FORMALIZED MATHEMATICS
Vol. 25, No. 3, Pages 171–179, 2017
DOI: 10.1515/forma-2017-0016

DE GRUYTER OPEN
degruyter.com/view/j/forma

Isomorphism Theorem on Vector Spaces over a Ring[1]

Yuichi Futa
Tokyo University of Technology
Tokyo, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article, we formalize some properties of vector spaces over a ring. We formally prove the first isomorphism theorem of vector spaces over a ring. We also formalize the product space of vector spaces. $\mathbb{Z}$-modules are useful for lattice problems such as LLL (Lenstra, Lenstra and Lovász) base reduction algorithm and cryptographic systems [10].

MSC: 03B35

Keywords:

MML identifier: VECTSP12, version: 8.1.06 5.44.1305

1. Bijective Linear Transformation

From now on K, F denote rings, V, W denote vector spaces over K, l denotes a linear combination of V, and T denotes a linear transformation from V to W.

Now we state the propositions:

(1) Let us consider a field K, finite dimensional vector spaces V, W over K, a subset A of V, a basis B of V, a linear transformation T from V to W, and a linear combination l of $B \setminus A$. Suppose A is a basis of $\ker T$ and $A \subseteq B$. Then $T(\sum l) = \sum(T \mathbin{@*} l)$.

(2) Let us consider a field F, vector spaces X, Y over F, a linear transformation T from X to Y, and a subset A of X. Suppose T is bijective. Then A is a basis of X if and only if $T^{\circ}A$ is a basis of Y.

[1]This work was supported by JSPS KAKENHI grant number JP15K00183.

ISSN 1426–2630(Print), 1898-9934(Online)

(3) Let us consider a field F, vector spaces X, Y over F, and a linear transformation T from X to Y. Suppose T is bijective. Then X is finite dimensional if and only if Y is finite dimensional.

(4) Let us consider a field F, a finite dimensional vector space X over F, a vector space Y over F, and a linear transformation T from X to Y. Suppose T is bijective. Then

(i) Y is finite dimensional, and

(ii) $\dim(X) = \dim(Y)$.

PROOF: For every basis I of X, $\dim(Y) = \overline{\overline{I}}$ by [1, (5), (33)], (2). □

(5) Let us consider a field F, vector spaces X, Y over F, a linear combination l of X, and a linear transformation T from X to Y. If T is one-to-one, then $T \,@\, l = T \,@*\, l$.

PROOF: For every element y of Y, $(T \,@\, l)(y) = \sum \mathrm{CFS}(l, T, y)$ by [13, (43)], [14, (8)], [4, (74), (59)]. □

2. Properties of Linear Combinations of Modules over a Ring

Now we state the proposition:

(6) Let us consider a field K, a vector space V over K, subspaces W_1, W_2 of V, a basis I_1 of W_1, and a basis I_2 of W_2. If V is the direct sum of W_1 and W_2, then $I_1 \cap I_2 = \emptyset$.

Let us consider a field K, a vector space V over K, subspaces W_1, W_2 of V, a basis I_1 of W_1, a basis I_2 of W_2, and a subset I of V. Now we state the propositions:

(7) Suppose V is the direct sum of W_1 and W_2 and $I = I_1 \cup I_2$. Then $\mathrm{Lin}(I) =$ the vector space structure of V.

PROOF: Reconsider $I_3 = I_1$ as a subset of V. Reconsider $I_4 = I_2$ as a subset of V. For every vector x of V, $x \in W_1 + W_2$ iff $x \in \mathrm{Lin}(I_3) + \mathrm{Lin}(I_4)$ by [16, (1)]. □

(8) If V is the direct sum of W_1 and W_2 and $I = I_1 \cup I_2$, then I is linearly independent.

PROOF: Consider l being a linear combination of I such that $\sum l = 0_V$ and the support of $l \neq \emptyset$. $I_1 \cap I_2 = \emptyset$. I_1 misses I_2. Reconsider $I_3 = I_1$, $I_4 = I_2$ as a subset of V. Consider l_1 being a linear combination of I_3, l_2 being a linear combination of I_4 such that $l = l_1 + l_2$. Reconsider $l_3 = l_1$ as a linear combination of I. Set $v_1 = \sum l_3$. $v_1 \neq 0_V$ by [18, (11)], [7, (25)]. □

Now we state the proposition:

(9) Let us consider a field K, a vector space V over K, subspaces W_1, W_2 of V, a basis I_1 of W_1, and a basis I_2 of W_2. If $W_1 \cap W_2 = \mathbf{0}_V$, then $I_1 \cup I_2$ is a basis of $W_1 + W_2$.

