

# Integral of Non Positive Functions

Noboru Endou  
National Institute of Technology, Gifu College  
2236-2 Kamimakuwa, Motosu, Gifu, Japan

**Summary.** In this article we formalize Lebesgue type integral and convergence theorems for non positive functions. Many theorems are based on our previous results [8], [10].

MSC: 03B35

Keywords:

MML identifier: MESFUN11, version: 8.1.06 5.44.1305

## 1. PRELIMINARIES

Let  $X$  be a non empty set and  $f$  be a non-negative partial function from  $X$  to  $\overline{\mathbb{R}}$ . Observe that  $-f$  is non-positive.

Let  $f$  be a non-positive partial function from  $X$  to  $\overline{\mathbb{R}}$ . One can check that  $-f$  is non-negative.

Now we state the propositions:

- (1) Let us consider a non empty set  $X$ , a non-positive partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ , and a set  $E$ . Then  $f \upharpoonright E$  is non-positive.
- (2) Let us consider a non empty set  $X$ , a set  $A$ , a real number  $r$ , and a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Then  $(r \cdot f) \upharpoonright A = r \cdot (f \upharpoonright A)$ .
- (3) Let us consider a non empty set  $X$ , a set  $A$ , and a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Then  $-f \upharpoonright A = (-f) \upharpoonright A$ . The theorem is a consequence of (2).
- (4) Let us consider a non empty set  $X$ , a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ , and a real number  $c$ . Suppose  $f$  is non-positive. Then
  - (i) if  $0 \leq c$ , then  $c \cdot f$  is non-positive, and
  - (ii) if  $c \leq 0$ , then  $c \cdot f$  is non-negative.

- (5) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , and a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Then
- (i)  $\max_+(f)$  is non-negative, and
  - (ii)  $\max_-(f)$  is non-negative, and
  - (iii)  $|f|$  is non-negative.
- (6) Let us consider a non empty set  $X$ , a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ , and an object  $x$ . Then
- (i)  $f(x) \leq (\max_+(f))(x)$ , and
  - (ii)  $f(x) \geq -(\max_-(f))(x)$ .
- (7) Let us consider a non empty set  $X$ , a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ , and a positive real number  $r$ . Then  $\text{LE-dom}(f, r) = \text{LE-dom}(\max_+(f), r)$ .
- (8) Let us consider a non empty set  $X$ , a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ , and a non positive real number  $r$ . Then  $\text{LE-dom}(f, r) = \text{GT-dom}(\max_-(f), -r)$ .
- (9) Let us consider a non empty set  $X$ , partial functions  $f, g$  from  $X$  to  $\overline{\mathbb{R}}$ , an extended real  $a$ , and a real number  $r$ . Suppose  $r \neq 0$  and  $g = r \cdot f$ . Then  $\text{EQ-dom}(f, a) = \text{EQ-dom}(g, a \cdot r)$ .
- (10) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ , and an element  $A$  of  $S$ . Suppose  $A \subseteq \text{dom } f$ . Then  $f$  is measurable on  $A$  if and only if  $\max_+(f)$  is measurable on  $A$  and  $\max_-(f)$  is measurable on  $A$ .

Let  $X$  be a non empty set,  $f$  be a function from  $X$  into  $\overline{\mathbb{R}}$ , and  $r$  be a real number. Note that the functor  $r \cdot f$  yields a function from  $X$  into  $\overline{\mathbb{R}}$ . Now we state the proposition:

- (11) Let us consider a non empty set  $X$ , a real number  $r$ , and a without  $+\infty$  function  $f$  from  $X$  into  $\overline{\mathbb{R}}$ . If  $r \geq 0$ , then  $r \cdot f$  is without  $+\infty$ .

Let  $X$  be a non empty set,  $f$  be a without  $+\infty$  function from  $X$  into  $\overline{\mathbb{R}}$ , and  $r$  be a non negative real number. Let us note that  $r \cdot f$  is without  $+\infty$  as a function from  $X$  into  $\overline{\mathbb{R}}$ .

Now we state the proposition:

- (12) Let us consider a non empty set  $X$ , a real number  $r$ , and a without  $+\infty$  function  $f$  from  $X$  into  $\overline{\mathbb{R}}$ . If  $r \leq 0$ , then  $r \cdot f$  is without  $-\infty$ .

Let  $X$  be a non empty set,  $f$  be a without  $+\infty$  function from  $X$  into  $\overline{\mathbb{R}}$ , and  $r$  be a non positive real number. One can check that  $r \cdot f$  is without  $-\infty$ .

Now we state the proposition:

- (13) Let us consider a non empty set  $X$ , a real number  $r$ , and a without  $-\infty$  function  $f$  from  $X$  into  $\overline{\mathbb{R}}$ . If  $r \geq 0$ , then  $r \cdot f$  is without  $-\infty$ .

Let  $X$  be a non empty set,  $f$  be a without  $-\infty$  function from  $X$  into  $\overline{\mathbb{R}}$ , and  $r$  be a non negative real number. One can check that  $r \cdot f$  is without  $-\infty$ .

Now we state the proposition:

- (14) Let us consider a non empty set  $X$ , a real number  $r$ , and a without  $-\infty$  function  $f$  from  $X$  into  $\overline{\mathbb{R}}$ . If  $r \leq 0$ , then  $r \cdot f$  is without  $+\infty$ .

Let  $X$  be a non empty set,  $f$  be a without  $-\infty$  function from  $X$  into  $\overline{\mathbb{R}}$ , and  $r$  be a non positive real number. One can check that  $r \cdot f$  is without  $+\infty$ .

Now we state the proposition:

- (15) Let us consider a non empty set  $X$ , a real number  $r$ , and a without  $-\infty$ , without  $+\infty$  function  $f$  from  $X$  into  $\overline{\mathbb{R}}$ . Then  $r \cdot f$  is without  $-\infty$  and without  $+\infty$ .

