

# Introduction to Stopping Time in Stochastic Finance Theory. Part II

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**Summary.** We start proceeding with stopping time theory in discrete time with the help of the Mizar system [2], [6]. We show, the necessity in [4, p.283], that in general the expression for two stopping times  $\tau_1, \tau_2$  not always implies a stoppingtime  $(\tau_1 + \tau_2)$ . This means you have to cut it -  $(\tau + \sigma) \wedge T$  (in the case of finity) -.

Next we introduce stopping time in continuous time. Here we are focused on the time intervals  $[0, r]$  where  $r$  in  $\mathbb{R}$  and  $[0, +\infty[$ . We prove some Borel-Sets, more general given in [5, p.12 1.8,(e)]. These Borel-Sets are necessary to define our next theorem. In the case of continuity we show, that the  $\sigma$ -algebra of the  $\tau$ -past, where  $\tau$  is a stopping time has a certain characteristic ([7, p.187, Definitions 9, 19 and Lemma 9.21]). Stopping Times can be used together with the conditional expected value. Look as an example to the Optional Sampling Theorem in [7, p.203].

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## 1. PRELIMINARIES

From now on  $\Omega$  denotes a non empty set,  $\Sigma$  denotes a  $\sigma$ -field of subsets of  $\Omega$ ,  $S$  denotes a non empty subset of  $\mathbb{R}$ ,  $r$  denotes a real number, and  $T$  denotes a natural number.

Let  $A$  be a non empty set,  $I$  be an extended real-membered set, and  $k_1, k_2$  be functions from  $A$  into  $I$ . We say that  $k_1 \leq k_2$  if and only if

(Def. 1) for every element  $w$  of  $A$ ,  $k_1(w) \leq k_2(w)$ .

Let  $f_1, f_2$  be extended real-valued functions. The functor  $f_1 + f_2$  yielding a function is defined by

(Def. 2)  $\text{dom } it = \text{dom } f_1 \cap \text{dom } f_2$  and for every object  $x$  such that  $x \in \text{dom } it$  holds  $it(x) = f_1(x) + f_2(x)$ .

One can check that the functor is commutative.

Let us note that  $f_1 + f_2$  is extended real-valued.

Let  $C$  be a set,  $D_1, D_2$  be extended real-membered, non empty sets,  $f_1$  be a function from  $C$  into  $D_1$ , and  $f_2$  be a function from  $C$  into  $D_2$ . One can verify that  $f_1 + f_2$  is total as a partial function from  $C$  to  $\overline{\mathbb{R}}$ .

Let  $D_1, D_2$  be extended real-membered sets,  $f_1$  be a partial function from  $C$  to  $D_1$ , and  $f_2$  be a partial function from  $C$  to  $D_2$ . Let us note that the functor  $f_1 + f_2$  yields a partial function from  $C$  to  $\overline{\mathbb{R}}$ . Now we state the propositions:

- (1) Let us consider non empty sets  $A, I, y$ , and a function  $F$  from  $A$  into  $I$ . Then  $\{z, \text{ where } z \text{ is an element of } A : F(z) \in y\} = F^{-1}(y)$ .
- (2) Let us consider a real number  $r$ . If  $r > 0$ , then there exists a natural number  $n$  such that  $\frac{1}{n} < r$  and  $n > 0$ .
- (3) Let us consider real numbers  $a, b$ . Then  $[-\infty, a] \cap [b, +\infty] = [b, a]$ .
- (4) Let us consider a real number  $r$ . Suppose  $r \geq 0$ . Then  $[0, +\infty] \setminus [0, r[ = [r, +\infty]$ .

Let  $r$  be an extended real. Observe that  $[r, +\infty]$  is non empty.

Now we state the proposition:

- (5) Let us consider an extended real  $k$ . Then  $\overline{\mathbb{R}} \setminus [-\infty, k] = ]k, +\infty]$ .  
Let  $a$  be a real number. One can check that  $]a, +\infty]$  is non empty.

## 2. THEORY OF STOPPING TIME IN DISCRETE TIME

Let us consider  $\Omega, \Sigma$ , and  $T$ . Let  $F_1$  be a filtration of  $\bigcup_{t \in \mathbb{N}: 0 \leq t \leq T} \{t\}$  and  $\Sigma$  and  $k$  be a function from  $\Omega$  into  $T_{\{+\infty\}}$ . We say that  $k$  is  $F_1$ -StoppingTime-like if and only if

(Def. 3)  $k$  is StoppingTime( $F_1, T$ ).

