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Group of Homography in Real Projective Plane

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Summary. Using the Mizar system [2], we formalized that homographies of the projective real plane (as defined in [5]), form a group.

Then, we prove that, using the notations of Borsuk and Szmielew in [3]

> "Consider in space $\mathbb{RP}^2$ points P_1, P_2, P_3, P_4 of which three points are not collinear and points Q_1, Q_2, Q_3, Q_4 each three points of which are also not collinear. There exists one homography h of space $\mathbb{RP}^2$ such that $h(P_i) = Q_i$ for $i = 1, 2, 3, 4$."

(Existence Statement 52 and Existence Statement 53) [3]. Or, using notations of Richter [11]

> "Let $[a]$, $[b]$, $[c]$, $[d]$ in $\mathbb{RP}^2$ be four points of which no three are collinear and let $[a']$,$[b']$,$[c']$,$[d']$ in $\mathbb{RP}^2$ be another four points of which no three are collinear, then there exists a 3×3 matrix M such that $[Ma] = [a']$, $[Mb] = [b']$, $[Mc] = [c']$, and $[Md] = [d']$"

Makarios has formalized the same results in Isabelle/Isar (the collineations form a group, lemma statement52-existence and lemma statement 53-existence) and published it in Archive of Formal Proofs[1] [10], [9].

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Keywords: projectivity; projective transformation; real projective plane; group of homography

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[1]http://isa-afp.org/entries/Tarskis_Geometry.shtml

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1. Preliminaries

From now on i, n denote natural numbers, r denotes a real number, r_1 denotes an element of $\mathbb{R}_{\rm F}$, a, b, c denote non zero elements of $\mathbb{R}_{\rm F}$, u, v denote elements of $\mathcal{E}^3_{\rm T}$, p_1 denotes a finite sequence of elements of $\mathbb{R}^1$, p_3, u_4 denote finite sequences of elements of $\mathbb{R}_{\rm F}$, N denotes a square matrix over $\mathbb{R}_{\rm F}$ of dimension 3, K denotes a field, and k denotes an element of K.

Now we state the propositions:

(1) $I^{3\times 3}_{\mathbb{R}_{\rm F}} = \langle\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle\rangle$.

(2) $r_1 \cdot N = r_1 \cdot I^{3\times 3}_{\mathbb{R}_{\rm F}} \cdot N$.

(3) If $r \neq 0$ and u is not zero, then $r \cdot u$ is not zero.
PROOF: $r \cdot u \neq 0_{\mathcal{E}^3_{\rm T}}$ by [4, (52), (49)]. $\square$

Let us consider elements a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33} of $\mathbb{R}_{\rm F}$ and a square matrix A over $\mathbb{R}_{\rm F}$ of dimension 3. Now we state the propositions:

(4) Suppose $A = \langle\langle a_{11}, a_{12}, a_{13}\rangle, \langle a_{21}, a_{22}, a_{23}\rangle, \langle a_{31}, a_{32}, a_{33}\rangle\rangle$. Then
   (i) $\mathrm{Line}(A, 1) = \langle a_{11}, a_{12}, a_{13}\rangle$, and
   (ii) $\mathrm{Line}(A, 2) = \langle a_{21}, a_{22}, a_{23}\rangle$, and
   (iii) $\mathrm{Line}(A, 3) = \langle a_{31}, a_{32}, a_{33}\rangle$.

(5) Suppose $A = \langle\langle a_{11}, a_{12}, a_{13}\rangle, \langle a_{21}, a_{22}, a_{23}\rangle, \langle a_{31}, a_{32}, a_{33}\rangle\rangle$. Then
   (i) $A_{\square,1} = \langle a_{11}, a_{21}, a_{31}\rangle$, and
   (ii) $A_{\square,2} = \langle a_{12}, a_{22}, a_{32}\rangle$, and
   (iii) $A_{\square,3} = \langle a_{13}, a_{23}, a_{33}\rangle$.

   The theorem is a consequence of (4).