PROOF: Set $I = I_1 \cup I_2$. Reconsider $W = W_1 + W_2$ as a strict subspace of V. Reconsider $W_3 = W_1$, $W_4 = W_2$ as a subspace of W. Reconsider $I_0 = I$ as a subset of W. For every object x, $x \in W_3 \cap W_4$ iff $x \in \mathbf{0}_V$ by [16, (3)]. For every object x, $x \in W$ iff $x \in W_3 + W_4$ by [16, (1), (2)], [15, (13)]. I_0 is base. □

3. First Isomophism Theorem

Let us consider a field K, a finite dimensional vector space V over K, and a subspace W of V. Now we state the propositions:

(10) There exists a linear complement S of W and there exists a linear transformation T from S to ${}^{V}/_{W}$ such that T is bijective and for every vector v of V such that $v \in S$ holds $T(v) = v + W$.

PROOF: Set S = the linear complement of W. Set $V_1 = {}^{V}/_{W}$. Define $\mathcal{P}[\text{vector of } V, \text{vector of } V_1] \equiv \$_2 = \$_1 + W$. Consider f_1 being a function from the carrier of V into the carrier of V_1 such that for every vector v of V, $\mathcal{P}[v, f_1(v)]$ from [5, Sch. 3]. Set $T = f_1 \restriction (\text{the carrier of } S)$. For every vector v of V such that $v \in S$ holds $T(v) = v + W$ by [4, (49)]. The carrier of $V_1 \subseteq \operatorname{rng} T$ by [8, (22)], [16, (1)], [8, (23)], [15, (49)]. For every objects x_1, x_2 such that x_1, $x_2 \in$ the carrier of S and $T(x_1) = T(x_2)$ holds $x_1 = x_2$ by [15, (42), (55)], [13, (28)], [9, (19)]. □

(11) (i) ${}^{V}/_{W}$ is finite dimensional, and

(ii) $\dim({}^{V}/_{W}) + \dim(W) = \dim(V)$.

The theorem is a consequence of (10) and (4).

Let K be a ring, V, U be vector spaces over K, W be a subspace of V, and f be a linear transformation from V to U. Assume the carrier of $W \subseteq$ the carrier of $\ker f$. The functor ${}^{f}/_{W}$ yielding a linear transformation from ${}^{V}/_{W}$ to U is defined by

(Def. 1) for every vector A of ${}^{V}/_{W}$ and for every vector a of V such that $A = a + W$ holds $it(A) = f(a)$.

The functor CQFunctional f yielding a linear transformation from ${}^{V}/_{\ker f}$ to U is defined by the term

(Def. 2) ${}^{f}/_{\ker f}$.

Observe that CQFunctional f is one-to-one.

Now we state the proposition:

(12) Let us consider a ring K, vector spaces V, U over K, and a linear transformation f from V to U. Then there exists a linear transformation T from ${}^{V}/_{\ker f}$ to $\operatorname{im} f$ such that

(i) $T = \text{CQFunctional}\, f$, and

(ii) T is bijective.

PROOF: Set $T = \text{CQFunctional}\, f$. For every object x, $x \in \operatorname{rng} T$ iff $x \in \operatorname{rng} f$ by [8, (22)], [5, (4)], [8, (23)]. □

Let K be a ring, V, U, W be vector spaces over K, f be a linear transformation from V to U, and g be a linear transformation from U to W. One can verify that the functor $g \cdot f$ yields a linear transformation from V to W.

4. The Product Space of Vector Spaces

Let K be a ring.

A VectorSpace-Sequence of K is a non empty finite sequence and is defined by

(Def. 3) for every set S such that $S \in \operatorname{rng} it$ holds S is a vector space over K.

Note that every VectorSpace-Sequence of K is Abelian group yielding.

