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Tarski Geometry Axioms. Part III

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Summary. czkam rzewnie

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1. Congruence Properties

From now on S denotes Tarski plane and a, b, c, d, e, f denote points of S. Now we state the proposition:

(1) 2.1 Satz:
$\overline{ab} \cong \overline{ab}$.

Now we state the proposition:

(2) 2.1 Satz bis:
Let us consider Tarski plane S, and points a, b of S. Then $\overline{ab} \cong \overline{ab}$.

Now we state the proposition:

(3) 2.2 Satz:
If $\overline{ab} \cong \overline{cd}$, then $\overline{cd} \cong \overline{ab}$. The theorem is a consequence of (1).

Now we state the proposition:

(4) 2.2 Satz bis:
Let us consider Tarski plane S, and points a, b, c, d of S. If $\overline{ab} \cong \overline{cd}$, then $\overline{cd} \cong \overline{ab}$. The theorem is a consequence of (2).

Now we state the proposition:

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(5) 2.3 Satz:
If $\overline{ab} \cong \overline{cd}$ and $\overline{cd} \cong \overline{ef}$, then $\overline{ab} \cong \overline{ef}$. The theorem is a consequence of (3).

Now we state the proposition:

(6) 2.4 Satz:
If $\overline{ab} \cong \overline{cd}$, then $\overline{ba} \cong \overline{cd}$. The theorem is a consequence of (5).

Now we state the proposition:

(7) 2.5 Satz:
If $\overline{ab} \cong \overline{cd}$, then $\overline{ab} \cong \overline{dc}$. The theorem is a consequence of (5).

Now we state the proposition:

(8) 2.8 Satz:
Let us consider Tarski plane S, and points a, b of S. Then $\overline{aa} \cong \overline{bb}$.

Let S be a Tarski plane. We say that S is satisfyingSSTA5 if and only if

(Def. 1) for every points a, b, c, d, a', b', c', d' of S such that $a \neq b$ and b lies between a and c and b' lies between a' and c' and $\overline{ab} \cong \overline{a'b'}$ and $\overline{bc} \cong \overline{b'c'}$ and $\overline{ad} \cong \overline{a'd'}$ and $\overline{bd} \cong \overline{b'd'}$ holds $\overline{cd} \cong \overline{c'd'}$.

Now we state the proposition:

(9) S satisfies the axiom of SAS if and only if S is satisfyingSSTA5. The theorem is a consequence of (6) and (7).

One can check that every Tarski plane which is satisfyingSSTA5 satisfies also the axiom of SAS and every Tarski plane which satisfies the axiom of SAS is also satisfyingSSTA5.

Let S be a Tarski plane and a, b, c, d, a', b', c', d' be points of S. We say that a, b, c, d AFS a', b', c', d' if and only if

(Def. 2) b lies between a and c and b' lies between a' and c' and $\overline{ab} \cong \overline{a'b'}$ and $\overline{bc} \cong \overline{b'c'}$ and $\overline{ad} \cong \overline{a'd'}$ and $\overline{bd} \cong \overline{b'd'}$.

Now we state the proposition:

(10) Let us consider Tarski plane S, and points a, b, c, d, a', b', c', d' of S. Suppose a, b, c, d AFS a', b', c', d' and $a \neq b$. Then $\overline{cd} \cong \overline{c'd'}$.

From now on S denotes Tarski plane and q, a, b, c, a', b', c', x_1, x_2 denote points of S.

Now we state the proposition:

(11) 2.11 Satz:
If b lies between a and c and b' lies between a' and c' and $\overline{ab} \cong \overline{a'b'}$ and $\overline{bc} \cong \overline{b'c'}$, then $\overline{ac} \cong \overline{a'c'}$. The theorem is a consequence of (6), (7), (8), and (3).

Now we state the proposition:

(12) 2.12 Satz:

Suppose $q \neq a$. If a lies between q and x_1 and $\overline{ax_1} \cong \overline{bc}$ and a lies between q and x_2 and $\overline{ax_2} \cong \overline{bc}$, then $x_1 = x_2$. The theorem is a consequence of (3), (5), (1), and (11).

2. Between Properties

Now we state the proposition:

(13) 3.1 Satz:

Let us consider Tarski plane S, and points a, b of S. Then b lies between a and b.

From now on S denotes Tarski plane and a, b, c, d denote points of S.

Now we state the proposition:

(14) 3.2 Satz:

If b lies between a and c, then b lies between c and a. The theorem is a consequence of (13).

Now we state the proposition:

(15) 3.3 Satz:

a lies between a and b.

Now we state the proposition:

(16) 3.4 Satz:

Let us consider Tarski plane S, and points a, b, c of S. If b lies between a and c and a lies between b and c, then $a = b$.

From now on S denotes Tarski plane and a, b, c, d denote points of S.

Now we state the proposition:

(17) 3.5 Satz:

If b lies between a and d and c lies between b and d, then b lies between a and c and c lies between a and d.

Now we state the proposition:

(18) 3.6 Satz:

If b lies between a and c and c lies between a and d, then c lies between b and d and b lies between a and d.

Now we state the proposition:

(19) 3.7 Satz:

If b lies between a and c and c lies between b and d and $b \neq c$, then c lies between a and d and b lies between a and d.

Let S be a Tarski plane and a, b, c, d be points of S. We say that between4a, b, c, d if and only if

(Def. 3) b lies between a and c and b lies between a and d and c lies between a and d and c lies between b and d.

Let S be a Tarski plane and a, b, c, d, e be points of S. We say that between5a, b, c, d, e if and only if

(Def. 4) b lies between a and c and b lies between a and d and b lies between a and e and c lies between a and d and c lies between a and e and d lies between a and e and c lies between b and d and c lies between b and e and d lies between b and e and d lies between c and e.