Let  $M_1$  be a filtration of  $\bigcup_{t \in \mathbb{N}: 0 \leq t \leq T} \{t\}$  and  $\Sigma$ . Note that there exists a function from  $\Omega$  into  $T_{\{+\infty\}}$  which is  $M_1$ -StoppingTime-like.

A StoppingTimeFunc of  $M_1$  is a  $M_1$ -StoppingTime-like function from  $\Omega$  into  $T_{\{+\infty\}}$ . Now we state the proposition:

- (6) Let us consider a non zero natural number  $T$ , and a filtration  $M_1$  of  $\bigcup_{t \in \mathbb{N}: 0 \leq t \leq T} \{t\}$  and  $\Sigma$ . Then there exist StoppingTimeFuncs  $k_1, k_2$  of  $M_1$  such that  $k_1 + k_2$  is not a StoppingTimeFunc of  $M_1$ .

PROOF: Reconsider  $M_2 = T$  as an element of  $T_{\{+\infty\}}$ . Consider  $k_1$  being a function from  $\Omega$  into  $T_{\{+\infty\}}$  such that  $k_1 = \Omega \mapsto M_2$  and  $k_1$  is  $\text{StoppingTime}(M_1, T)$ . Consider  $k_2$  being a function from  $\Omega$  into  $T_{\{+\infty\}}$  such that  $k_2 = \Omega \mapsto M_2$  and  $k_2$  is  $\text{StoppingTime}(M_1, T)$ . There exists an element  $w$  of  $\text{dom}(k_1 + k_2)$  such that  $w \in \text{dom}(k_1 + k_2)$  and  $(k_1 + k_2)(w) \notin T_{\{+\infty\}}$  by [9, (7)], [1, (16)].  $\square$

3. THEORY OF FOR STOPPING TIME IN CONTINUOUS TIME

Let  $r$  be a real number.

**A TheEvent of  $r$**  is a subset of  $\mathbb{R}$  and is defined by

- (Def. 4) (i)  $it = [0, +\infty[$ , if  $r \leq 0$ ,
- (ii)  $it = [0, r]$ , otherwise.

Let us note that every TheEvent of  $r$  is non empty.

In the sequel  $I$  denotes a TheEvent of  $r$ .

Now we state the proposition:

- (7)  $I$  is an event of the Borel sets.

4. BOREL-SETS

Let us consider  $r$  and  $I$ . Let  $A$  be an element of the Borel sets. The functor **Borelsubsets-help( $A, I$ )** yielding an element of the Borel sets is defined by the term

- (Def. 5)  $A \cap I$ .

The functor **BorelSubsets( $I$ )** yielding a  $\sigma$ -field of subsets of  $I$  is defined by the term

- (Def. 6) the set of all Borelsubsets-help( $A, I$ ) where  $A$  is an element of the Borel sets.

Let us consider  $\Omega$  and  $\Sigma$ . Let  $M_1$  be a function and  $k$  be a random variable of  $\Sigma$  and BorelSubsets( $I$ ). We say that  **$k$  isStoppingTimewrt ( $M_1$ )** if and only if

- (Def. 7) for every element  $t$  of  $I$ ,  $\{w, \text{ where } w \text{ is an element of } \Omega : k(w) \leq t\} \in M_1(t)$ .

Now we state the proposition:

- (8) Let us consider a filtration  $M_1$  of  $I$  and  $\Sigma$ , and an element  $t_1$  of  $I$ . Then there exists a random variable  $q$  of  $\Sigma$  and BorelSubsets( $I$ ) such that
  - (i)  $q = \Omega \mapsto t_1$ , and

(ii)  $q$  isStoppingTimewrt  $(M_1)$ .

PROOF: For every element  $t$  of  $I$ ,  $\{w, \text{ where } w \text{ is an element of } \Omega : (\Omega \mapsto t_1)(w) \leq t\} \in M_1(t)$  by [9, (7)], [8, (5), (4)]. Set  $O = \Omega \mapsto t_1$ . For every set  $x$ ,  $O^{-1}(x) \in \Sigma$  by [9, (7)], (1), [8, (4), (5)].  $\square$

Let us consider  $\Omega, \Sigma, r$ , and  $I$ . Let  $F_1$  be a filtration of  $I$  and  $\Sigma$  and  $k$  be a random variable of  $\Sigma$  and BorelSubsets( $I$ ). We say that  **$k$  is  $F_1$ -StoppingTime-like** if and only if

(Def. 8)  $k$  isStoppingTimewrt  $(F_1)$ .