Let  $X$  be a non empty set,  $f$  be a without  $-\infty$ , without  $+\infty$  function from  $X$  into  $\overline{\mathbb{R}}$ , and  $r$  be a real number. Note that  $r \cdot f$  is without  $-\infty$  and without  $+\infty$ .

Now we state the propositions:

- (16) Let us consider a non empty set  $X$ , a positive real number  $r$ , and a function  $f$  from  $X$  into  $\overline{\mathbb{R}}$ . Then  $f$  is without  $+\infty$  if and only if  $r \cdot f$  is without  $+\infty$ .
- (17) Let us consider a non empty set  $X$ , a negative real number  $r$ , and a function  $f$  from  $X$  into  $\overline{\mathbb{R}}$ . Then  $f$  is without  $+\infty$  if and only if  $r \cdot f$  is without  $-\infty$ .
- (18) Let us consider a non empty set  $X$ , a positive real number  $r$ , and a function  $f$  from  $X$  into  $\overline{\mathbb{R}}$ . Then  $f$  is without  $-\infty$  if and only if  $r \cdot f$  is without  $-\infty$ .
- (19) Let us consider a non empty set  $X$ , a negative real number  $r$ , and a function  $f$  from  $X$  into  $\overline{\mathbb{R}}$ . Then  $f$  is without  $-\infty$  if and only if  $r \cdot f$  is without  $+\infty$ .
- (20) Let us consider a non empty set  $X$ , a non zero real number  $r$ , and a function  $f$  from  $X$  into  $\overline{\mathbb{R}}$ . Then  $f$  is without  $-\infty$  and without  $+\infty$  if and only if  $r \cdot f$  is without  $-\infty$  and without  $+\infty$ . The theorem is a consequence of (16), (18), (17), and (19).
- (21) Let us consider non empty sets  $X, Y$ , a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ , and a real number  $r$ . Suppose  $f = Y \mapsto r$ . Then  $f$  is without  $-\infty$  and without  $+\infty$ .
- (22) Let us consider a non empty set  $X$ , and a function  $f$  from  $X$  into  $\overline{\mathbb{R}}$ .

Then

- (i)  $0 \cdot f = X \mapsto 0$ , and  
(ii)  $0 \cdot f$  is without  $-\infty$  and without  $+\infty$ .

PROOF: For every element  $x$  of  $X$ ,  $(0 \cdot f)(x) = (X \mapsto 0)(x)$  by [13, (7)].  
 $\square$

(23) Let us consider a non empty set  $X$ , and partial functions  $f, g$  from  $X$  to  $\overline{\mathbb{R}}$ . Suppose  $f$  is without  $-\infty$  and without  $+\infty$ . Then

- (i)  $\text{dom}(f + g) = \text{dom } f \cap \text{dom } g$ , and
- (ii)  $\text{dom}(f - g) = \text{dom } f \cap \text{dom } g$ , and
- (iii)  $\text{dom}(g - f) = \text{dom } f \cap \text{dom } g$ .

Let us consider a non empty set  $X$  and functions  $f_1, f_2$  from  $X$  into  $\overline{\mathbb{R}}$ . Now we state the propositions:

(24) Suppose  $f_2$  is without  $-\infty$  and without  $+\infty$ . Then

- (i)  $f_1 + f_2$  is a function from  $X$  into  $\overline{\mathbb{R}}$ , and
- (ii) for every element  $x$  of  $X$ ,  $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ .

The theorem is a consequence of (23).

(25) Suppose  $f_1$  is without  $-\infty$  and without  $+\infty$ . Then

- (i)  $f_1 - f_2$  is a function from  $X$  into  $\overline{\mathbb{R}}$ , and
- (ii) for every element  $x$  of  $X$ ,  $(f_1 - f_2)(x) = f_1(x) - f_2(x)$ .

The theorem is a consequence of (23).

(26) Suppose  $f_2$  is without  $-\infty$  and without  $+\infty$ . Then

- (i)  $f_1 - f_2$  is a function from  $X$  into  $\overline{\mathbb{R}}$ , and
- (ii) for every element  $x$  of  $X$ ,  $(f_1 - f_2)(x) = f_1(x) - f_2(x)$ .

The theorem is a consequence of (23).

Now we state the proposition:

(27) Let us consider non empty sets  $X, Y$ , and partial functions  $f_1, f_2$  from  $X$  to  $\overline{\mathbb{R}}$ . Suppose  $\text{dom } f_1 \subseteq Y$  and  $f_2 = Y \mapsto 0$ . Then

- (i)  $f_1 + f_2 = f_1$ , and
- (ii)  $f_1 - f_2 = f_1$ , and
- (iii)  $f_2 - f_1 = -f_1$ .

The theorem is a consequence of (21) and (23).

Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , and partial functions  $f, g$  from  $X$  to  $\overline{\mathbb{R}}$ . Now we state the propositions:

(28) If  $f$  is simple function in  $S$  and  $g$  is simple function in  $S$ , then  $f + g$  is simple function in  $S$ .

PROOF: Consider  $F$  being a finite sequence of separated subsets of  $S$ ,  $a$  being a finite sequence of elements of  $\overline{\mathbb{R}}$  such that  $F$  and  $a$  are representation of  $f$ . Consider  $G$  being a finite sequence of separated subsets of  $S$ ,

$b$  being a finite sequence of elements of  $\overline{\mathbb{R}}$  such that  $G$  and  $b$  are representation of  $g$ . Set  $l_1 = \text{len } a$ . Set  $l_2 = \text{len } b$ . Define  $\mathcal{F}_\Delta(\text{natural number}) = F((\$1 -' 1 \text{ div } l_2) + 1) \cap G((\$1 -' 1 \text{ mod } l_2) + 1)$ . Consider  $F_3$  being a finite sequence such that  $\text{len } F_3 = l_1 \cdot l_2$  and for every natural number  $k$  such that  $k \in \text{dom } F_3$  holds  $F_3(k) = F_4(k)$  from [2, Sch. 2]. For every natural numbers  $k, l$  such that  $k, l \in \text{dom } F_3$  and  $k \neq l$  holds  $F_3(k)$  misses  $F_3(l)$  by [14, (25)], [1, (14)], [11, (15), (24)].  $\text{dom}(f + g) = \bigcup \text{rng } F_3$  by [14, (25)], [1, (13), (14)], [11, (15), (24)]. For every natural number  $k$  and for every elements  $x, y$  of  $X$  such that  $k \in \text{dom } F_3$  and  $x, y \in F_3(k)$  holds  $(f + g)(x) = (f + g)(y)$  by [1, (11)], [14, (25)], [1, (14)], [11, (15), (24)].  $\square$

(29) If  $f$  is simple function in  $S$  and  $g$  is simple function in  $S$ , then  $f - g$  is simple function in  $S$ . The theorem is a consequence of (28).