From now on S denotes Tarski plane and a, b, c, d, e denote points of S.

Now we state the proposition:

(20) 3.9 Satz (n = 3):
If b lies between a and c, then b lies between c and a.

Now we state the proposition:

(21) 3.9 Satz (n = 4):
If between4a, b, c, d, then between4d, c, b, a.

Now we state the proposition:

(22) 3.9 Satz (n = 5):
If between5a, b, c, d, e, then between5e, d, c, b, a.

Now we state the proposition:

(23) 3.10 Satz (n = 4):
Let us consider Tarski plane S, and points a, b, c, d of S. Suppose between4a, b, c, d. Then

(i) b lies between a and c, and

(ii) b lies between a and d, and

(iii) c lies between a and d, and

(iv) c lies between b and d.

Now we state the proposition:

(24) 3.10 Satz (n = 5):
Suppose between5a, b, c, d, e. Then

(i) b lies between a and c, and

(ii) b lies between a and d, and

(iii) b lies between a and e, and

(iv) c lies between a and d, and

(v) c lies between a and e, and

(vi) d lies between a and e, and

(vii) c lies between b and d, and

(viii) c lies between b and e, and

(ix) d lies between b and e, and

(x) d lies between c and e, and

(xi) between4a, b, c, d, and

(xii) between4a, b, c, e, and

(xiii) between4a, c, d, e, and

(xiv) between4b, c, d, e.

From now on S denotes Tarski plane and a, b, c, d, p denote points of S.

Now we state the proposition:

(25) 3.11 Satz (n = 3, l = 1):
If b lies between a and c and p lies between a and b, then between4a, p, b, c.

Now we state the proposition:

(26) 3.11 Satz (n = 3, l = 2):
If b lies between a and c and p lies between b and c, then between4a, b, p, c.

Now we state the proposition:

(27) 3.11 Satz (n = 3, l = 1):
If between4a, b, c, d and p lies between a and b, then between5a, p, b, c, d.

Now we state the proposition:

(28) 3.11 Satz (n = 3, l = 2):
If between4a, b, c, d and p lies between b and c, then between5a, b, p, c, d.

Now we state the proposition:

(29) 3.11 Satz (n = 3, l = 3):
If between4a, b, c, d and p lies between c and d, then between5a, b, c, p, d.

Now we state the proposition:

(30) 3.12 Satz (n = 3, l = 1):
If b lies between a and c and c lies between a and p, then between4a, b, c, p and if $a \neq c$, then between4a, b, c, p.

Now we state the proposition:

(31) 3.12 Satz (n = 3, l = 2):
If b lies between a and c and c lies between b and p, then c lies between b and p and if $b \neq c$, then between4a, b, c, p.

Now we state the proposition:

(32) 3.12 Satz (n = 4, l = 1):
If between4a, b, c, d and d lies between a and p, then between5a, b, c, d, p and if $a \neq d$, then between5a, b, c, d, p.

Now we state the proposition:

(33) 3.12 Satz (n = 4, l = 2):
If between4a, b, c, d and d lies between b and p, then between4b, c, d, p and if $b \neq d$, then between5a, b, c, d, p.

Now we state the proposition:

(34) 3.12 Satz (n = 4, l = 3):
If between4a, b, c, d and d lies between c and p, then d lies between c and p and if $c \neq d$, then between5a, b, c, d, p.

Let us note that there exists Tarski plane which satisfies Lower Dimension Axiom.

Now we state the proposition:

(35) 3.13 Satz:
Let us consider Tarski plane S. Then there exist points a, b, c of S such that

(i) b does not lie between a and c, and

(ii) c does not lie between b and a, and

(iii) a does not lie between c and b, and

(iv) $a \neq b$, and

(v) $b \neq c$, and

(vi) $c \neq a$.

The theorem is a consequence of (13).

Now we state the proposition:

(36) 3.14 Satz:
Let us consider Tarski plane S, and points a, b of S. Then there exists a point c of S such that

(i) b lies between a and c, and

(ii) $b \neq c$.

The theorem is a consequence of (35) and (3).

Now we state the proposition:

(37) 3.15 Satz (n = 3):
Let us consider Tarski plane S, and points a_1, a_2 of S. Suppose $a_1 \neq a_2$. Then there exists a point a_3 of S such that

(i) a_2 lies between a_1 and a_3, and

(ii) a_1, a_2, a_3 are mutually different.

The theorem is a consequence of (36).

Now we state the proposition:

(38) 3.15 SATZ (N = 4):

Let us consider Tarski plane S, and points a_1, a_2 of S. Suppose $a_1 \neq a_2$. Then there exist points a_3, a_4 of S such that

(i) between4a_1, a_2, a_3, a_4, and

(ii) a_1, a_2, a_3, a_4 are mutually different.

The theorem is a consequence of (37).

Now we state the proposition:

(39) 3.15 SATZ (N = 5):

Let us consider Tarski plane S, and points a_1, a_2 of S. Suppose $a_1 \neq a_2$. Then there exist points a_3, a_4, a_5 of S such that

(i) between5a_1, a_2, a_3, a_4, a_5, and

(ii) a_1,a_2,a_3,a_4,a_5 are mutually different.

The theorem is a consequence of (38) and (37).

Now we state the proposition:

(40) 3.17 SATZ:

Let us consider Tarski plane S, and points a, b, c, p, a', b', c' of S. Suppose b lies between a and c and b' lies between a' and c and p lies between a and a'. Then there exists a point q of S such that

(i) q lies between p and c, and

(ii) q lies between b and b'.

The theorem is a consequence of (14).

