

Fubini's Theorem on Measure

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Summary. The purpose of this article is to show Fubini's theorem on measure [16], [4], [7], [15], [18]. Some theorems have the possibility of slight generalization, but we have priority to avoid the complexity of the description. First of all, for the product measure constructed in [14], we show some theorems. Then we introduce the section which plays an important role in Fubini's theorem, and prove the relevant proposition. Finally we show Fubini's theorem on measure.

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1. PRELIMINARIES

Now we state the propositions:

- (1) Let us consider a disjoint valued finite sequence F , and natural numbers n, m . If $n < m$, then $\bigcup \text{rng}(F \upharpoonright n)$ misses $F(m)$.
- (2) Let us consider a finite sequence F , and natural numbers m, n . Suppose $m \leq n$. Then $\text{len}(F \upharpoonright m) \leq \text{len}(F \upharpoonright n)$.
- (3) Let us consider a finite sequence F , and a natural number n . Then $\bigcup \text{rng}(F \upharpoonright n) \cup F(n+1) = \bigcup \text{rng}(F \upharpoonright (n+1))$. The theorem is a consequence of (2).
- (4) Let us consider a disjoint valued finite sequence F , and a natural number n . Then $\bigcup (F \upharpoonright n)$ misses $F(n+1)$.
- (5) Let us consider a set P , and a finite sequence F . Suppose P is \cup -closed and $\emptyset \in P$ and for every natural number n such that $n \in \text{dom } F$ holds $F(n) \in P$. Then $\bigcup F \in P$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \bigcup \text{rng}(F \upharpoonright \mathbb{N}_1) \in P$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [2, (13)], [3, (59)], [19, (55)], [3, (82)]. For every natural number k , $\mathcal{P}[k]$ from [2, Sch. 2]. \square

Let A, X be sets. Observe that the functor $\chi_{A,X}$ yields a function from X into $\overline{\mathbb{R}}$. Let X be a non empty set, S be a σ -field of subsets of X , and F be a finite sequence of elements of S . Let us observe that the functor $\bigcup F$ yields an element of S . Let F be a sequence of S . Let us note that the functor $\bigcup F$ yields an element of S . Let F be a finite sequence of elements of $X \rightarrow \overline{\mathbb{R}}$ and x be an element of X . The functor $F \# x$ yielding a finite sequence of elements of $\overline{\mathbb{R}}$ is defined by

(Def. 1) $\text{dom } it = \text{dom } F$ and for every element n of \mathbb{N} such that $n \in \text{dom } it$ holds $it(n) = F(n)(x)$.

Now we state the proposition:

(6) Let us consider a non empty set X , a non empty family S of subsets of X , a finite sequence f of elements of S , and a finite sequence F of elements of $X \rightarrow \overline{\mathbb{R}}$. Suppose $\text{dom } f = \text{dom } F$ and f is disjoint valued and for every natural number n such that $n \in \text{dom } F$ holds $F(n) = \chi_{f(n),X}$. Let us consider an element x of X . Then $\chi_{\bigcup f, X}(x) = \sum(F \# x)$.

2. PRODUCT MEASURE AND PRODUCT σ -MEASURE

Now we state the proposition:

(7) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , and a σ -field S_2 of subsets of X_2 . Then $\sigma(\text{DisUnion MeasRect}(S_1, S_2)) = \sigma(\text{MeasRect}(S_1, S_2))$.

Let X_1, X_2 be non empty sets, S_1 be a σ -field of subsets of X_1 , S_2 be a σ -field of subsets of X_2 , M_1 be a σ -measure on S_1 , and M_2 be a σ -measure on S_2 . The functor $\text{ProdMeas}(M_1, M_2)$ yielding an induced measure of $\text{MeasRect}(S_1, S_2)$ and $\text{ProdpreMeas}(M_1, M_2)$ is defined by

(Def. 2) for every set E such that $E \in$ the field generated by $\text{MeasRect}(S_1, S_2)$ for every disjoint valued finite sequence F of elements of $\text{MeasRect}(S_1, S_2)$ such that $E = \bigcup F$ holds $it(E) = \sum(\text{ProdpreMeas}(M_1, M_2) \cdot F)$.

The functor $\text{Prod } \sigma\text{-Meas}(M_1, M_2)$ yielding an induced σ -measure of $\text{MeasRect}(S_1, S_2)$ and $\text{ProdMeas}(M_1, M_2)$ is defined by the term

(Def. 3) $\sigma\text{-Meas}(\text{the Caratheodory measure determined by } \text{ProdMeas}(M_1, M_2)) \upharpoonright \sigma(\text{MeasRe}$

Now we state the propositions:

(8) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , and a σ -measure M_2 on

- S_2 . Then $\text{Prod } \sigma\text{-Meas}(M_1, M_2)$ is a σ -measure on $\sigma(\text{MeasRect}(S_1, S_2))$. The theorem is a consequence of (7).
- (9) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a set sequence F_1 of S_1 , a set sequence F_2 of S_2 , and a natural number n . Then $F_1(n) \times F_2(n)$ is an element of $\sigma(\text{MeasRect}(S_1, S_2))$. The theorem is a consequence of (7).
- (10) Let us consider sets X_1, X_2 , a sequence F_1 of subsets of X_1 , a sequence F_2 of subsets of X_2 , and a natural number n . Suppose F_1 is non descending and F_2 is non descending. Then $F_1(n) \times F_2(n) \subseteq F_1(n+1) \times F_2(n+1)$.
- (11) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , an element A of S_1 , and an element B of S_2 . Then $(\text{ProdMeas}(M_1, M_2))(A \times B) = M_1(A) \cdot M_2(B)$.
- (12) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , a set sequence F_1 of S_1 , a set sequence F_2 of S_2 , and a natural number n . Then $(\text{ProdMeas}(M_1, M_2))(F_1(n) \times F_2(n)) = M_1(F_1(n)) \cdot M_2(F_2(n))$. The theorem is a consequence of (11).
- (13) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , a finite sequence F_1 of elements of S_1 , a finite sequence F_2 of elements of S_2 , and a natural number n . Suppose $n \in \text{dom } F_1$ and $n \in \text{dom } F_2$. Then $(\text{ProdMeas}(M_1, M_2))(F_1(n) \times F_2(n)) = M_1(F_1(n)) \cdot M_2(F_2(n))$.
- (14) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , and a subset E of $X_1 \times X_2$. Then (the Caratheodory measure determined by $\text{ProdMeas}(M_1, M_2)$)(E) = $\inf \text{Svc}(\text{ProdMeas}(M_1, M_2), E)$.
- (15) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , and a σ -measure M_2 on S_2 . Then $\sigma(\text{MeasRect}(S_1, S_2)) \subseteq \sigma\text{-Field}(\text{the Caratheodory measure determined by } \text{ProdMeas}(M_1, M_2))$. The theorem is a consequence of (7).
- (16) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , an element E of $\sigma(\text{MeasRect}(S_1, S_2))$, an element A of S_1 , and an element B of S_2 . Suppose $E = A \times B$. Then $(\text{Prod } \sigma\text{-Meas}(M_1, M_2))(E) = M_1(A) \cdot M_2(B)$. The theorem is a consequence of (15) and (11).
- (17) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , a set sequence F_1 of S_1 , a set sequence F_2 of S_2 , and a natural number

n . Then $(\text{Prod } \sigma\text{-Meas}(M_1, M_2))(F_1(n) \times F_2(n)) = M_1(F_1(n)) \cdot M_2(F_2(n))$. The theorem is a consequence of (9), (15), and (12).

- (18) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , and elements E_1, E_2 of $\sigma(\text{MeasRect}(S_1, S_2))$. Suppose E_1 misses E_2 . Then $(\text{Prod } \sigma\text{-Meas}(M_1, M_2))(E_1 \cup E_2) = (\text{Prod } \sigma\text{-Meas}(M_1, M_2))(E_1) + (\text{Prod } \sigma\text{-Meas}(M_1, M_2))(E_2)$. The theorem is a consequence of (8).
- (19) Let us consider sets X_1, X_2, A, B , a sequence F_1 of subsets of X_1 , a sequence F_2 of subsets of X_2 , and a sequence F of subsets of $X_1 \times X_2$. Suppose F_1 is non descending and $\lim F_1 = A$ and F_2 is non descending and $\lim F_2 = B$ and for every natural number n , $F(n) = F_1(n) \times F_2(n)$. Then $\lim F = A \times B$. The theorem is a consequence of (10).

3. SECTIONS

Let X be a set, Y be a non empty set, E be a subset of $X \times Y$, and x be a set. The functor $\text{Xsection}(E, x)$ yielding a subset of Y is defined by the term

(Def. 4) $\{y, \text{ where } y \text{ is an element of } Y : \langle x, y \rangle \in E\}$.

Let X be a non empty set, Y be a set, and y be a set. The functor $\text{Ysection}(E, y)$ yielding a subset of X is defined by the term

(Def. 5) $\{x, \text{ where } x \text{ is an element of } X : \langle x, y \rangle \in E\}$.

Now we state the propositions:

- (20) Let us consider a set X , a non empty set Y , subsets E_1, E_2 of $X \times Y$, and a set p . Suppose $E_1 \subseteq E_2$. Then $\text{Xsection}(E_1, p) \subseteq \text{Xsection}(E_2, p)$.
- (21) Let us consider a non empty set X , a set Y , subsets E_1, E_2 of $X \times Y$, and a set p . Suppose $E_1 \subseteq E_2$. Then $\text{Ysection}(E_1, p) \subseteq \text{Ysection}(E_2, p)$.
- (22) Let us consider non empty sets X, Y , a subset A of X , a subset B of Y , and a set p . Then
- (i) if $p \in A$, then $\text{Xsection}(A \times B, p) = B$, and
 - (ii) if $p \notin A$, then $\text{Xsection}(A \times B, p) = \emptyset$, and
 - (iii) if $p \in B$, then $\text{Ysection}(A \times B, p) = A$, and
 - (iv) if $p \notin B$, then $\text{Ysection}(A \times B, p) = \emptyset$.
- (23) Let us consider non empty sets X, Y , a subset E of $X \times Y$, and a set p . Then
- (i) if $p \notin X$, then $\text{Xsection}(E, p) = \emptyset$, and
 - (ii) if $p \notin Y$, then $\text{Ysection}(E, p) = \emptyset$.

(24) Let us consider non empty sets X , Y , and a set p . Then

- (i) $X\text{section}(\emptyset_{X \times Y}, p) = \emptyset$, and
- (ii) $Y\text{section}(\emptyset_{X \times Y}, p) = \emptyset$, and
- (iii) if $p \in X$, then $X\text{section}(\Omega_{X \times Y}, p) = Y$, and
- (iv) if $p \in Y$, then $Y\text{section}(\Omega_{X \times Y}, p) = X$.

The theorem is a consequence of (22).

(25) Let us consider non empty sets X , Y , a subset E of $X \times Y$, and a set p . Then

- (i) if $p \in X$, then $X\text{section}(X \times Y \setminus E, p) = Y \setminus X\text{section}(E, p)$, and
- (ii) if $p \in Y$, then $Y\text{section}(X \times Y \setminus E, p) = X \setminus Y\text{section}(E, p)$.