Let  $M_1$  be a filtration of  $I$  and  $\Sigma$ . One can check that there exists a random variable of  $\Sigma$  and BorelSubsets( $I$ ) which is  $M_1$ -StoppingTime-like.

**A StoppingTimeFunc of  $M_1$**  is a  $M_1$ -StoppingTime-like random variable of  $\Sigma$  and BorelSubsets( $I$ ).

### 5. $\sigma$ -ALGEBRA OF THE $\tau$ -PAST

Let us consider  $\Omega, \Sigma, r$ , and  $I$ . Let  $M_1$  be a filtration of  $I$  and  $\Sigma$ ,  $\tau$  be a StoppingTimeFunc of  $M_1$ , and  $A_1$  be a sequence of subsets of  $\Omega$ . Assume  $\text{rng } A_1 \subseteq \{A, \text{ where } A \text{ is an element of } \Sigma : \text{ for every element } t_1 \text{ of } I, A \cap \{w, \text{ where } w \text{ is an element of } \Omega : \tau(w) \leq t_1\} \in M_1(t_1)\}$ . Let  $t$  be an element of  $I$  and  $n$  be a natural number. The functor **Sigma-tauhelp2( $\tau, A_1, n, t$ )** yielding an element of the  $t$ - $\mathcal{EF}$  of  $M_1$  is defined by the term

(Def. 9)  $(\text{Complement } A_1)(n) \cap \{w, \text{ where } w \text{ is an element of } \Omega : \tau(w) \leq t\}$ .

Let  $A$  be a sequence of subsets of  $\Omega$ . The functor **Sigma-tauhelp3( $\tau, A, t$ )** yielding a sequence of subsets of the  $t$ - $\mathcal{EF}$  of  $M_1$  is defined by

(Def. 10) for every natural number  $n$ ,  $it(n) = \text{Sigma-tauhelp2}(\tau, A, n, t)$ .

The functor **Sigma-tau( $\tau$ )** yielding a  $\sigma$ -field of subsets of  $\Omega$  is defined by the term

(Def. 11)  $\{A, \text{ where } A \text{ is an element of } \Sigma : \text{ for every element } t \text{ of } I, A \cap \{w, \text{ where } w \text{ is an element of } \Omega : \tau(w) \leq t\} \in M_1(t)\}$ .

Now we state the proposition:

(9) Let us consider a filtration  $M_1$  of  $I$  and  $\Sigma$ , and StoppingTimeFuncs  $k_1, k_2$  of  $M_1$ . Suppose  $k_1 \leq k_2$ . Then  $\text{Sigma-tau}(k_1) \subseteq \text{Sigma-tau}(k_2)$ .

PROOF: Consider  $A$  being an element of  $\Sigma$  such that  $x = A$  and for every element  $t$  of  $I$ ,  $A \cap \{w_1, \text{ where } w_1 \text{ is an element of } \Omega : k_1(w_1) \leq t\} \in M_1(t)$ .  $x \in \{A, \text{ where } A \text{ is an element of } \Sigma : \text{ for every element } t \text{ of } I, A \cap \{w_1, \text{ where } w_1 \text{ is an element of } \Omega : k_2(w_1) \leq t\} \in M_1(t)\}$  by [8, (19)].  $\square$

The functor **Ext-Family-of-halflines** yielding a family of subsets of  $\overline{\mathbb{R}}$  is defined by the term

(Def. 12) the set of all  $[-\infty, r]$  where  $r$  is a real number.

The functor **Ext-Borel-Sets** yielding a  $\sigma$ -field of subsets of  $\overline{\mathbb{R}}$  is defined by the term

(Def. 13)  $\sigma(\text{Ext-Family-of-halflines})$ .

Now we state the proposition:

(10) Let us consider a real number  $k$ . Then

(i)  $]k, +\infty]$  is an element of Ext-Borel-Sets, and

(ii)  $[-\infty, k]$  is an element of Ext-Borel-Sets.

The theorem is a consequence of (5).