Let G be a VectorSpace-Sequence of K and j be an element of $\operatorname{dom} G$. One can check that the functor $G(j)$ yields a vector space over K. Let j be an element of $\operatorname{dom} \overline{G}$. One can verify that the functor $G(j)$ yields a vector space over K. The functor $\operatorname{multop} G$ yielding a multi-operation of the carrier of K and $\overline{G}$ is defined by

(Def. 4) $\operatorname{len} it = \operatorname{len} \overline{G}$ and for every element j of $\operatorname{dom} \overline{G}$, $it(j) =$ the left multiplication of $G(j)$.

The functor $\prod G$ yielding a strict, non empty vector space structure over K is defined by the term

(Def. 5) $\langle \prod \overline{G}, \prod^{\circ} \langle +_{G_i} \rangle_i, \langle 0_{G_i} \rangle_i, \prod^{\circ} \operatorname{multop} G \rangle$.

Let us note that $\prod G$ is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital.

5. Cartesian Product of Vector Spaces

From now on K denotes a ring.

Let K be a ring and G, F be non empty vector space structures over K. The functor $\operatorname{prodmlt}(G, F)$ yielding a function from (the carrier of K) $\times$ ((the carrier

of G) $\times$ (the carrier of F)) into (the carrier of G) $\times$ (the carrier of F) is defined by

(Def. 6) for every element r of K and for every vector g of G and for every vector f of F, $it(r, \langle g, f \rangle) = \langle r \cdot g, r \cdot f \rangle$.

The functor $G \times F$ yielding a strict, non empty vector space structure over K is defined by the term

(Def. 7) $\langle$(the carrier of G)$\times$(the carrier of F), prodadd(G, F), prodzero(G, F), prodmlt($G,$

Let G, F be Abelian, non empty vector space structures over K. Note that $G \times F$ is Abelian.

Let G, F be add-associative, non empty vector space structures over K. One can verify that $G \times F$ is add-associative.

Let G, F be right zeroed, non empty vector space structures over K. One can verify that $G \times F$ is right zeroed.

Let G, F be right complementable, non empty vector space structures over K. One can check that $G \times F$ is right complementable.

Now we state the propositions:

(13) Let us consider non empty vector space structures G, F over K. Then

(i) for every set x, x is a vector of $G \times F$ iff there exists a vector x_1 of G and there exists a vector x_2 of F such that $x = \langle x_1, x_2 \rangle$, and

(ii) for every vectors x, y of $G \times F$ and for every vectors x_1, y_1 of G and for every vectors x_2, y_2 of F such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $x + y = \langle x_1 + y_1, x_2 + y_2 \rangle$, and

(iii) $0_{G \times F} = \langle 0_G, 0_F \rangle$, and

(iv) for every vector x of $G \times F$ and for every vector x_1 of G and for every vector x_2 of F and for every element a of K such that $x = \langle x_1, x_2 \rangle$ holds $a \cdot x = \langle a \cdot x_1, a \cdot x_2 \rangle$.

(14) Let us consider add-associative, right zeroed, right complementable, non empty vector space structures G, F over K, a vector x of $G \times F$, a vector x_1 of G, and a vector x_2 of F. Suppose $x = \langle x_1, x_2 \rangle$. Then $-x = \langle -x_1, -x_2 \rangle$.

Let K be a ring and G, F be vector distributive, non empty vector space structures over K. Let us note that $G \times F$ is vector distributive.

Let G, F be scalar distributive, non empty vector space structures over K. One can check that $G \times F$ is scalar distributive.

Let G, F be scalar associative, non empty vector space structures over K. Let us note that $G \times F$ is scalar associative.

Let G, F be scalar unital, non empty vector space structures over K. Let us observe that $G \times F$ is scalar unital.

Let G be a vector space over K. One can check that the functor $\langle G \rangle$ yields a VectorSpace-Sequence of K. Let G, F be vector spaces over K. Let us note that the functor $\langle G, F \rangle$ yields a VectorSpace-Sequence of K. Now we state the proposition:

(15) Let us consider a vector space X over K. Then there exists a function I from X into $\prod \langle X \rangle$ such that

(i) I is one-to-one and onto, and

(ii) for every vector x of X, $I(x) = \langle x \rangle$, and

(iii) for every vectors v, w of X, $I(v + w) = I(v) + I(w)$, and

(iv) for every vector v of X and for every element r of the carrier of K, $I(r \cdot v) = r \cdot I(v)$, and

(v) $I(0_X) = 0_{\prod \langle X \rangle}$.