Let us consider non empty sets X , Y , subsets E_1, E_2 of $X \times Y$, and a set p . Now we state the propositions:

- (26) (i) $X\text{section}(E_1 \cup E_2, p) = X\text{section}(E_1, p) \cup X\text{section}(E_2, p)$, and
- (ii) $Y\text{section}(E_1 \cup E_2, p) = Y\text{section}(E_1, p) \cup Y\text{section}(E_2, p)$.
- (27) (i) $X\text{section}(E_1 \cap E_2, p) = X\text{section}(E_1, p) \cap X\text{section}(E_2, p)$, and
- (ii) $Y\text{section}(E_1 \cap E_2, p) = Y\text{section}(E_1, p) \cap Y\text{section}(E_2, p)$.

Now we state the propositions:

- (28) Let us consider a set X , a non empty set Y , a finite sequence F of elements of $2^{X \times Y}$, a finite sequence F_4 of elements of 2^Y , and a set p . Suppose $\text{dom } F = \text{dom } F_4$ and for every natural number n such that $n \in \text{dom } F_4$ holds $F_4(n) = X\text{section}(F(n), p)$. Then $X\text{section}(\bigcup \text{rng } F, p) = \bigcup \text{rng } F_4$.
- (29) Let us consider a non empty set X , a set Y , a finite sequence F of elements of $2^{X \times Y}$, a finite sequence F_3 of elements of 2^X , and a set p . Suppose $\text{dom } F = \text{dom } F_3$ and for every natural number n such that $n \in \text{dom } F_3$ holds $F_3(n) = Y\text{section}(F(n), p)$. Then $Y\text{section}(\bigcup \text{rng } F, p) = \bigcup \text{rng } F_3$.

Let us consider a set X , a non empty set Y , a set p , a sequence F of subsets of $X \times Y$, and a sequence F_4 of subsets of Y . Now we state the propositions:

- (30) If for every natural number n , $F_4(n) = X\text{section}(F(n), p)$, then $X\text{section}(\bigcup \text{rng } F, p) = \bigcup \text{rng } F_4$.
- (31) If for every natural number n , $F_4(n) = X\text{section}(F(n), p)$, then $X\text{section}(\bigcap \text{rng } F, p) = \bigcap \text{rng } F_4$.

Let us consider a non empty set X , a set Y , a set p , a sequence F of subsets of $X \times Y$, and a sequence F_3 of subsets of X . Now we state the propositions:

- (32) If for every natural number n , $F_3(n) = Y\text{section}(F(n), p)$, then $Y\text{section}(\bigcup \text{rng } F, p) = \bigcup \text{rng } F_3$.

- (33) If for every natural number n , $F_3(n) = \text{Ysection}(F(n), p)$, then $\text{Ysection}(\bigcap \text{rng } F, p) \cap \text{rng } F_3$.

Now we state the propositions:

- (34) Let us consider non empty sets X, Y , sets x, y , and a subset E of $X \times Y$. Then
- (i) $\chi_{E, X \times Y}(x, y) = \chi_{\text{Xsection}(E, x), Y}(y)$, and
 - (ii) $\chi_{E, X \times Y}(x, y) = \chi_{\text{Ysection}(E, y), X}(x)$.
- (35) Let us consider non empty sets X, Y , subsets E_1, E_2 of $X \times Y$, and a set p . Suppose E_1 misses E_2 . Then
- (i) $\text{Xsection}(E_1, p)$ misses $\text{Xsection}(E_2, p)$, and
 - (ii) $\text{Ysection}(E_1, p)$ misses $\text{Ysection}(E_2, p)$.
- (36) Let us consider non empty sets X, Y , a disjoint valued finite sequence F of elements of $2^{X \times Y}$, and a set p . Then
- (i) there exists a disjoint valued finite sequence F_4 of elements of 2^X such that $\text{dom } F = \text{dom } F_4$ and for every natural number n such that $n \in \text{dom } F_4$ holds $F_4(n) = \text{Ysection}(F(n), p)$, and
 - (ii) there exists a disjoint valued finite sequence F_3 of elements of 2^Y such that $\text{dom } F = \text{dom } F_3$ and for every natural number n such that $n \in \text{dom } F_3$ holds $F_3(n) = \text{Xsection}(F(n), p)$.

PROOF: Define $\{\infty(\text{natural number}) = \text{Ysection}(F(\$1), p)$. Define $\{\in(\text{natural number}) = \text{Xsection}(F(\$1), p)$. There exists a disjoint valued finite sequence F_4 of elements of 2^X such that $\text{dom } F = \text{dom } F_4$ and for every natural number n such that $n \in \text{dom } F_4$ holds $F_4(n) = \text{Ysection}(F(n), p)$ by (35), [19, (29)]. There exists a disjoint valued finite sequence F_3 of elements of 2^Y such that $\text{dom } F = \text{dom } F_3$ and for every natural number n such that $n \in \text{dom } F_3$ holds $F_3(n) = \text{Xsection}(F(n), p)$ by (35), [19, (29)]. \square

- (37) Let us consider non empty sets X, Y , a disjoint valued sequence F of subsets of $X \times Y$, and a set p . Then
- (i) there exists a disjoint valued sequence F_4 of subsets of X such that for every natural number n , $F_4(n) = \text{Ysection}(F(n), p)$, and
 - (ii) there exists a disjoint valued sequence F_3 of subsets of Y such that for every natural number n , $F_3(n) = \text{Xsection}(F(n), p)$.

PROOF: There exists a disjoint valued sequence F_4 of subsets of X such that for every natural number n , $F_4(n) = \text{Ysection}(F(n), p)$. Define $\{(\text{natural number}) = \text{Xsection}(F(\$1), p)$. Consider F_3 being a sequence of subsets of Y such that for every element n of \mathbb{N} , $F_3(n) = f(n)$ from [11, Sch. 4]. \square

(38) Let us consider non empty sets X, Y , sets x, y , and subsets E_1, E_2 of $X \times Y$. Suppose E_1 misses E_2 . Then

(i) $\chi_{E_1 \cup E_2, X \times Y}(x, y) = \chi_{X \text{section}(E_1, x), Y}(y) + \chi_{X \text{section}(E_2, x), Y}(y)$, and

(ii) $\chi_{E_1 \cup E_2, X \times Y}(x, y) = \chi_{Y \text{section}(E_1, y), X}(x) + \chi_{Y \text{section}(E_2, y), X}(x)$.

The theorem is a consequence of (35), (34), and (26).

(39) Let us consider a set X , a non empty set Y , a set x , a sequence E of subsets of $X \times Y$, and a sequence G of subsets of Y . Suppose E is non descending and for every natural number n , $G(n) = X \text{section}(E(n), x)$. Then G is non descending. The theorem is a consequence of (20).

(40) Let us consider a non empty set X , a set Y , a set x , a sequence E of subsets of $X \times Y$, and a sequence G of subsets of X . Suppose E is non descending and for every natural number n , $G(n) = Y \text{section}(E(n), x)$. Then G is non descending. The theorem is a consequence of (21).

(41) Let us consider a set X , a non empty set Y , a set x , a sequence E of subsets of $X \times Y$, and a sequence G of subsets of Y . Suppose E is non ascending and for every natural number n , $G(n) = X \text{section}(E(n), x)$. Then G is non ascending. The theorem is a consequence of (20).

(42) Let us consider a non empty set X , a set Y , a set x , a sequence E of subsets of $X \times Y$, and a sequence G of subsets of X . Suppose E is non ascending and for every natural number n , $G(n) = Y \text{section}(E(n), x)$. Then G is non ascending. The theorem is a consequence of (21).

(43) Let us consider a set X , a non empty set Y , a sequence E of subsets of $X \times Y$, and a set x . Suppose E is non descending. Then there exists a sequence G of subsets of Y such that

(i) G is non descending, and

(ii) for every natural number n , $G(n) = X \text{section}(E(n), x)$.

PROOF: Define $\mathcal{F}(\text{natural number}) = X \text{section}(E(\$1), x)$. Consider G being a function from \mathbb{N} into 2^Y such that for every element n of \mathbb{N} , $G(n) = \mathcal{F}(n)$ from [11, Sch. 4]. For every natural number n , $G(n) = X \text{section}(E(n), x)$. \square

(44) Let us consider a non empty set X , a set Y , a sequence E of subsets of $X \times Y$, and a set x . Suppose E is non descending. Then there exists a sequence G of subsets of X such that

(i) G is non descending, and

(ii) for every natural number n , $G(n) = Y \text{section}(E(n), x)$.

PROOF: Define $\mathcal{F}(\text{natural number}) = Y \text{section}(E(\$1), x)$. Consider G being a function from \mathbb{N} into 2^X such that for every element n of \mathbb{N} , $G(n) =$

$\mathcal{F}(n)$ from [11, Sch. 4]. For every natural number n , $G(n) = \text{Ysection}(E(n), x)$.
□

(45) Let us consider a set X , a non empty set Y , a sequence E of subsets of $X \times Y$, and a set x . Suppose E is non ascending. Then there exists a sequence G of subsets of Y such that

- (i) G is non ascending, and
- (ii) for every natural number n , $G(n) = \text{Xsection}(E(n), x)$.

PROOF: Define $\mathcal{F}(\text{natural number}) = \text{Xsection}(E(\$1), x)$. Consider G being a function from \mathbb{N} into 2^Y such that for every element n of \mathbb{N} , $G(n) = \mathcal{F}(n)$ from [11, Sch. 4]. For every natural number n , $G(n) = \text{Xsection}(E(n), x)$.
□

(46) Let us consider a non empty set X , a set Y , a sequence E of subsets of $X \times Y$, and a set x . Suppose E is non ascending. Then there exists a sequence G of subsets of X such that

- (i) G is non ascending, and
- (ii) for every natural number n , $G(n) = \text{Ysection}(E(n), x)$.

PROOF: Define $\mathcal{F}(\text{natural number}) = \text{Ysection}(E(\$1), x)$. Consider G being a function from \mathbb{N} into 2^X such that for every element n of \mathbb{N} , $G(n) = \mathcal{F}(n)$ from [11, Sch. 4]. For every natural number n , $G(n) = \text{Ysection}(E(n), x)$.
□

4. MEASURABLE SECTIONS

Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , an element E of $\sigma(\text{MeasRect}(S_1, S_2))$, and a set K . Now we state the propositions:

(47) Suppose $K = \{C, \text{ where } C \text{ is a subset of } X_1 \times X_2 : \text{ for every set } p, \text{Xsection}(C, p) \in S_2\}$. Then

- (i) the field generated by $\text{MeasRect}(S_1, S_2) \subseteq K$, and
- (ii) K is a σ -field of subsets of $X_1 \times X_2$.

PROOF: For every set x , $\text{Xsection}(\emptyset_{X_1 \times X_2}, x) \in S_2$ by (24), [5, (7)]. For every subset C of $X_1 \times X_2$ such that $C \in K$ holds $C^c \in K$ by [17, (5), (6)], (25), (23). □

(48) Suppose $K = \{C, \text{ where } C \text{ is a subset of } X_1 \times X_2 : \text{ for every set } p, \text{Ysection}(C, p) \in S_1\}$. Then

- (i) the field generated by $\text{MeasRect}(S_1, S_2) \subseteq K$, and

(ii) K is a σ -field of subsets of $X_1 \times X_2$.