Now we state the propositions:

(30) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , and a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . If  $f$  is simple function in  $S$ , then  $-f$  is simple function in  $S$ .

(31) Let us consider a non empty set  $X$ , and a non-negative partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Then  $f = \max_+(f)$ .

PROOF: For every element  $x$  of  $X$  such that  $x \in \text{dom } f$  holds  $f(x) = (\max_+(f))(x)$  by [3, (51)].  $\square$

(32) Let us consider a non empty set  $X$ , and a non-positive partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Then  $f = -\max_-(f)$ .

PROOF: For every element  $x$  of  $X$  such that  $x \in \text{dom } f$  holds  $f(x) = (-\max_-(f))(x)$  by [8, (8)], [9, (20)].  $\square$

(33) Let us consider a non empty set  $C$ , a partial function  $f$  from  $C$  to  $\overline{\mathbb{R}}$ , and a real number  $c$ . Suppose  $c \leq 0$ . Then

(i)  $\max_+(c \cdot f) = (-c) \cdot \max_-(f)$ , and

(ii)  $\max_-(c \cdot f) = (-c) \cdot \max_+(f)$ .

PROOF: For every element  $x$  of  $C$  such that  $x \in \text{dom } \max_+(c \cdot f)$  holds  $(\max_+(c \cdot f))(x) = ((-c) \cdot \max_-(f))(x)$ . For every element  $x$  of  $C$  such that  $x \in \text{dom } \max_-(c \cdot f)$  holds  $(\max_-(c \cdot f))(x) = ((-c) \cdot \max_+(f))(x)$  by [8, (6)].  $\square$

(34) Let us consider a non empty set  $X$ , and a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Then  $\max_+(f) = \max_-( -f)$ . The theorem is a consequence of (33).

(35) Let us consider a non empty set  $X$ , a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ , and real numbers  $r_1, r_2$ . Then  $r_1 \cdot (r_2 \cdot f) = (r_1 \cdot r_2) \cdot f$ .

(36) Let us consider a non empty set  $X$ , and partial functions  $f, g$  from  $X$  to  $\overline{\mathbb{R}}$ . If  $f = -g$ , then  $g = -f$ . The theorem is a consequence of (35).

Let  $X$  be a non empty set,  $F$  be a sequence of partial functions from  $X$  into  $\overline{\mathbb{R}}$ , and  $r$  be a real number. The functor  $r \cdot F$  yielding a sequence of partial functions from  $X$  into  $\overline{\mathbb{R}}$  is defined by

(Def. 1) for every natural number  $n$ ,  $it(n) = r \cdot F(n)$ .

The functor  $-F$  yielding a sequence of partial functions from  $X$  into  $\overline{\mathbb{R}}$  is defined by the term

(Def. 2)  $(-1) \cdot F$ .

Now we state the proposition:

(37) Let us consider a non empty set  $X$ , a sequence  $F$  of partial functions from  $X$  into  $\overline{\mathbb{R}}$ , and a natural number  $n$ . Then  $(-F)(n) = -F(n)$ .

Let us consider a non empty set  $X$ , a sequence  $F$  of partial functions from  $X$  into  $\overline{\mathbb{R}}$ , and an element  $x$  of  $X$ . Now we state the propositions:

(38)  $(-F)\#x = -F\#x$ . The theorem is a consequence of (37).

(39) (i)  $F\#x$  is convergent to  $+\infty$  iff  $(-F)\#x$  is convergent to  $-\infty$ , and

(ii)  $F\#x$  is convergent to  $-\infty$  iff  $(-F)\#x$  is convergent to  $+\infty$ , and

(iii)  $F\#x$  is convergent to a finite limit iff  $(-F)\#x$  is convergent to a finite limit, and

(iv)  $F\#x$  is convergent iff  $(-F)\#x$  is convergent, and

(v) if  $F\#x$  is convergent, then  $\lim(((-F)\#x)) = -\lim(F\#x)$ .

The theorem is a consequence of (38).

Let us consider a non empty set  $X$  and a sequence  $F$  of partial functions from  $X$  into  $\overline{\mathbb{R}}$ . Now we state the propositions:

(40) If  $F$  has the same dom, then  $-F$  has the same dom. The theorem is a consequence of (37).

(41) If  $F$  is additive, then  $-F$  is additive. The theorem is a consequence of (37).

Now we state the propositions:

(42) Let us consider a non empty set  $X$ , a sequence  $F$  of partial functions from  $X$  into  $\overline{\mathbb{R}}$ , and a natural number  $n$ . Then  $(\sum_{\alpha=0}^{\kappa} (-F)(\alpha))_{\kappa \in \mathbb{N}}(n) = -(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$ .

PROOF: Define  $\mathcal{P}[\text{natural number}] \equiv (\sum_{\alpha=0}^{\kappa} (-F)(\alpha))_{\kappa \in \mathbb{N}}(\$1) = -(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(\$1)$ . For every natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k+1]$  by (37), [7, (64)]. For every natural number  $k$ ,  $\mathcal{P}[k]$  from [1, Sch. 2].  $\square$

(43) Let us consider a sequence  $s$  of extended reals, and a natural number  $n$ . Then  $(\sum_{\alpha=0}^{\kappa} (-s)(\alpha))_{\kappa \in \mathbb{N}}(n) = -(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n)$ .

