, an element j of $\operatorname{dom}(X \frown \langle Y \rangle)$, an element x_i of $X(i)$, and a point y of Y. Suppose $i = j$ and $z = x \frown \langle y \rangle$. Then $(\operatorname{reproj}(j, z))(x_i) = (\operatorname{reproj}(i, x))(x_i) \frown \langle y \rangle$.
PROOF: Reconsider $x_j = x_i$ as an element of $(X \frown \langle Y \rangle)(j)$. For every object k such that $k \in \operatorname{dom}((\operatorname{reproj}(i, x))(x_i) \frown \langle y \rangle)$ holds $((\operatorname{reproj}(i, x))(x_i) \frown \langle y \rangle)(k) = (\operatorname{reproj}(j, z))(x_j)(k)$. $\square$

(32) Let us consider a real norm space sequence X, an element x of $\prod X$, a real normed space Y, an element z of $\prod(X \frown \langle Y \rangle)$, an element j of $\operatorname{dom}(X \frown \langle Y \rangle)$, an element y of Y, and a point y_0 of Y. Suppose $z = x \frown \langle y_0 \rangle$ and $j = \operatorname{len} x + 1$. Then $(\operatorname{reproj}(j, z))(y) = x \frown \langle y \rangle$.
PROOF: Reconsider $y_1 = y$ as an element of $(X \frown \langle Y \rangle)(j)$. For every object k such that $k \in \operatorname{dom}((\operatorname{reproj}(j, z))(y_1))$ holds $(\operatorname{reproj}(j, z))(y_1)(k) = (x \frown \langle y \rangle)(k)$. $\square$

(33) Let us consider a real norm space sequence X, an element x of $\prod X$, a real normed space Y, and a point y of Y. Then $x \frown \langle y \rangle$ is a point of $\prod(X \frown \langle Y \rangle)$.
PROOF: Set $C_1 = \overline{X}$. Set $C_2 =$ the carrier of Y. The carrier of $\prod(X \frown \langle Y \rangle) = \prod(\overline{X} \frown \overline{\langle Y \rangle})$. For every object i such that $i \in \operatorname{dom}(C_1 \frown \langle C_2 \rangle)$ holds $(x \frown \langle y \rangle)(i) \in (C_1 \frown \langle C_2 \rangle)(i)$. $\square$

(34) Let us consider a real norm space sequence X, an element x of $\prod X$, a real normed space Y, an element z of $\prod(X \frown \langle Y \rangle)$, and a point y of Y. Suppose $z = x \frown \langle y \rangle$. Then $\operatorname{NrProduct} z = \|y\| \cdot (\operatorname{NrProduct} x)$.
PROOF: Consider n_4 being a finite sequence of elements of $\mathbb{R}$ such that $\operatorname{dom} n_4 = \operatorname{dom}(X \frown \langle Y \rangle)$ and for every element i of $\operatorname{dom}(X \frown \langle Y \rangle)$, $n_4(i) = \|z(i)\|$ and $\operatorname{NrProduct} z = \prod n_4$. Set $n_3 = n_4 \restriction \operatorname{len} x$. Set $C_1 = \overline{X}$. Consider x_1 being a function such that $x = x_1$ and $\operatorname{dom} x_1 = \operatorname{dom} C_1$ and for every object i such that $i \in \operatorname{dom} C_1$ holds $x_1(i) \in C_1(i)$. For every element i of $\operatorname{dom} X$, $n_3(i) = \|x(i)\|$. $0 \leqslant \prod n_3$ by [7, (42)]. For every object i such that $i \in \operatorname{dom}(n_3 \frown \langle \|y\| \rangle)$ holds $(n_3 \frown \langle \|y\| \rangle)(i) = n_4(i)$. $\square$

(35) Let us consider real normed spaces X, Z, and a real norm space sequence Y. Then there exists a Lipschitzian linear operator I from the real norm space of bounded linear operators from X into NormSpaceOfBoundedMultOpers$_{\mathbb{R}}(Y, Z)$ into NormSpaceOfBoundedMultOpers$_{\mathbb{R}}(Y \frown \langle X \rangle, Z)$ such that

(i) I is one-to-one, onto, and isometric, and

(ii) for every point u of the real norm space of bounded linear operators from X into NormSpaceOfBoundedMultOpers$_{\mathbb{R}}(Y, Z)$, $\|u\| = \|I(u)\|$ and for every point y of $\prod Y$ and for every point x of X, $I(u)(y \frown \langle x \rangle) = u(x)(y)$.

