

Introduction to Stopping Time in Stochastic Finance Theory

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Summary. We start with the definition of stopping time according to [4], p.283. We prove, that different definitions for stopping time can coincide. We give examples of stopping times using constant-functions or functions defined with the operator max or min (defined in [6], pp.37–38). Last we give an example with some given filtration. Stopping times are very important for stochastic finance. A stopping time is the moment, where a certain event occurs ([7], p.372) and can be used together with stochastic processes ([4], p.283). Look at the following example: we install a function $ST: \{1,2,3,4\} \rightarrow \{0,1,2\} \vee \{+\infty\}$, we define:

a. $ST(1)=1, ST(2)=1, ST(3)=2, ST(4)=2$.

b. The set $\{0,1,2\}$ consists of time points: 0=now, 1=tomorrow, 2=the day after tomorrow.

We can prove:

c. $\{w, \text{ where } w \text{ is Element of } \Omega: ST.w=0\}=\{\}$ & $\{w, \text{ where } w \text{ is Element of } \Omega: ST.w=1\}=\{1,2\}$ & $\{w, \text{ where } w \text{ is Element of } \Omega: ST.w=2\}=\{3,4\}$ and ST is a stopping time.

We use a function $Filt$ as Filtration of $\{0,1,2\}$, Σ where $Filt(0)=\Omega_{now}$, $Filt(1)=\Omega_{fut1}$ and $Filt(2)=\Omega_{fut2}$. From a., b. and c. we know that:

d. $\{w, \text{ where } w \text{ is Element of } \Omega: ST.w=0\}$ in Ω_{now} and

$\{w, \text{ where } w \text{ is Element of } \Omega: ST.w=1\}$ in Ω_{fut1} and

$\{w, \text{ where } w \text{ is Element of } \Omega: ST.w=2\}$ in Ω_{fut2} .

The sets in d. are events, which occur at the time points 0(=now), 1(=tomorrow) or 2(=the day after tomorrow), see also [7], p.371. Suppose we have $ST(1)=+\infty$, then this means that for 1 the corresponding event never occurs.

As an interpretation for our installed functions consider the given adapted stochastic process in the article [5].

$ST(1)=1$ means, that the given element 1 in $\{1,2,3,4\}$ is stopped in 1 (=tomorrow). That tells us, that we have to look at the value $f_2(1)$ which is equal to 80. Same argumentation for the element 2 in $\{1,2,3,4\}$.

ST(3)=2 means, that the given element 3 in $\{1,2,3,4\}$ is stopped in 2 (=the day after tomorrow). That tells us, that we have to look at the value $f_3(3)$ which is equal to 100.

ST(4)=2 means, that the given element 4 in $\{1,2,3,4\}$ is stopped in 2 (=the day after tomorrow). That tells us, that we have to look at the value $f_3(4)$ which is equal to 120.

In real world, these functions can be used for questions like: when does the share price exceed a certain limit ? see [7], p.372.

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1. PRELIMINARIES

From now on Ω denotes a non empty set, Σ denotes a σ -field of subsets of Ω , and T denotes a natural number.

Now we state the proposition:

- (1) Let us consider a non empty set X , an object t , and a function f . Suppose $\text{dom } f = X$. Then $\{w, \text{ where } w \text{ is an element of } X : f(w) = t\} = \text{Coim}(f, t)$.

PROOF: Set $A = \{w, \text{ where } w \text{ is an element of } X : f(w) = t\}$. $A \subseteq \text{Coim}(f, t)$ by [2, (1)]. Consider y being an object such that $\langle x, y \rangle \in f$ and $y \in \{t\}$. \square

Let I be an extended real-membered set. The functor $I_{\{+\infty\}}$ yielding a subset of $\overline{\mathbb{R}}$ is defined by the term

(Def. 1) $I \cup \{+\infty\}$.

Let us note that $I_{\{+\infty\}}$ is non empty.

2. DEFINITION OF STOPPING TIME

Let T be a natural number. The functor $\bigcup_{t \in \mathbb{N}: 0 \leq t \leq T} \{t\}$ yielding a subset of \mathbb{R} is defined by the term

(Def. 2) $\{t, \text{ where } t \text{ is an element of } \mathbb{N} : 0 \leq t \leq T\}$.

Let us note that $\bigcup_{t \in \mathbb{N}: 0 \leq t \leq T} \{t\}$ is non empty.

The functor $T_{\{+\infty\}}$ yielding a subset of $\overline{\mathbb{R}}$ is defined by the term

(Def. 3) $\bigcup_{t \in \mathbb{N}: 0 \leq t \leq T} \{t\} \cup \{+\infty\}$.

Let us note that $T_{\{+\infty\}}$ is non empty.

In the sequel T_1 denotes an element of $T_{\{+\infty\}}$, MF denotes a filtration of $\bigcup_{t \in \mathbb{N}: 0 \leq t \leq T} \{t\}$ and Σ , and k, k_1, k_2 denote functions from Ω into $T_{\{+\infty\}}$.

Let T be a natural number, F be a function, and R be a binary relation. We say that R is **StoppingTime(F, T)** if and only if

(Def. 4) for every element t of $\bigcup_{t \in \mathbb{N}: 0 \leq t \leq T} \{t\}$, $\text{Coim}(R, t) \in F(t)$.

Let Ω be a non empty set, MF be a function, and k be a function from Ω into $T_{\{+\infty\}}$. Let us observe that k is **StoppingTime(MF, T)** if and only if the condition (Def. 5) is satisfied.