PROOF: Define  $\mathcal{P}[\text{natural number}] \equiv (\sum_{\alpha=0}^{\kappa} (-s)(\alpha))_{\kappa \in \mathbb{N}} (\$1) = -(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}} (\$1)$ .  
 For every natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k+1]$ . For every natural number  $k$ ,  $\mathcal{P}[k]$  from [1, Sch. 2].  $\square$

Let us consider a sequence  $s$  of extended reals. Now we state the propositions:

- (44)  $(\sum_{\alpha=0}^{\kappa} (-s)(\alpha))_{\kappa \in \mathbb{N}} = -(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$ . The theorem is a consequence of (43).
- (45) If  $s$  is summable, then  $-s$  is summable. The theorem is a consequence of (44).

Let us consider a non empty set  $X$  and a sequence  $F$  of partial functions from  $X$  into  $\overline{\mathbb{R}}$ . Now we state the propositions:

- (46) If for every natural number  $n$ ,  $F(n)$  is without  $+\infty$ , then  $F$  is additive.
- (47) If for every natural number  $n$ ,  $F(n)$  is without  $-\infty$ , then  $F$  is additive.

Now we state the propositions:

- (48) Let us consider a non empty set  $X$ , a sequence  $F$  of partial functions from  $X$  into  $\overline{\mathbb{R}}$ , and an element  $x$  of  $X$ . Suppose  $F \# x$  is summable. Then
  - (i)  $(-F) \# x$  is summable, and
  - (ii)  $\sum((-F) \# x) = -\sum(F \# x)$ .

The theorem is a consequence of (45), (38), and (44).

- (49) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , and a sequence  $F$  of partial functions from  $X$  into  $\overline{\mathbb{R}}$ . Suppose  $F$  is additive and has the same dom and for every element  $x$  of  $X$  such that  $x \in \text{dom}(F(0))$  holds  $F \# x$  is summable. Then  $\lim(\sum_{\alpha=0}^{\kappa} (-F)(\alpha))_{\kappa \in \mathbb{N}} = -\lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}$ .  
 PROOF: Set  $G = -F$ . For every element  $n$  of  $\mathbb{N}$ ,  $(\sum_{\alpha=0}^{\kappa} G(\alpha))_{\kappa \in \mathbb{N}}(n) = -(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$ . For every element  $x$  of  $X$  such that  $x \in \text{dom} \lim(\sum_{\alpha=0}^{\kappa} G(\alpha))$  holds  $(\lim(\sum_{\alpha=0}^{\kappa} G(\alpha))_{\kappa \in \mathbb{N}})(x) = (-\lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})(x)$  by [10, (33)], (38), [6, (17)].  $\square$

- (50) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , sequences  $F, G$  of partial functions from  $X$  into  $\overline{\mathbb{R}}$ , and an element  $E$  of  $S$ . Suppose  $E \subseteq \text{dom}(F(0))$  and  $F$  is additive and has the same dom and for every natural number  $n$ ,  $G(n) = F(n) \upharpoonright E$ . Then  $\lim(\sum_{\alpha=0}^{\kappa} G(\alpha))_{\kappa \in \mathbb{N}} = \lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \upharpoonright E$ .

PROOF: For every element  $x$  of  $X$  such that  $x \in E$  holds  $F \# x = G \# x$  by [10, (18)], [4, (47)]. Set  $P_1 = (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}$ . Set  $P_2 = (\sum_{\alpha=0}^{\kappa} G(\alpha))_{\kappa \in \mathbb{N}}$ . For every element  $x$  of  $X$  such that  $x \in \text{dom} \lim P_2$  holds  $(\lim P_2)(x) = (\lim P_1)(x)$  by [10, (29)], [15, (62)], [10, (32)], [5, (63)]. For every element  $x$  of  $X$  such that  $x \in \text{dom}(\lim P_2 \upharpoonright E)$  holds  $(\lim P_2 \upharpoonright E)(x) = (\lim P_1 \upharpoonright E)(x)$  by [4, (47)].  $\square$

## 2. INTEGRAL OF NON POSITIVE MEASURABLE FUNCTIONS

Now we state the propositions:

- (51) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , and a non-negative partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Then  $\int' \max_-( -f) dM = \int' f dM$ . The theorem is a consequence of (32), (36), and (35).
- (52) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ , and an element  $A$  of  $S$ . Suppose  $A = \text{dom } f$  and  $f$  is measurable on  $A$ . Then  $\int -f dM = -\int f dM$ . The theorem is a consequence of (36), (10), (5), and (34).
- (53) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , a non-negative partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ , and an element  $E$  of  $S$ . Suppose  $E = \text{dom } f$  and  $f$  is measurable on  $E$ . Then
- (i)  $\int \max_-(f) dM = 0$ , and
  - (ii)  $\int^+ \max_-(f) dM = 0$ .

PROOF:  $\max_-(f)$  is measurable on  $E$ . For every object  $x$  such that  $x \in \text{dom } \max_-(f)$  holds  $(\max_-(f))(x) = 0$  by [3, (51)], [9, (18), (19)].  $\square$

Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ , and an element  $E$  of  $S$ . Now we state the propositions:

- (54) If  $E = \text{dom } f$  and  $f$  is measurable on  $E$ , then  $\int f dM = \int \max_+(f) dM - \int \max_-(f) dM$ . The theorem is a consequence of (10) and (5).
- (55) If  $E \subseteq \text{dom } f$  and  $f$  is measurable on  $E$ , then  $\int (-f) \upharpoonright E dM = -\int f \upharpoonright E dM$ . The theorem is a consequence of (3) and (52).

