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Basel Problem[1]

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Summary. A rigorous elementary proof of the Basel problem

$$\Sigma_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

is formalized in the Mizar system [2]. This theorem is item `#14` from the "Formalizing 100 Theorems" list maintained by Freek Wiedijk at `http://www.cs.ru.nl/F.Wiedijk/100/` .

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1. Preliminaries

From now on k, m, n denote natural numbers, R denotes a commutative ring, p, q denote polynomials over R, and z_0, z_1 denote elements of R.

Let L be a right zeroed, non empty double loop structure. Let us consider n. Let us note that $n \cdot 0_L$ reduces to 0_L.

Now we state the proposition:

(1) Let us consider a complex z, and an element e of $\mathbb{C}_{\mathrm{F}}$. If $z = e$, then $n \cdot z = n \cdot e$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv \$_1 \cdot z = \$_1 \cdot e$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i, $\mathcal{P}[i]$ from [1, Sch. 2]. $\square$

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Let e be an element of $\mathbb{C}_{\mathrm{F}}$ and z be a complex. Let us consider n. We identify $n \cdot z$ with $n \cdot e$. Now we state the propositions:

(2) Let us consider a complex-valued finite sequence Z, and a finite sequence E of elements of $\mathbb{C}_{\mathrm{F}}$. If $E = Z$, then $\sum Z = \sum E$.
PROOF: Consider f being a sequence of $\mathbb{C}_{\mathrm{F}}$ such that $\sum E = f(\operatorname{len} E)$ and $f(0) = 0_{\mathbb{C}_{\mathrm{F}}}$ and for every natural number j and for every element v of $\mathbb{C}_{\mathrm{F}}$ such that $j < \operatorname{len} E$ and $v = E(j+1)$ holds $f(j+1) = f(j) + v$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant \operatorname{len} Z$, then $\sum(Z{\restriction}\$_1) = f(\$_1)$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$ by [1, (11)], [13, (25)], [4, (10)], [1, (13)]. $\mathcal{P}[n]$ from [1, Sch. 2]. $\square$

(3) $(\mathbf{1}_{\mathbb{C}_{\mathrm{F}}})^n = \mathbf{1}_{\mathbb{C}_{\mathrm{F}}}$.

(4) Let us consider a left zeroed, right zeroed, non empty additive loop structure L, and elements z_0, z_1 of L. Then $\langle z_0, z_1\rangle = \langle z_0\rangle + \langle 0_L, z_1\rangle$.

(5) Let us consider an add-associative, right zeroed, right complementable, distributive, non empty double loop structure L, and elements a, b, c, d of L. Then $\langle a, b\rangle * \langle c, d\rangle = \langle a \cdot c, a \cdot d + b \cdot c, b \cdot d\rangle$.

(6) Let us consider an Abelian, add-associative, right zeroed, right complementable, well unital, commutative, distributive, non empty double loop structure L. Then $\langle 0_L, 0_L, 1_L\rangle = \langle 0_L, 1_L\rangle^2$. The theorem is a consequence of (5).

(7) Let us consider a right zeroed, add-associative, right complementable, right distributive, non empty double loop structure L, an element z of L, and a polynomial p over L. Then $(p * \langle z\rangle)(n) = p(n) \cdot z$.
PROOF: Set $Z = \langle z\rangle$. Consider r being a finite sequence of elements of the carrier of L such that $\operatorname{len} r = n+1$ and $(p*\langle z\rangle)(n) = \sum r$ and for every element k of $\mathbb{N}$ such that $k \in \operatorname{dom} r$ holds $r(k) = p(k -' 1) \cdot Z(n+1 -' k)$. Set $l = \operatorname{len} r$. For every element k of $\mathbb{N}$ such that $k \in \operatorname{dom} r$ and $k \neq l$ holds $r_k = 0_L$ by [13, (25)], [1, (14)], [9, (32)]. $\square$

(8) Let us consider an Abelian, add-associative, right zeroed, right complementable, well unital, associative, commutative, distributive, non empty double loop structure L, and an element x of L. Then $\langle x\rangle^n = \langle x^n\rangle$.
PROOF: Set $X = \langle x\rangle$. Define $\mathcal{P}[\text{natural number}] \equiv X^{\$_1} = \langle x^{\$_1}\rangle$. $\mathcal{P}[0]$ by [11, (8)], [1, (14)], [9, (32)], [7, (30)]. For every n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [9, (19)], [1, (14)], [9, (32)], [11, (8)]. For every n, $\mathcal{P}[n]$ from [1, Sch. 2]. $\square$