3. COLLINEARITY

Let S be a Tarski plane and a, b, c, d, a', b', c', d' be points of S. We say that a, b, c, d IFS a', b', c', d' if and only if

(Def. 5) b lies between a and c and b' lies between a' and c' and $\overline{ac} \cong \overline{a'c'}$ and $\overline{bc} \cong \overline{b'c'}$ and $\overline{ad} \cong \overline{a'd'}$ and $\overline{cd} \cong \overline{c'd'}$.

From now on S denotes Tarski plane and a, b, c, d, a', b', c', d' denote points of S.

Now we state the proposition:

(41) 4.2 SATZ:

If a, b, c, d IFS a', b', c', d', then $\overline{bd} \cong \overline{b'd'}$. The theorem is a consequence of (3), (6), (7), and (14).

Now we state the proposition:

(42) 4.3 Satz:
If b lies between a and c and b' lies between a' and c' and $\overline{ac} \cong \overline{a'c'}$ and $\overline{bc} \cong \overline{b'c'}$, then $\overline{ab} \cong \overline{a'b'}$. The theorem is a consequence of (6), (8), (7), and (41).

Now we state the proposition:

(43) 4.5 Satz:
If b lies between a and c and $\overline{ac} \cong \overline{a'c'}$, then there exists b' such that b' lies between a' and c' and $\triangle abc \cong \triangle a'b'c'$. The theorem is a consequence of (3), (8), (13), (14), (11), and (12).

Now we state the proposition:

(44) 4.6 Satz:
If b lies between a and c and $\triangle abc \cong \triangle a'b'c'$, then b' lies between a' and c'. The theorem is a consequence of (43), (3), (5), (6), (1), (7), and (41).

Now we state the proposition:

(45) 4.11 Satz:
Let us consider Tarski plane S, and points a, b, c of S. Suppose a, b and c are collinear. Then

(i) b, c and a are collinear, and

(ii) c, a and b are collinear, and

(iii) c, b and a are collinear, and

(iv) b, a and c are collinear, and

(v) a, c and b are collinear.

Now we state the propositions:

(46) 4.12 Sazz:
Let us consider Tarski plane S, and points a, b of S. Then a, a and b are collinear.

(47) Let us consider Tarski plane S, and points a, b, c, a', b', c' of S. Suppose $\triangle abc \cong \triangle a'b'c'$. Then $\triangle bca \cong \triangle b'c'a'$. The theorem is a consequence of (6) and (7).

Now we state the proposition:

(48) 4.13 Satz:
Let us consider Tarski plane S, and points a, b, c, a', b', c' of S. Suppose a, b and c are collinear and $\triangle abc \cong \triangle a'b'c'$. Then a', b' and c' are collinear. The theorem is a consequence of (47) and (44).

Let us consider Tarski plane S and points a, b, c, a', b', c' of S. Now we state the propositions:

(49) If $\triangle bac \cong \triangle b'a'c'$, then $\triangle abc \cong \triangle a'b'c'$. The theorem is a consequence of (6) and (7).

(50) If $\triangle acb \cong \triangle a'c'b'$, then $\triangle abc \cong \triangle a'b'c'$. The theorem is a consequence of (6) and (7).

From now on S denotes Tarski plane and a, b, c, d, a', b', c', d', p, q denote points of S.

Now we state the proposition:

(51) 4.14 SATZ:
If a, b and c are collinear and $\overline{ab} \cong \overline{a'b'}$, then there exists a point c' of S such that $\triangle abc \cong \triangle a'b'c'$. The theorem is a consequence of (3), (11), (14), (6), (7), (49), (43), and (50).

Let S be a Tarski plane and a, b, c, d, a', b', c', d' be points of S. We say that $a, b, c, d \,\mathrm{FS}\, a', b', c', d'$ if and only if

(Def. 6) a, b and c are collinear and $\triangle abc \cong \triangle a'b'c'$ and $\overline{ad} \cong \overline{a'd'}$ and $\overline{bd} \cong \overline{b'd'}$.

Now we state the proposition:

(52) 4.16 SATZ:
If $a, b, c, d \,\mathrm{FS}\, a', b', c', d'$ and $a \neq b$, then $\overline{cd} \cong \overline{c'd'}$. The theorem is a consequence of (44), (47), (41), (14), and (49).

Now we state the proposition:

(53) 4.17 SATZ:
If $a \neq b$ and a, b and c are collinear and $\overline{ap} \cong \overline{aq}$ and $\overline{bp} \cong \overline{bq}$, then $\overline{cp} \cong \overline{cq}$. The theorem is a consequence of (1) and (52).

Now we state the proposition:

(54) 4.18 SATZ:
If $a \neq b$ and a, b and c are collinear and $\overline{ac} \cong \overline{ac'}$ and $\overline{bc} \cong \overline{bc'}$, then $c = c'$. The theorem is a consequence of (53) and (3).

Now we state the proposition:

(55) 4.19 SATZ:
If c lies between a and b and $\overline{ac} \cong \overline{ac'}$ and $\overline{bc} \cong \overline{bc'}$, then $c = c'$. The theorem is a consequence of (3), (14), and (54).

4. BETWEEN TRANSITIVITY LE??

From now on S denotes Tarski plane and a, b, c, d, e, f, a', b', c', d' denote points of S.

Now we state the proposition:

(56) 5.1 Satz:
If $a \neq b$ and b lies between a and c and b lies between a and d, then c lies between a and d or d lies between a and c.

Now we state the proposition:

(57) 5.2 Satz:
If $a \neq b$ and b lies between a and c and b lies between a and d, then c lies between b and d or d lies between b and c. The theorem is a consequence of (56).

Now we state the proposition:

(58) 5.3 Satz:
If b lies between a and d and c lies between a and d, then b lies between a and c or c lies between a and b. The theorem is a consequence of (13), (14), (3), and (57).