Now we state the propositions:

- (56) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , and a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Suppose there exists an element  $A$  of  $S$  such that  $A = \text{dom } f$  and  $f$  is measurable on  $A$  and ( $f$  **qua** extended real-valued function) is non-positive. Then there exists a sequence  $F$  of partial functions from  $X$  into  $\overline{\mathbb{R}}$  such that
- (i) for every natural number  $n$ ,  $F(n)$  is simple function in  $S$  and  $\text{dom}(F(n)) = \text{dom } f$ , and
  - (ii) for every natural number  $n$ ,  $F(n)$  is non-positive, and
  - (iii) for every natural numbers  $n, m$  such that  $n \leq m$  for every element  $x$  of  $X$  such that  $x \in \text{dom } f$  holds  $F(n)(x) \geq F(m)(x)$ , and



- (iv) for every element  $x$  of  $X$  such that  $x \in \text{dom } f$  holds  $F\#x$  is convergent and  $\lim(F\#x) = f(x)$ .

The theorem is a consequence of (37), (30), and (39).

- (57) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , an element  $E$  of  $S$ , and a non-positive partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Suppose there exists an element  $A$  of  $S$  such that  $A = \text{dom } f$  and  $f$  is measurable on  $A$ . Then

- (i)  $\int f \, dM = -f^+ \max_-(f) \, dM$ , and
- (ii)  $\int f \, dM = -f^+ - f \, dM$ , and
- (iii)  $\int f \, dM = -f - f \, dM$ .

PROOF: Consider  $A$  being an element of  $S$  such that  $A = \text{dom } f$  and  $f$  is measurable on  $A$ .  $f = -\max_-(f)$ .  $-f = \max_-(f)$ . For every element  $x$  of  $X$  such that  $x \in \text{dom } \max_+(f)$  holds  $(\max_+(f))(x) = 0$  by [9, (21)].  $\square$

- (58) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , and a non-positive partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Suppose  $f$  is simple function in  $S$ . Then

- (i)  $\int f \, dM = -f' - f \, dM$ , and
- (ii)  $\int f \, dM = -f' \max_-(f) \, dM$ .

The theorem is a consequence of (30), (57), (32), and (36).

Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ , and a real number  $c$ . Now we state the propositions:

- (59) If  $f$  is simple function in  $S$  and  $f$  is non-negative, then  $\int c \cdot f \, dM = c \cdot \int f \, dM$ .
- (60) Suppose  $f$  is simple function in  $S$  and  $f$  is non-positive. Then
  - (i)  $\int c \cdot f \, dM = -c \cdot \int' -f \, dM$ , and
  - (ii)  $\int c \cdot f \, dM = -(c \cdot \int' -f \, dM)$ .

The theorem is a consequence of (35), (30), and (59).

Now we state the propositions:

- (61) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , and a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Suppose there exists an element  $A$  of  $S$  such that  $A = \text{dom } f$  and  $f$  is measurable on  $A$  and  $f$  is non-positive. Then  $0 \geq \int f \, dM$ . The theorem is a consequence of (57).
- (62) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ , and elements  $A, B, E$

of  $S$ . Suppose  $E = \text{dom } f$  and  $f$  is measurable on  $E$  and  $f$  is non-positive and  $A$  misses  $B$ . Then  $\int f \upharpoonright (A \cup B) dM = \int f \upharpoonright A dM + \int f \upharpoonright B dM$ . The theorem is a consequence of (3) and (52).

- (63) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ , and elements  $A, E$  of  $S$ . Suppose  $E = \text{dom } f$  and  $f$  is measurable on  $E$  and  $f$  is non-positive. Then  $0 \geq \int f \upharpoonright A dM$ . The theorem is a consequence of (61) and (1).
- (64) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ , and elements  $A, B, E$  of  $S$ . Suppose  $E = \text{dom } f$  and  $f$  is measurable on  $E$  and  $f$  is non-positive and  $A \subseteq B$ . Then  $\int f \upharpoonright A dM \geq \int f \upharpoonright B dM$ . The theorem is a consequence of (3) and (52).

### 3. CONVERGENCE THEOREMS FOR NON POSITIVE FUNCTION'S INTEGRATION

Now we state the propositions:

- (65) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , an element  $E$  of  $S$ , and a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Suppose  $E = \text{dom } f$  and  $f$  is measurable on  $E$  and  $f$  is non-positive and  $M(E \cap \text{EQ-dom}(f, -\infty)) \neq 0$ . Then  $\int f dM = -\infty$ . The theorem is a consequence of (9) and (52).
- (66) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , an element  $E$  of  $S$ , and partial functions  $f, g$  from  $X$  to  $\overline{\mathbb{R}}$ . Suppose  $E \subseteq \text{dom } f$  and  $E \subseteq \text{dom } g$  and  $f$  is measurable on  $E$  and  $g$  is measurable on  $E$  and  $f$  is non-positive and for every element  $x$  of  $X$  such that  $x \in E$  holds  $g(x) \leq f(x)$ . Then  $\int g \upharpoonright E dM \leq \int f \upharpoonright E dM$ . The theorem is a consequence of (3) and (52).
- (67) Let us consider a non empty set  $X$ , a sequence  $F$  of partial functions from  $X$  into  $\overline{\mathbb{R}}$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , an element  $E$  of  $S$ , and a natural number  $m$ . Suppose  $F$  has the same dom and  $E = \text{dom}(F(0))$  and for every natural number  $n$ ,  $F(n)$  is measurable on  $E$  and  $F(n)$  is without  $+\infty$ . Then  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$  is measurable on  $E$ . The theorem is a consequence of (37), (42), and (46).
- (68) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , a sequence  $F$  of partial functions from  $X$  into  $\overline{\mathbb{R}}$ , an element  $E$  of  $S$ , a sequence  $I$  of extended reals, and a natural number  $m$ . Suppose  $E = \text{dom}(F(0))$  and  $F$  is additive and has the same dom and for every natural number  $n$ ,  $F(n)$  is measurable on  $E$  and  $F(n)$  is

non-positive and  $I(n) = \int F(n) \, dM$ . Then  $\int (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m) \, dM = (\sum_{\alpha=0}^{\kappa} I(\alpha))_{\kappa \in \mathbb{N}}(m)$ .