PROOF: Set C_3 = the carrier of X. Consider I being a function from C_3 into $\prod \langle C_3 \rangle$ such that I is one-to-one and onto and for every object x such that $x \in C_3$ holds $I(x) = \langle x \rangle$. For every vectors v, w of X, $I(v + w) = I(v) + I(w)$ by [3, (40), (2)]. For every vector v of X and for every element r of the carrier of K, $I(r \cdot v) = r \cdot I(v)$ by [3, (40), (2)]. □

Let K be a ring and G, F be VectorSpace-Sequences of K. One can verify that the functor $G \frown F$ yields a VectorSpace-Sequence of K. Now we state the propositions:

(16) Let us consider vector spaces X, Y over K. Then there exists a function I from $X \times Y$ into $\prod \langle X, Y \rangle$ such that

(i) I is one-to-one and onto, and

(ii) for every vector x of X and for every vector y of Y, $I(x, y) = \langle x, y \rangle$, and

(iii) for every vectors v, w of $X \times Y$, $I(v + w) = I(v) + I(w)$, and

(iv) for every vector v of $X \times Y$ and for every element r of K, $I(r \cdot v) = r \cdot I(v)$, and

(v) $I(0_{X \times Y}) = 0_{\prod \langle X, Y \rangle}$.

PROOF: Set C_3 = the carrier of X. Set C_4 = the carrier of Y. Consider I being a function from $C_3 \times C_4$ into $\prod \langle C_3, C_4 \rangle$ such that I is one-to-one and onto and for every objects x, y such that $x \in C_3$ and $y \in C_4$ holds $I(x, y) = \langle x, y \rangle$. For every vectors v, w of $X \times Y$, $I(v + w) = I(v) + I(w)$ by [17, (43)], [3, (44)]. For every vector v of $X \times Y$ and for every element r of K, $I(r \cdot v) = r \cdot I(v)$ by [17, (43)], [3, (44)]. □

(17) Let us consider VectorSpace-Sequences X, Y of K. Then there exists a function I from $\prod X \times \prod Y$ into $\prod (X \frown Y)$ such that

(i) I is one-to-one and onto, and

(ii) for every vector x of $\prod X$ and for every vector y of $\prod Y$, there exist finite sequences x_1, y_1 such that $x = x_1$ and $y = y_1$ and $I(x, y) = x_1 \frown y_1$, and

(iii) for every vectors v, w of $\prod X \times \prod Y$, $I(v + w) = I(v) + I(w)$, and

(iv) for every vector v of $\prod X \times \prod Y$ and for every element r of the carrier of K, $I(r \cdot v) = r \cdot I(v)$, and

(v) $I(0_{\prod X \times \prod Y}) = 0_{\prod (X \frown Y)}$.

PROOF: Reconsider $C_1 = \overline{X}$, $C_2 = \overline{Y}$ as a non-empty, non empty finite sequence. Consider I being a function from $\prod C_1 \times \prod C_2$ into $\prod (C_1 \frown C_2)$ such that I is one-to-one and onto and for every finite sequences x, y such that $x \in \prod C_1$ and $y \in \prod C_2$ holds $I(x, y) = x \frown y$. Set $P_1 = \prod X$. Set $P_2 = \prod Y$. For every natural number k such that $k \in \operatorname{dom} \overline{X \frown Y}$ holds $\overline{X \frown Y}(k) = (C_1 \frown C_2)(k)$ by [12, (29)], [3, (25)]. For every vector x of $\prod X$ and for every vector y of $\prod Y$, there exist finite sequences x_1, y_1 such that $x = x_1$ and $y = y_1$ and $I(x, y) = x_1 \frown y_1$ by [11, (9)]. For every vectors v, w of $P_1 \times P_2$, $I(v + w) = I(v) + I(w)$ by [17, (43)], [11, (9)], [2, (9)], [3, (25)]. For every vector v of $P_1 \times P_2$ and for every element r of the carrier of K, $I(r \cdot v) = r \cdot I(v)$ by [17, (43)], [11, (9)], [2, (9)], [3, (25)]. □

(18) Let us consider vector spaces G, F over K. Then

(i) for every set x, x is a vector of $\prod \langle G, F \rangle$ iff there exists a vector x_1 of G and there exists a vector x_2 of F such that $x = \langle x_1, x_2 \rangle$, and