Proof: Set $C_1 =$ the carrier of X. Set $C_2 = \overline{Y}$. Set $C_3 =$ the carrier of Z. Consider J being a function from $({C_3}^{\prod C_2})^{C_1}$ into ${C_3}^{\prod (C_2 \frown \langle C_1 \rangle)}$ such that J is bijective and for every function f from C_1 into ${C_3}^{\prod C_2}$ and for every finite sequence y and for every object x such that $y \in \prod C_2$ and $x \in C_1$ holds $J(f)(y \frown \langle x \rangle) = f(x)(y)$. Set $L_1 =$ the carrier of the real norm space of bounded linear operators from X into NormSpaceOfBoundedMultOpers$_{\mathbb{R}}$ (Y, Z). Set $B_1 =$ the carrier of NormSpaceOfBoundedMultOpers$_{\mathbb{R}}(Y \frown \langle X \rangle, Z)$. Set $L_2 =$ the carrier of NormSpaceOfBoundedMultOpers$_{\mathbb{R}}(Y, Z)$. The carrier of $\prod \langle X \rangle = \prod \langle$the carrier of $X \rangle$. The carrier of $\prod (Y \frown \langle X \rangle) = \prod (\overline{Y} \frown \overline{\langle X \rangle})$. ${L_2}^{C_1} \subseteq ({C_3}^{\prod C_2})^{C_1}$. Reconsider $I = J \restriction L_1$ as a function from L_1 into ${C_3}^{\prod (C_2 \frown \langle C_1 \rangle)}$.

For every element f of L_1, for every point x of X, there exists a Lipschitzian multilinear operator g from Y into Z such that $g = f(x)$ and for every point y of $\prod Y$, $I(f)(y \frown \langle x \rangle) = g(y)$ and $I(f)$ is a Lipschitzian multilinear operator from $Y \frown \langle X \rangle$ into Z and $I(f) \in B_1$ and there exists a point I_f of NormSpaceOfBoundedMultOpers$_{\mathbb{R}}(Y \frown \langle X \rangle, Z)$ such that $I_f = I(f)$ and $\|f\| = \|I_f\|$. For every elements f_1, f_2 of L_1, $I(f_1 + f_2) = I(f_1) + I(f_2)$. For every element f_1 of L_1 and for every real number a, $I(a \cdot f_1) = a \cdot I(f_1)$ by [6, (2)], (11), [5, (49)]. For every point u of the real norm space of bounded linear operators from X into NormSpaceOfBoundedMultOpers$_{\mathbb{R}}(Y, Z)$, $\|u\| = \|I(u)\|$ and for every point y of $\prod Y$ and for every point x of X, $I(u)(y \frown \langle x \rangle) = u(x)(y)$. For every object I_f such that $I_f \in B_1$ there exists an object f such that $f \in L_1$ and $I_f = I(f)$. $\square$

Let Y be a real normed space and X be a real norm space sequence. The functor NestingLB(X, Y) yielding a real normed space is defined by

(Def. 3) there exists a function f such that $\operatorname{dom} f = \mathbb{N}$ and $it = f(\operatorname{len} X)$ and $f(0) = Y$ and for every natural number i such that $i < \operatorname{len} X$ there exists a real normed space f_i and there exists an element j of $\operatorname{dom} X$ such that

$f_i = f(i)$ and $i + 1 = j$ and $f(i+1) =$ the real norm space of bounded linear operators from $X(j)$ into f_i.

Let us consider real normed spaces X, Y, Z and a Lipschitzian linear operator I from Y into Z. Now we state the propositions:

(36) Suppose I is one-to-one, onto, and isometric. Then there exists a Lipschitzian linear operator L from the real norm space of bounded linear operators from X into Y into the real norm space of bounded linear operators from X into Z such that

(i) L is one-to-one, onto, and isometric, and

(ii) for every point f of the real norm space of bounded linear operators from X into Y, $L(f) = I \cdot f$.

