FORMALIZED MATHEMATICS
Vol. 25, No. 3, Pages 243–248, 2017
DOI: 10.1515/forma-2017-0023

DE GRUYTER OPEN
degruyter.com/view/j/forma

# Formal Introduction to Fuzzy Implications

Adam Grabowski
Institute of Informatics
University of Białystok
Poland

**Summary.** In the article we present in the Mizar system [1] the catalogue of nine basic fuzzy implications, used especially in the theory of fuzzy sets [3]. This work is a continuation of the development of fuzzy sets in Mizar [2]; it could be used to give a variety of more general operations on fuzzy sets.

MSC: 03B35

Keywords:

MML identifier: FUZIMPL1, version: 8.1.06 5.44.1305

## 1. Preliminaries

Let us consider elements $a$, $b$ of $[0,1]$. Now we state the propositions:

(1) $\max(b, \min(1-a, 1-b)) \in [0,1]$.

(2) $\min(1, 1-a+b) \in [0,1]$.

(3) $1-a+(a \cdot b) \in [0,1]$.

(4) $\max(1-a, b) \in [0,1]$.

(5) If $a > 0$ or $b > 0$, then $b^a \in [0,1]$.

(6) If $a > b$, then $\frac{b}{a} \in [0,1]$.

## 2. Basic Attributes Defining Fuzzy Implications

Let $f$ be a binary operation on $[0,1]$. We say that $f$ is decreasing-on-1st if and only if




ISSN 1426–2630(Print), 1898-9934(Online)

(Def. 1) for every elements $x_1$, $x_2$, $y$ of $[0,1]$ such that $x_1 \leqslant x_2$ holds $f(x_1, y) \geqslant f(x_2, y)$.

We say that $f$ is increasing-on-2nd if and only if

(Def. 2) for every elements $x$, $y_1$, $y_2$ of $[0,1]$ such that $y_1 \leqslant y_2$ holds $f(x, y_1) \leqslant f(x, y_2)$.

We say that $f$ is 00-dominant if and only if

(Def. 3) $f(0,0) = 1$.

We say that $f$ is 11-dominant if and only if

(Def. 4) $f(1,1) = 1$.

We say that $f$ is 10-weak if and only if

(Def. 5) $f(1,0) = 0$.

We say that $f$ is 01-dominant if and only if

(Def. 6) $f(0,1) = 1$.

We say that $f$ is with-properties-of-fuzzy-implication if and only if

(Def. 7) $f$ is decreasing-on-1st, increasing-on-2nd, 00-dominant, 11-dominant, and 10-weak.

We say that $f$ is with-properties-of-classical-implication if and only if

(Def. 8) $f$ is 00-dominant, 01-dominant, 11-dominant, and 10-weak.

## 3. Examples Showing Independence of Axioms

The functor $\mathrm{I}_1$ yielding a binary operation on $[0,1]$ is defined by

(Def. 9) for every elements $x$, $y$ of $[0,1]$, $it(x,y) = \max(1-x, \min(x,y))$.

One can verify that $\mathrm{I}_1$ is increasing-on-2nd, 00-dominant, 11-dominant, and 10-weak.

The functor $\mathrm{I}_2$ yielding a binary operation on $[0,1]$ is defined by

(Def. 10) for every elements $x$, $y$ of $[0,1]$, $it(x,y) = \max(y, \min(1-x, 1-y))$.

Let us note that $\mathrm{I}_2$ is decreasing-on-1st, 00-dominant, 11-dominant, and 10-weak.

The functor $\mathrm{I}_3$ yielding a binary operation on $[0,1]$ is defined by

(Def. 11) for every elements $x$, $y$ of $[0,1]$, if $y < 1$, then $it(x,y) = 0$ and if $y = 1$, then $it(x,y) = 1$.

Let us observe that $\mathrm{I}_3$ is decreasing-on-1st, increasing-on-2nd, non 00-dominant, 11-dominant, and 10-weak.