(ii) for every vectors x, y of $\prod \langle G, F \rangle$ and for every vectors x_1, y_1 of G and for every vectors x_2, y_2 of F such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $x + y = \langle x_1 + y_1, x_2 + y_2 \rangle$, and

(iii) $0_{\prod \langle G, F \rangle} = \langle 0_G, 0_F \rangle$, and

(iv) for every vector x of $\prod \langle G, F \rangle$ and for every vector x_1 of G and for every vector x_2 of F such that $x = \langle x_1, x_2 \rangle$ holds $-x = \langle -x_1, -x_2 \rangle$, and

(v) for every vector x of $\prod \langle G, F \rangle$ and for every vector x_1 of G and for every vector x_2 of F and for every element a of K such that $x = \langle x_1, x_2 \rangle$ holds $a \cdot x = \langle a \cdot x_1, a \cdot x_2 \rangle$.

PROOF: Consider I being a function from $G \times F$ into $\prod \langle G, F \rangle$ such that I is one-to-one and onto and for every vector x of G and for every vector y of F, $I(x, y) = \langle x, y \rangle$ and for every vectors v, w of $G \times F$, $I(v + w) = I(v) + I(w)$ and for every vector v of $G \times F$ and for every element r of K, $I(r \cdot v) = r \cdot I(v)$ and $0_{\prod \langle G, F \rangle} = I(0_{G \times F})$. For every set x, x is a vector of $\prod \langle G, F \rangle$ iff there

exists a vector x_1 of G and there exists a vector x_2 of F such that $x = \langle x_1, x_2 \rangle$ by [5, (113)], [17, (43)], [6, (87)], [5, (112)]. For every vectors x, y of $\prod\langle G, F\rangle$ and for every vectors x_1, y_1 of G and for every vectors x_2, y_2 of F such that $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$ holds $x + y = \langle x_1 + y_1, x_2 + y_2\rangle$ by [6, (87)]. $0_{\prod\langle G,F\rangle} = \langle 0_G, 0_F\rangle$. For every vector x of $\prod\langle G, F\rangle$ and for every vector x_1 of G and for every vector x_2 of F such that $x = \langle x_1, x_2\rangle$ holds $-x = \langle -x_1, -x_2\rangle$. For every vector x of $\prod\langle G, F\rangle$ and for every vector x_1 of G and for every vector x_2 of F and for every element a of K such that $x = \langle x_1, x_2\rangle$ holds $a \cdot x = \langle a \cdot x_1, a \cdot x_2\rangle$ by [6, (87)]. □

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**): 55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[7] Yuichi Futa, Hiroyuki Okazaki, and Yasunari Shidama. Submodule of free $\mathbb{Z}$-module. *Formalized Mathematics*, 21(**4**):273–282, 2013. doi:10.2478/forma-2013-0029.

[8] Jarosław Kotowicz. Quotient vector spaces and functionals. *Formalized Mathematics*, 11 (**1**):59–68, 2003.

[9] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[10] Daniele Micciancio and Shafi Goldwasser. Complexity of lattice problems: a cryptographic perspective. *The International Series in Engineering and Computer Science*, 2002.

[11] Yasunari Shidama. Differentiable functions on normed linear spaces. *Formalized Mathematics*, 20(**1**):31–40, 2012. doi:10.2478/v10037-012-0005-1.

[12] Wojciech A. Trybulec. Non-contiguous substrings and one-to-one finite sequences. *Formalized Mathematics*, 1(**3**):569–573, 1990.

[13] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[14] Wojciech A. Trybulec. Linear combinations in real linear space. *Formalized Mathematics*, 1(**3**):581–588, 1990.

[15] Wojciech A. Trybulec. Subspaces and cosets of subspaces in vector space. *Formalized Mathematics*, 1(**5**):865–870, 1990.

[16] Wojciech A. Trybulec. Operations on subspaces in vector space. *Formalized Mathematics*, 1(**5**):871–876, 1990.

[17] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[18] Mariusz Żynel. The Steinitz theorem and the dimension of a vector space. *Formalized Mathematics*, 5(**3**):423–428, 1996.

Received August 30, 2017

The English version of this volume of Formalized Mathematics was financed under agreement 548/P-DUN/2016 with the funds from the Polish Minister of Science and Higher Education for the dissemination of science.