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Introduction to Stopping Time in Stochastic Finance Theory. Part II

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Summary. We start proceeding with the stopping time theory in discrete time with the help of the Mizar system [1], [4]. We prove, that the expression for two stopping times k_1 and k_2 not always implies a stopping time $(k_1 + k_2)$ (see Theorem 6 in this paper). If you want to get a stopping time, you have to cut the function e.g. $(k_1 + k_2) \cap T$ (see [2, p. 283 Remark 6.14]).

Next we introduce the stopping time in continuous time. We are focused on the intervals $[0, r]$ where $r \in \mathbb{R}$. We prove, that for $I = [0, r]$ or $I = [0, +\infty[$ the set $\{A \cap I : A \in \text{Borel-Sets}\}$ is a σ-algebra of I (see Definition 6 in this paper, and more general given in [3, p.12 1.8e]). The interval I can be considered as a timeline from now to some point in the future.

This set is necessary to define our next lemma. We prove the existence of the σ-algebra of the τ-past, where τ is a stopping time (see Definition 11 in this paper and [6, p.187, Definition 9.19]). If τ_1 and τ_2 are stopping times with τ_1 is smaller or equal than τ_2 we can prove, that the σ-algebra of the τ_1-past is a subset of the σ-algebra of the τ_2-past (see Theorem 9 in this paper and [6, p.187 Lemma 9.21]).

Suppose, that you want to use Lemma 9.21 with some events, that never occur, see as a comparison the paper [5] and the example for ST(1)=$\{+\infty\}$ in the Summary. We don't have the element $+\infty$ in our above-mentioned time intervals $[0, r[$ and $[0,+\infty[$. This is only possible if we construct a new σ-algebra on $\mathbb{R} \cup \{-\infty, +\infty\}$. This construction is similar to the Borel-Sets and we call this σ-algebra extended Borel sets (see Definition 13 in this paper and [3, p. 21]). It can be proved, that $\{+\infty\}$ is an Element of extended Borel sets (see Theorem 21 in this paper). Now we use the interval $[0, +\infty]$ as a basis. We construct a σ-algebra on $[0, +\infty]$ similar to the book ([3, p. 12 18e]), see Definition 18 in this paper, and call it extended Borel subsets. We prove for stopping times with this given σ-algebra, that for τ_1 and τ_2 are stopping times with τ_1 is smaller or equal than τ_2 we have the σ-algebra of the τ_1-past is a subset of the σ-algebra of the τ_2-past, see Theorem 25 in this paper. It is obvious, that $\{+\infty\} \in$ extended Borel subsets.

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In general, Lemma 9.21 is important for the proof of the Optional Sampling Theorem, see 10.11 Proof of (i) in [6, p. 203].

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1. Preliminaries

From now on Ω denotes a non empty set, Σ denotes a σ-field of subsets of Ω, S denotes a non empty subset of $\mathbb{R}$, r denotes a real number, and T denotes a natural number.

Let A be a non empty set, I be an extended real-membered set, and k_1, k_2 be functions from A into I. We say that $k_1 \leqslant k_2$ if and only if

(Def. 1) for every element w of A, $k_1(w) \leqslant k_2(w)$.

Let f_1, f_2 be extended real-valued functions. The functor $f_1 + f_2$ yielding a function is defined by

(Def. 2) $\operatorname{dom} it = \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and for every object x such that $x \in \operatorname{dom} it$ holds $it(x) = f_1(x) + f_2(x)$.

One can check that the functor is commutative.

Let us note that $f_1 + f_2$ is extended real-valued.

Let C be a set, D_1, D_2 be extended real-membered, non empty sets, f_1 be a function from C into D_1, and f_2 be a function from C into D_2. One can verify that $f_1 + f_2$ is total as a partial function from C to $\overline{\mathbb{R}}$.

Let D_1, D_2 be extended real-membered sets, f_1 be a partial function from C to D_1, and f_2 be a partial function from C to D_2. Let us note that the functor $f_1 + f_2$ yields a partial function from C to $\overline{\mathbb{R}}$. Now we state the propositions:

(1) Let us consider non empty sets A, I, y, and a function F from A into I. Then $\{z$, where z is an element of $A : F(z) \in y\} = F^{-1}(y)$.

(2) Let us consider a real number r. If $r > 0$, then there exists a natural number n such that $\frac{1}{n} < r$ and $n > 0$.

(3) Let us consider real numbers a, b. Then $[-\infty, a] \cap [b, +\infty] = [b, a]$.