Let  $b$  be a real number. The functor **ext-half-open-sets( $b$ )** yielding a sequence of subsets of  $\overline{\mathbb{R}}$  is defined by

(Def. 14)  $it(0) = ]b - 1, +\infty]$  and for every natural number  $n$ ,  $it(n + 1) = ]b - \frac{1}{n+1}, +\infty]$ .

Let us consider a real number  $b$ . Now we state the propositions:

(11) Intersection ext-half-open-sets( $b$ ) is an element of Ext-Borel-Sets.

PROOF: For every natural number  $n$ , (Complement ext-half-open-sets( $b$ ))( $n$ ) is an element of Ext-Borel-Sets by [1, (6)], (10).  $\square$

(12) Intersection ext-half-open-sets( $b$ ) =  $[b, +\infty]$ .

PROOF: For every object  $c$ ,  $c \in$  Intersection ext-half-open-sets( $b$ ) iff  $c \in [b, +\infty]$  by [8, (13)], (2), [1, (13), (20)].  $\square$

Now we state the propositions:

(13) Let us consider real numbers  $a, b$ . Then  $[b, a]$  is an element of Ext-Borel-Sets.

PROOF:  $[-\infty, a]$  is an element of Ext-Borel-Sets.  $[-\infty, a] \cap [b, +\infty]$  is an element of Ext-Borel-Sets by (12), (11), [8, (19)].  $\square$

(14) Let us consider a real number  $a$ . Then  $\{a\}$  is an element of Ext-Borel-Sets.

The theorem is a consequence of (13).

(15) Let us consider a real number  $r$ . Then  $[r, +\infty]$  is an event of Ext-Borel-Sets.

The theorem is a consequence of (11) and (12).

Let  $b$  be a real number. The functor **ext-right-closed-sets( $b$ )** yielding a sequence of subsets of  $\overline{\mathbb{R}}$  is defined by

(Def. 15) for every natural number  $n$ ,  $it(n) = [-\infty, b - n]$ .

Now we state the propositions:

(16) Let us consider a real number  $b$ . Then Intersection ext-right-closed-sets( $b$ ) is an element of Ext-Borel-Sets. The theorem is a consequence of (10).

(17) Intersection ext-right-closed-sets(0) =  $\{-\infty\}$ .

PROOF: For every object  $c$ ,  $c \in$  Intersection ext-right-closed-sets(0) iff  $c \in \{-\infty\}$  by [8, (13)].  $\square$

(18)  $\{-\infty\}$  is an element of Ext-Borel-Sets.

Let  $b$  be a real number. The functor **ext-left-closed-sets( $b$ )** yielding a sequence of subsets of  $\overline{\mathbb{R}}$  is defined by

(Def. 16) for every natural number  $n$ ,  $it(n) = [b + n, +\infty]$ .

Now we state the propositions:

(19) Let us consider a real number  $b$ . Then Intersection ext-left-closed-sets( $b$ ) is an element of Ext-Borel-Sets. The theorem is a consequence of (15).

(20) Intersection ext-left-closed-sets(0) =  $\{+\infty\}$ .

PROOF: For every object  $c$ ,  $c \in$  Intersection ext-left-closed-sets(0) iff  $c \in \{+\infty\}$  by [8, (13)], [3, (31)].  $\square$

(21)  $\{+\infty\}$  is an element of Ext-Borel-Sets.

(22)  $\mathbb{R}$  is an element of Ext-Borel-Sets. The theorem is a consequence of (19), (20), (16), (17), and (2).

(23) Halflines  $\subseteq$  Ext-Borel-Sets. The theorem is a consequence of (10), (14), (16), and (17).

Let  $A$  be an element of Ext-Borel-Sets. The functor **Ext-Borelsubsets-help( $A$ )** yielding an element of Ext-Borel-Sets is defined by the term

(Def. 17)  $A \cap [0, +\infty]$ .

The functor **ExtBorelsubsets** yielding a  $\sigma$ -field of subsets of  $[0, +\infty]$  is defined by the term

(Def. 18) the set of all Ext-Borelsubsets-help( $A$ ) where  $A$  is an element of Ext-Borel-Sets.

Let us consider  $\Omega$  and  $\Sigma$ . Let  $M_1$  be a function,  $S$  be a non empty, extended real-membered set, and  $k$  be a random variable of  $\Sigma$  and ExtBorelsubsets. We say that  $k$  is StoppingTime( $M_1, S$ ) if and only if

(Def. 19) for every element  $t$  of  $S$ ,  $\{w, \text{ where } w \text{ is an element of } \Omega : k(w) \leq t\} \in M_1(t)$ .