Let S be a Tarski plane and a, b, c, d be points of S. We say that $a, b \leqslant c, d$ if and only if

(Def. 7) there exists a point y of S such that y lies between c and d and $\overline{ab} \cong \overline{cy}$.

Now we state the proposition:

(59) 5.5 Satz:
$a, b \leqslant c, d$ if and only if there exists a point x of S such that b lies between a and x and $\overline{ax} \cong \overline{cd}$. The theorem is a consequence of (3), (51), (44), (6), and (7).

Now we state the proposition:

(60) 5.6 Satz:
If $a, b \leqslant c, d$ and $\overline{ab} \cong \overline{a'b'}$ and $\overline{cd} \cong \overline{c'd'}$, then $a', b' \leqslant c', d'$. The theorem is a consequence of (59), (51), (3), (5), and (44).

Now we state the proposition:

(61) 5.7 Satz:
$a, b \leqslant a, b$. The theorem is a consequence of (13) and (1).

Now we state the proposition:

(62) 5.8 Satz:
If $a, b \leqslant c, d$ and $c, d \leqslant e, f$, then $a, b \leqslant e, f$. The theorem is a consequence of (59), (3), (51), (44), and (5).

Now we state the proposition:

(63) 5.9 Satz:
If $a, b \leqslant c, d$ and $c, d \leqslant a, b$, then $\overline{ab} \cong \overline{cd}$. The theorem is a consequence of (59), (14), (3), (12), and (16).

Now we state the proposition:

(64) 5.10 SATZ:

(i) $a, b \leqslant c, d$, or

(ii) $c, d \leqslant a, b$.

The theorem is a consequence of (3), (59), (14), and (56).

Now we state the proposition:

(65) 5.11 SATZ:

$a, a \leqslant b, c$. The theorem is a consequence of (59).

Now we state the proposition:

(66) 5.12 LEMMA 1:

Let us consider Tarski plane S, and points a, b, c, d of S. If $a, b \leqslant c, d$, then $b, a \leqslant c, d$.

Now we state the proposition:

(67) 5.12 LEMMA 2:

If $a, b \leqslant c, d$, then $a, b \leqslant d, c$. The theorem is a consequence of (59) and (7).

Now we state the proposition:

(68) 5.12 LEMMA 3:

If b lies between a and c and $\overline{ac} \cong \overline{ab}$, then $c = b$. The theorem is a consequence of (14), (6), (3), (7), (44), and (16).

Now we state the proposition:

(69) METAMATH: ENDOFSEGIDAND:

If c lies between a and b and $a, b \leqslant a, c$, then $b = c$. The theorem is a consequence of (59) and (68).

Now we state the proposition:

(70) 5.12 SATZ:

If a, b and c are collinear, then b lies between a and c iff $a, b \leqslant a, c$ and $b, c \leqslant a, c$. The theorem is a consequence of (1), (14), (6), (67), (69), and (13).

5. OUT LINES

Let S be a Tarski plane and a, b, p be points of S. We say that p out a, b if and only if

(Def. 8) $p \neq a$ and $p \neq b$ and (a lies between p and b or b lies between p and a).

From now on p denotes a point of S.

Now we state the proposition:

(71) 6.2 Satz:
If $a \neq p$ and $b \neq p$ and $c \neq p$ and p lies between a and c, then p lies between b and c iff p out a, b. The theorem is a consequence of (14) and (57).

Now we state the proposition:

(72) 6.3 Satz:
p out a, b if and only if $a \neq p$ and $b \neq p$ and there exists c such that $c \neq p$ and p lies between a and c and p lies between b and c. The theorem is a consequence of (3) and (71).

Now we state the proposition:

(73) 6.4 Satz:
p out a, b if and only if a, p and b are collinear and p does not lie between a and b. The theorem is a consequence of (14), (16), and (13).

Now we state the proposition:

(74) 6.5 Satz:
If $a \neq p$, then p out a, a.

Now we state the proposition:

(75) 6.6 Satz:
If p out a, b, then p out b, a.

Now we state the proposition:

(76) 6.7 Satz:
If p out a, b and p out b, c, then p out a, c.

Now we state the proposition:

(77) METAMATH: segcon2:
There exists a point x of S such that

(i) a lies between p and x or x lies between p and a, and

(ii) $\overline{px} \cong \overline{bc}$.

The theorem is a consequence of (3), (14), and (57).

In the sequel r denotes a point of S.

Now we state the proposition:

(78) 6.11 Satz a):
If $r \neq a$ and $b \neq c$, then there exists a point x of S such that a out x, r and $\overline{ax} \cong \overline{bc}$. The theorem is a consequence of (77) and (3).

Let S be a Tarski plane and a, p be points of S. The functor half $-$ line(p, a) yielding a set is defined by the term

(Def. 9) $\{x$, where x is a point of $S : p$ out $x, a\}$.

From now on x, y denote points of S.

Now we state the proposition:

(79) 6.11 SATZ B):
If $r \neq a$ and $b \neq c$ and a out x, r and $\overline{ax} \cong \overline{bc}$ and a out y, r and $\overline{ay} \cong \overline{bc}$, then $x = y$. The theorem is a consequence of (72), (14), (12), and (57).

Now we state the proposition:

(80) 6.13 SATZ:
If p out a, b, then $p, a \leqslant p, b$ iff a lies between p and b. The theorem is a consequence of (1), (79), and (70).

Let S be a non empty Tarski plane and p, q be points of S. The functor $\mathrm{Line}(p, q)$ yielding a subset of S is defined by the term

(Def. 10) $\{x$, where x is a point of $S : p$, q and x are collinear$\}$.