PROOF: Set  $G = -F$ . Set  $J = -I$ .  $G(0) = -F(0)$ .  $G$  has the same dom. For every natural number  $n$ ,  $F(n)$  is measurable on  $E$  and  $F(n)$  is without  $+\infty$ . For every natural number  $n$ ,  $G(n)$  is measurable on  $E$  and  $G(n)$  is non-negative and  $J(n) = \int G(n) \, dM$  by (37), [7, (63)], (52).

$$\begin{aligned} \int (\sum_{\alpha=0}^{\kappa} G(\alpha))_{\kappa \in \mathbb{N}}(m) \, dM &= (\sum_{\alpha=0}^{\kappa} J(\alpha))_{\kappa \in \mathbb{N}}(m) \cdot \int (-\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m) \, dM = \\ &= (\sum_{\alpha=0}^{\kappa} J(\alpha))_{\kappa \in \mathbb{N}}(m) \cdot \int (-\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m) \, dM = -(\sum_{\alpha=0}^{\kappa} I(\alpha))_{\kappa \in \mathbb{N}}(m) \cdot \\ &\int -(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m) \, dM = -(\sum_{\alpha=0}^{\kappa} I(\alpha))_{\kappa \in \mathbb{N}}(m) \cdot -\int (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m) \, dM = \\ &= -(\sum_{\alpha=0}^{\kappa} I(\alpha))_{\kappa \in \mathbb{N}}(m). \quad \square \end{aligned}$$

- (69) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , a sequence  $F$  of partial functions from  $X$  into  $\overline{\mathbb{R}}$ , an element  $E$  of  $S$ , and a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Suppose  $E \subseteq \text{dom } f$  and  $f$  is non-positive and  $f$  is measurable on  $E$  and for every natural number  $n$ ,  $F(n)$  is simple function in  $S$  and  $F(n)$  is non-positive and  $E \subseteq \text{dom}(F(n))$  and for every element  $x$  of  $X$  such that  $x \in E$  holds  $F\#x$  is summable and  $f(x) = \sum(F\#x)$ . Then there exists a sequence  $I$  of extended reals such that

- (i) for every natural number  $n$ ,  $I(n) = \int F(n)\upharpoonright E \, dM$ , and
- (ii)  $I$  is summable, and
- (iii)  $\int f\upharpoonright E \, dM = \sum I$ .

PROOF: Set  $g = -f$ . Set  $G = -F$ .  $G$  is additive. For every natural number  $n$ ,  $G(n)$  is simple function in  $S$  and  $G(n)$  is non-negative and  $E \subseteq \text{dom}(G(n))$  by [8, (39)], [9, (9)], (37). For every element  $x$  of  $X$  such that  $x \in E$  holds  $G\#x$  is summable and  $g(x) = \sum(G\#x)$ . Consider  $J$  being a sequence of extended reals such that for every natural number  $n$ ,  $J(n) = \int G(n)\upharpoonright E \, dM$  and  $J$  is summable and  $\int g\upharpoonright E \, dM = \sum J$ . For every natural number  $n$ ,  $I(n) = \int F(n)\upharpoonright E \, dM$  by [9, (34)], (37), (36), (55).  $\int g\upharpoonright E \, dM = -\int f\upharpoonright E \, dM$ .  $\lim(\sum_{\alpha=0}^{\kappa} I(\alpha))_{\kappa \in \mathbb{N}} = -\int g\upharpoonright E \, dM$ .  $\square$

- (70) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , an element  $E$  of  $S$ , and a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Suppose  $E \subseteq \text{dom } f$  and  $f$  is non-positive and  $f$  is measurable on  $E$ . Then there exists a sequence  $F$  of partial functions from  $X$  into  $\overline{\mathbb{R}}$  such that

- (i)  $F$  is additive, and
- (ii) for every natural number  $n$ ,  $F(n)$  is simple function in  $S$  and  $F(n)$  is non-positive and  $F(n)$  is measurable on  $E$ , and

- (iii) for every element  $x$  of  $X$  such that  $x \in E$  holds  $F\#x$  is summable and  $f(x) = \sum(F\#x)$ , and
- (iv) there exists a sequence  $I$  of extended reals such that for every natural number  $n$ ,  $I(n) = \int F(n)\upharpoonright E \, dM$  and  $I$  is summable and  $\int f\upharpoonright E \, dM = \sum I$ .

PROOF: Set  $g = -f$ . Consider  $G$  being a sequence of partial functions from  $X$  into  $\overline{\mathbb{R}}$  such that  $G$  is additive and for every natural number  $n$ ,  $G(n)$  is simple function in  $S$  and  $G(n)$  is non-negative and  $G(n)$  is measurable on  $E$  and for every element  $x$  of  $X$  such that  $x \in E$  holds  $G\#x$  is summable and  $g(x) = \sum(G\#x)$  and there exists a sequence  $J$  of extended reals such that for every natural number  $n$ ,  $J(n) = \int G(n)\upharpoonright E \, dM$  and  $J$  is summable and  $\int g\upharpoonright E \, dM = \sum J$ . For every natural number  $n$ ,  $F(n)$  is simple function in  $S$  and  $F(n)$  is non-positive and  $F(n)$  is measurable on  $E$  by [8, (39)], [9, (9)], (37), [9, (34)]. For every element  $x$  of  $X$  such that  $x \in E$  holds  $F\#x$  is summable and  $f(x) = \sum(F\#x)$ . There exists a sequence  $I$  of extended reals such that for every natural number  $n$ ,  $I(n) = \int F(n)\upharpoonright E \, dM$  and  $I$  is summable and  $\int f\upharpoonright E \, dM = \sum I$  by (37), (3), [9, (31)], [15, (61)].  $\square$

Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , a sequence  $F$  of partial functions from  $X$  into  $\overline{\mathbb{R}}$ , and an element  $E$  of  $S$ . Now we state the propositions:

- (71) Suppose  $E = \text{dom}(F(0))$  and  $F$  has the same dom and for every natural number  $n$ ,  $F(n)$  is non-positive and  $F(n)$  is measurable on  $E$ . Then there exists a sequence  $F_2$  of  $(X \rightarrow \overline{\mathbb{R}})^{\mathbb{N}}$  such that for every natural number  $n$ , for every natural number  $m$ ,  $F_2(n)(m)$  is simple function in  $S$  and  $\text{dom}(F_2(n)(m)) = \text{dom}(F(n))$  and for every natural number  $m$ ,  $F_2(n)(m)$  is non-positive and for every natural numbers  $j, k$  such that  $j \leq k$  for every element  $x$  of  $X$  such that  $x \in \text{dom}(F(n))$  holds  $F_2(n)(j)(x) \geq F_2(n)(k)(x)$  and for every element  $x$  of  $X$  such that  $x \in \text{dom}(F(n))$  holds  $F_2(n)\#x$  is convergent and  $\lim(F_2(n)\#x) = F(n)(x)$ .

PROOF: Define  $\mathcal{Q}[\text{element of } \mathbb{N}, \text{set}] \equiv$  for every sequence  $G$  of partial functions from  $X$  into  $\overline{\mathbb{R}}$  such that  $\mathcal{S}_2 = G$  holds for every natural number  $m$ ,  $G(m)$  is simple function in  $S$  and  $\text{dom}(G(m)) = \text{dom}(F(\mathcal{S}_1))$  and for every natural number  $m$ ,  $G(m)$  is non-positive and for every natural numbers  $j, k$  such that  $j \leq k$  for every element  $x$  of  $X$  such that  $x \in \text{dom}(F(\mathcal{S}_1))$  holds  $G(j)(x) \geq G(k)(x)$  and for every element  $x$  of  $X$  such that  $x \in \text{dom}(F(\mathcal{S}_1))$  holds  $G\#x$  is convergent and  $\lim(G\#x) = F(\mathcal{S}_1)(x)$ . For every element  $n$  of  $\mathbb{N}$ , there exists a sequence  $G$  of partial functions from  $X$  into  $\overline{\mathbb{R}}$  such that for every natural number  $m$ ,  $G(m)$  is simple function in  $S$  and  $\text{dom}(G(m)) = \text{dom}(F(n))$  and for every natural number

$m$ ,  $G(m)$  is non-positive and for every natural numbers  $j, k$  such that  $j \leq k$  for every element  $x$  of  $X$  such that  $x \in \text{dom}(F(n))$  holds  $G(j)(x) \geq G(k)(x)$  and for every element  $x$  of  $X$  such that  $x \in \text{dom}(F(n))$  holds  $G\#x$  is convergent and  $\lim(G\#x) = F(n)(x)$ . For every element  $n$  of  $\mathbb{N}$ , there exists an element  $G$  of  $(X \rightarrow \overline{\mathbb{R}})^{\mathbb{N}}$  such that  $\mathcal{Q}[n, G]$  by [5, (8)]. Consider  $F_2$  being a sequence of  $(X \rightarrow \overline{\mathbb{R}})^{\mathbb{N}}$  such that for every element  $n$  of  $\mathbb{N}$ ,  $\mathcal{Q}[n, F_2(n)]$  from [5, Sch. 3]. For every natural number  $n$ , for every natural number  $m$ ,  $F_2(n)(m)$  is simple function in  $S$  and  $\text{dom}(F_2(n)(m)) = \text{dom}(F(n))$  and for every natural number  $m$ ,  $F_2(n)(m)$  is non-positive and for every natural numbers  $j, k$  such that  $j \leq k$  for every element  $x$  of  $X$  such that  $x \in \text{dom}(F(n))$  holds  $F_2(n)(j)(x) \geq F_2(n)(k)(x)$  and for every element  $x$  of  $X$  such that  $x \in \text{dom}(F(n))$  holds  $F_2(n)\#x$  is convergent and  $\lim(F_2(n)\#x) = F(n)(x)$ .  $\square$

(72) Suppose  $E = \text{dom}(F(0))$  and  $F$  is additive and has the same dom and for every natural number  $n$ ,  $F(n)$  is measurable on  $E$  and  $F(n)$  is non-positive. Then there exists a sequence  $I$  of extended reals such that for every natural number  $n$ ,  $I(n) = \int F(n) dM$  and  $\int (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n) dM = (\sum_{\alpha=0}^{\kappa} I(\alpha))_{\kappa \in \mathbb{N}}(n)$ .

PROOF: Set  $G = -F$ .  $G(0) = -F(0)$ .  $G$  has the same dom. For every natural number  $n$ ,  $G(n)$  is measurable on  $E$  and  $G(n)$  is non-negative by [7, (63)], (37). Consider  $J$  being a sequence of extended reals such that for every natural number  $n$ ,  $J(n) = \int G(n) dM$  and  $\int (\sum_{\alpha=0}^{\kappa} G(\alpha))_{\kappa \in \mathbb{N}}(n) dM = (\sum_{\alpha=0}^{\kappa} J(\alpha))_{\kappa \in \mathbb{N}}(n)$ . For every natural number  $n$ ,  $F(n)$  is measurable on  $E$  and  $F(n)$  is without  $+\infty$ .  $\square$

(73) Suppose  $E \subseteq \text{dom}(F(0))$  and  $F$  is additive and has the same dom and for every natural number  $n$ ,  $F(n)$  is non-positive and  $F(n)$  is measurable on  $E$  and for every element  $x$  of  $X$  such that  $x \in E$  holds  $F\#x$  is summable. Then there exists a sequence  $I$  of extended reals such that

- (i) for every natural number  $n$ ,  $I(n) = \int F(n)\upharpoonright E dM$ , and
- (ii)  $I$  is summable, and
- (iii)  $\int \lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}\upharpoonright E dM = \sum I$ .