PROOF: For every set y , $Y_{\text{section}}(\emptyset_{X_1 \times X_2}, y) \in S_1$ by (24), [5, (7)]. For every subset C of $X_1 \times X_2$ such that $C \in K$ holds $C^c \in K$ by [17, (5), (6)], (25), (23). \square

Now we state the proposition:

(49) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , and an element E of $\sigma(\text{MeasRect}(S_1, S_2))$. Then

- (i) for every set p , $X_{\text{section}}(E, p) \in S_2$, and
- (ii) for every set p , $Y_{\text{section}}(E, p) \in S_1$.

The theorem is a consequence of (47) and (48).

Let X_1, X_2 be non empty sets, S_1 be a σ -field of subsets of X_1 , S_2 be a σ -field of subsets of X_2 , E be an element of $\sigma(\text{MeasRect}(S_1, S_2))$, and x be a set. The functor **MeasurableXsection(E, x)** yielding an element of S_2 is defined by the term

(Def. 6) $X_{\text{section}}(E, x)$.

Let y be a set. The functor **MeasurableYsection(E, y)** yielding an element of S_1 is defined by the term

(Def. 7) $Y_{\text{section}}(E, y)$.

Now we state the propositions:

(50) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a finite sequence F of elements of $\sigma(\text{MeasRect}(S_1, S_2))$, a finite sequence F_4 of elements of S_2 , and a set p . Suppose $\text{dom } F = \text{dom } F_4$ and for every natural number n such that $n \in \text{dom } F_4$ holds $F_4(n) = \text{MeasurableXsection}(F(n), p)$. Then $\text{MeasurableXsection}(\bigcup F, p) = \bigcup F_4$. The theorem is a consequence of (28).

(51) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a finite sequence F of elements of $\sigma(\text{MeasRect}(S_1, S_2))$, a finite sequence F_3 of elements of S_1 , and a set p . Suppose $\text{dom } F = \text{dom } F_3$ and for every natural number n such that $n \in \text{dom } F_3$ holds $F_3(n) = \text{MeasurableYsection}(F(n), p)$. Then $\text{MeasurableYsection}(\bigcup F, p) = \bigcup F_3$. The theorem is a consequence of (29).

(52) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_2 on S_2 , an element A of S_1 , an element B of S_2 , and an element x of X_1 . Then $M_2(B) \cdot \chi_{A, X_1}(x) = \int \text{curry}(\chi_{A \times B, X_1 \times X_2}, x) dM_2$.

PROOF: For every element y of X_2 , $(\text{curry}(\chi_{A \times B, X_1 \times X_2}, x))(y) = \chi_{A, X_1}(x) \cdot \chi_{B, X_2}(y)$ by [?, (2)]. \square

- (53) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_2 on S_2 , an element E of $\sigma(\text{MeasRect}(S_1, S_2))$, an element A of S_1 , an element B of S_2 , and an element x of X_1 . Suppose $E = A \times B$. Then $M_2(\text{MeasurableXsection}(E, x)) = M_2(B) \cdot \chi_{A, X_1}(x)$. The theorem is a consequence of (22).
- (54) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , an element A of S_1 , an element B of S_2 , and an element y of X_2 . Then $M_1(A) \cdot \chi_{B, X_2}(y) = \int \text{curry}'(\chi_{A \times B, X_1 \times X_2}, y) dM_1$.
 PROOF: For every element x of X_1 , $(\text{curry}'(\chi_{A \times B, X_1 \times X_2}, y))(x) = \chi_{A, X_1}(x) \cdot \chi_{B, X_2}(y)$ by [? , (2)]. \square
- (55) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , an element E of $\sigma(\text{MeasRect}(S_1, S_2))$, an element A of S_1 , an element B of S_2 , and an element y of X_2 . Suppose $E = A \times B$. Then $M_1(\text{MeasurableYsection}(E, y)) = M_1(A) \cdot \chi_{B, X_2}(y)$. The theorem is a consequence of (22).

5. FINITE SEQUENCE OF FUNCTIONS

Let X, Y be non empty sets, G be a non empty set of functions from X to Y , F be a finite sequence of elements of G , and n be a natural number. Observe that the functor F_n yields an element of G . Let X be a set and F be a finite sequence of elements of $\overline{\mathbb{R}}^X$. We say that **F is (without $+\infty$)-valued** if and only if

(Def. 8) for every natural number n such that $n \in \text{dom } F$ holds $F(n)$ is without $+\infty$.

We say that **F is (without $-\infty$)-valued** if and only if

(Def. 9) for every natural number n such that $n \in \text{dom } F$ holds $F(n)$ is without $-\infty$.

Now we state the proposition:

(56) Let us consider a non empty set X . Then

- (i) $\langle X \mapsto 0 \rangle$ is a finite sequence of elements of $\overline{\mathbb{R}}^X$, and
- (ii) for every natural number n such that $n \in \text{dom} \langle X \mapsto 0 \rangle$ holds $\langle X \mapsto 0 \rangle(n)$ is without $+\infty$, and
- (iii) for every natural number n such that $n \in \text{dom} \langle X \mapsto 0 \rangle$ holds $\langle X \mapsto 0 \rangle(n)$ is without $-\infty$.

Let X be a non empty set. One can verify that there exists a finite sequence of elements of $\overline{\mathbb{R}}^X$ which is (without $+\infty$)-valued and (without $-\infty$)-valued.

Now we state the propositions:

- (57) Let us consider a non empty set X , a (without $+\infty$)-valued finite sequence F of elements of $\overline{\mathbb{R}}^X$, and a natural number n . If $n \in \text{dom } F$, then $(F_n)^{-1}(\{+\infty\}) = \emptyset$.
- (58) Let us consider a non empty set X , a (without $-\infty$)-valued finite sequence F of elements of $\overline{\mathbb{R}}^X$, and a natural number n . If $n \in \text{dom } F$, then $(F_n)^{-1}(\{-\infty\}) = \emptyset$.
- (59) Let us consider a non empty set X , and a finite sequence F of elements of $\overline{\mathbb{R}}^X$. Suppose F is (without $+\infty$)-valued or (without $-\infty$)-valued. Let us consider natural numbers n, m . If $n, m \in \text{dom } F$, then $\text{dom}(F_n + F_m) = X$. The theorem is a consequence of (57) and (58).

Let X be a non empty set and F be a finite sequence of elements of $\overline{\mathbb{R}}^X$. We say that F is summable if and only if

(Def. 10) F is (without $+\infty$)-valued or (without $-\infty$)-valued.

Observe that there exists a finite sequence of elements of $\overline{\mathbb{R}}^X$ which is summable.

Let F be a summable finite sequence of elements of $\overline{\mathbb{R}}^X$. The functor $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}$ yielding a finite sequence of elements of $\overline{\mathbb{R}}^X$ is defined by

(Def. 11) $\text{len } F = \text{len } it$ and $F(1) = it(1)$ and for every natural number n such that $1 \leq n < \text{len } F$ holds $it(n+1) = it_n + F_{n+1}$.

One can check that every finite sequence of elements of $\overline{\mathbb{R}}^X$ which is (without $+\infty$)-valued is also summable and every finite sequence of elements of $\overline{\mathbb{R}}^X$ which is (without $-\infty$)-valued is also summable.

Now we state the propositions:

- (60) Let us consider a non empty set X , and a (without $+\infty$)-valued finite sequence F of elements of $\overline{\mathbb{R}}^X$. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}$ is (without $+\infty$)-valued.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$1 \in \text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}$, then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(\$1)$ is without $+\infty$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [19, (29)], [2, (14)], [19, (25)], [2, (13)]. For every natural number n , $\mathcal{P}[n]$ from [2, Sch. 2]. \square

- (61) Let us consider a non empty set X , and a (without $-\infty$)-valued finite sequence F of elements of $\overline{\mathbb{R}}^X$. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}$ is (without $-\infty$)-valued.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$1 \in \text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}$, then

$(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(\mathcal{S}_1)$ is without $-\infty$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [19, (29)], [2, (14)], [19, (25)], [2, (13)]. For every natural number n , $\mathcal{P}[n]$ from [2, Sch. 2]. \square

(62) Let us consider a non empty set X , a set A , an extended real e , and a function f from X into $\overline{\mathbb{R}}$. Suppose for every element x of X , $f(x) = e \cdot \chi_{A,X}(x)$. Then

- (i) if $e = +\infty$, then $f = \overline{\chi}_{A,X}$, and
- (ii) if $e = -\infty$, then $f = -\overline{\chi}_{A,X}$, and
- (iii) if $e \neq +\infty$ and $e \neq -\infty$, then there exists a real number r such that $r = e$ and $f = r \cdot \chi_{A,X}$.

(63) Let us consider a non empty set X , a σ -field S of subsets of X , a partial function f from X to $\overline{\mathbb{R}}$, and an element A of S . Suppose f is measurable on A and $A \subseteq \text{dom } f$. Then $-f$ is measurable on A .

Let X be a non empty set and f be a without $-\infty$ partial function from X to $\overline{\mathbb{R}}$. Observe that $-f$ is without $+\infty$.

Let f be a without $+\infty$ partial function from X to $\overline{\mathbb{R}}$. One can check that $-f$ is without $-\infty$.

Let f_1, f_2 be without $+\infty$ partial functions from X to $\overline{\mathbb{R}}$. Let us note that the functor $f_1 + f_2$ yields a without $+\infty$ partial function from X to $\overline{\mathbb{R}}$. Let f_1, f_2 be without $-\infty$ partial functions from X to $\overline{\mathbb{R}}$. Note that the functor $f_1 + f_2$ yields a without $-\infty$ partial function from X to $\overline{\mathbb{R}}$. Let f_1 be a without $+\infty$ partial function from X to $\overline{\mathbb{R}}$ and f_2 be a without $-\infty$ partial function from X to $\overline{\mathbb{R}}$. One can verify that the functor $f_1 - f_2$ yields a without $+\infty$ partial function from X to $\overline{\mathbb{R}}$. Let f_1 be a without $-\infty$ partial function from X to $\overline{\mathbb{R}}$ and f_2 be a without $+\infty$ partial function from X to $\overline{\mathbb{R}}$. Observe that the functor $f_1 - f_2$ yields a without $-\infty$ partial function from X to $\overline{\mathbb{R}}$. Now we state the propositions:

(64) Let us consider a non empty set X , and partial functions f, g from X to $\overline{\mathbb{R}}$. Then

- (i) $-(f + g) = -f + -g$, and
- (ii) $-(f - g) = -f + g$, and
- (iii) $-(f - g) = g - f$, and
- (iv) $-(-f + g) = f - g$, and
- (v) $-(-f + g) = f + -g$.

(65) Let us consider a non empty set X , a σ -field S of subsets of X , without $+\infty$ partial functions f, g from X to $\overline{\mathbb{R}}$, and an element A of S . Suppose f is measurable on A and g is measurable on A and $A \subseteq \text{dom}(f + g)$. Then $f + g$ is measurable on A . The theorem is a consequence of (63) and (64).