In the sequel S denotes a non empty Tarski plane satisfying seven Tarski's geometry axioms and p, q, r, s denote points of S.

Now we state the proposition:

(81) 6.15 SATZ:
If $p \neq q$ and $p \neq r$ and p lies between q and r, then $\mathrm{Line}(p, q) = (\mathrm{half}-\mathrm{line}(p, q) \cup \{p\}) \cup \mathrm{half}-\mathrm{line}(p, r)$. The theorem is a consequence of (14), (57), and (13).

Let S be a non empty Tarski plane and A be a subset of S. We say that $A \mathrm{is}_\mathrm{l}\mathrm{ine}$ if and only if

(Def. 11) there exist points p, q of S such that $p \neq q$ and $A = \mathrm{Line}(p, q)$.

Now we state the proposition:

(82) 6.16 SATZ:
If $p \neq q$ and $s \neq p$ and $s \in \mathrm{Line}(p, q)$, then $\mathrm{Line}(p, q) = \mathrm{Line}(p, s)$. The theorem is a consequence of (56), (14), (58), and (57).

In the sequel S denotes a non empty Tarski plane satisfying the axiom of congruence identity, the axiom of segment construction, the axiom of betweenness identity, and the axiom of Pasch and a, b, p, q denote points of S.

Now we state the proposition:

(83) 6.17 SATZ:

(i) p, $q \in \mathrm{Line}(p, q)$, and

(ii) $\mathrm{Line}(p, q) = \mathrm{Line}(q, p)$.

The theorem is a consequence of (13) and (14).

In the sequel S denotes a non empty Tarski plane satisfying seven Tarski's geometry axioms, A, B denote subsets of S, and a, b, c, p, q, r, s denote points of S.

Now we state the proposition:

(84) Let us consider Tarski plane S, and elements a, b, c of S. Then $a \neq b$ and a, b and c are collinear if and only if c lies on the line passing through a and b.

Let us consider a non empty Tarski plane S satisfying seven Tarski's geometry axioms and points a, b, x, y of S. Now we state the propositions:

(85) If the line passing through a and b is equal to the line passing through x and y, then $\mathrm{Line}(a, b) = \mathrm{Line}(x, y)$. The theorem is a consequence of (84).

(86) If $a \neq b$ and $x \neq y$ and $\mathrm{Line}(a, b) = \mathrm{Line}(x, y)$, then the line passing through a and b is equal to the line passing through x and y.

Now we state the proposition:

(87) 6.18 Satz:
If $A$is$_{\mathrm{l}}$ine and $a \neq b$ and a, $b \in A$, then $A = \mathrm{Line}(a, b)$. The theorem is a consequence of (85).

Now we state the proposition:

(88) 6.19 Satz:
If $a \neq b$ and $A$is$_{\mathrm{l}}$ine and a, $b \in A$ and $B$is$_{\mathrm{l}}$ine and a, $b \in B$, then $A = B$. The theorem is a consequence of (87).

Now we state the proposition:

(89) 6.21 Satz:
If $A$is$_{\mathrm{l}}$ine and $B$is$_{\mathrm{l}}$ine and $A \neq B$ and $a \in A$ and $a \in B$ and $b \in A$ and $b \in B$, then $a = b$.

Now we state the proposition:

(90) 6.23 Satz:
If there exists p and there exists q such that $p \neq q$, then a, b and c are collinear iff there exists A such that $A$is$_{\mathrm{l}}$ine and a, b, $c \in A$. The theorem is a consequence of (87) and (13).

Now we state the proposition:

(91) 6.24 Satz:
Let us consider Tarski plane S. Then there exist points a, b, c of S such that a, b and c are not collinear.

Now we state the propositions:

(92) 6.25 Satz:
Let us consider a non empty Tarski plane S satisfying seven Tarski's geometry axioms, and points a, b of S. Suppose S satisfies (A8) and $a \neq b$.

Then there exists a point c of S such that a, b and c are not collinear. The theorem is a consequence of (91), (13), and (87).

(93) Let us consider Tarski plane S, and points p, a, b of S. If p out a, b and $p, a \leqslant p, b$, then a lies between p and b.

(94) Let us consider Tarski plane S, and elements a, b, c, d, e, f, g, h of S. Suppose $c, d \not\leqslant a, b$ and $\overline{ab} \cong \overline{ef}$ and $\overline{cd} \cong \overline{gh}$. Then $e, f \leqslant g, h$. The theorem is a consequence of (64) and (60).

Now we state the proposition:

(95) 6.28 Satz, introduced by Beeson:
Let us consider Tarski plane S, and elements a, b, c, a_1, b_1, c_1 of S. Suppose b out a, c and b_1 out a_1, c_1 and $\overline{ba} \cong \overline{b_1a_1}$ and $\overline{bc} \cong \overline{b_1c_1}$. Then $\overline{ac} \cong \overline{a_1c_1}$. The theorem is a consequence of (7), (6), (42), (94), (93), and (14).

6. Midpoint

Let S be a Tarski plane and a, b, m be points of S. We say that Middlea, m, b if and only if

(Def. 12) m lies between a and b and $\overline{ma} \cong \overline{mb}$.

From now on S denotes Tarski plane and a, b, m denote points of S.

Now we state the proposition:

(96) 7.2 Satz:
If Middlea, m, b, then Middleb, m, a.

From now on S denotes Tarski plane and a, b, m denote points of S.

Now we state the proposition:

(97) 7.3 Satz:
Middlea, m, a if and only if $m = a$.

Now we state the proposition:

(98) 7.4 Existence:
Let us consider a point p of S. Then there exists a point p' of S such that Middlep, a, p'. The theorem is a consequence of (7), (3), and (97).

From now on S denotes Tarski plane and a denotes a point of S.