(4) Let us consider a real number r. Suppose $r \geqslant 0$. Then $[0, +\infty] \setminus [0, r[= [r, +\infty]$.

Let r be an extended real. Observe that $[r, +\infty]$ is non empty.

(5) Let us consider an extended real k. Then $\overline{\mathbb{R}} \setminus [-\infty, k] =]k, +\infty]$.

Let a be a real number. One can check that $]a, +\infty]$ is non empty.

2. Stopping Time in Discrete Time

Let us consider Ω, Σ, and T. Let F_1 be a filtration of $\bigcup_{t \in \mathbb{N}: 0 \leqslant t \leqslant T}\{t\}$ and Σ and k be a function from Ω into $T_{\{+\infty\}}$. We say that k is like stopping time of F_1 if and only if

(Def. 3) k is StoppingTime(F_1,T).

Let M_1 be a filtration of $\bigcup_{t \in \mathbb{N}: 0 \leqslant t \leqslant T}\{t\}$ and Σ. Note that there exists a function from Ω into $T_{\{+\infty\}}$ which is like stopping time of M_1.

A stopping time of M_1 is a like stopping time of M_1 function from Ω into $T_{\{+\infty\}}$. Now we state the proposition:

(6) Let us consider a non zero natural number T, and a filtration M_1 of $\bigcup_{t \in \mathbb{N}: 0 \leqslant t \leqslant T}\{t\}$ and Σ. Then there exist stopping times k_1, k_2 of M_1 such that $k_1 + k_2$ is not a stopping time of M_1.

Proof: Reconsider $M_2 = T$ as an element of $T_{\{+\infty\}}$. Consider k_1 being a function from Ω into $T_{\{+\infty\}}$ such that $k_1 = \Omega \longmapsto M_2$ and k_1 is StoppingTime(M_1,T). Consider k_2 being a function from Ω into $T_{\{+\infty\}}$ such that $k_2 = \Omega \longmapsto M_2$ and k_2 is StoppingTime(M_1,T). There exists an element w of dom($k_1 + k_2$) such that $w \in$ dom($k_1 + k_2$) and $(k_1 + k_2)(w) \notin T_{\{+\infty\}}$. □

3. Stopping Time in Continuous Time

Let r be a real number.

A stopping event of r is a subset of $\mathbb{R}$ defined by

(Def. 4) (i) $it = [0, +\infty[$, **if** $r \leqslant 0$,

(ii) $it = [0, r]$, **otherwise**.

Let us note that every stopping event of r is non empty.

In the sequel I denotes a stopping event of r.

Now we state the proposition:

(7) I is an event of the Borel sets.

4. Borel-Sets

Let us consider r and I. Let A be an element of the Borel sets. The intersection of A and I yielding an element of the Borel sets is defined by

(Def. 5) $A \cap I$.

The first Borel subsets with I yielding a σ-field of subsets of I is defined by

(Def. 6) the set of all the intersection of A and I where A is an element of the Borel sets.

Let us consider Ω and Σ. Let M_1 be a function and k be a random variable of Σ and the first Borel subsets with I. We say that k is stopping time of M_1 if and only if

(Def. 7) for every element t of I, $\{w$, where w is an element of $\Omega : k(w) \leqslant t\} \in M_1(t)$.

(8) Let us consider a filtration M_1 of I and Σ, and an element t_1 of I. Then there exists a random variable q of Σ and the first Borel subsets with I such that

(i) $q = \Omega \longmapsto t_1$, and

(ii) q is stopping time of M_1.

Proof: For every element t of I, $\{w$, where w is an element of $\Omega : (\Omega \longmapsto t_1)(w) \leqslant t\} \in M_1(t)$. Set $O = \Omega \longmapsto t_1$. For every set x, $O^{-1}(x) \in \Sigma$. □

Let us consider Ω, Σ, r, and I. Let F_1 be a filtration of I and Σ and k be a random variable of Σ and the first Borel subsets with I. We say that k is like stopping time of F_1 if and only if

(Def. 8) k is stopping time of F_1.

Let M_1 be a filtration of I and Σ. One can check that there exists a random variable of Σ and the first Borel subsets with I which is like stopping time of M_1.

A stopping time of M_1 is a like stopping time of M_1 random variable of Σ and the first Borel subsets with I.