Now we state the propositions:

(6) Let us consider elements a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}, b_{11}, b_{12}, b_{13}, b_{21}, b_{22}, b_{23}, b_{31}, b_{32}, b_{33} of $\mathbb{R}_{\rm F}$, and square matrices A, B over $\mathbb{R}_{\rm F}$ of dimension 3. Suppose $A = \langle\langle a_{11}, a_{12}, a_{13}\rangle, \langle a_{21}, a_{22}, a_{23}\rangle, \langle a_{31}, a_{32}, a_{33}\rangle\rangle$ and $B = \langle\langle b_{11}, b_{12}, b_{13}\rangle, \langle b_{21}, b_{22}, b_{23}\rangle, \langle b_{31}, b_{32}, b_{33}\rangle\rangle$. Then $A \cdot B = \langle\langle a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31}, a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32}, a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33}\rangle$, $\langle a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31}, a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32}, a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33}\rangle, \langle a_{31} \cdot b_{11} + a_{32} \cdot b_{21} + a_{33} \cdot b_{31}, a_{31} \cdot b_{12} + a_{32} \cdot b_{22} + a_{33} \cdot b_{32}$, $a_{31} \cdot b_{13} + a_{32} \cdot b_{23} + a_{33} \cdot b_{33}\rangle\rangle$. The theorem is a consequence of (4) and (5).

(7) Let us consider elements a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}, b_1, b_2, b_3 of $\mathbb{R}_{\rm F}$, a matrix A over $\mathbb{R}_{\rm F}$ of dimension 3×3, and a matrix B over $\mathbb{R}_{\rm F}$ of dimension 3×1. Suppose $A = \langle\langle a_{11}, a_{12}, a_{13}\rangle, \langle a_{21}, a_{22}, a_{23}\rangle, \langle a_{31}, a_{32}, a_{33}\rangle\rangle$ and $B = \langle\langle b_1\rangle, \langle b_2\rangle, \langle b_3\rangle\rangle$. Then $A \cdot B = \langle\langle a_{11} \cdot b_1 + a_{12} \cdot b_2 + a_{13} \cdot b_3\rangle$,

$\langle a_{21} \cdot b_1 + a_{22} \cdot b_2 + a_{23} \cdot b_3 \rangle, \langle a_{31} \cdot b_1 + a_{32} \cdot b_2 + a_{33} \cdot b_3 \rangle \rangle$. The theorem is a consequence of (4).

(8) Let us consider non zero elements a, b, c of $\mathbb{R}_{\mathrm{F}}$, and square matrices M_1, M_2 over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose $M_1 = \langle \langle a, 0, 0 \rangle, \langle 0, b, 0 \rangle, \langle 0, 0, c \rangle \rangle$ and $M_2 = \langle \langle \frac{1}{a}, 0, 0 \rangle, \langle 0, \frac{1}{b}, 0 \rangle, \langle 0, 0, \frac{1}{c} \rangle \rangle$. Then

(i) $M_1 \cdot M_2 = I^{3 \times 3}_{\mathbb{R}_{\mathrm{F}}}$, and

(ii) $M_2 \cdot M_1 = I^{3 \times 3}_{\mathbb{R}_{\mathrm{F}}}$.

The theorem is a consequence of (1).

(9) Let us consider non zero elements a, b, c of $\mathbb{R}_{\mathrm{F}}$. Then $\langle \langle a, 0, 0 \rangle, \langle 0, b, 0 \rangle, \langle 0, 0, c \rangle \rangle$ is an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. The theorem is a consequence of (8).

(10) (i) $[1, 0, 0]$ is not zero, and

(ii) $[0, 1, 0]$ is not zero, and

(iii) $[0, 0, 1]$ is not zero, and

(iv) $[1, 1, 1]$ is not zero.

(11) (i) $[1, 0, 0] \neq 0_{\mathcal{E}^3_{\mathrm{T}}}$, and

(ii) $[0, 1, 0] \neq 0_{\mathcal{E}^3_{\mathrm{T}}}$, and

(iii) $[0, 0, 1] \neq 0_{\mathcal{E}^3_{\mathrm{T}}}$, and

(iv) $[1, 1, 1] \neq 0_{\mathcal{E}^3_{\mathrm{T}}}$.