Now we state the proposition:

(99) 7.4 Uniqueness:
Let us consider points p, p_1, p_2 of S. If Middlep, a, p_1 and Middlep, a, p_2, then $p_1 = p_2$. The theorem is a consequence of (3) and (12).

Let S be Tarski plane and a, p be points of S. The functor reflection(a, p) yielding a point of S is defined by

(Def. 13) Middlep, a, it.

From now on S denotes Tarski plane and a, p, p' denote points of S.

Now we state the proposition:

(100) 7.6 Satz:
reflection$(a, p) = p'$ if and only if Middlep, a, p'.

From now on S denotes Tarski plane and a, p, p' denote points of S.

Now we state the proposition:

(101) 7.7 Satz:
reflection$(a, (\text{reflection}(a, p))) = p$. The theorem is a consequence of (14) and (3).

Now we state the proposition:

(102) 7.8 Satz:
There exists p such that reflection$(a, p) = p'$. The theorem is a consequence of (101).

Now we state the proposition:

(103) 7.9 Satz:
If reflection$(a, p) =$ reflection(a, p'), then $p = p'$. The theorem is a consequence of (101).

From now on S denotes Tarski plane and a, p denote points of S.

Now we state the proposition:

(104) 7.10 Satz:
reflection$(a, p) = p$ if and only if $p = a$. The theorem is a consequence of (13) and (1).

From now on S denotes Tarski plane and a, b, c, d, m, p, p', q, r, s denote points of S.

Now we state the proposition:

(105) 7.13 Satz:
$\overline{pq} \cong \overline{\text{reflection}(a, p)\,\text{reflection}(a, q)}$. The theorem is a consequence of (104), (14), (26), (28), (3), (6), (7), (11), (5), (1), and (41).

Now we state the proposition:

(106) 7.15 Satz:
q lies between p and r if and only if reflection(a, q) lies between reflection(a, p) and reflection(a, r). The theorem is a consequence of (101).

Now we state the proposition:

(107) 7.16 Satz:
$\overline{pq} \cong \overline{rs}$ if and only if $\overline{\text{reflection}(a, p)\,\text{reflection}(a, q)} \cong \overline{\text{reflection}(a, r)\,\text{reflection}(a, s)}$. The theorem is a consequence of (101).

Now we state the proposition:

(108) 7.17 SATZ:

If Middlep, a, p' and Middlep, b, p', then $a = b$. The theorem is a consequence of (105), (101), (5), (6), (7), (55), and (104).

Now we state the proposition:

(109) 7.18 SATZ:

If reflection(a, p) = reflection(b, p), then $a = b$. The theorem is a consequence of (108).

Now we state the proposition:

(110) 7.19 SATZ:

reflection$(b,$ (reflection$(a, p)))$ = reflection$(a,$ (reflection$(b, p)))$ if and only if $a = b$. The theorem is a consequence of (106), (107), (101), (108), and (104).

Now we state the proposition:

(111) 7.20 SATZ:

If a, m and b are collinear and $\overline{ma} \cong \overline{mb}$, then $a = b$ or Middlea, m, b. The theorem is a consequence of (14), (13), (7), (6), (1), (42), and (3).

From now on S denotes a non empty Tarski plane satisfying seven Tarski's geometry axioms and a, b, c, d, p denote points of S.

Now we state the proposition:

(112) 7.21 SATZ:

Suppose a, b and c are not collinear and $b \neq d$ and $\overline{ab} \cong \overline{cd}$ and $\overline{bc} \cong \overline{da}$ and a, p and c are collinear and b, p and d are collinear. Then

(i) Middlea, p, c, and

(ii) Middleb, p, d.

The theorem is a consequence of (14), (51), (48), (7), (6), (3), (52), (13), (83), (88), and (111).

From now on a_1, a_2, b_1, b_2, m_1, m_2 denote points of S.

Now we state the proposition:

(113) 7.22 SATZ, PART 1:

Suppose c lies between a_1 and a_2 and c lies between b_1 and b_2 and $\overline{ca_1} \cong \overline{cb_1}$ and $\overline{ca_2} \cong \overline{cb_2}$ and Middlea_1, m_1, b_1 and Middlea_2, m_2, b_2 and $c, a_1 \leqslant c, a_2$. Then c lies between m_1 and m_2. The theorem is a consequence of (59), (3), (13), (1), (105), (104), (60), (14), (103), (56), (80), (106), (40), (107), (7), (6), (41), (53), and (108).

Now we state the proposition:

(114) 7.22 SATZ, PART 2:

Suppose c lies between a_1 and a_2 and c lies between b_1 and b_2 and $\overline{ca_1} \cong \overline{cb_1}$ and $\overline{ca_2} \cong \overline{cb_2}$ and Middlea_1, m_1, b_1 and Middlea_2, m_2, b_2 and $c, a_2 \leqslant c, a_1$.

Then c lies between m_1 and m_2. The theorem is a consequence of (59), (3), (13), (14), (1), (105), (104), (60), (103), (56), (80), (106), (40), (107), (7), (6), (41), (53), and (108).

Now we state the proposition:

(115) 7.22 Satz: Krippenlemma, (Gupta 1965, 3.45 Theorem): Suppose c lies between a_1 and a_2 and c lies between b_1 and b_2 and $\overline{ca_1} \cong \overline{cb_1}$ and $\overline{ca_2} \cong \overline{cb_2}$ and Middlea_1, m_1, b_1 and Middlea_2, m_2, b_2. Then c lies between m_1 and m_2. The theorem is a consequence of (64), (113), and (114).