(9) (i) $\langle z_0, z_1\rangle^0(0) = 1_R$, and
(ii) if $n > 0$, then $\langle 0_R, z_1\rangle^n(n) = {z_1}^n$, and
(iii) if $k \neq n$, then $\langle 0_R, z_1\rangle^n(k) = 0_R$.
PROOF: Set $P = \langle 0_R, z_1\rangle$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 > 0$, then $P^{\$_1}(\$_1) = {z_1}^{\$_1}$ and for every k such that $k \neq \$_1$ holds $P^{\$_1}(k) = 0_R$. $\mathcal{P}[0]$

by [9, (15)], [7, (30)]. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [9, (19), (16), (38)], [11, (8)]. For every natural number i, $\mathcal{P}[i]$ from [1, Sch. 2]. $\square$

(10) (i) $\langle 0_R, 0_R, \mathbf{1}_R \rangle^n (2 \cdot n) = \mathbf{1}_R$, and

(ii) for every k such that $k \neq 2 \cdot n$ holds $\langle 0_R, 0_R, \mathbf{1}_R \rangle^n (k) = 0_R$.

Proof: Set $x_1 = \langle 0_R, \mathbf{1}_R \rangle$. Set $x_2 = \langle 0_R, 0_R, \mathbf{1}_R \rangle$. Define $\mathcal{P}$[natural number] $\equiv {x_2}^{\$_1} = {x_1}^{2 \cdot \$_1}$. If $\mathcal{P}[k]$, then $\mathcal{P}[k+1]$ by (6), [9, (17), (19)], [7, (33)]. $\mathcal{P}[k]$ from [1, Sch. 2]. Define $\mathcal{Q}$[natural number] $\equiv (\mathbf{1}_R)^{\$_1} = \mathbf{1}_R$. If $\mathcal{Q}[k]$, then $\mathcal{Q}[k+1]$. $\mathcal{Q}[k]$ from [1, Sch. 2]. $\square$

(11) Let us consider an integral domain L, and a non-zero polynomial p over L. Then $\overline{\overline{\mathrm{Roots}(p)}} < \mathrm{len}\, p$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every non-zero polynomial p over L such that $\mathrm{len}\, p = \$_1$ holds $\overline{\overline{\mathrm{Roots}(p)}} < \mathrm{len}\, p$. For every natural number n such that $n \geqslant 1$ and $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [10, (47)], [8, (3)], [10, (50), (23), (48)]. For every natural number n such that $n \geqslant 1$ holds $\mathcal{P}[n]$ from [1, Sch. 8]. $\square$

Let L be an add-associative, right zeroed, right complementable, distributive, non empty double loop structure and a be a polynomial over L. The functor ${}^{@}a$ yielding an element of PolyRing(L) is defined by the term

(Def. 1) a.

Let n be a natural number. The functor $n \cdot a$ yielding a polynomial over L is defined by the term

(Def. 2) $n \cdot {}^{@}a$.

Now we state the propositions:

(12) Let us consider an add-associative, right zeroed, right complementable, distributive, non empty double loop structure L, and a polynomial a over L. Then $(n \cdot a)(k) = n \cdot a(k)$.

Proof: Define $\mathcal{P}$[natural number] $\equiv (\$_1 \cdot a)(k) = \$_1 \cdot a(k)$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i, $\mathcal{P}[i]$ from [1, Sch. 2]. $\square$

(13) $\langle z_0, z_1 \rangle^n (k) = \binom{n}{k} \cdot ({z_1}^k \cdot {z_0}^{n -' k})$.