PROOF: $[1, 0, 0] \neq [0, 0, 0]$ by [7, (2)]. $[0, 1, 0] \neq [0, 0, 0]$ by [7, (2)]. $[0, 0, 1] \neq [0, 0, 0]$ by [7, (2)]. $[1, 1, 1] \neq [0, 0, 0]$ by [7, (2)]. □

(12) (i) $e_1 \neq 0_{\mathcal{E}^3_{\mathrm{T}}}$, and

(ii) $e_2 \neq 0_{\mathcal{E}^3_{\mathrm{T}}}$, and

(iii) $e_3 \neq 0_{\mathcal{E}^3_{\mathrm{T}}}$.

PROOF: $[1, 0, 0] \neq [0, 0, 0]$ by [7, (2)]. $[0, 1, 0] \neq [0, 0, 0]$ by [7, (2)]. $[0, 0, 1] \neq [0, 0, 0]$ by [7, (2)]. □

Let n be a natural number. Note that $I^{n \times n}_{\mathbb{R}_{\mathrm{F}}}$ is invertible.

Let M be an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension n. One can verify that $M^{\smile}$ is invertible.

Let K be a field and N_1, N_2 be invertible square matrices over K of dimension n. One can check that $N_1 \cdot N_2$ is invertible.

2. Group of Homography

From now on N, N_1, N_2 denote invertible square matrices over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 and P, P_1, P_2, P_3 denote points of the projective space over $\mathcal{E}^3_{\mathrm{T}}$.

(13) (The homography of N_1)((the homography of N_2)(P)) $=$ (the homography of $N_1 \cdot N_2$)(P).

PROOF: Consider u_{12}, v_{12} being elements of $\mathcal{E}^3_{\mathrm{T}}$, u_8 being a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$, p_{12} being a finite sequence of elements of $\mathbb{R}^1$ such that $P =$ the direction of u_{12} and u_{12} is not zero and $u_{12} = u_8$ and $p_{12} = N_1 \cdot N_2 \cdot u_8$ and $v_{12} = \mathrm{M2F}(p_{12})$ and v_{12} is not zero and (the homography of $N_1 \cdot N_2$)(P) $=$ the direction of v_{12}. Consider u_2, v_2 being elements of $\mathcal{E}^3_{\mathrm{T}}$, u_6 being a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$, p_2 being a finite sequence of elements of $\mathbb{R}^1$ such that $P =$ the direction of u_2 and u_2 is not zero and $u_2 = u_6$ and $p_2 = N_2 \cdot u_6$ and $v_2 = \mathrm{M2F}(p_2)$ and v_2 is not zero and (the homography of N_2)(P) $=$ the direction of v_2. Consider u_1, v_1 being elements of $\mathcal{E}^3_{\mathrm{T}}$, u_7 being a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$, p_1 being a finite sequence of elements of $\mathbb{R}^1$ such that (the homography of N_2)(P) $=$ the direction of u_1 and u_1 is not zero and $u_1 = u_7$ and $p_1 = N_1 \cdot u_7$ and $v_1 = \mathrm{M2F}(p_1)$ and v_1 is not zero and (the homography of N_1)((the homography of N_2)(P)) $=$ the direction of v_1. Consider a being a real number such that $a \neq 0$ and $u_2 = a \cdot u_{12}$. Consider b being a real number such that $b \neq 0$ and $u_1 = b \cdot v_2$. $v_1 = \langle (N_1 \cdot \langle u_7 \rangle^{\mathrm{T}})_{1,1}, (N_1 \cdot \langle u_7 \rangle^{\mathrm{T}})_{2,1}, (N_1 \cdot \langle u_7 \rangle^{\mathrm{T}})_{3,1} \rangle$ by [1, (1), (40)]. $v_2 = \langle (N_2 \cdot \langle u_6 \rangle^{\mathrm{T}})_{1,1}, (N_2 \cdot \langle u_6 \rangle^{\mathrm{T}})_{2,1}, (N_2 \cdot \langle u_6 \rangle^{\mathrm{T}})_{3,1} \rangle$ by [1, (1), (40)]. $v_{12} = \langle (N_1 \cdot N_2 \cdot \langle u_8 \rangle^{\mathrm{T}})_{1,1}, (N_1 \cdot N_2 \cdot \langle u_8 \rangle^{\mathrm{T}})_{2,1}, (N_1 \cdot N_2 \cdot \langle u_8 \rangle^{\mathrm{T}})_{3,1} \rangle$ by [1, (1), (40)]. Reconsider $v_6 = v_2$ as a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$. Reconsider $i_4 = \frac{1}{b}$ as a real number. $v_6 = i_4 \cdot u_1$ by [4, (49), (52)]. Reconsider $l_{11} = \mathrm{Line}(N_2, 1)(1)$, $l_{12} = \mathrm{Line}(N_2, 1)(2)$, $l_{13} = \mathrm{Line}(N_2, 1)(3)$, $l_{21} = \mathrm{Line}(N_2, 2)(1)$, $l_{22} = \mathrm{Line}(N_2, 2)(2)$, $l_{23} = \mathrm{Line}(N_2, 2)(3)$, $l_{31} = \mathrm{Line}(N_2, 3)(1)$, $l_{32} = \mathrm{Line}(N_2, 3)(2)$, $l_{33} = \mathrm{Line}(N_2, 3)(3)$ as an element of $\mathbb{R}_{\mathrm{F}}$. $N_{2\square,1} = \langle l_{11}, l_{21}, l_{31} \rangle$ and $N_{2\square,2} = \langle l_{12}, l_{22}, l_{32} \rangle$ and $N_{2\square,3} = \langle l_{13}, l_{23}, l_{33} \rangle$ by [1, (1), (45)]. The direction of $v_1 =$ the direction of v_{12} by [5, (7)], [1, (45)], [5, (93)], [7, (8)]. $\square$