Let S be a Tarski plane and a_1, a_2, b_1, b_2, c, m_1, m_2 be points of S. We say that Krippenfigur$a_1, m_1, b_1, c, b_2, m_2, a_2$ if and only if

(Def. 14) c lies between a_1 and a_2 and c lies between b_1 and b_2 and $\overline{ca_1} \cong \overline{cb_1}$ and $\overline{ca_2} \cong \overline{cb_2}$ and Middlea_1, m_1, b_1 and Middlea_2, m_2, b_2.

Now we state the proposition:

(116) Krippenfigur: If Krippenfigur$a_1, m_1, b_1, c, b_2, m_2, a_2$, then c lies between m_1 and m_2.

Let us observe that there exists Tarski plane which is non empty.

In the sequel S denotes a non empty Tarski plane satisfying Lower Dimension Axiom and seven Tarski's geometry axioms and a, b, c, p, q, r denote points of S.

Now we state the proposition:

(117) If $\overline{ca} \cong \overline{cb}$, then there exists a point x of S such that Middlea, x, b. The theorem is a consequence of (14), (111), (13), (1), (36), (3), (7), (10), (6), (43), (41), (48), (88), (83), (87), and (53).

7. Note about Makarios's Simplification of Tarski's Axiom of Geometry

Let S be a Tarski plane. We say that S is (RE) if and only if

(Def. 15) for every points a, b of S, $\overline{ab} \cong \overline{ba}$.

We say that S is (TE) if and only if

(Def. 16) for every points a, b, p, q, r, s of S such that $\overline{ab} \cong \overline{pq}$ and $\overline{ab} \cong \overline{rs}$ holds $\overline{pq} \cong \overline{rs}$.

We say that S is (IE) if and only if

(Def. 17) for every points a, b, c of S such that $\overline{ab} \cong \overline{cc}$ holds $a = b$.

We say that S is (SC) if and only if

(Def. 18) for every points a, b, c, q of S, there exists a point x of S such that a lies between q and x and $\overline{ax} \cong \overline{bc}$.

We say that **S is (FS)** if and only if

(Def. 19) for every points a, b, c, d, a', b', c', d' of S such that $a \neq b$ and b lies between a and c and b' lies between a' and c' and $\overline{ab} \cong \overline{a'b'}$ and $\overline{bc} \cong \overline{b'c'}$ and $\overline{ad} \cong \overline{a'd'}$ and $\overline{bd} \cong \overline{b'd'}$ holds $\overline{cd} \cong \overline{c'd'}$.

We say that **S is (IB)** if and only if

(Def. 20) for every points a, b of S such that b lies between a and a holds $a = b$.

We say that **S is (Pa)** if and only if

(Def. 21) for every points a, b, c, p, q of S such that p lies between a and c and q lies between b and c there exists a point x of S such that x lies between p and b and x lies between q and a.

We say that **S is (Lo2)** if and only if

(Def. 22) there exist points a, b, c of S such that b does not lie between a and c and c does not lie between b and a and a does not lie between c and b.

We say that **S is (Up2)** if and only if

(Def. 23) for every points a, b, c, p, q of S such that $p \neq q$ and $\overline{ap} \cong \overline{aq}$ and $\overline{bp} \cong \overline{bq}$ and $\overline{cp} \cong \overline{cq}$ holds b lies between a and c or c lies between b and a or a lies between c and b.

We say that **S is (Eu)** if and only if

(Def. 24) for every points a, b, c, d, t of S such that d lies between a and t and d lies between b and c and $a \neq d$ there exist points x, y of S such that b lies between a and x and c lies between a and y and t lies between x and y.

We say that **S is (Co)** if and only if

(Def. 25) for every sets X, Y such that there exists a point a of S such that for every points x, y of S such that $x \in X$ and $y \in Y$ holds x lies between a and y there exists a point b of S such that for every points x, y of S such that $x \in X$ and $y \in Y$ holds b lies between x and y.

We say that **S is (FS')** if and only if

(Def. 26) for every points a, b, c, d, a', b', c', d' of S such that $a \neq b$ and b lies between a and c and b' lies between a' and c' and $\overline{ab} \cong \overline{a'b'}$ and $\overline{bc} \cong \overline{b'c'}$ and $\overline{ad} \cong \overline{a'd'}$ and $\overline{bd} \cong \overline{b'd'}$ holds $\overline{dc} \cong \overline{c'd'}$.

In the sequel S denotes a Tarski plane.

Now we state the propositions:

(118) S satisfies the axiom of congruence symmetry if and only if S is (RE).

(119) S satisfies the axiom of congruence equivalence relation if and only if S is (TE).

(120) S satisfies the axiom of congruence identity if and only if S is (IE).

(121) S satisfies the axiom of segment construction if and only if S is (SC).

(122) S satisfies the axiom of betweenness identity if and only if S is (IB).

(123) S satisfies the axiom of Pasch if and only if S is (Pa).

(124) S satisfies Lower Dimension Axiom if and only if S is (Lo2).

(125) S satisfies Upper Dimension Axiom if and only if S is (Up2).

(126) S satisfies Euclid Axiom if and only if S is (Eu).

(127) Let us consider Tarski plane S. Then S satisfies the axiom of SAS if and only if S is (FS).

(128) Let us consider a non empty Tarski plane S. Then S satisfies Continuity Axiom if and only if S is (Co).