- (66) Let us consider a non empty set X , a σ -field S of subsets of X , an element A of S , a without $+\infty$ partial function f from X to $\overline{\mathbb{R}}$, and a without $-\infty$ partial function g from X to $\overline{\mathbb{R}}$. Suppose f is measurable on A and g is measurable on A and $A \subseteq \text{dom}(f - g)$. Then $f - g$ is measurable on A . The theorem is a consequence of (63) and (64).
- (67) Let us consider a non empty set X , a σ -field S of subsets of X , an element A of S , a without $-\infty$ partial function f from X to $\overline{\mathbb{R}}$, and a without $+\infty$ partial function g from X to $\overline{\mathbb{R}}$. Suppose f is measurable on A and g is measurable on A and $A \subseteq \text{dom}(f - g)$. Then $f - g$ is measurable on A . The theorem is a consequence of (64), (63), and (65).
- (68) Let us consider a non empty set X , a σ -field S of subsets of X , an element P of S , and a summable finite sequence F of elements of $\overline{\mathbb{R}}^X$. Suppose for every natural number n such that $n \in \text{dom} F$ holds F_n is measurable on P . Let us consider a natural number n . Suppose $n \in \text{dom} F$. Then $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_n$ is measurable on P . The theorem is a consequence of (60), (65), and (61).

6. SOME PROPERTIES OF INTEGRAL

Now we state the propositions:

- (69) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , an element E of $\sigma(\text{MeasRect}(S_1, S_2))$, an element A of S_1 , an element B of S_2 , an element x of X_1 , and an element y of X_2 . Suppose $E = A \times B$. Then
- (i) $\int \text{curry}(\chi_{E, X_1 \times X_2}, x) dM_2 = M_2(\text{MeasurableXsection}(E, x)) \cdot \chi_{A, X_1}(x)$,
and
 - (ii) $\int \text{curry}'(\chi_{E, X_1 \times X_2}, y) dM_1 = M_1(\text{MeasurableYsection}(E, y)) \cdot \chi_{B, X_2}(y)$.

The theorem is a consequence of (52), (53), (54), and (55).

- (70) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , and an element E of $\sigma(\text{MeasRect}(S_1, S_2))$. Suppose $E \in$ the field generated by $\text{MeasRect}(S_1, S_2)$. Then there exists a disjoint valued finite sequence f of elements of $\text{MeasRect}(S_1, S_2)$ and there exists a finite sequence A of elements of S_1 and there exists a finite sequence B of elements of S_2 such that $\text{len } f = \text{len } A$ and $\text{len } f = \text{len } B$ and $E = \bigcup f$ and for every natural number n such that $n \in \text{dom } f$ holds $\pi_1(f(n)) = A(n)$ and $\pi_2(f(n)) = B(n)$ and for every natural number n

and for every sets x, y such that $n \in \text{dom } f$ and $x \in X_1$ and $y \in X_2$ holds $\chi_{f(n), X_1 \times X_2}(x, y) = \chi_{A(n), X_1}(x) \cdot \chi_{B(n), X_2}(y)$.

PROOF: Consider E_1 being a subset of $X_1 \times X_2$ such that $E = E_1$ and there exists a disjoint valued finite sequence f of elements of $\text{MeasRect}(S_1, S_2)$ such that $E_1 = \bigcup f$. Consider f being a disjoint valued finite sequence of elements of $\text{MeasRect}(S_1, S_2)$ such that $E_1 = \bigcup f$. Define \mathcal{S} [natural number, object] $\equiv \mathcal{S}_2 = \pi_1(f(\mathcal{S}_1))$. For every natural number i such that $i \in \text{Seg len } f$ there exists an element A_1 of S_1 such that $\mathcal{S}[i, A_1]$ by [12, (4)], [1, (9)], [5, (7)]. Consider A being a finite sequence of elements of S_1 such that $\text{dom } A = \text{Seg len } f$ and for every natural number i such that $i \in \text{Seg len } f$ holds $\mathcal{S}[i, A(i)]$ from [3, Sch. 5]. Define \mathcal{T} [natural number, object] $\equiv \mathcal{T}_2 = \pi_2(f(\mathcal{S}_1))$. For every natural number i such that $i \in \text{Seg len } f$ there exists an element B_1 of S_2 such that $\mathcal{T}[i, B_1]$ by [12, (4)], [1, (9)], [5, (7)]. Consider B being a finite sequence of elements of S_2 such that $\text{dom } B = \text{Seg len } f$ and for every natural number i such that $i \in \text{Seg len } f$ holds $\mathcal{T}[i, B(i)]$ from [3, Sch. 5]. For every natural number n such that $n \in \text{dom } f$ holds $\pi_1(f(n)) = A(n)$ and $\pi_2(f(n)) = B(n)$. Consider A_2 being an element of S_1 , B_2 being an element of S_2 such that $f(n) = A_2 \times B_2$. \square

- (71) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , an element E of $\sigma(\text{MeasRect}(S_1, S_2))$, an element x of X_1 , an element y of X_2 , an element U of S_1 , and an element V of S_2 . Then

$$(i) M_1(\text{MeasurableYsection}(E, y) \cap U) = \int \text{curry}'(\chi_{E \cap (U \times X_2), X_1 \times X_2}, y) dM_1,$$

and

$$(ii) M_2(\text{MeasurableXsection}(E, x) \cap V) = \int \text{curry}(\chi_{E \cap (X_1 \times V), X_1 \times X_2}, x) dM_2.$$

The theorem is a consequence of (34), (27), and (22).

- (72) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , an element E of $\sigma(\text{MeasRect}(S_1, S_2))$, an element x of X_1 , and an element y of X_2 . Then

$$(i) M_1(\text{MeasurableYsection}(E, y)) = \int \text{curry}'(\chi_{E, X_1 \times X_2}, y) dM_1, \text{ and}$$

$$(ii) M_2(\text{MeasurableXsection}(E, x)) = \int \text{curry}(\chi_{E, X_1 \times X_2}, x) dM_2.$$

The theorem is a consequence of (71).

- (73) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_2 on S_2 , a disjoint valued finite sequence f of elements of $\text{MeasRect}(S_1, S_2)$, an element x of X_1 , a natural number n , an element E_2 of $\sigma(\text{MeasRect}(S_1, S_2))$, an element A_2 of S_1 , and

an element B_2 of S_2 . Suppose $n \in \text{dom } f$ and $f(n) = E_2$ and $E_2 = A_2 \times B_2$. Then $\int \text{curry}(\chi_{f(n), X_1 \times X_2}, x) \, dM_2 = M_2(\text{MeasurableXsection}(E_2, x)) \cdot \chi_{A_2, X_1}(x)$.

- (74) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , and an element E of $\sigma(\text{MeasRect}(S_1, S_2))$. Suppose $E \in$ the field generated by $\text{MeasRect}(S_1, S_2)$ and $E \neq \emptyset$. Then there exists a disjoint valued finite sequence f of elements of $\text{MeasRect}(S_1, S_2)$ and there exists a finite sequence A of elements of S_1 and there exists a finite sequence B of elements of S_2 and there exists a summable finite sequence X_3 of elements of $\overline{\mathbb{R}}^{X_1 \times X_2}$ such that $E = \bigcup f$ and $\text{len } f \in \text{dom } f$ and $\text{len } f = \text{len } A$ and $\text{len } f = \text{len } B$ and $\text{len } f = \text{len } X_3$ and for every natural number n such that $n \in \text{dom } f$ holds $f(n) = A(n) \times B(n)$ and for every natural number n such that $n \in \text{dom } X_3$ holds $X_3(n) = \chi_{f(n), X_1 \times X_2}$ and $(\sum_{\alpha=0}^{\kappa} X_3(\alpha))_{\kappa \in \mathbb{N}(\text{len } X_3)} = \chi_{E, X_1 \times X_2}$ and for every natural number n and for every sets x, y such that $n \in \text{dom } X_3$ and $x \in X_1$ and $y \in X_2$ holds $X_3(n)(x, y) = \chi_{A(n), X_1}(x) \cdot \chi_{B(n), X_2}(y)$ and for every element x of X_1 , $\text{curry}(\chi_{E, X_1 \times X_2}, x) = \text{curry}(((\sum_{\alpha=0}^{\kappa} X_3(\alpha))_{\kappa \in \mathbb{N}(\text{len } X_3)}, x)$ and for every element y of X_2 , $\text{curry}'(\chi_{E, X_1 \times X_2}, y) = \text{curry}'(((\sum_{\alpha=0}^{\kappa} X_3(\alpha))_{\kappa \in \mathbb{N}(\text{len } X_3)}, y)$.

PROOF: Consider f being a disjoint valued finite sequence of elements of $\text{MeasRect}(S_1, S_2)$, A being a finite sequence of elements of S_1 , B being a finite sequence of elements of S_2 such that $\text{len } f = \text{len } A$ and $\text{len } f = \text{len } B$ and $E = \bigcup f$ and for every natural number n such that $n \in \text{dom } f$ holds $\pi_1(f(n)) = A(n)$ and $\pi_2(f(n)) = B(n)$ and for every natural number n and for every sets x, y such that $n \in \text{dom } f$ and $x \in X_1$ and $y \in X_2$ holds $\chi_{f(n), X_1 \times X_2}(x, y) = \chi_{A(n), X_1}(x) \cdot \chi_{B(n), X_2}(y)$. Define $\mathcal{F}(\text{set}) = \chi_{f(\$1), X_1 \times X_2}$. Consider X_3 being a finite sequence such that $\text{len } X_3 = \text{len } f$ and for every natural number n such that $n \in \text{dom } X_3$ holds $X_3(n) = \mathcal{F}(n)$ from [3, Sch. 2]. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$1 \in \text{dom } f$, then $(\sum_{\alpha=0}^{\kappa} X_3(\alpha))_{\kappa \in \mathbb{N}(\$1)} = \chi_{\bigcup(f \upharpoonright \$1), X_1 \times X_2}$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [9, (20)], [3, (39)], [13, (25)], [2, (14)]. For every natural number n , $\mathcal{P}[n]$ from [2, Sch. 2]. For every natural number n such that $n \in \text{dom } f$ holds $f(n) = A(n) \times B(n)$ by [12, (4)], [13, (90)], [1, (9)]. For every natural number n and for every sets x, y such that $n \in \text{dom } X_3$ and $x \in X_1$ and $y \in X_2$ holds $X_3(n)(x, y) = \chi_{A(n), X_1}(x) \cdot \chi_{B(n), X_2}(y)$. For every element x of X_1 , $\text{curry}(\chi_{E, X_1 \times X_2}, x) = \text{curry}(((\sum_{\alpha=0}^{\kappa} X_3(\alpha))_{\kappa \in \mathbb{N}(\text{len } X_3)}, x)$. \square

- (75) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , and a finite sequence F of elements of $\text{MeasRect}(S_1, S_2)$. Then $\bigcup F \in \sigma(\text{MeasRect}(S_1, S_2))$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$1 \leq \text{len } F$, then $\bigcup \text{rng}(F \upharpoonright \$1) \in$

$\sigma(\text{MeasRect}(S_1, S_2))$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [2, (11)], [19, (25)], [8, (11)], [3, (59)]. For every natural number k , $\mathcal{P}[k]$ from [2, Sch. 2]. \square