Proof: Set $Z_0 = \langle z_0 \rangle$. Set $Z_1 = \langle 0_R, z_1 \rangle$. Set $C = \binom{n}{k} \cdot ({z_1}^k \cdot {z_0}^{n -' k})$. Set $P_2 = \mathrm{PolyRing}(R)$. $\langle z_0, z_1 \rangle = Z_0 + Z_1$. Consider F being a finite sequence of elements of PolyRing(R) such that $\langle z_0, z_1 \rangle^n = \sum F$ and $\mathrm{len}\, F = n+1$ and for every natural number k such that $k \leqslant n$ holds $F(k+1) = \binom{n}{k} \cdot ({Z_1}^k * {Z_0}^{n -' k})$. For every natural number i such that $i \leqslant n$ for every polynomial F_1 over R such that $F_1 = F(i+1)$ holds if $k \neq i$, then $F_1(k) = 0_R$ and if $k = i$, then $F_1(k) = C$ by (12), (8), (7), (9). Consider f being a sequence of the carrier of P_2 such that $\sum F = f(\operatorname{len} F)$ and $f(0) = 0_{P_2}$ and for every natural number j and for every element v of P_2 such that $j < \operatorname{len} F$ and $v = F(j+1)$ holds $f(j+1) = f(j) + v$. For every polynomial p over R such that $p = f(0)$ holds $p(k) = 0_R$ by [12, (7)]. $\square$

2. Imaginary Complex Numbers

Let z be a complex. We say that z is imaginary if and only if

(Def. 3) $\Re(z) = 0$.

Note that i is imaginary and every complex which is real and imaginary is also zero and every complex which is zero is also imaginary.

Let z_1, z_2 be imaginary complexes. One can verify that $z_1 \cdot z_2$ is real and $z_1 + z_2$ is imaginary.

Let z be an imaginary complex and r be a real complex. Note that $z \cdot r$ is imaginary and $0_{\mathbb{C}_\mathrm{F}}$ is real and imaginary and there exists an element of $\mathbb{C}_\mathrm{F}$ which is real and imaginary.

Let z be a real element of $\mathbb{C}_\mathrm{F}$ and n be a natural number. Observe that $n \cdot z$ is real.

Let z be an imaginary element of $\mathbb{C}_\mathrm{F}$. Observe that $n \cdot z$ is imaginary.

Let z be an imaginary complex and n be an even natural number. Let us observe that $\operatorname{power}_{\mathbb{C}_\mathrm{F}}(z, n)$ is real.

Let n be an odd natural number. One can check that $\operatorname{power}_{\mathbb{C}_\mathrm{F}}(z, n)$ is imaginary as a complex.

Let r be a real element of $\mathbb{C}_\mathrm{F}$ and n be a natural number. Let us note that $\operatorname{power}_{\mathbb{C}_\mathrm{F}}(r, n)$ is real and every element of $\mathbb{C}_\mathrm{F}$ which is zero is also imaginary and real.

Let p be a sequence of $\mathbb{C}_\mathrm{F}$. We say that p is imaginary if and only if

(Def. 4) for every natural number i, $p(i)$ is imaginary.

Let i_1 be an imaginary element of $\mathbb{C}_\mathrm{F}$. One can check that $\langle i_1 \rangle$ is imaginary.

Let i_2 be an imaginary element of $\mathbb{C}_\mathrm{F}$. Observe that $\langle i_1, i_2 \rangle$ is imaginary and there exists a polynomial over $\mathbb{C}_\mathrm{F}$ which is imaginary.

Now we state the propositions:

(14) Let us consider an imaginary polynomial I over $\mathbb{C}_\mathrm{F}$, and a real element r of $\mathbb{C}_\mathrm{F}$. Then $\operatorname{eval}(I, r)$ is imaginary.

Proof: Consider H being a finite sequence of elements of $\mathbb{C}_\mathrm{F}$ such that $\operatorname{eval}(I, r) = \sum H$ and $\operatorname{len} H = \operatorname{len} I$ and for every element n of $\mathbb{N}$ such that $n \in \operatorname{dom} H$ holds $H(n) = I(n -' 1) \cdot \operatorname{power}_{\mathbb{C}_\mathrm{F}}(r, n -' 1)$. Consider h being a sequence of the carrier of $\mathbb{C}_\mathrm{F}$ such that $\sum H = h(\operatorname{len} H)$ and $h(0) = 0_{\mathbb{C}_\mathrm{F}}$ and for every natural number j and for every element v of $\mathbb{C}_\mathrm{F}$ such that

$j < \operatorname{len} H$ and $v = H(j+1)$ holds $h(j+1) = h(j) + v$. Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \leqslant \operatorname{len} H$, then $h(\$_1)$ is imaginary. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$ by [1, (11)], [13, (25)], [1, (13)]. $\mathcal{P}[n]$ from [1, Sch. 2]. $\square$

(15) Let us consider a real polynomial R over $\mathbb{C}_{\mathrm{F}}$, and a real element r of $\mathbb{C}_{\mathrm{F}}$. Then $\operatorname{eval}(R, r)$ is real.

