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Klein-Beltrami Model. Part I

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Summary. Makarios (with Isabelle/HOL[1]) and John Harrison (with HOL-Light [2]) shown that “the Klein-Beltrami model of the hyperbolic plane satisfy all of Tarski’s axioms except his Euclidean axiom” [4], [5], [22], [6].

With the Mizar system [3], [13] we use some ideas are taken from Tim Makarios’ MSc thesis [21] for formalized some definitions (like the absolute) and lemmas necessary for the verification of the independence of the parallel postulate. Note that the model presented here, may also be called “Beltrami-Klein Model”, “Klein disk model”, and the “Cayley-Klein model” [1].

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1. Preliminaries

From now on a, b, c, d, e, f denote real numbers, g denotes a positive real number, x, y denote complexes, S, T denote elements of $\mathcal{R}^2$, and u, v, w denote elements of $\mathcal{E}^3_{\mathrm{T}}$.

Now we state the propositions:

(1) Let us consider elements P_1, P_2, P_3 of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose u is not zero and v is not zero and w is not zero and $P_1 =$ the direction of u and $P_2 =$ the direction of v and $P_3 =$ the direction of w. Then P_1, P_2 and P_3 are collinear if and only if $\langle|u, v, w|\rangle = 0$.

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[1] `https://www.isa-afp.org/entries/Tarskis_Geometry.html`

[2] `https://github.com/jrh13/hol-light/blob/master/100/independence.ml`

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(2) If ($a \neq 0$ or $b \neq 0$) and $a \cdot d = b \cdot c$, then there exists e such that $c = e \cdot a$ and $d = e \cdot b$.

(3) If $a^{\mathbf{2}} + b^{\mathbf{2}} = 1$ and $c \cdot a^{\mathbf{2}} + c \cdot b^{\mathbf{2}} = 1$, then $c = 1$ or $c = -1$.

(4) $a \cdot u + (-a) \cdot u = 0_{\mathcal{E}^{3}_{\mathrm{T}}}$.

(5) If $0 \leqslant a$ and $c < 0$ and $\Delta(a, b, c) = 0$, then $a = 0$.
Proof: $0 \leqslant b^{\mathbf{2}}$ by [23, (12)]. □

(6) $\sum(^{2}(T - S)) = \sum(^{2}(S - T))$.

(7) If $a^{\mathbf{2}} + b^{\mathbf{2}} = 1$ and $c^{\mathbf{2}} + d^{\mathbf{2}} = 1$ and $c \cdot a + d \cdot b = 1$, then $b \cdot c = a \cdot d$.

(8) If $a^{\mathbf{2}} + b^{\mathbf{2}} = 1$ and $a = 0$, then $b = 1$ or $b = -1$.

(9) $0 \leqslant a^{\mathbf{2}}$.

(10) If $a \cdot b^{\mathbf{2}} + b^{\mathbf{2}} = 1$, then $b = \frac{1}{\sqrt{1+a^{\mathbf{2}}}}$ or $b = \frac{-1}{\sqrt{1+a^{\mathbf{2}}}}$.

(11) If $a \neq 0$ and $b^{\mathbf{2}} = 1 + a \cdot a$, then $a \cdot (\frac{1}{b}) \cdot a \cdot (\frac{-1}{b}) + (\frac{1}{b}) \cdot (\frac{-1}{b}) = -1$.
Proof: $b \neq 0$ by [23, (12)]. □

(12) $a^{\mathbf{2}} \cdot (\frac{1}{b^{\mathbf{2}}}) = (\frac{a}{b})^{\mathbf{2}}$.

(13) $a^{\mathbf{2}} + b^{\mathbf{2}} = 1$ if and only if $[a, b] \in \mathrm{circle}(0, 0, 1)$.

(14) $a^{\mathbf{2}} + b^{\mathbf{2}} = g^{\mathbf{2}}$ if and only if $[a, b] \in \mathrm{circle}(0, 0, g)$.

(15) If $a \neq 0$ and $-1 < a < 1$ and $b = \frac{2+\sqrt{\Delta(a \cdot a, -2, 1)}}{2 \cdot a \cdot a}$, then $(1 + a \cdot a) \cdot b \cdot b - 2 \cdot b + 1 - b \cdot b = 0$.
Proof: $0 \leqslant 1 - a^{\mathbf{2}}$ by [23, (53), (41)]. $\Delta(a \cdot a, -2, 1) \geqslant 0$. □

(16) Suppose $a^{\mathbf{2}} + b^{\mathbf{2}} = 1$ and $-1 < c < 1$. Then there exists d and there exists e and there exists f such that $e = d \cdot c \cdot a + (1 - d) \cdot (-b)$ and $f = d \cdot c \cdot b + (1 - d) \cdot a$ and $e^{\mathbf{2}} + f^{\mathbf{2}} = d^{\mathbf{2}}$.

(17) If $a^{\mathbf{2}} + b^{\mathbf{2}} < 1$ and $c^{\mathbf{2}} + d^{\mathbf{2}} = 1$, then $(\frac{a+c}{2})^{\mathbf{2}} + (\frac{b+d}{2})^{\mathbf{2}} < 1$.

(18) If $|S|^{\mathbf{2}} \leqslant 1$, then $0 \leqslant \Delta(\sum(^{2}(T - S)), b, \sum(^{2}S) - 1)$.

(19) If $a^{\mathbf{2}} + b^{\mathbf{2}}$ is negative, then $a = 0$ and $b = 0$.

(20) If $u = [a, b, 1]$ and $v = [c, d, 1]$ and $w = [\frac{a+c}{2}, \frac{b+d}{2}, 1]$, then $\langle|u, v, w|\rangle = 0$.