- (76) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , and an element E of $\sigma(\text{MeasRect}(S_1, S_2))$. Suppose $E \in$ the field generated by $\text{MeasRect}(S_1, S_2)$ and $E \neq \emptyset$. Then there exists a disjoint valued finite sequence F of elements of $\text{MeasRect}(S_1, S_2)$ and there exists a finite sequence A of elements of S_1 and there exists a finite sequence B of elements of S_2 and there exists a summable finite sequence C of elements of $\overline{\mathbb{R}}^{X_1 \times X_2}$ and there exists a summable finite sequence I of elements of $\overline{\mathbb{R}}^{X_1}$ and there exists a summable finite sequence J of elements of $\overline{\mathbb{R}}^{X_2}$ such that $E = \bigcup F$ and $\text{len } F \in \text{dom } F$ and $\text{len } F = \text{len } A$ and $\text{len } F = \text{len } B$ and $\text{len } F = \text{len } C$ and $\text{len } F = \text{len } I$ and $\text{len } F = \text{len } J$ and for every natural number n such that $n \in \text{dom } C$ holds $C(n) = \chi_{F(n), X_1 \times X_2}$ and $((\sum_{\alpha=0}^{\kappa} C(\alpha))_{\kappa \in \mathbb{N}})_{\text{len } C} = \chi_{E, X_1 \times X_2}$ and for every element x of X_1 and for every natural number n such that $n \in \text{dom } I$ holds $I(n)(x) = \int \text{curry}(C_n, x) \, dM_2$ and for every natural number n and for every element P of S_1 such that $n \in \text{dom } I$ holds I_n is measurable on P and for every element x of X_1 , $\int \text{curry}(((\sum_{\alpha=0}^{\kappa} C(\alpha))_{\kappa \in \mathbb{N}})_{\text{len } C}, x) \, dM_2 = ((\sum_{\alpha=0}^{\kappa} I(\alpha))_{\kappa \in \mathbb{N}})_{\text{len } I}(x)$ and for every element y of X_2 and for every natural number n such that $n \in \text{dom } J$ holds $J(n)(y) = \int \text{curry}'(C_n, y) \, dM_1$ and for every natural number n and for every element P of S_2 such that $n \in \text{dom } J$ holds J_n is measurable on P and for every element y of X_2 , $\int \text{curry}'(((\sum_{\alpha=0}^{\kappa} C(\alpha))_{\kappa \in \mathbb{N}})_{\text{len } C}, y) \, dM_1 = ((\sum_{\alpha=0}^{\kappa} J(\alpha))_{\kappa \in \mathbb{N}})_{\text{len } J}(y)$.

PROOF: Consider F being a disjoint valued finite sequence of elements of $\text{MeasRect}(S_1, S_2)$, A being a finite sequence of elements of S_1 , B being a finite sequence of elements of S_2 , C being a summable finite sequence of elements of $\overline{\mathbb{R}}^{X_1 \times X_2}$ such that $E = \bigcup F$ and $\text{len } F \in \text{dom } F$ and $\text{len } F = \text{len } A$ and $\text{len } F = \text{len } B$ and $\text{len } F = \text{len } C$ and for every natural number n such that $n \in \text{dom } F$ holds $F(n) = A(n) \times B(n)$ and for every natural number n such that $n \in \text{dom } C$ holds $C(n) = \chi_{F(n), X_1 \times X_2}$ and $(\sum_{\alpha=0}^{\kappa} C(\alpha))_{\kappa \in \mathbb{N}}(\text{len } C) = \chi_{E, X_1 \times X_2}$ and for every natural number n and for every sets x, y such that $n \in \text{dom } C$ and $x \in X_1$ and $y \in X_2$ holds $C(n)(x, y) = \chi_{A(n), X_1}(x) \cdot \chi_{B(n), X_2}(y)$ and for every element x of X_1 , $\text{curry}(\chi_{E, X_1 \times X_2}, x) = \text{curry}(((\sum_{\alpha=0}^{\kappa} C(\alpha))_{\kappa \in \mathbb{N}})_{\text{len } C}, x)$ and for every element y of X_2 , $\text{curry}'(\chi_{E, X_1 \times X_2}, y) = \text{curry}'(((\sum_{\alpha=0}^{\kappa} C(\alpha))_{\kappa \in \mathbb{N}})_{\text{len } C}, y)$. Define $\mathcal{S}[\text{natural number, object}] \equiv$ there exists a function f from X_1 into $\overline{\mathbb{R}}$ such that $f = \mathcal{S}_2$ and for every element x of X_1 , $f(x) = \int \text{curry}(C_{\mathcal{S}_1}, x) \, dM_2$. For every natural number n such that $n \in \text{Seg len } F$ there exists an ob-

ject z such that $\mathcal{S}[n, z]$. Consider I being a finite sequence such that $\text{dom } I = \text{Seg len } F$ and for every natural number n such that $n \in \text{Seg len } F$ holds $\mathcal{S}[n, I(n)]$ from [3, Sch. 1]. For every element x of X_1 and for every natural number n such that $n \in \text{dom } I$ holds $I(n)(x) = \int \text{curry}(C_n, x) dM_2$ by [12, (4)]. Define $\mathcal{T}[\text{natural number, object}] \equiv$ there exists a function f from X_2 into $\overline{\mathbb{R}}$ such that $f = \mathcal{S}_2$ and for every element x of X_2 , $f(x) = \int \text{curry}'(C_{\mathcal{S}_1}, x) dM_1$. For every natural number n such that $n \in \text{Seg len } F$ there exists an object z such that $\mathcal{T}[n, z]$. Consider J being a finite sequence such that $\text{dom } J = \text{Seg len } F$ and for every natural number n such that $n \in \text{Seg len } F$ holds $\mathcal{T}[n, J(n)]$ from [3, Sch. 1]. For every element x of X_2 and for every natural number n such that $n \in \text{dom } J$ holds $J(n)(x) = \int \text{curry}'(C_n, x) dM_1$ by [12, (4)]. For every natural number n and for every element P of S_1 such that $n \in \text{dom } I$ holds I_n is measurable on P by [12, (4)], (69), (22), [? , (32)]. For every element x of X_1 , $\int \text{curry}(((\sum_{\alpha=0}^{\kappa} C(\alpha))_{\kappa \in \mathbb{N}})_{\text{len } C}, x) dM_2 = ((\sum_{\alpha=0}^{\kappa} I(\alpha))_{\kappa \in \mathbb{N}})_{\text{len } I}(x)$ by [19, (24), (25)], [2, (13)], [9, (20)]. For every natural number n and for every element P of S_2 such that $n \in \text{dom } J$ holds J_n is measurable on P by [12, (4)], (69), (22), [? , (32)]. For every element x of X_2 , $\int \text{curry}'(((\sum_{\alpha=0}^{\kappa} C(\alpha))_{\kappa \in \mathbb{N}})_{\text{len } C}, x) dM_1 = ((\sum_{\alpha=0}^{\kappa} J(\alpha))_{\kappa \in \mathbb{N}})_{\text{len } J}(x)$ by [19, (24), (25)], [2, (13)], [9, (20)]. \square

Let X_1, X_2 be non empty sets, S_1 be a σ -field of subsets of X_1 , S_2 be a σ -field of subsets of X_2 , F be a set sequence of $\sigma(\text{MeasRect}(S_1, S_2))$, and n be a natural number. One can verify that the functor $F(n)$ yields an element of $\sigma(\text{MeasRect}(S_1, S_2))$. Let F be a function from $\mathbb{N} \times \sigma(\text{MeasRect}(S_1, S_2))$ into $\sigma(\text{MeasRect}(S_1, S_2))$, n be an element of \mathbb{N} , and E be an element of $\sigma(\text{MeasRect}(S_1, S_2))$. Let us observe that the functor $F(n, E)$ yields an element of $\sigma(\text{MeasRect}(S_1, S_2))$. Now we state the propositions:

(77) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , an element E of $\sigma(\text{MeasRect}(S_1, S_2))$, and an element V of S_2 . Suppose $E \in$ the field generated by $\text{MeasRect}(S_1, S_2)$. Then there exists a function F from X_1 into $\overline{\mathbb{R}}$ such that

(i) for every element x of X_1 , $F(x) = M_2(\text{MeasurableXsection}(E, x) \cap V)$, and

(ii) for every element P of S_1 , F is measurable on P .

The theorem is a consequence of (22), (27), (24), (76), (71), and (68).

(78) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , an element E of $\sigma(\text{MeasRect}(S_1, S_2))$, and an element V of S_1 .

Suppose $E \in$ the field generated by $\text{MeasRect}(S_1, S_2)$. Then there exists a function F from X_2 into $\overline{\mathbb{R}}$ such that

- (i) for every element x of X_2 , $F(x) = M_1(\text{MeasurableYsection}(E, x) \cap V)$, and
- (ii) for every element P of S_2 , F is measurable on P .

The theorem is a consequence of (22), (27), (24), (76), (71), and (68).

(79) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_2 on S_2 , and an element E of $\sigma(\text{MeasRect}(S_1, S_2))$. Suppose $E \in$ the field generated by $\text{MeasRect}(S_1, S_2)$. Let us consider an element B of S_2 . Then $E \in \{E, \text{ where } E \text{ is an element of } \sigma(\text{MeasRect}(S_1, S_2)) : \text{ there exists a function } F \text{ from } X_1 \text{ into } \overline{\mathbb{R}} \text{ such that for every element } x \text{ of } X_1, F(x) = M_2(\text{MeasurableXsection}(E, x) \cap B) \text{ and for every element } V \text{ of } S_1, F \text{ is measurable on } V\}$. The theorem is a consequence of (77).

(80) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , and an element E of $\sigma(\text{MeasRect}(S_1, S_2))$. Suppose $E \in$ the field generated by $\text{MeasRect}(S_1, S_2)$. Let us consider an element B of S_1 . Then $E \in \{E, \text{ where } E \text{ is an element of } \sigma(\text{MeasRect}(S_1, S_2)) : \text{ there exists a function } F \text{ from } X_2 \text{ into } \overline{\mathbb{R}} \text{ such that for every element } x \text{ of } X_2, F(x) = M_1(\text{MeasurableYsection}(E, x) \cap B) \text{ and for every element } V \text{ of } S_2, F \text{ is measurable on } V\}$. The theorem is a consequence of (78).

(81) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_2 on S_2 , and an element B of S_2 . Then the field generated by $\text{MeasRect}(S_1, S_2) \subseteq \{E, \text{ where } E \text{ is an element of } \sigma(\text{MeasRect}(S_1, S_2)) : \text{ there exists a function } F \text{ from } X_1 \text{ into } \overline{\mathbb{R}} \text{ such that for every element } x \text{ of } X_1, F(x) = M_2(\text{MeasurableXsection}(E, x) \cap B) \text{ and for every element } V \text{ of } S_1, F \text{ is measurable on } V\}$. The theorem is a consequence of (7) and (79).