Proof: Consider H being a finite sequence of elements of $\mathbb{C}_{\mathrm{F}}$ such that $\operatorname{eval}(I, r) = \sum H$ and $\operatorname{len} H = \operatorname{len} I$ and for every element n of $\mathbb{N}$ such that $n \in \operatorname{dom} H$ holds $H(n) = I(n -' 1) \cdot \operatorname{power}_{\mathbb{C}_{\mathrm{F}}}(r, n -' 1)$. Consider h being a sequence of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $\sum H = h(\operatorname{len} H)$ and $h(0) = 0_{\mathbb{C}_{\mathrm{F}}}$ and for every natural number j and for every element v of $\mathbb{C}_{\mathrm{F}}$ such that $j < \operatorname{len} H$ and $v = H(j+1)$ holds $h(j+1) = h(j) + v$. Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \leqslant \operatorname{len} H$, then $h(\$_1)$ is real. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$ by [1, (11)], [13, (25)], [1, (13)]. $\mathcal{P}[n]$ from [1, Sch. 2]. $\square$

Let us consider an imaginary element i_3 of $\mathbb{C}_{\mathrm{F}}$ and a real element r of $\mathbb{C}_{\mathrm{F}}$. Now we state the propositions:

(16) If n is even, then the even part of $\langle i_3, r\rangle^n$ is real and the odd part of $\langle i_3, r\rangle^n$ is imaginary. The theorem is a consequence of (13).

(17) If n is odd, then the even part of $\langle i_3, r\rangle^n$ is imaginary and the odd part of $\langle i_3, r\rangle^n$ is real. The theorem is a consequence of (13).

Now we state the proposition:

(18) Let us consider a non empty zero structure L, and a polynomial p over L. Suppose len(the even part of p) $\neq 0$. Then len(the even part of p) is odd.

Proof: Set E = the even part of p. Consider n such that $2 \cdot n = \operatorname{len} E$. Reconsider $n_1 = n - 1$ as a natural number. the length of E is at most $n + n_1$ by [1, (13)]. $\square$

3. Main Fcts

Let L be a non empty set, p be a sequence of L, and m be a natural number. The functor $\boxed{\operatorname{sieve}_m(p)}$ yielding a sequence of L is defined by

(Def. 5) for every natural number i, $it(i) = p(m \cdot i)$.

Let L be a non empty zero structure, p be a finite-Support sequence of L, and m be a non zero natural number. Let us observe that $\operatorname{sieve}_m(p)$ is finite-Support.

Now we state the propositions:

(19) Let us consider a non empty zero structure L, and a sequence p of L. Then $\operatorname{sieve}_{(2 \cdot k)}(p) = \operatorname{sieve}_{(2 \cdot k)}$(the even part of p).

(20) Let us consider a non empty zero structure L, and a polynomial p over L. Suppose len(the even part of p) is odd. Then $2 \cdot \operatorname{len} \operatorname{sieve}_2(p) = $ len(the even part of p) $+ 1$.
PROOF: Set $E =$ the even part of p. Set $C = \operatorname{sieve}_2(E)$. Consider n such that $\operatorname{len} E = 2 \cdot n + 1$. Set $n_1 = n + 1$. the length of C is at most n_1 by [1, (13)]. For every natural number m such that the length of C is at most m holds $n_1 \leqslant m$ by [1, (13)]. $C = \operatorname{sieve}_{(2 \cdot 1)}(p)$. □

(21) Let us consider a non empty zero structure L, and a polynomial p over L. Suppose len(the even part of p) $= 0$. Let us consider a non zero natural number n. Then $\operatorname{len} \operatorname{sieve}_{(2 \cdot n)}(p) = 0$.

(22) Let us consider a field L, and a polynomial p over L. Then the even part of $p = (\operatorname{sieve}_2(p))[\langle 0_L, 0_L, \mathbf{1}_L \rangle]$. The theorem is a consequence of (10), (18), (20), and (21).

(23) $(\operatorname{sieve}_2(\langle i_{\mathbb{C}_\mathrm{F}}, 1_{\mathbb{C}_\mathrm{F}} \rangle^{2 \cdot n+1}))(n) = \binom{2 \cdot n+1}{1} \cdot i_{\mathbb{C}_\mathrm{F}}$. The theorem is a consequence of (3) and (13).