Proof: Consider J being a linear operator from Z into Y such that $J = I^{-1}$ and J is one-to-one and onto and J is isometric. Set $F =$ the carrier of the real norm space of bounded linear operators from X into Y. Set $G =$ the carrier of the real norm space of bounded linear operators from X into Z. Define $\mathcal{P}[\text{function}, \text{function}] \equiv \$_2 = I \cdot \$_1$. For every element f of F, there exists an element g of G such that $\mathcal{P}[f, g]$. Consider L being a function from F into G such that for every element f of F, $\mathcal{P}[f, L(f)]$.

For every objects f_1, f_2 such that f_1, $f_2 \in F$ and $L(f_1) = L(f_2)$ holds $f_1 = f_2$. For every object g such that $g \in G$ there exists an object f such that $f \in F$ and $g = L(f)$ by [10, (2)]. For every points f_1, f_2 of the real norm space of bounded linear operators from X into Y, $L(f_1 + f_2) = L(f_1) + L(f_2)$. For every point f of the real norm space of bounded linear operators from X into Y and for every real number a, $L(a \cdot f) = a \cdot L(f)$. For every element f of the real norm space of bounded linear operators from X into Y, $\|L(f)\| = \|f\|$ by [3, (7)]. $\square$

(37) Suppose I is one-to-one, onto, and isometric. Then there exists a Lipschitzian linear operator L from the real norm space of bounded linear operators from Y into X into the real norm space of bounded linear operators from Z into X such that

(i) L is one-to-one, onto, and isometric, and

(ii) for every point f of the real norm space of bounded linear operators from Y into X, $L(f) = f \cdot (I^{-1})$.

Proof: Consider J being a linear operator from Z into Y such that $J = I^{-1}$ and J is one-to-one and onto and J is isometric. Set $F =$ the carrier of the real norm space of bounded linear operators from Y into X. Set $G =$ the carrier of the real norm space of bounded linear operators from Z into X. Define $\mathcal{P}[\text{function}, \text{function}] \equiv \$_2 = \$_1 \cdot J$. For every element f

of F, there exists an element g of G such that $\mathcal{P}[f, g]$. Consider L being a function from F into G such that for every element f of F, $\mathcal{P}[f, L(f)]$.

For every objects f_1, f_2 such that f_1, $f_2 \in F$ and $L(f_1) = L(f_2)$ holds $f_1 = f_2$. For every object g such that $g \in G$ there exists an object f such that $f \in F$ and $g = L(f)$. For every points f_1, f_2 of the real norm space of bounded linear operators from Y into X, $L(f_1 + f_2) = L(f_1) + L(f_2)$. For every point f of the real norm space of bounded linear operators from Y into X and for every real number a, $L(a \cdot f) = a \cdot L(f)$. For every element f of the real norm space of bounded linear operators from Y into X, $\|L(f)\| = \|f\|$. □

(38) Let us consider real normed spaces X, Y. Then there exists a Lipschitzian linear operator I from the real norm space of bounded linear operators from X into Y into the real norm space of bounded linear operators from $\prod\langle X\rangle$ into Y such that

(i) I is one-to-one, onto, and isometric, and

(ii) for every point u of the real norm space of bounded linear operators from X into Y and for every point x of X, $I(u)(\langle x\rangle) = u(x)$, and

(iii) for every point u of the real norm space of bounded linear operators from X into Y, $\|u\| = \|I(u)\|$.

Proof: Set $J = \mathrm{IsoCPNrSP}(X)$. Consider I being a Lipschitzian linear operator from the real norm space of bounded linear operators from X into Y into the real norm space of bounded linear operators from $\prod\langle X\rangle$ into Y such that I is one-to-one, onto, and isometric and for every point x of the real norm space of bounded linear operators from X into Y, $I(x) = x \cdot (J^{-1})$. For every point u of the real norm space of bounded linear operators from X into Y and for every point x of X, $I(u)(\langle x\rangle) = u(x)$. □

(39) Let us consider real normed spaces X, Y, Z, W, a Lipschitzian linear operator I from X into Z, and a Lipschitzian linear operator J from Y into W. Suppose I is one-to-one, onto, and isometric and J is one-to-one, onto, and isometric.