(82) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , and an element B of S_1 . Then the field generated by $\text{MeasRect}(S_1, S_2) \subseteq \{E, \text{ where } E \text{ is an element of } \sigma(\text{MeasRect}(S_1, S_2)) : \text{ there exists a function } F \text{ from } X_2 \text{ into } \overline{\mathbb{R}} \text{ such that for every element } y \text{ of } X_2, F(y) = M_1(\text{MeasurableYsection}(E, y) \cap B) \text{ and for every element } V \text{ of } S_2, F \text{ is measurable on } V\}$. The theorem is a consequence of (7) and (80).

7. σ -FINITE MEASURE

Let X be a non empty set, S be a σ -field of subsets of X , and M be a σ -measure on S . We say that M is σ -finite if and only if

(Def. 12) there exists a set sequence E of S such that for every natural number n , $M(E(n)) < +\infty$ and $\bigcup E = X$.

Now we state the propositions:

- (83) Let us consider a non empty set X , a σ -field S of subsets of X , and a σ -measure M on S . Then M is σ -finite if and only if there exists a set sequence F of S such that F is non descending and for every natural number n , $M(F(n)) < +\infty$ and $\lim F = X$.
- (84) Let us consider a set X , a semialgebra S of sets of X , a pre-measure P of S , and an induced measure M of S and P . Then $M =$ (the Caratheodory measure determined by M) \upharpoonright (the field generated by S).

8. FUBINI'S THEOREM ON MEASURE

Now we state the propositions:

- (85) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_2 on S_2 , and an element B of S_2 . Suppose $M_2(B) < +\infty$. Then $\{E, \text{ where } E \text{ is an element of } \sigma(\text{MeasRect}(S_1, S_2)) : \text{ there exists a function } F \text{ from } X_1 \text{ into } \overline{\mathbb{R}} \text{ such that for every element } x \text{ of } X_1, F(x) = M_2(\text{MeasurableXsection}(E, x) \cap B) \text{ and for every element } V \text{ of } S_1, F \text{ is measurable on } V\}$ is a monotone class of $X_1 \times X_2$.

PROOF: Set $Z = \{E, \text{ where } E \text{ is an element of } \sigma(\text{MeasRect}(S_1, S_2)) : \text{ there exists a function } F \text{ from } X_1 \text{ into } \overline{\mathbb{R}} \text{ such that for every element } x \text{ of } X_1, F(x) = M_2(\text{MeasurableXsection}(E, x) \cap B) \text{ and for every element } V \text{ of } S_1, F \text{ is measurable on } V\}$. For every sequence A_1 of subsets of $X_1 \times X_2$ such that A_1 is monotone and $\text{rng } A_1 \subseteq Z$ holds $\lim A_1 \in Z$ by [10, (3)], [5, (35)], [21, (63)], [12, (45)]. \square

- (86) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , and an element B of S_1 . Suppose $M_1(B) < +\infty$. Then $\{E, \text{ where } E \text{ is an element of } \sigma(\text{MeasRect}(S_1, S_2)) : \text{ there exists a function } F \text{ from } X_2 \text{ into } \overline{\mathbb{R}} \text{ such that for every element } y \text{ of } X_2, F(y) = M_1(\text{MeasurableYsection}(E, y) \cap B) \text{ and for every element } V \text{ of } S_2, F \text{ is measurable on } V\}$ is a monotone class of $X_1 \times X_2$.

PROOF: Set $Z = \{E, \text{ where } E \text{ is an element of } \sigma(\text{MeasRect}(S_1, S_2)) : \text{ there exists a function } F \text{ from } X_2 \text{ into } \overline{\mathbb{R}} \text{ such that for every element } y \text{ of } X_2, F(y) = M_1(\text{MeasurableYsection}(E, y) \cap B) \text{ and for every element } V$

- of S_2, F is measurable on V }. For every sequence A_1 of subsets of $X_1 \times X_2$ such that A_1 is monotone and $\text{rng } A_1 \subseteq Z$ holds $\lim A_1 \in Z$ by [10, (3)], [5, (35)], [21, (63)], [12, (45)]. \square
- (87) Let us consider a non empty set X , a field F of subsets of X , and a sequence L of subsets of X . Suppose $\text{rng } L$ is a monotone class of X and $F \subseteq \text{rng } L$. Then
- (i) $\sigma(F) = \text{monotone-class}(F)$, and
 - (ii) $\sigma(F) \subseteq \text{rng } L$.
- (88) Let us consider a non empty set X , a field F of subsets of X , and a family K of subsets of X . Suppose K is a monotone class of X and $F \subseteq K$. Then
- (i) $\sigma(F) = \text{monotone-class}(F)$, and
 - (ii) $\sigma(F) \subseteq K$.
- (89) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_2 on S_2 , and an element B of S_2 . Suppose $M_2(B) < +\infty$. Then $\sigma(\text{MeasRect}(S_1, S_2)) \subseteq \{E, \text{ where } E \text{ is an element of } \sigma(\text{MeasRect}(S_1, S_2)) : \text{ there exists a function } F \text{ from } X_1 \text{ into } \overline{\mathbb{R}} \text{ such that for every element } x \text{ of } X_1, F(x) = M_2(\text{MeasurableXsection}(E, x) \cap B) \text{ and for every element } V \text{ of } S_1, F \text{ is measurable on } V\}$. The theorem is a consequence of (85), (81), (7), and (88).
- (90) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , and an element B of S_1 . Suppose $M_1(B) < +\infty$. Then $\sigma(\text{MeasRect}(S_1, S_2)) \subseteq \{E, \text{ where } E \text{ is an element of } \sigma(\text{MeasRect}(S_1, S_2)) : \text{ there exists a function } F \text{ from } X_2 \text{ into } \overline{\mathbb{R}} \text{ such that for every element } y \text{ of } X_2, F(y) = M_1(\text{MeasurableYsection}(E, y) \cap B) \text{ and for every element } V \text{ of } S_2, F \text{ is measurable on } V\}$. The theorem is a consequence of (86), (82), (7), and (88).
- (91) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_2 on S_2 , and an element E of $\sigma(\text{MeasRect}(S_1, S_2))$. Suppose M_2 is σ -finite. Then there exists a function F from X_1 into $\overline{\mathbb{R}}$ such that
- (i) for every element x of X_1 , $F(x) = M_2(\text{MeasurableXsection}(E, x))$, and
 - (ii) for every element V of S_1 , F is measurable on V .

PROOF: Consider B being a set sequence of S_2 such that B is non descending and for every natural number n , $M_2(B(n)) < +\infty$ and $\lim B = X_2$. Define $\mathcal{P}[\text{natural number, object}] \equiv \text{there exists a function } f_1 \text{ from } X_1 \text{ into } \overline{\mathbb{R}} \text{ such that } \mathcal{S}_2 = f_1 \text{ and for every element } x \text{ of } X_1, f_1(x) =$

$M_2(\text{MeasurableXsection}(E, x) \cap B(\$1))$ and for every element V of S_1 , f_1 is measurable on V . For every element n of \mathbb{N} , there exists an element f of $X_1 \dot{\rightarrow} \overline{\mathbb{R}}$ such that $\mathcal{P}[n, f]$ by (89), [12, (45)]. Consider f being a function from \mathbb{N} into $X_1 \dot{\rightarrow} \overline{\mathbb{R}}$ such that for every element n of \mathbb{N} , $\mathcal{P}[n, f(n)]$ from [11, Sch. 3]. For every natural number n , $f(n)$ is a function from X_1 into $\overline{\mathbb{R}}$ and for every element x of X_1 , $f(n)(x) = M_2(\text{MeasurableXsection}(E, x) \cap B(n))$ and for every element V of S_1 , $f(n)$ is measurable on V . For every natural numbers n, m , $\text{dom}(f(n)) = \text{dom}(f(m))$. For every element x of X_1 such that $x \in X_1$ holds $f \# x$ is convergent by [5, (11), (31)], [20, (7), (37)]. Reconsider $F = \lim f$ as a function from X_1 into $\overline{\mathbb{R}}$. For every element x of X_1 , $F(x) = M_2(\text{MeasurableXsection}(E, x))$ by [21, (80)], [22, (92)], (49), [5, (11)]. \square

(92) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , and an element E of $\sigma(\text{MeasRect}(S_1, S_2))$. Suppose M_1 is σ -finite. Then there exists a function F from X_2 into $\overline{\mathbb{R}}$ such that

- (i) for every element y of X_2 , $F(y) = M_1(\text{MeasurableYsection}(E, y))$, and
- (ii) for every element V of S_2 , F is measurable on V .

PROOF: Consider B being a set sequence of S_1 such that B is non descending and for every natural number n , $M_1(B(n)) < +\infty$ and $\lim B = X_1$. Define $\mathcal{P}[\text{natural number, object}] \equiv$ there exists a function f_1 from X_2 into $\overline{\mathbb{R}}$ such that $\$2 = f_1$ and for every element y of X_2 , $f_1(y) = M_1(\text{MeasurableYsection}(E, y) \cap B(\$1))$ and for every element V of S_2 , f_1 is measurable on V . For every element n of \mathbb{N} , there exists an element f of $X_2 \dot{\rightarrow} \overline{\mathbb{R}}$ such that $\mathcal{P}[n, f]$ by (90), [12, (45)]. Consider f being a function from \mathbb{N} into $X_2 \dot{\rightarrow} \overline{\mathbb{R}}$ such that for every element n of \mathbb{N} , $\mathcal{P}[n, f(n)]$ from [11, Sch. 3]. For every natural number n , $f(n)$ is a function from X_2 into $\overline{\mathbb{R}}$ and for every element y of X_2 , $f(n)(y) = M_1(\text{MeasurableYsection}(E, y) \cap B(n))$ and for every element V of S_2 , $f(n)$ is measurable on V . For every natural numbers n, m , $\text{dom}(f(n)) = \text{dom}(f(m))$. For every element y of X_2 such that $y \in X_2$ holds $f \# y$ is convergent by [5, (11), (31)], [20, (7), (37)]. Reconsider $F = \lim f$ as a function from X_2 into $\overline{\mathbb{R}}$. For every element y of X_2 , $F(y) = M_1(\text{MeasurableYsection}(E, y))$ by [21, (80)], [22, (92)], (49), [5, (11)]. \square

Let X_1, X_2 be non empty sets, S_1 be a σ -field of subsets of X_1 , S_2 be a σ -field of subsets of X_2 , M_2 be a σ -measure on S_2 , and E be an element of $\sigma(\text{MeasRect}(S_1, S_2))$. Assume M_2 is σ -finite. The functor $\text{Yvol}(E, M_2)$ yielding a non-negative function from X_1 into $\overline{\mathbb{R}}$ is defined by

(Def. 13) for every element x of X_1 , $it(x) = M_2(\text{MeasurableXsection}(E, x))$ and for every element V of S_1 , it is measurable on V .

Let M_1 be a σ -measure on S_1 . Assume M_1 is σ -finite. The functor $\text{Xvol}(E, M_1)$

yielding a non-negative function from X_2 into $\overline{\mathbb{R}}$ is defined by

(Def. 14) for every element y of X_2 , $it(y) = M_1(\text{MeasurableYsection}(E, y))$ and for every element V of S_2 , it is measurable on V .