(24) Suppose $n \geqslant 1$. Then $(\operatorname{sieve}_2(\langle i_{\mathbb{C}_\mathrm{F}}, 1_{\mathbb{C}_\mathrm{F}} \rangle^{2 \cdot n+1}))(n-1) = \binom{2 \cdot n+1}{3} \cdot (-i_{\mathbb{C}_\mathrm{F}})$. The theorem is a consequence of (3) and (13).

(25) $\operatorname{len} \operatorname{sieve}_2(\langle i_{\mathbb{C}_\mathrm{F}}, 1_{\mathbb{C}_\mathrm{F}} \rangle^{2 \cdot n+1}) = n + 1$.
PROOF: Set $P_1 = \langle i_{\mathbb{C}_\mathrm{F}}, 1_{\mathbb{C}_\mathrm{F}} \rangle^{2 \cdot n+1}$. the length of $\operatorname{sieve}_2(P_1)$ is at most $n+1$. For every m such that the length of $\operatorname{sieve}_2(P_1)$ is at most m holds $n+1 \leqslant m$ by [1, (13)], (23). □

Let n be a natural number. Let us note that $\operatorname{sieve}_2(\langle i_{\mathbb{C}_\mathrm{F}}, 1_{\mathbb{C}_\mathrm{F}} \rangle^{2 \cdot n+1})$ is non-zero.

Now we state the propositions:

(26) $\operatorname{rng}({}^2\cot \text{x-r-seq}(n)) \subseteq \operatorname{Roots}(\operatorname{sieve}_2(\langle i_{\mathbb{C}_\mathrm{F}}, 1_{\mathbb{C}_\mathrm{F}} \rangle^{2 \cdot n+1}))$.
PROOF: Set $f = \text{x-r-seq}(n)$. Set $f_1 = {}^2\cot f$. Set $P_1 = \langle i_{\mathbb{C}_\mathrm{F}}, 1_{\mathbb{C}_\mathrm{F}} \rangle^{2 \cdot n+1}$. Consider x being an object such that $x \in \operatorname{dom} f_1$ and $f_1(x) = y$. Reconsider $c = \cot(f(x))$ as an element of $\mathbb{C}_\mathrm{F}$. Set $N = 2 \cdot n + 1$. $(\cot(f(x)) + i)^N$ is real by [5, (21)], [13, (29), (25)], [5, (23)]. eval(the even part of $P_1, c) = 0$ by [6, (74)], [3, (6)], [6, (8)], (17). Set $X_2 = \langle 0_{\mathbb{C}_\mathrm{F}}, 0_{\mathbb{C}_\mathrm{F}}, \mathbf{1}_{\mathbb{C}_\mathrm{F}} \rangle$. The even part of $P_1 = (\operatorname{sieve}_2(P_1))[X_2]$. □

(27) $\operatorname{Roots}(\operatorname{sieve}_2(\langle i_{\mathbb{C}_\mathrm{F}}, 1_{\mathbb{C}_\mathrm{F}} \rangle^{2 \cdot n+1})) = \operatorname{rng}({}^2\cot \text{x-r-seq}(n))$. The theorem is a consequence of (26), (11), and (25).

(28) $\sum({}^2\cot \text{x-r-seq}(m)) = \frac{2 \cdot m \cdot (2 \cdot m-1)}{6}$. The theorem is a consequence of (25), (27), (23), (24), and (2).

(29) $\sum({}^2\operatorname{cosec} \text{x-r-seq}(m)) = \frac{2 \cdot m \cdot (2 \cdot m+2)}{6}$. The theorem is a consequence of (28).

(30) $(\text{Basel-seq}^1)(m) \leqslant \sum_{\kappa=0}^{m} \text{Basel-seq}(\kappa)$. The theorem is a consequence of

(28).

(31) $\sum_{\kappa=0}^{m} \text{Basel-seq}(\kappa) \leqslant (\text{Basel-seq}^2)(m)$. The theorem is a consequence of (29).

Now we state the proposition:

(32) Basel problem:
$\sum \text{Basel-seq} = \frac{\pi^2}{6}$. The theorem is a consequence of (30) and (31).

Note that $(\sum_{\alpha=0}^{\kappa} (\text{Basel-seq})(\alpha))_{\kappa \in \mathbb{N}}$ is non summable as a sequence of real numbers.

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