Then there exists a Lipschitzian linear operator K from the real norm space of bounded linear operators from X into Y into the real norm space of bounded linear operators from Z into W such that

(i) K is one-to-one, onto, and isometric, and

(ii) for every point x of the real norm space of bounded linear operators from X into Y, $K(x) = J \cdot (x \cdot (I^{-1}))$.

Proof: Consider H being a Lipschitzian linear operator from the real norm space of bounded linear operators from X into Y into the real norm space of bounded linear operators from Z into Y such that H is one-to-one, onto, and isometric and for every point x of the real norm space of bounded linear operators from X into Y, $H(x) = x \cdot (I^{-1})$. Consider L being a Lipschitzian linear operator from the real norm space of bounded linear operators from Z into Y into the real norm space of bounded linear operators from Z into W such that L is one-to-one, onto, and isometric and for every point x of the real norm space of bounded linear operators from Z into Y, $L(x) = J \cdot x$.

Reconsider $K = L \cdot H$ as a Lipschitzian linear operator from the real norm space of bounded linear operators from X into Y into the real norm space of bounded linear operators from Z into W. For every point x of the real norm space of bounded linear operators from X into Y, $\|K(x)\| = \|x\|$. □

(40) Let us consider a natural number n, real norm space sequences A, B, and real normed spaces X, Y. Suppose $\operatorname{len} A = n + 1$ and $A{\restriction}n = B$ and $X = A(n+1)$. Then NestingLB(A, Y) = the real norm space of bounded linear operators from X into NestingLB(B, Y).

PROOF: Consider f being a function such that $\operatorname{dom} f = \mathbb{N}$ and NestingLB $(A, Y) = f(\operatorname{len} A)$ and $f(0) = Y$ and for every natural number j such that $j < \operatorname{len} A$ there exists a real normed space V and there exists an element k of $\operatorname{dom} A$ such that $V = f(j)$ and $j + 1 = k$ and $f(j + 1)$ = the real norm space of bounded linear operators from $A(k)$ into V.

Consider V being a real normed space, k being an element of $\operatorname{dom} A$ such that $V = f(\operatorname{len} B)$ and $\operatorname{len} B+1 = k$ and $f(\operatorname{len} B+1)$ = the real norm space of bounded linear operators from $A(k)$ into V. For every natural number j such that $j < \operatorname{len} B$ there exists a real normed space V and there exists an element k of $\operatorname{dom} B$ such that $V = f(j)$ and $j + 1 = k$ and $f(j + 1)$ = the real norm space of bounded linear operators from $B(k)$ into V. □

Let Y be a real normed space and X be a real norm space sequence. Let us observe that NestingLB(X, Y) is constituted functions.

The functor NestMult(X, Y) yielding a Lipschitzian linear operator from NestingLB(X, Y) into NormSpaceOfBoundedMultOpers$_{\mathbb{R}}(X, Y)$ is defined by

(Def. 4) *it* is one-to-one, onto, and isometric and for every element u of NestingLB (X, Y), $\|it(u)\| = \|u\|$ and for every point u of NestingLB(X, Y) and for every point x of $\prod X$, there exists a finite sequence g such that $\operatorname{len} g = \operatorname{len} X$ and $g(1) = u$ and for every element i of $\mathbb{N}$ such that $1 \leqslant i < \operatorname{len} X$ there exists a real norm space sequence X_2.

There exists a point h of NestingLB(X_2, Y) such that $X_2 = X{\restriction}(\operatorname{len} X -' i+1)$ and $h = g(i)$ and $g(i+1) = h(x(\operatorname{len} X -' i+1))$ and there exists a real norm space sequence X_1 and there exists a point h of NestingLB(X_1, Y) such that $X_1 = \langle X(1) \rangle$ and $h = g(\operatorname{len} X)$ and $(it(u))(x) = h(x(1))$.

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