Now we state the propositions:

(93) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_2 on S_2 , and elements E_1, E_2 of $\sigma(\text{MeasRect}(S_1, S_2))$. Suppose M_2 is σ -finite and E_1 misses E_2 . Then $\text{Yvol}(E_1 \cup E_2, M_2) = \text{Yvol}(E_1, M_2) + \text{Yvol}(E_2, M_2)$.

PROOF: For every element x of X_1 such that $x \in \text{dom Yvol}(E_1 \cup E_2, M_2)$ holds $(\text{Yvol}(E_1 \cup E_2, M_2))(x) = (\text{Yvol}(E_1, M_2) + \text{Yvol}(E_2, M_2))(x)$ by (26), (35), [5, (30)]. \square

(94) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , and elements E_1, E_2 of $\sigma(\text{MeasRect}(S_1, S_2))$. Suppose M_1 is σ -finite and E_1 misses E_2 . Then $\text{Xvol}(E_1 \cup E_2, M_1) = \text{Xvol}(E_1, M_1) + \text{Xvol}(E_2, M_1)$.

PROOF: For every element x of X_2 such that $x \in \text{dom Xvol}(E_1 \cup E_2, M_1)$ holds $(\text{Xvol}(E_1 \cup E_2, M_1))(x) = (\text{Xvol}(E_1, M_1) + \text{Xvol}(E_2, M_1))(x)$ by (26), (35), [5, (30)]. \square

Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , and elements E_1, E_2 of $\sigma(\text{MeasRect}(S_1, S_2))$. Now we state the propositions:

(95) Suppose M_2 is σ -finite and E_1 misses E_2 . Then $\int \text{Yvol}(E_1 \cup E_2, M_2) dM_1 = \int \text{Yvol}(E_1, M_2) dM_1 + \int \text{Yvol}(E_2, M_2) dM_1$. The theorem is a consequence of (93).

(96) Suppose M_1 is σ -finite and E_1 misses E_2 . Then $\int \text{Xvol}(E_1 \cup E_2, M_1) dM_2 = \int \text{Xvol}(E_1, M_1) dM_2 + \int \text{Xvol}(E_2, M_1) dM_2$. The theorem is a consequence of (94).

Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , an element E of $\sigma(\text{MeasRect}(S_1, S_2))$, an element A of S_1 , and an element B of S_2 . Now we state the propositions:

(97) Suppose $E = A \times B$ and M_2 is σ -finite. Then

(i) if $M_2(B) = +\infty$, then $\text{Yvol}(E, M_2) = \bar{\chi}_{A, X_1}$, and

(ii) if $M_2(B) \neq +\infty$, then there exists a real number r such that $r = M_2(B)$ and $\text{Yvol}(E, M_2) = r \cdot \chi_{A, X_1}$.

The theorem is a consequence of (53).

(98) Suppose $E = A \times B$ and M_1 is σ -finite. Then

(i) if $M_1(A) = +\infty$, then $X\text{vol}(E, M_1) = \bar{\chi}_{B, X_2}$, and

(ii) if $M_1(A) \neq +\infty$, then there exists a real number r such that $r = M_1(A)$ and $X\text{vol}(E, M_1) = r \cdot \chi_{B, X_2}$.

The theorem is a consequence of (55).

Now we state the proposition:

(99) Let us consider a non empty set X , a σ -field S of subsets of X , a σ -measure M on S , an element A of S , and a real number r . If $r \geq 0$, then $\int r \cdot \chi_{A, X} dM = r \cdot M(A)$.

Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , a finite sequence F of elements of $\sigma(\text{MeasRect}(S_1, S_2))$, and a natural number n . Now we state the propositions:

(100) Suppose M_2 is σ -finite and F is a finite sequence of elements of $\text{MeasRect}(S_1, S_2)$.

Then $(\text{Prod } \sigma\text{-Meas}(M_1, M_2))(F(n)) = \int Y\text{vol}(F(n), M_2) dM_1$. The theorem is a consequence of (16), (97), and (99).

(101) Suppose M_1 is σ -finite and F is a finite sequence of elements of $\text{MeasRect}(S_1, S_2)$.

Then $(\text{Prod } \sigma\text{-Meas}(M_1, M_2))(F(n)) = \int X\text{vol}(F(n), M_1) dM_2$. The theorem is a consequence of (16), (98), and (99).

Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , a disjoint valued finite sequence F of elements of $\sigma(\text{MeasRect}(S_1, S_2))$, and a natural number n . Now we state the propositions:

(102) Suppose M_2 is σ -finite and F is a finite sequence of elements of $\text{MeasRect}(S_1, S_2)$.

Then $(\text{Prod } \sigma\text{-Meas}(M_1, M_2))(\cup F) = \int Y\text{vol}(\cup F, M_2) dM_1$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (\text{Prod } \sigma\text{-Meas}(M_1, M_2))(\cup(F \upharpoonright \$_1)) = \int Y\text{vol}(\cup(F \upharpoonright \$_1), M_2) dM_1$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [2, (13)], [3, (59)], [19, (55)], [3, (82)]. For every natural number k , $\mathcal{P}[k]$ from [2, Sch. 2]. \square

(103) Suppose M_1 is σ -finite and F is a finite sequence of elements of $\text{MeasRect}(S_1, S_2)$.

Then $(\text{Prod } \sigma\text{-Meas}(M_1, M_2))(\cup F) = \int X\text{vol}(\cup F, M_1) dM_2$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (\text{Prod } \sigma\text{-Meas}(M_1, M_2))(\cup(F \upharpoonright \$_1)) = \int X\text{vol}(\cup(F \upharpoonright \$_1), M_1) dM_2$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [2, (13)], [3, (59)], [19, (55)], [3, (82)]. For every natural number k , $\mathcal{P}[k]$ from [2, Sch. 2]. \square

Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , an element

E of $\sigma(\text{MeasRect}(S_1, S_2))$, an element V of $\sigma(\text{MeasRect}(S_1, S_2))$, an element A of S_1 , and an element B of S_2 . Now we state the propositions:

- (104) Suppose $E \in$ the field generated by $\text{MeasRect}(S_1, S_2)$ and M_2 is σ -finite. Then suppose $V = A \times B$. Then $E \in \{E, \text{ where } E \text{ is an element of } \sigma(\text{MeasRect}(S_1, S_2)) : \int Y\text{vol}(E \cap V, M_2) dM_1 = (\text{Prod } \sigma\text{-Meas}(M_1, M_2))(E \cap V)\}$. The theorem is a consequence of (102).
- (105) Suppose $E \in$ the field generated by $\text{MeasRect}(S_1, S_2)$ and M_1 is σ -finite. Then suppose $V = A \times B$. Then $E \in \{E, \text{ where } E \text{ is an element of } \sigma(\text{MeasRect}(S_1, S_2)) : \int X\text{vol}(E \cap V, M_1) dM_2 = (\text{Prod } \sigma\text{-Meas}(M_1, M_2))(E \cap V)\}$. The theorem is a consequence of (103).

Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , an element V of $\sigma(\text{MeasRect}(S_1, S_2))$, an element A of S_1 , and an element B of S_2 . Now we state the propositions:

- (106) Suppose M_2 is σ -finite and $V = A \times B$. Then the field generated by $\text{MeasRect}(S_1, S_2) \subseteq \{E, \text{ where } E \text{ is an element of } \sigma(\text{MeasRect}(S_1, S_2)) : \int Y\text{vol}(E \cap V, M_2) dM_1 = (\text{Prod } \sigma\text{-Meas}(M_1, M_2))(E \cap V)\}$. The theorem is a consequence of (7) and (104).
- (107) Suppose M_1 is σ -finite and $V = A \times B$. Then the field generated by $\text{MeasRect}(S_1, S_2) \subseteq \{E, \text{ where } E \text{ is an element of } \sigma(\text{MeasRect}(S_1, S_2)) : \int X\text{vol}(E \cap V, M_1) dM_2 = (\text{Prod } \sigma\text{-Meas}(M_1, M_2))(E \cap V)\}$. The theorem is a consequence of (7) and (105).

Now we state the propositions:

- (108) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_2 on S_2 , elements E, V of $\sigma(\text{MeasRect}(S_1, S_2))$, a set sequence P of $\sigma(\text{MeasRect}(S_1, S_2))$, and an element x of X_1 . Suppose P is non descending and $\lim P = E$. Then there exists a sequence K of subsets of S_2 such that
- (i) K is non descending, and
 - (ii) for every natural number n , $K(n) = \text{MeasurableXsection}(P(n), x) \cap \text{MeasurableXsection}(V, x)$, and
 - (iii) $\lim K = \text{MeasurableXsection}(E, x) \cap \text{MeasurableXsection}(V, x)$.

The theorem is a consequence of (43), (49), and (30).

- (109) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , elements E, V of $\sigma(\text{MeasRect}(S_1, S_2))$, a set sequence P of $\sigma(\text{MeasRect}(S_1, S_2))$, and an element y of X_2 . Suppose P is non descending and $\lim P = E$. Then there exists a sequence K of subsets of S_1 such that

- (i) K is non descending, and
- (ii) for every natural number n , $K(n) = \text{MeasurableYsection}(P(n), y) \cap \text{MeasurableYsection}(V, y)$, and
- (iii) $\lim K = \text{MeasurableYsection}(E, y) \cap \text{MeasurableYsection}(V, y)$.

The theorem is a consequence of (44), (49), and (32).

- (110) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_2 on S_2 , elements E, V of $\sigma(\text{MeasRect}(S_1, S_2))$, a set sequence P of $\sigma(\text{MeasRect}(S_1, S_2))$, and an element x of X_1 . Suppose P is non ascending and $\lim P = E$. Then there exists a sequence K of subsets of S_2 such that

- (i) K is non ascending, and
- (ii) for every natural number n , $K(n) = \text{MeasurableXsection}(P(n), x) \cap \text{MeasurableXsection}(V, x)$, and
- (iii) $\lim K = \text{MeasurableXsection}(E, x) \cap \text{MeasurableXsection}(V, x)$.

The theorem is a consequence of (45), (49), and (31).

- (111) Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , elements E, V of $\sigma(\text{MeasRect}(S_1, S_2))$, a set sequence P of $\sigma(\text{MeasRect}(S_1, S_2))$, and an element y of X_2 . Suppose P is non ascending and $\lim P = E$. Then there exists a sequence K of subsets of S_1 such that

- (i) K is non ascending, and
- (ii) for every natural number n , $K(n) = \text{MeasurableYsection}(P(n), y) \cap \text{MeasurableYsection}(V, y)$, and
- (iii) $\lim K = \text{MeasurableYsection}(E, y) \cap \text{MeasurableYsection}(V, y)$.

The theorem is a consequence of (46), (49), and (33).

Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , an element V of $\sigma(\text{MeasRect}(S_1, S_2))$, an element A of S_1 , and an element B of S_2 . Now we state the propositions:

- (112) Suppose M_2 is σ -finite and $V = A \times B$ and $(\text{Prod } \sigma\text{-Meas}(M_1, M_2))(V) < +\infty$ and $M_2(B) < +\infty$. Then $\{E, \text{ where } E \text{ is an element of } \sigma(\text{MeasRect}(S_1, S_2)) : \int Y \text{vol}(E \cap V, M_2) dM_1 = (\text{Prod } \sigma\text{-Meas}(M_1, M_2))(E \cap V)\}$ is a monotone class of $X_1 \times X_2$.