5. σ-Algebra of the τ-Past

Let us consider Ω, Σ, r, and I. Let M_1 be a filtration of I and Σ, τ be a stopping time of M_1, and A_1 be a sequence of subsets of Ω. Assume $\operatorname{rng} A_1 \subseteq \{A$, where A is an element of $\Sigma :$ for every element t_1 of $I, A \cap \{w$, where w is an element of $\Omega : \tau(w) \leqslant t_1\} \in M_1(t_1)\}$. Let t be an element of I and n be a natural number. The first set for σ-tau of τ, A_1, n and t yielding an element of the t-$\mathcal{EF}$ of M_1 is defined by the term

(Def. 9) $(\text{Complement } A_1)(n) \cap \{w$, where w is an element of $\Omega : \tau(w) \leqslant t\}$.

Let A be a sequence of subsets of Ω. The second set for σ-tau of τ, A and t yielding a sequence of subsets of the t-$\mathcal{EF}$ of M_1 is defined by

(Def. 10) for every natural number n, $it(n) =$ the first set for σ-tau of τ, A, n and t.

The functor Σ-tau(τ) yielding a σ-field of subsets of Ω is defined by the term

(Def. 11) $\{A$, where A is an element of Σ : for every element t of I, $A \cap \{w$, where w is an element of $\Omega : \tau(w) \leqslant t\} \in M_1(t)\}$.

Now we state the proposition:

(9) Let us consider a filtration M_1 of I and Σ, and stopping times k_1, k_2 of M_1. Suppose $k_1 \leqslant k_2$. Then Σ-tau(k_1) $\subseteq$ Σ-tau(k_2).

Proof: Consider A being an element of Σ such that $x = A$ and for every element t of I, $A \cap \{w_1$, where w_1 is an element of Ω : $k_1(w_1) \leqslant t\} \in M_1(t)$. $x \in \{A$, where A is an element of Σ : for every element t of $I, A \cap \{w_1$, where w_1 is an element of $\Omega : k_2(w_1) \leqslant t\} \in M_1(t)\}$. □

The extended family of halflines yielding a family of subsets of $\overline{\mathbb{R}}$ is defined by the term

(Def. 12) the set of all $[-\infty, r]$ where r is a real number.

The extended Borel sets yielding a σ-field of subsets of $\overline{\mathbb{R}}$ is defined by the term

(Def. 13) σ(the extended family of halflines).

Now we state the proposition:

(10) Let us consider a real number k. Then

(i) $]k, +\infty]$ is an element of the extended Borel sets, and

(ii) $[-\infty, k]$ is an element of the extended Borel sets.

The theorem is a consequence of (5).

Let b be a real number. The extended half open sets of b yielding a sequence of subsets of $\overline{\mathbb{R}}$ is defined by

(Def. 14) $it(0) =]b-1, +\infty]$ and for every natural number n, $it(n+1) =]b - \frac{1}{n+1}, +\infty]$.

Let us consider a real number b. Now we state the propositions:

(11) Intersection(the extended half open sets of b) is an element of the extended Borel sets.

Proof: For every natural number n, (Complement(the extended half open sets of b))(n) is an element of the extended Borel sets. □

(12) Intersection(the extended half open sets of b) $= [b, +\infty]$.

Proof: For every object c, $c \in$ Intersection(the extended half open sets of b) iff $c \in [b, +\infty]$. □

(13) Let us consider real numbers a, b. Then $[b, a]$ is an element of the extended Borel sets.

Proof: $[-\infty, a]$ is an element of the extended Borel sets. $[-\infty, a] \cap [b, +\infty]$ is an element of the extended Borel sets by (12), (11), [7, (19)]. □

(14) Let us consider a real number a. Then $\{a\}$ is an element of the extended Borel sets. The theorem is a consequence of (13).

(15) Let us consider a real number r. Then $[r, +\infty]$ is an event of the extended Borel sets. The theorem is a consequence of (11) and (12).

Let b be a real number. The extended right closed sets of b yielding a sequence of subsets of $\overline{\mathbb{R}}$ is defined by

(Def. 15) for every natural number n, $it(n) = [-\infty, b - n]$.

Now we state the propositions:

(16) Let us consider a real number b. Then Intersection(the extended right closed sets of b) is an element of the extended Borel sets. The theorem is a consequence of (10).

(17) Intersection(the extended right closed sets of 0) $= \{-\infty\}$.
PROOF: For every object c, $c \in$ Intersection(the extended right closed sets of 0) iff $c \in \{-\infty\}$. □

(18) $\{-\infty\}$ is an element of the extended Borel sets.

Let b be a real number. The extended left closed sets of b yielding a sequence of subsets of $\overline{\mathbb{R}}$ is defined by

(Def. 16) for every natural number n, $it(n) = [b + n, +\infty]$.