(14) (The homography of $I^{3 \times 3}_{\mathbb{R}_{\mathrm{F}}}$)($P$) $= P$.

(15) (i) (the homography of N)((the homography of $N^{\smile}$)(P)) $= P$, and

(ii) (the homography of $N^{\smile}$)((the homography of N)(P)) $= P$.

The theorem is a consequence of (13) and (14).

(16) If (the homography of N)(P_1) $=$ (the homography of N)(P_2), then $P_1 = P_2$. The theorem is a consequence of (15).

(17) Let us consider a non zero element a of $\mathbb{R}_{\mathrm{F}}$. Suppose $a \cdot I^{3 \times 3}_{\mathbb{R}_{\mathrm{F}}} = N$. Then (the homography of N)(P) $= P$.

The functor EnsHomography3 yielding a set is defined by the term

(Def. 1) the set of all the homography of N where N is an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3.

One can check that EnsHomography3 is non empty.

Let h_1, h_2 be elements of EnsHomography3. The functor $h_1 \circ h_2$ yielding an element of EnsHomography3 is defined by

(Def. 2) there exist invertible square matrices N_1, N_2 over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that $h_1 =$ the homography of N_1 and $h_2 =$ the homography of N_2 and $it =$ the homography of $N_1 \cdot N_2$.

(18) Let us consider elements h_1, h_2 of EnsHomography3. Suppose $h_1 =$ the homography of N_1 and $h_2 =$ the homography of N_2. Then the homography of $N_1 \cdot N_2 = h_1 \circ h_2$. The theorem is a consequence of (13).

(19) Let us consider elements x, y, z of EnsHomography3. Then $(x \circ y) \circ z = x \circ (y \circ z)$. The theorem is a consequence of (18).

The functor BinOpHomography3 yielding a binary operation on EnsHomography3 is defined by

(Def. 3) for every elements h_1, h_2 of EnsHomography3, $it(h_1, h_2) = h_1 \circ h_2$.

The functor GroupHomography3 yielding a strict multiplicative magma is defined by the term

(Def. 4) $\langle$EnsHomography3, BinOpHomography3$\rangle$.

Note that GroupHomography3 is non empty, associative, and group-like.

(20) $\mathbf{1}_{\mathrm{GroupHomography3}} =$ the homography of $I^{3 \times 3}_{\mathbb{R}_{\mathrm{F}}}$.