(Def. 5) for every element t of $\bigcup_{t \in \mathbb{N}: 0 \leq t \leq T} \{t\}$, $\{w, \text{ where } w \text{ is an element of } \Omega : k(w) = t\} \in MF(t)$.

Now we state the proposition:

(2) k is **StoppingTime(MF, T)** if and only if for every element t of $\bigcup_{t \in \mathbb{N}: 0 \leq t \leq T} \{t\}$, $\{w, \text{ where } w \text{ is an element of } \Omega : k(w) \leq t\} \in MF(t)$.

PROOF: If k is **StoppingTime(MF, T)**, then for every element t of $\bigcup_{t \in \mathbb{N}: 0 \leq t \leq T} \{t\}$, $\{w, \text{ where } w \text{ is an element of } \Omega : k(w) \leq t\} \in MF(t)$ by [1, (8), (12), (13)], [8, (21)]. For every element t of $\bigcup_{t \in \mathbb{N}: 0 \leq t \leq T} \{t\}$, $\{w, \text{ where } w \text{ is an element of } \Omega : k(w) = t\} \in MF(t)$ by [1, (13)], [8, (22), (24)], [1, (22)]. \square

3. EXAMPLES OF STOPPING TIMES

Now we state the proposition:

(3) $\Omega \mapsto T_1$ is **StoppingTime(MF, T)**.

PROOF: Set $c = \Omega \mapsto T_1$. For every element t of $\bigcup_{t \in \mathbb{N}: 0 \leq t \leq T} \{t\}$, $\{w, \text{ where } w \text{ is an element of } \Omega : c(w) = t\} \in MF(t)$ by [9, (7)], [8, (5), (4)]. \square

Let us consider Ω, T, k_1 , and k_2 . The functor $\max(k_1, k_2)$ yielding a function from Ω into $\overline{\mathbb{R}}$ is defined by

(Def. 6) for every element w of Ω , $it(w) = \max(k_1(w), k_2(w))$.

The functor $\min(k_1, k_2)$ yielding a function from Ω into $\overline{\mathbb{R}}$ is defined by

(Def. 7) for every element w of Ω , $it(w) = \min(k_1(w), k_2(w))$.

Now we state the propositions:

(4) Suppose k_1 is **StoppingTime(MF, T)** and k_2 is **StoppingTime(MF, T)**.

Then there exists a function k_3 from Ω into $T_{\{+\infty\}}$ such that

(i) $k_3 = \max(k_1, k_2)$, and

(ii) k_3 is **StoppingTime(MF, T)**.

PROOF: Set $k_3 = \max(k_1, k_2)$. k_3 is a function from Ω into $T_{\{+\infty\}}$ by [2, (3)], [3, (2)]. k_3 is **StoppingTime(MF, T)** by (2), [8, (19)]. \square

- (5) Suppose k_1 is $\text{StoppingTime}(MF, T)$ and k_2 is $\text{StoppingTime}(MF, T)$. Then there exists a function k_3 from Ω into $T_{\{+\infty\}}$ such that
- (i) $k_3 = \min(k_1, k_2)$, and
 - (ii) k_3 is $\text{StoppingTime}(MF, T)$.

PROOF: Set $k_3 = \min(k_1, k_2)$. k_3 is a function from Ω into $T_{\{+\infty\}}$ by [2, (3)], [3, (2)]. k_3 is $\text{StoppingTime}(MF, T)$ by (2), [8, (3)]. \square

Let t be an object. **The special element of $t_{\{+\infty\}}$** yielding an element of $2_{\{+\infty\}}$ is defined by the term
(Def. 8) $\text{IFIN}(t, \{1, 2\}, 1, 2)$.

Now we state the proposition:

- (6) Suppose $\Omega = \{1, 2, 3, 4\}$. Let us consider a filtration MF of $\bigcup_{t \in \mathbb{N}: 0 \leq t \leq 2} \{t\}$ and Σ . Suppose $MF(0) = \Omega_{\text{now}}$ and $MF(1) = \Omega_{\text{fut1}}$ and $MF(2) = \Omega_{\text{fut2}}$. Then there exists a function S from Ω into $2_{\{+\infty\}}$ such that
- (i) S is $\text{StoppingTime}(MF, 2)$, and
 - (ii) $S(1) = 1$, and
 - (iii) $S(2) = 1$, and
 - (iv) $S(3) = 2$, and
 - (v) $S(4) = 2$, and
 - (vi) $\{w, \text{ where } w \text{ is an element of } \Omega : S(w) = 0\} = \emptyset$, and
 - (vii) $\{w, \text{ where } w \text{ is an element of } \Omega : S(w) = 1\} = \{1, 2\}$, and
 - (viii) $\{w, \text{ where } w \text{ is an element of } \Omega : S(w) = 2\} = \{3, 4\}$.

PROOF: Define $\mathcal{U}(\text{element of } \Omega) = \text{the special element of } \mathbb{S}_{1\{+\infty\}}$. Consider f being a function from Ω into $2_{\{+\infty\}}$ such that for every element d of Ω , $f(d) = \mathcal{U}(d)$ from [3, Sch. 4]. $f(1) = 1$ and $f(2) = 1$ and $f(3) = 2$ and $f(4) = 2$. f is $\text{StoppingTime}(MF, 2)$ and $\{w, \text{ where } w \text{ is an element of } \Omega : f(w) = 0\} = \emptyset$ and $\{w, \text{ where } w \text{ is an element of } \Omega : f(w) = 1\} = \{1, 2\}$ and $\{w, \text{ where } w \text{ is an element of } \Omega : f(w) = 2\} = \{3, 4\}$ by [1, (9)], [8, (4)], [5, (24)]. \square

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