Now we state the propositions:

(19) Let us consider a real number b. Then Intersection(the extended left closed sets of b) is an element of the extended Borel sets. The theorem is a consequence of (15).

(20) Intersection(the extended left closed sets of 0) $= \{+\infty\}$.
PROOF: For every object c, $c \in$ Intersection(the extended left closed sets of 0) iff $c \in \{+\infty\}$. □

(21) $\{+\infty\}$ is an element of the extended Borel sets.

(22) $\mathbb{R}$ is an element of the extended Borel sets. The theorem is a consequence of (19), (20), (16), (17), and (2).

(23) Halflines $\subseteq$ the extended Borel sets. The theorem is a consequence of (10), (14), (16), and (17).

Let A be an element of the extended Borel sets. The positive subset of A yielding an element of the extended Borel sets is defined by the term

(Def. 17) $A \cap [0, +\infty]$.

The extended Borel subsets yielding a σ-field of subsets of $[0, +\infty]$ is defined by the term

(Def. 18) the set of all the positive subset of A where A is an element of the extended Borel sets.

Now we state the proposition:

(24) $\{+\infty\}$ is an element of the extended Borel subsets. The theorem is a consequence of (21).

Let us consider Ω and Σ. Let M_1 be a function, S be a non empty, extended real-membered set, and k be a random variable of Σ and the extended Borel subsets. We say that k is StoppingTime(M_1,S) if and only if

(Def. 19) for every element t of S, $\{w$, where w is an element of $\Omega : k(w) \leqslant t\} \in M_1(t)$.

Now we state the proposition:

(25) Let us consider a filtration M_1 of S and Σ, and an element t_1 of $[0, +\infty]$. Then there exists a random variable q of Σ and the extended Borel subsets such that

(i) $q = \Omega \longmapsto t_1$, and

(ii) q is StoppingTime(M_1,S).

PROOF: For every element t of S, $\{w$, where w is an element of $\Omega : (\Omega \longmapsto t_1)(w) \leqslant t\} \in M_1(t)$. Set $O = \Omega \longmapsto t_1$. For every set x, $O^{-1}(x) \in \Sigma$. $\square$

Let us consider Ω, Σ, and S. Let F_1 be a filtration of S and Σ and k be a random variable of Σ and the extended Borel subsets. We say that k is like stopping time of F_1 if and only if

(Def. 20) k is StoppingTime(F_1,S).

Let M_1 be a filtration of S and Σ. Observe that there exists a random variable of Σ and the extended Borel subsets which is like stopping time of M_1.

A stopping time of Σ and M_1 is a like stopping time of M_1 random variable of Σ and the extended Borel subsets. Let τ be a stopping time of Σ and M_1 and A_1 be a sequence of subsets of Ω. Assume rng $A_1 \subseteq \{A$, where A is an element of $\Sigma :$ for every element t_1 of $S, A \cap \{w$, where w is an element of $\Omega : \tau(w) \leqslant t_1\} \in M_1(t_1)\}$. Let t be an element of S and n be a natural number. The first set for σ-tau of M_1, τ, A_1, n and t yielding an element of the t-$\mathcal{EF}$ of M_1 is defined by the term

(Def. 21) (Complement $A_1)(n) \cap \{w$, where w is an element of $\Omega : \tau(w) \leqslant t\}$.

The second set for σ-tau of M_1, τ, A_1 and t yielding a sequence of subsets of the t-$\mathcal{EF}$ of M_1 is defined by

(Def. 22) for every natural number n, $it(n) =$ the first set for σ-tau of M_1, τ, A_1, n and t.

The functor Σ-tau(M_1, τ) yielding a σ-field of subsets of Ω is defined by the term

(Def. 23) $\{A$, where A is an element of $\Sigma :$ for every element t of S, $A \cap \{w$, where w is an element of $\Omega : \tau(w) \leqslant t\} \in M_1(t)\}$.

Now we state the proposition:

(26) Let us consider a filtration M_1 of S and Σ, and stopping times k_1, k_2 of Σ and M_1. Suppose $k_1 \leqslant k_2$. Then $\Sigma\text{-tau}(M_1, k_1) \subseteq \Sigma\text{-tau}(M_1, k_2)$.

Proof: Consider A being an element of Σ such that $x = A$ and for every element t of S, $A \cap \{w_1$, where w_1 is an element of $\Omega : k_1(w_1) \leqslant t\} \in M_1(t)$. For every element t of S, $x \cap \{w_1$, where w_1 is an element of $\Omega : k_2(w_1) \leqslant t\} \in M_1(t)$. $\square$

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