(21) Let us consider elements h, g of GroupHomography3, and invertible square matrices N, N_{10} over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose $h =$ the homography of N and $g =$ the homography of N_{10} and $N_{10} = N^{\smile}$. Then $g = h^{-1}$. The theorem is a consequence of (20).

3. Main Results

The functors: Dir100, Dir010, Dir001, and Dir111 yielding points of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ are defined by terms

(Def. 5) the direction of $[1, 0, 0]$,

(Def. 6) the direction of $[0, 1, 0]$,

(Def. 7) the direction of $[0, 0, 1]$,

(Def. 8) the direction of $[1, 1, 1]$,

respectively. Now we state the proposition:

(22) (i) Dir100 $\neq$ Dir010, and

(ii) Dir100 $\neq$ Dir001, and

(iii) Dir100 $\neq$ Dir111, and

(iv) Dir010 $\neq$ Dir001, and

(v) Dir010 $\neq$ Dir111, and

(vi) Dir001 $\neq$ Dir111.

Let a be a non zero element of $\mathbb{R}_{\mathrm{F}}$. Let us consider N. Note that $a \cdot N$ is invertible as a square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3.

(23) Let us consider a non zero element a of $\mathbb{R}_{\mathrm{F}}$. Then (the homography of $a \cdot N_1)(P) =$ (the homography of $N_1)(P)$. The theorem is a consequence of (2), (13), and (17).

(24) Suppose P_1, P_2 and P_3 are not collinear. Then there exists an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that

(i) (the homography of $N)(P_1) =$ Dir100, and

(ii) (the homography of $N)(P_2) =$ Dir010, and

(iii) (the homography of $N)(P_3) =$ Dir001.

PROOF: Consider u_1 being an element of $\mathcal{E}^3_{\mathrm{T}}$ such that u_1 is not zero and $P_1 =$ the direction of u_1. Consider u_2 being an element of $\mathcal{E}^3_{\mathrm{T}}$ such that u_2 is not zero and $P_2 =$ the direction of u_2. Consider u_3 being an element of $\mathcal{E}^3_{\mathrm{T}}$ such that u_3 is not zero and $P_3 =$ the direction of u_3. Reconsider $p_3 = u_1$, $q_1 = u_2$, $r_2 = u_3$ as a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$. Consider N being a square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that N is invertible and $N \cdot p_3 = \mathrm{F2M}(e_1)$ and $N \cdot q_1 = \mathrm{F2M}(e_2)$ and $N \cdot r_2 = \mathrm{F2M}(e_3)$. (The homography of $N)(P_1) =$ Dir100 by [8, (22), (1)], [6, (22)], [5, (75)]. (The homography of $N)(P_2) =$ Dir010 by [8, (22), (1)], [6, (22)], [5, (75)]. (The homography of $N)(P_3) =$ Dir001 by [8, (22), (1)], [6, (22)], [5, (75)]. $\square$

(25) Let us consider non zero elements a, b, c of $\mathbb{R}_{\mathrm{F}}$. Suppose $N = \langle\langle a, 0, 0\rangle, \langle 0, b, 0\rangle, \langle 0, 0, c\rangle\rangle$. Then

(i) (the homography of $N)(\mathrm{Dir100}) =$ Dir100, and

(ii) (the homography of $N)(\mathrm{Dir010}) =$ Dir010, and

(iii) (the homography of $N)(\mathrm{Dir001}) =$ Dir001.

PROOF: (The homography of $N)(\mathrm{Dir100}) =$ Dir100 by (12), [8, (22), (1)], [7, (8), (2)]. (The homography of $N)(\mathrm{Dir010}) =$ Dir010 by (12), [8, (22), (1)], [7, (8), (2)]. (The homography of $N)(\mathrm{Dir001}) =$ Dir001 by (12), [8, (22), (1)], [7, (8), (2)]. $\square$

Let us consider a point P of the projective space over $\mathcal{E}^3_{\rm T}$. Now we state the propositions:

(26) There exist elements a, b, c of $\mathbb{R}_{\rm F}$ such that

(i) $P =$ the direction of $[a, b, c]$, and

(ii) $a \neq 0$ or $b \neq 0$ or $c \neq 0$.