The functor $\mathrm{I}_4$ yielding a binary operation on $[0,1]$ is defined by

(Def. 12) for every elements $x$, $y$ of $[0,1]$, if $x = 0$, then $it(x,y) = 1$ and if $x > 0$, then $it(x,y) = 0$.

Observe that $\mathrm{I}_4$ is decreasing-on-1st, increasing-on-2nd, 00-dominant, non 11-dominant, and 10-weak.

The functor $\mathrm{I}_5$ yielding a binary operation on $[0,1]$ is defined by

(Def. 13) for every elements $x$, $y$ of $[0,1]$, $it(x,y) = 1$.

Observe that $\mathrm{I}_5$ is decreasing-on-1st, increasing-on-2nd, 00-dominant, 11-dominant, and non 10-weak.

## 4. Catalogue of Fuzzy Implications

The functor Lukasiewicz-implication yielding a binary operation on $[0,1]$ is defined by

(Def. 14) for every elements $x$, $y$ of $[0,1]$, $it(x,y) = \min(1, 1-x+y)$.

Note that Lukasiewicz-implication is decreasing-on-1st, increasing-on-2nd, 00-dominant, 11-dominant, and 10-weak and there exists a binary operation on $[0,1]$ which is with-properties-of-fuzzy-implication and every binary operation on $[0,1]$ which is with-properties-of-fuzzy-implication is also decreasing-on-1st, increasing-on-2nd, 00-dominant, 11-dominant, and 10-weak and every binary operation on $[0,1]$ which is decreasing-on-1st, increasing-on-2nd, 00-dominant, 01-dominant, 11-dominant, and 10-weak is also with-properties-of-fuzzy-implication and every binary operation on $[0,1]$ which is with-properties-of-classical-implication is also 00-dominant, 01-dominant, 11-dominant, and 10-weak and every binary operation on $[0,1]$ which is 00-dominant, 01-dominant, 11-dominant, and 10-weak is also with-properties-of-classical-implication and every binary operation on $[0,1]$ which is with-properties-of-fuzzy-implication is also with-properties-of-classical-implication.

A Fuzzy-Implication is a decreasing-on-1st, increasing-on-2nd, 00-dominant, 11-dominant, 10-weak binary operation on $[0,1]$. The functor FI yielding a set is defined by the term

(Def. 15) the set of all $f$ where $f$ is a Fuzzy-Implication.

The functor Goedel-implication yielding a binary operation on $[0,1]$ is defined by

(Def. 16) for every elements $x$, $y$ of $[0,1]$, if $x \leqslant y$, then $it(x,y) = 1$ and if $x > y$, then $it(x,y) = y$.

Let us note that Goedel-implication is decreasing-on-1st, increasing-on-2nd, 00-dominant, 11-dominant, and 10-weak.

The functor Reichenbach-implication yielding a binary operation on $[0,1]$ is defined by

(Def. 17) for every elements $x$, $y$ of $[0,1]$, $it(x,y) = 1 - x + (x \cdot y)$.

Let us note that Reichenbach-implication is decreasing-on-1st, increasing-on-2nd, 00-dominant, 11-dominant, and 10-weak.

The functor Kleene-Dienes-implication yielding a binary operation on $[0, 1]$ is defined by

(Def. 18) for every elements $x$, $y$ of $[0, 1]$, $it(x, y) = \max(1 - x, y)$.

Let us observe that Kleene-Dienes-implication is decreasing-on-1st, increasing-on-2nd, 00-dominant, 11-dominant, and 10-weak.

The functor Goguen-implication yielding a binary operation on $[0, 1]$ is defined by

(Def. 19) for every elements $x$, $y$ of $[0, 1]$, if $x \leqslant y$, then $it(x, y) = 1$ and if $x > y$, then $it(x, y) = \frac{y}{x}$.

One can verify that Goguen-implication is decreasing-on-1st, increasing-on-2nd, 00-dominant, 11-dominant, and 10-weak.

The functor Rescher-implication yielding a binary operation on $[0, 1]$ is defined by

(Def. 20) for every elements $x$, $y$ of $[0, 1]$, if $x \leqslant y$, then $it(x, y) = 1$ and if $x > y$, then $it(x, y) = 0$.