PROOF: Set $Z = \{E, \text{ where } E \text{ is an element of } \sigma(\text{MeasRect}(S_1, S_2)) : \int Y \text{vol}(E \cap V, M_2) dM_1 = (\text{Prod } \sigma\text{-Meas}(M_1, M_2))(E \cap V)\}$. For every sequence A_1 of subsets of $X_1 \times X_2$ such that A_1 is monotone and $\text{rng } A_1 \subseteq Z$ holds $\lim A_1 \in Z$ by [10, (3)], [5, (35)], [21, (63)], [12, (45)]. \square

(113) Suppose M_1 is σ -finite and $V = A \times B$ and $(\text{Prod } \sigma\text{-Meas}(M_1, M_2))(V) < +\infty$ and $M_1(A) < +\infty$. Then $\{E, \text{ where } E \text{ is an element of } \sigma(\text{MeasRect}(S_1, S_2)) : \int X\text{vol}(E \cap V, M_1) dM_2 = (\text{Prod } \sigma\text{-Meas}(M_1, M_2))(E \cap V)\}$ is a monotone class of $X_1 \times X_2$.

PROOF: Set $Z = \{E, \text{ where } E \text{ is an element of } \sigma(\text{MeasRect}(S_1, S_2)) : \int X\text{vol}(E \cap V, M_1) dM_2 = (\text{Prod } \sigma\text{-Meas}(M_1, M_2))(E \cap V)\}$. For every sequence A_1 of subsets of $X_1 \times X_2$ such that A_1 is monotone and $\text{rng } A_1 \subseteq Z$ holds $\lim A_1 \in Z$ by [10, (3)], [5, (35)], [21, (63)], [12, (45)]. \square

(114) Suppose M_2 is σ -finite and $V = A \times B$ and $(\text{Prod } \sigma\text{-Meas}(M_1, M_2))(V) < +\infty$ and $M_2(B) < +\infty$. Then $\sigma(\text{MeasRect}(S_1, S_2)) \subseteq \{E, \text{ where } E \text{ is an element of } \sigma(\text{MeasRect}(S_1, S_2)) : \int Y\text{vol}(E \cap V, M_2) dM_1 = (\text{Prod } \sigma\text{-Meas}(M_1, M_2))(E \cap V)\}$. The theorem is a consequence of (112), (106), (7), and (88).

(115) Suppose M_1 is σ -finite and $V = A \times B$ and $(\text{Prod } \sigma\text{-Meas}(M_1, M_2))(V) < +\infty$ and $M_1(A) < +\infty$. Then $\sigma(\text{MeasRect}(S_1, S_2)) \subseteq \{E, \text{ where } E \text{ is an element of } \sigma(\text{MeasRect}(S_1, S_2)) : \int X\text{vol}(E \cap V, M_1) dM_2 = (\text{Prod } \sigma\text{-Meas}(M_1, M_2))(E \cap V)\}$. The theorem is a consequence of (113), (107), (7), and (88).

Now we state the proposition:

(116) Let us consider sets X, Y , a sequence A of subsets of X , a sequence B of subsets of Y , and a sequence C of subsets of $X \times Y$. Suppose A is non descending and B is non descending and for every natural number n , $C(n) = A(n) \times B(n)$. Then

(i) C is non descending and convergent, and

(ii) $\bigcup C = \bigcup A \times \bigcup B$.

PROOF: For every natural numbers n, m such that $n \leq m$ holds $C(n) \subseteq C(m)$ by [13, (96)]. \square

Now we state the proposition:

(117) FUBINI'S THEOREM:

Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , and an element E of $\sigma(\text{MeasRect}(S_1, S_2))$. Suppose M_1 is σ -finite and M_2 is σ -finite. Then $\int Y\text{vol}(E, M_2) dM_1 = (\text{Prod } \sigma\text{-Meas}(M_1, M_2))(E)$.

PROOF: Consider A being a set sequence of S_1 such that A is non descending and for every natural number n , $M_1(A(n)) < +\infty$ and $\lim A = X_1$. Consider B being a set sequence of S_2 such that B is non descending and for every natural number n , $M_2(B(n)) < +\infty$ and $\lim B = X_2$. Define $\mathcal{C}(\text{element of } \mathbb{N}) = A(\$1) \times B(\$1)$. Consider C being a function from \mathbb{N} into $2^{X_1 \times X_2}$ such that for every element n of \mathbb{N} , $C(n) = \mathcal{C}(n)$ from [11, Sch. 4]. For every natural number n , $C(n) = A(n) \times$

$B(n)$. For every natural number n , $C(n) \in \sigma(\text{MeasRect}(S_1, S_2))$. For every natural numbers n, m such that $n \leq m$ holds $C(n) \subseteq C(m)$ by [13, (96)]. For every natural number n , $(\text{Prod } \sigma\text{-Meas}(M_1, M_2))(C(n)) < +\infty$ by (16), [6, (51)]. Set $C_1 = E \cap C$. For every object n such that $n \in \mathbb{N}$ holds $C_1(n) \in \sigma(\text{MeasRect}(S_1, S_2))$. For every natural number n , $\int \text{Yvol}(E \cap C(n), M_2) dM_1 = (\text{Prod } \sigma\text{-Meas}(M_1, M_2))(E \cap C(n))$. Define $\mathcal{P}[\text{element of } \mathbb{N}, \text{object}] \equiv \mathcal{P}[n, \text{object}] \equiv \mathcal{P}[n, F(n)]$ from [11, Sch. 3]. For every natural number n , $F(n) = \text{Yvol}(E \cap C(n), M_2)$. Reconsider $X_3 = X_1$ as an element of S_1 . For every natural number n and for every element x of X_1 , $(F\#x)(n) = (\text{Yvol}(E \cap C(n), M_2))(x)$. For every natural numbers n, m , $\text{dom}(F(n)) = \text{dom}(F(m))$. For every natural number n , $F(n)$ is measurable on X_3 . For every natural numbers n, m such that $n \leq m$ for every element x of X_1 such that $x \in X_3$ holds $F(n)(x) \leq F(m)(x)$ by (20), [5, (31)]. For every element x of X_1 such that $x \in X_3$ holds $F\#x$ is convergent by [20, (7), (37)]. Consider I being a sequence of extended reals such that for every natural number n , $I(n) = \int F(n) dM_1$ and I is convergent and $\int \lim F dM_1 = \lim I$. For every element x of X_1 such that $x \in \text{dom } \lim F$ holds $(\lim F)(x) = (\text{Yvol}(E, M_2))(x)$ by (116), (108), (27), [10, (13)]. Set $J = E \cap C$. For every object n such that $n \in \mathbb{N}$ holds $J(n) \in \sigma(\text{MeasRect}(S_1, S_2))$. $\text{Prod } \sigma\text{-Meas}(M_1, M_2)$ is a σ -measure on $\sigma(\text{MeasRect}(S_1, S_2))$. For every element n of \mathbb{N} , $I(n) = (\text{Prod } \sigma\text{-Meas}(M_1, M_2)*J)(n)$ by [10, (13)]. \square

Now we state the proposition:

(118) FUBINI'S THEOREM:

Let us consider non empty sets X_1, X_2 , a σ -field S_1 of subsets of X_1 , a σ -field S_2 of subsets of X_2 , a σ -measure M_1 on S_1 , a σ -measure M_2 on S_2 , and an element E of $\sigma(\text{MeasRect}(S_1, S_2))$. Suppose M_1 is σ -finite and M_2 is σ -finite. Then $\int \text{Xvol}(E, M_1) dM_2 = (\text{Prod } \sigma\text{-Meas}(M_1, M_2))(E)$.

PROOF: Consider A being a set sequence of S_1 such that A is non descending and for every natural number n , $M_1(A(n)) < +\infty$ and $\lim A = X_1$. Consider B being a set sequence of S_2 such that B is non descending and for every natural number n , $M_2(B(n)) < +\infty$ and $\lim B = X_2$. Define $\mathcal{C}(\text{element of } \mathbb{N}) = A(\$_1) \times B(\$_1)$. Consider C being a function from \mathbb{N} into $2^{X_1 \times X_2}$ such that for every element n of \mathbb{N} , $C(n) = \mathcal{C}(n)$ from [11, Sch. 4]. For every natural number n , $C(n) = A(n) \times B(n)$. For every natural number n , $C(n) \in \sigma(\text{MeasRect}(S_1, S_2))$. For every natural numbers n, m such that $n \leq m$ holds $C(n) \subseteq C(m)$ by

[13, (96)]. For every natural number n , $(\text{Prod } \sigma\text{-Meas}(M_1, M_2))(C(n)) < +\infty$ by (16), [6, (51)]. Set $C_1 = E \cap C$. For every object n such that $n \in \mathbb{N}$ holds $C_1(n) \in \sigma(\text{MeasRect}(S_1, S_2))$. For every natural number n , $\int \text{Xvol}(E \cap C(n), M_1) dM_2 = (\text{Prod } \sigma\text{-Meas}(M_1, M_2))(E \cap C(n))$. Define $\mathcal{P}[\text{element of } \mathbb{N}, \text{object}] \equiv \mathcal{P}_2 = \text{Xvol}(E \cap C(\mathcal{P}_1), M_1)$. For every element n of \mathbb{N} , there exists an element f of $X_2 \rightarrow \overline{\mathbb{R}}$ such that $\mathcal{P}[n, f]$ by [12, (45)]. Consider F being a function from \mathbb{N} into $X_2 \rightarrow \overline{\mathbb{R}}$ such that for every element n of \mathbb{N} , $\mathcal{P}[n, F(n)]$ from [11, Sch. 3]. For every natural number n , $F(n) = \text{Xvol}(E \cap C(n), M_1)$. Reconsider $X_3 = X_2$ as an element of S_2 . For every natural number n and for every element x of X_2 , $(F\#x)(n) = (\text{Xvol}(E \cap C(n), M_1))(x)$. For every natural numbers n, m , $\text{dom}(F(n)) = \text{dom}(F(m))$. For every natural number n , $F(n)$ is measurable on X_3 . For every natural numbers n, m such that $n \leq m$ for every element x of X_2 such that $x \in X_3$ holds $F(n)(x) \leq F(m)(x)$ by (21), [5, (31)]. For every element x of X_2 such that $x \in X_3$ holds $F\#x$ is convergent by [20, (7), (37)]. Consider I being a sequence of extended reals such that for every natural number n , $I(n) = \int F(n) dM_2$ and I is convergent and $\int \lim F dM_2 = \lim I$. For every element x of X_2 such that $x \in \text{dom } \lim F$ holds $(\lim F)(x) = (\text{Xvol}(E, M_1))(x)$ by (116), (109), (27), [10, (13)]. Set $J = E \cap C$. For every object n such that $n \in \mathbb{N}$ holds $J(n) \in \sigma(\text{MeasRect}(S_1, S_2))$. $\text{Prod } \sigma\text{-Meas}(M_1, M_2)$ is a σ -measure on $\sigma(\text{MeasRect}(S_1, S_2))$. For every element n of \mathbb{N} , $I(n) = (\text{Prod } \sigma\text{-Meas}(M_1, M_2)_* J)(n)$ by [10, (13)]. \square

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