(27) Suppose Dir100, Dir010 and P are not collinear and Dir100, Dir001 and P are not collinear and Dir010, Dir001 and P are not collinear. Then there exist non zero elements a, b, c of $\mathbb{R}_{\rm F}$ such that $P =$ the direction of $[a, b, c]$. The theorem is a consequence of (26).

(28) Let us consider non zero elements a, b, c, i_1, i_2, i_3 of $\mathbb{R}_{\rm F}$, a point P of the projective space over $\mathcal{E}^3_{\rm T}$, and an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3. Suppose $P =$ the direction of $[a, b, c]$ and $i_1 = \frac{1}{a}$ and $i_2 = \frac{1}{b}$ and $i_3 = \frac{1}{c}$ and $N = \langle\langle i_1, 0, 0\rangle, \langle 0, i_2, 0\rangle, \langle 0, 0, i_3\rangle\rangle$. Then (the homography of $N)(P) =$ the direction of $[1, 1, 1]$.

Proof: Consider u, v being elements of $\mathcal{E}^3_{\rm T}$, u_4 being a finite sequence of elements of $\mathbb{R}_{\rm F}$, p being a finite sequence of elements of $\mathbb{R}^1$ such that $P =$ the direction of u and u is not zero and $u = u_4$ and $p = N \cdot u_4$ and $v =$ M2F(p) and v is not zero and (the homography of $N)(P) =$ the direction of v. $[a, b, c]$ is not zero by [7, (4)], [1, (78)]. Consider d being a real number such that $d \neq 0$ and $u = d \cdot [a, b, c]$. Reconsider $z = 0$, $d_1 = d \cdot a$, $d_2 = d \cdot b$, $d_3 = d \cdot c$ as an element of $\mathbb{R}_{\rm F}$. $v = [i_1 \cdot d_1, i_2 \cdot d_2, i_3 \cdot d_3]$ by [1, (45)]. $\square$

(29) Let us consider a point P of the projective space over $\mathcal{E}^3_{\rm T}$. Suppose Dir100, Dir010 and P are not collinear and Dir100, Dir001 and P are not collinear and Dir010, Dir001 and P are not collinear. Then there exist non zero elements a, b, c of $\mathbb{R}_{\rm F}$ such that for every invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3 such that $N = \langle\langle a, 0, 0\rangle, \langle 0, b, 0\rangle, \langle 0, 0, c\rangle\rangle$ holds (the homography of $N)(P) =$ Dir111. The theorem is a consequence of (27) and (28).

(30) Let us consider points P_1, P_2, P_3, P_4 of the projective space over $\mathcal{E}^3_{\rm T}$. Suppose P_1, P_2 and P_3 are not collinear and P_1, P_2 and P_4 are not collinear and P_1, P_3 and P_4 are not collinear and P_2, P_3 and P_4 are not collinear. Then there exists an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3 such that

(i) (the homography of $N)(P_1) =$ Dir100, and

(ii) (the homography of $N)(P_2) =$ Dir010, and

(iii) (the homography of $N)(P_3) =$ Dir001, and

(iv) (the homography of $N)(P_4) =$ Dir111.

The theorem is a consequence of (24), (29), (9), (25), and (13).

(31) Let us consider points P_1, P_2, P_3, P_4, Q_1, Q_2, Q_3, Q_4 of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose P_1, P_2 and P_3 are not collinear and P_1, P_2 and P_4 are not collinear and P_1, P_3 and P_4 are not collinear and P_2, P_3 and P_4 are not collinear and Q_1, Q_2 and Q_3 are not collinear and Q_1, Q_2 and Q_4 are not collinear and Q_1, Q_3 and Q_4 are not collinear and Q_2, Q_3 and Q_4 are not collinear. Then there exists an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that

(i) (the homography of N)(P_1) $= Q_1$, and

(ii) (the homography of N)(P_2) $= Q_2$, and

(iii) (the homography of N)(P_3) $= Q_3$, and

(iv) (the homography of N)(P_4) $= Q_4$.

The theorem is a consequence of (30), (15), and (13).

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