Now we state the proposition:

(24) Let us consider a filtration  $M_1$  of  $S$  and  $\Sigma$ , and an element  $t_1$  of  $[0, +\infty]$ . Then there exists a random variable  $q$  of  $\Sigma$  and ExtBorelsubsets such that

- (i)  $q = \Omega \mapsto t_1$ , and
- (ii)  $q$  is StoppingTime( $M_1, S$ ).

PROOF: For every element  $t$  of  $S$ ,  $\{w, \text{ where } w \text{ is an element of } \Omega : (\Omega \mapsto t_1)(w) \leq t\} \in M_1(t)$  by [9, (7)], [8, (5), (4)]. Set  $O = \Omega \mapsto t_1$ . For every set  $x$ ,  $O^{-1}(x) \in \Sigma$  by [9, (7)], (1), [8, (4), (5)].  $\square$

Let us consider  $\Omega$ ,  $\Sigma$ , and  $S$ . Let  $F_1$  be a filtration of  $S$  and  $\Sigma$  and  $k$  be a random variable of  $\Sigma$  and ExtBorelsubsets. We say that  $k$  is  $F_1$ -StoppingTime-like if and only if

(Def. 20)  $k$  is StoppingTime( $F_1, S$ ).

Let  $M_1$  be a filtration of  $S$  and  $\Sigma$ . Observe that there exists a random variable of  $\Sigma$  and ExtBorelsubsets which is  $M_1$ -StoppingTime-like.

A StoppingTimeFunc of  $\Sigma$ ,  $M_1$  is a  $M_1$ -StoppingTime-like random variable of  $\Sigma$  and ExtBorelsubsets. Let  $\tau$  be a StoppingTimeFunc of  $\Sigma$ ,  $M_1$  and  $A_1$  be a sequence of subsets of  $\Omega$ . Assume  $\text{rng } A_1 \subseteq \{A, \text{ where } A \text{ is an element of } \Sigma : \text{ for every element } t_1 \text{ of } S, A \cap \{w, \text{ where } w \text{ is an element of } \Omega : \tau(w) \leq t_1\} \in M_1(t_1)\}$ . Let  $t$  be an element of  $S$  and  $n$  be a natural number. The functor Sigma-tauh2( $M_1, \tau, A_1, n, t$ ) yielding an element of the  $t$ - $\mathcal{EF}$  of  $M_1$  is defined by the term

(Def. 21)  $(\text{Complement } A_1)(n) \cap \{w, \text{ where } w \text{ is an element of } \Omega : \tau(w) \leq t\}$ .

The functor Sigma-tauh3( $M_1, \tau, A_1, t$ ) yielding a sequence of subsets of the  $t$ - $\mathcal{EF}$  of  $M_1$  is defined by

(Def. 22) for every natural number  $n$ ,  $it(n) = \text{Sigma-tauh2}(M_1, \tau, A_1, n, t)$ .

The functor Sigma-tau( $M_1, \tau$ ) yielding a  $\sigma$ -field of subsets of  $\Omega$  is defined by the term

(Def. 23)  $\{A, \text{ where } A \text{ is an element of } \Sigma : \text{ for every element } t \text{ of } S, A \cap \{w, \text{ where } w \text{ is an element of } \Omega : \tau(w) \leq t\} \in M_1(t)\}$ .

Now we state the proposition:

- (25) Let us consider a filtration  $M_1$  of  $S$  and  $\Sigma$ , and StoppingTimeFuncs  $k_1, k_2$  of  $\Sigma$ ,  $M_1$ . Suppose  $k_1 \leq k_2$ . Then  $\text{Sigma-tau}(M_1, k_1) \subseteq \text{Sigma-tau}(M_1, k_2)$ .  
 PROOF: Consider  $A$  being an element of  $\Sigma$  such that  $x = A$  and for every element  $t$  of  $S$ ,  $A \cap \{w_1, \text{ where } w_1 \text{ is an element of } \Omega : k_1(w_1) \leq t\} \in M_1(t)$ . For every element  $t$  of  $S$ ,  $x \cap \{w_1, \text{ where } w_1 \text{ is an element of } \Omega : k_2(w_1) \leq t\} \in M_1(t)$  by [8, (19)].  $\square$

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