One can verify that every Tarski plane which is (RE) satisfies also the axiom of congruence symmetry and every Tarski plane which is (TE) satisfies also the axiom of congruence equivalence relation and every Tarski plane which is (IE) satisfies also the axiom of congruence identity and every Tarski plane which is (SC) satisfies also the axiom of segment construction and every Tarski plane which is (IB) satisfies also the axiom of betweenness identity and every Tarski plane which is (Pa) satisfies also the axiom of Pasch and every Tarski plane which is (Lo2) satisfies also Lower Dimension Axiom and every Tarski plane which is (Up2) satisfies also Upper Dimension Axiom and every Tarski plane which is (Eu) satisfies also Euclid Axiom and every Tarski plane which is (Co) satisfies also Continuity Axiom and every Tarski plane which satisfies the axiom of congruence symmetry is also (RE) and every Tarski plane which satisfies the axiom of congruence equivalence relation is also (TE) and every Tarski plane which satisfies the axiom of congruence identity is also (IE) and every Tarski plane which satisfies the axiom of segment construction is also (SC) and every Tarski plane which satisfies the axiom of betweenness identity is also (IB) and every Tarski plane which satisfies the axiom of Pasch is also (Pa) and every Tarski plane which satisfies Lower Dimension Axiom is also (Lo2) and every Tarski plane which satisfies Upper Dimension Axiom is also (Up2) and every Tarski plane which satisfies Euclid Axiom is also (Eu) and every non empty Tarski plane which satisfies Continuity Axiom is also (Co) and there exists a Tarski plane which is (RE) and (TE).

Now we state the proposition:

(129) Let us consider a (RE), (TE) Tarski plane S. Then S satisfies the axiom of SAS if and only if S is (FS).

One can check that every (RE), (TE) Tarski plane which is (FS) satisfies also the axiom of SAS and there exists a (RE), (TE) Tarski plane which is (FS).

From now on S denotes a Tarski plane.

Now we state the propositions:

(130) Makarios: Lemma 6:
Let us consider a Tarski plane S. If S is (RE) and (TE), then S is (FS) iff S is (FS').

(131) Let us consider a (RE), (TE) Tarski plane S. Then S is (FS) if and only if S is (FS').

Let us note that every (RE), (TE) Tarski plane which is (FS') is also (FS) and there exists a Tarski plane which is (TE) and (SC) and there exists a (RE), (TE) Tarski plane which is (FS') and there exists a (RE), (TE), (FS') Tarski plane which is (SC).

Now we state the propositions:

(132) Let us consider a (TE), (SC) Tarski plane S, and points a, b of S. Then $\overline{ab} \cong \overline{ab}$.

(133) Let us consider a (IE), (SC) Tarski plane S, and points a, b of S. Then b lies between a and b.

(134) Let us consider a (TE), (SC) Tarski plane S, and points a, b, c, d of S. If $\overline{ab} \cong \overline{cd}$, then $\overline{cd} \cong \overline{ab}$.

(135) Let us consider a (TE), (SC), (FS') Tarski plane S, and points a, b, c, d, e, f of S. Suppose $a \neq b$ and a lies between b and c and a lies between d and e and $\overline{ba} \cong \overline{da}$ and $\overline{ac} \cong \overline{ae}$ and $\overline{bf} \cong \overline{df}$. Then $\overline{fc} \cong \overline{ef}$. The theorem is a consequence of (2).

Let S be a Tarski plane. We say that S is (RE') if and only if

(Def. 27) for every points a, b, c, d of S such that $a \neq b$ and a lies between b and c holds $\overline{dc} \cong \overline{cd}$.

Now we state the proposition:

(136) Every (TE), (SC), (FS') Tarski plane is (RE'). The theorem is a consequence of (2) and (135).

Let us note that every Tarski plane which is (TE), (SC), and (FS') is also (RE') and there exists a (IE) Tarski plane which is (RE') and there exists a (RE'), (IE) Tarski plane which is (SC) and there exists a (IE), non empty Tarski plane which is trivial and there exists a (IE), (SC), non empty Tarski plane which is trivial.

Now we state the proposition:

(137) Every trivial, (IE), (SC), non empty Tarski plane is (RE). The theorem is a consequence of (8).

One can verify that there exists a (TE), (IE), (SC), non empty Tarski plane which is (RE').

Now we state the proposition:

(138) Every (RE'), (TE), (IE), (SC), non empty Tarski plane is (RE). The theorem is a consequence of (8), (13), and (4).

Note that there exists a (TE), (IE), (SC), non empty Tarski plane which is (FS').

Now we state the propositions:

(139) Every (TE), (IE), (SC), (FS'), non empty Tarski plane is (RE).

(140) Every (TE), (IE), (SC), (FS'), non empty Tarski plane is (FS). The theorem is a consequence of (138).

8. Main Results and Corollaries

Let us note that every Tarski plane which is (RE), (TE), and (FS) is also (FS') and every non empty Tarski plane which is (TE), (IE), (SC), and (FS') is also (FS) and every non empty Tarski plane which is (TE), (IE), (SC), and (FS') is also (RE) and every non empty Tarski plane which is (TE), (IE), (SC), and (FS') satisfies also the axiom of SAS and there exists a non empty Tarski plane which is (RE), (TE), (IE), (SC), (FS), (IB), (Pa), (Lo2), (Up2), (Eu), and (Co).

A MakariosCE2 is a (RE), (TE), (IE), (SC), (FS), (IB), (Pa), (Lo2), (Up2), (Eu), (Co), non empty Tarski plane.

A MakariosCE'2 is a (TE), (IE), (SC), (FS'), (IB), (Pa), (Lo2), (Up2), (Eu), (Co), non empty Tarski plane. Now we state the propositions:

(141) Every MakariosCE2 is a MakariosCE'2.

(142) Every MakariosCE'2 is a MakariosCE2.

(143) Every MakariosCE2 satisfies seven Tarski's geometry axioms, Lower Dimension Axiom, Upper Dimension Axiom, Euclid Axiom, and Continuity Axiom.

(144) Every MakariosCE'2 satisfies seven Tarski's geometry axioms, Lower Dimension Axiom, Upper Dimension Axiom, Euclid Axiom, and Continuity Axiom.

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