Let us note that Rescher-implication is decreasing-on-1st, increasing-on-2nd, 00-dominant, 11-dominant, and 10-weak.

The functor Yager-implication yielding a binary operation on $[0, 1]$ is defined by

(Def. 21) for every elements $x$, $y$ of $[0, 1]$, if $x = y = 0$, then $it(x, y) = 1$ and if $x > 0$ or $y > 0$, then $it(x, y) = y^x$.

One can check that Yager-implication is decreasing-on-1st, increasing-on-2nd, 00-dominant, 11-dominant, and 10-weak.

The functor Weber-implication yielding a binary operation on $[0, 1]$ is defined by

(Def. 22) for every elements $x$, $y$ of $[0, 1]$, if $x < 1$, then $it(x, y) = 1$ and if $x = 1$, then $it(x, y) = y$.

Let us note that Weber-implication is decreasing-on-1st, increasing-on-2nd, 00-dominant, 11-dominant, and 10-weak.

The functor Fodor-implication yielding a binary operation on $[0, 1]$ is defined by

(Def. 23) for every elements $x$, $y$ of $[0, 1]$, if $x \leqslant y$, then $it(x, y) = 1$ and if $x > y$, then $it(x, y) = \max(1 - x, y)$.

One can check that Fodor-implication is decreasing-on-1st, increasing-on-2nd, 00-dominant, 11-dominant, and 10-weak.

## 5. Boundary Fuzzy Implications

The functor $\mathrm{I}_0$ yielding a binary operation on $[0,1]$ is defined by

(Def. 24) for every elements $x$, $y$ of $[0,1]$, if $x = 0$ or $y = 1$, then $it(x,y) = 1$ and if $x > 0$ and $y < 1$, then $it(x,y) = 0$.

One can verify that $\mathrm{I}_0$ is decreasing-on-1st, increasing-on-2nd, 00-dominant, 11-dominant, and 10-weak.

The functor $\mathrm{I}_1$ yielding a binary operation on $[0,1]$ is defined by

(Def. 25) for every elements $x$, $y$ of $[0,1]$, if $x < 1$ or $y > 0$, then $it(x,y) = 1$ and if $x = 1$ and $y = 0$, then $it(x,y) = 0$.

One can verify that $\mathrm{I}_1$ is decreasing-on-1st, increasing-on-2nd, 00-dominant, 11-dominant, and 10-weak.

Let $f$ be a binary operation on $[0,1]$. We say that $f$ is satisfying-(LB) if and only if

(Def. 26) for every element $y$ of $[0,1]$, $f(0,y) = 1$.

We say that $f$ is satisfying-(RB) if and only if

(Def. 27) for every element $x$ of $[0,1]$, $f(x,1) = 1$.

Now we state the propositions:

(7) Let us consider a Fuzzy-Implication $f_1$, and an element $y$ of $[0,1]$. Then $f_1(0,y) = 1$.

(8) Let us consider a Fuzzy-Implication $f_1$, and an element $x$ of $[0,1]$. Then $f_1(x,1) = 1$.

Observe that every Fuzzy-Implication is satisfying-(LB) and satisfying-(RB).

Let us consider a Fuzzy-Implication $f_1$. Now we state the propositions:

(9) $\mathrm{I}_0 \leqslant f_1$. The theorem is a consequence of (7) and (8).

(10) $f_1 \leqslant \mathrm{I}_1$.

## References

[1] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.

[2] Adam Grabowski. On the computer certification of fuzzy numbers. In M. Ganzha, L. Maciaszek, and M. Paprzycki, editors, *2013 Federated Conference on Computer Science and Information Systems (FedCSIS)*, Federated Conference on Computer Science and Information Systems, pages 51–54, 2013.

[3] Lotfi Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

*Received September 3, 2017*

The English version of this volume of Formalized Mathematics was financed under agreement 548/P-DUN/2016 with the funds from the Polish Minister of Science and Higher Education for the dissemination of science.