PROOF: Set  $G = -F$ .  $G(0) = -F(0)$ .  $G$  is additive.  $G$  has the same dom. For every natural number  $n$ ,  $G(n)$  is non-negative and  $G(n)$  is measurable on  $E$  by (37), [7, (63)]. For every element  $x$  of  $X$  such that  $x \in E$  holds  $G\#x$  is summable. Consider  $J$  being a sequence of extended reals such that for every natural number  $n$ ,  $J(n) = \int G(n)\upharpoonright E dM$  and  $J$  is summable and  $\int \lim(\sum_{\alpha=0}^{\kappa} G(\alpha))_{\kappa \in \mathbb{N}}\upharpoonright E dM = \sum J$ . For every natural number  $n$ ,  $I(n) = \int F(n)\upharpoonright E dM$ . Define  $\mathcal{F}_{\infty}$ (natural number) =  $F(\$1)\upharpoonright E$ . Consider  $F_1$  being a sequence of partial functions from  $X$  in-

to  $\overline{\mathbb{R}}$  such that for every natural number  $n$ ,  $F_1(n) = F_1(n)$  from [12, Sch. 1].  $\lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}} = \lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \upharpoonright E$ . Define  $\mathcal{G}_{\infty}$ (natural number) =  $G(\mathcal{S}_1) \upharpoonright E$ . Consider  $G_1$  being a sequence of partial functions from  $X$  into  $\overline{\mathbb{R}}$  such that for every natural number  $n$ ,  $G_1(n) = G_1(n)$  from [12, Sch. 1].  $\lim(\sum_{\alpha=0}^{\kappa} G_1(\alpha))_{\kappa \in \mathbb{N}} = \lim(\sum_{\alpha=0}^{\kappa} G(\alpha))_{\kappa \in \mathbb{N}} \upharpoonright E$ . For every element  $n$  of  $\mathbb{N}$ ,  $F_1(n) = (-G_1)(n)$  by (37), [6, (2)], (3).  $\lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}} = -\lim(\sum_{\alpha=0}^{\kappa} G_1(\alpha))_{\kappa \in \mathbb{N}}$ . For every natural number  $n$ ,  $G_1(n)$  is measurable on  $E$  and  $G_1(n)$  is without  $-\infty$  by (40), [10, (20)], [8, (15)].  $\int(-\lim(\sum_{\alpha=0}^{\kappa} G_1(\alpha))_{\kappa \in \mathbb{N}}) - \int \lim(\sum_{\alpha=0}^{\kappa} G_1(\alpha))_{\kappa \in \mathbb{N}} \upharpoonright E \, dM$ .  $\square$

(74) Suppose  $E = \text{dom}(F(0))$  and  $F(0)$  is non-positive and  $F$  has the same dom and for every natural number  $n$ ,  $F(n)$  is measurable on  $E$  and for every natural numbers  $n, m$  such that  $n \leq m$  for every element  $x$  of  $X$  such that  $x \in E$  holds  $F(n)(x) \geq F(m)(x)$  and for every element  $x$  of  $X$  such that  $x \in E$  holds  $F \# x$  is convergent. Then there exists a sequence  $I$  of extended reals such that

- (i) for every natural number  $n$ ,  $I(n) = \int F(n) \, dM$ , and
- (ii)  $I$  is convergent, and
- (iii)  $\int \lim F \, dM = \lim I$ .

PROOF: Set  $G = -F$ .  $G(0) = -F(0)$ . For every natural number  $n$ ,  $G(n)$  is measurable on  $E$  by [7, (63)], (37). For every natural numbers  $n, m$  such that  $n \leq m$  for every element  $x$  of  $X$  such that  $x \in E$  holds  $G(n)(x) \leq G(m)(x)$ . For every element  $x$  of  $X$  such that  $x \in E$  holds  $G \# x$  is convergent by [6, (17)], (38). Consider  $J$  being a sequence of extended reals such that for every natural number  $n$ ,  $J(n) = \int G(n) \, dM$  and  $J$  is convergent and  $\int \lim G \, dM = \lim J$ . Set  $I = -J$ . For every natural number  $n$ ,  $I(n) = \int F(n) \, dM$ . For every element  $x$  of  $X$  such that  $x \in \text{dom} \lim G$  holds  $(\lim G)(x) = (-\lim F)(x)$  by (38), [6, (17)].  $\int \lim G \, dM = -\int \lim F \, dM$ .  $\square$

## REFERENCES

- [1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [3] Józef Białas. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(1):173–183, 1991.
- [4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1): 55–65, 1990.
- [5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.

- [6] Noboru Endou. Extended real-valued double sequence and its convergence. *Formalized Mathematics*, 23(3):253–277, 2015. doi:10.1515/forma-2015-0021.
- [7] Noboru Endou. Fubini’s theorem on measure. *Formalized Mathematics*, 25(1):1–29, 2017. doi:10.1515/forma-2017-0001.
- [8] Noboru Endou and Yasunari Shidama. Integral of measurable function. *Formalized Mathematics*, 14(2):53–70, 2006. doi:10.2478/v10037-006-0008-x.
- [9] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. The measurability of extended real valued functions. *Formalized Mathematics*, 9(3):525–529, 2001.
- [10] Noboru Endou, Keiko Narita, and Yasunari Shidama. The Lebesgue monotone convergence theorem. *Formalized Mathematics*, 16(2):167–175, 2008. doi:10.2478/v10037-008-0023-1.
- [11] Robert Milewski. Natural numbers. *Formalized Mathematics*, 7(1):19–22, 1998.
- [12] Beata Perkowska. Functional sequence from a domain to a domain. *Formalized Mathematics*, 3(1):17–21, 1992.
- [13] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(2):329–334, 1990.
- [14] Wojciech A. Trybulec. Non-contiguous substrings and one-to-one finite sequences. *Formalized Mathematics*, 1(3):569–573, 1990.
- [15] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.

*Received September 3, 2017*

---



The English version of this volume of *Formalized Mathematics* was financed under agreement 548/P-DUN/2016 with the funds from the Polish Minister of Science and Higher Education for the dissemination of science.