(21) (i) $a \cdot |(u, v)| = |(a \cdot u, v)|$, and
  (ii) $a \cdot |(u, v)| = |(u, a \cdot v)|$.

In the sequel a, b, c denote elements of $\mathbb{R}_{\mathrm{F}}$ and M, N denote square matrices over $\mathbb{R}_{\mathrm{F}}$ of dimension 3.

Now we state the propositions:

(22) If $M = \mathrm{symmetric3}(0, 0, 0, 0, 0, 0)$, then $\mathrm{Det}\, M = 0_{\mathbb{R}_{\mathrm{F}}}$.

(23) Suppose $N = \langle\langle a, 0, 0\rangle, \langle 0, b, 0\rangle, \langle 0, 0, c\rangle\rangle$. Then
  (i) $M^{\mathrm{T}} \cdot (N \cdot M)_{1,1} = a \cdot (M_{1,1}) \cdot (M_{1,1}) + b \cdot (M_{2,1}) \cdot (M_{2,1}) + c \cdot (M_{3,1}) \cdot (M_{3,1})$, and

(ii) $M^{\mathrm{T}} \cdot (N \cdot M)_{1,2} = a \cdot (M_{1,1}) \cdot (M_{1,2}) + b \cdot (M_{2,1}) \cdot (M_{2,2}) + c \cdot (M_{3,1}) \cdot (M_{3,2})$, and

(iii) $M^{\mathrm{T}} \cdot (N \cdot M)_{1,3} = a \cdot (M_{1,1}) \cdot (M_{1,3}) + b \cdot (M_{2,1}) \cdot (M_{2,3}) + c \cdot (M_{3,1}) \cdot (M_{3,3})$, and

(iv) $M^{\mathrm{T}} \cdot (N \cdot M)_{2,1} = a \cdot (M_{1,2}) \cdot (M_{1,1}) + b \cdot (M_{2,2}) \cdot (M_{2,1}) + c \cdot (M_{3,2}) \cdot (M_{3,1})$, and

(v) $M^{\mathrm{T}} \cdot (N \cdot M)_{2,2} = a \cdot (M_{1,2}) \cdot (M_{1,2}) + b \cdot (M_{2,2}) \cdot (M_{2,2}) + c \cdot (M_{3,2}) \cdot (M_{3,2})$, and

(vi) $M^{\mathrm{T}} \cdot (N \cdot M)_{2,3} = a \cdot (M_{1,2}) \cdot (M_{1,3}) + b \cdot (M_{2,2}) \cdot (M_{2,3}) + c \cdot (M_{3,2}) \cdot (M_{3,3})$, and

(vii) $M^{\mathrm{T}} \cdot (N \cdot M)_{3,1} = a \cdot (M_{1,3}) \cdot (M_{1,1}) + b \cdot (M_{2,3}) \cdot (M_{2,1}) + c \cdot (M_{3,3}) \cdot (M_{3,1})$, and

(viii) $M^{\mathrm{T}} \cdot (N \cdot M)_{3,2} = a \cdot (M_{1,3}) \cdot (M_{1,2}) + b \cdot (M_{2,3}) \cdot (M_{2,2}) + c \cdot (M_{3,3}) \cdot (M_{3,2})$, and

(ix) $M^{\mathrm{T}} \cdot (N \cdot M)_{3,3} = a \cdot (M_{1,3}) \cdot (M_{1,3}) + b \cdot (M_{2,3}) \cdot (M_{2,3}) + c \cdot (M_{3,3}) \cdot (M_{3,3})$.

(24) Let us consider natural numbers m, n, a square matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension m, and a matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension $m \times n$. Suppose $m > 0$. Then $M \cdot N$ is a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $m \times n$.

In the sequel D denotes a non empty set, d_1, d_2, d_3 denote elements of D, A denotes a matrix over D of dimension 1×3, and B denotes a matrix over D of dimension 3×1.

Now we state the propositions:

(25) Let us consider a square matrix M over D of dimension 1. Then $M^{\mathrm{T}} = M$.

(26) A^{T} is 3,1-size.

(27) $\langle\langle d_1, d_2, d_3 \rangle\rangle$ is a matrix over D of dimension 1×3.

(28) $\langle\langle d_1 \rangle, \langle d_2 \rangle, \langle d_3 \rangle\rangle$ is a matrix over D of dimension 3×1.

(29) $A = \langle\langle A_{1,1}, A_{1,2}, A_{1,3} \rangle\rangle$.
PROOF: Reconsider $B = \langle\langle A_{1,1}, A_{1,2}, A_{1,3} \rangle\rangle$ as a matrix over D of dimension 1×3. For every natural numbers i, j such that $\langle i, j \rangle \in$ the indices of A holds $A_{i,j} = B_{i,j}$ by [9, (87)], [2, (2)], [24, (1)], [2, (40), (45)]. □

(30) $B = \langle\langle B_{1,1} \rangle, \langle B_{2,1} \rangle, \langle B_{3,1} \rangle\rangle$.
PROOF: Reconsider $C = \langle\langle B_{1,1} \rangle, \langle B_{2,1} \rangle, \langle B_{3,1} \rangle\rangle$ as a matrix over D of dimension 3×1. For every natural numbers i, j such that $\langle i, j \rangle \in$ the indices of B holds $B_{i,j} = C_{i,j}$ by [9, (87)], [2, (2)], [24, (1)], [2, (45), (40)]. □

(31) $A^{\mathrm{T}} = \langle\langle A_{1,1} \rangle, \langle A_{1,2} \rangle, \langle A_{1,3} \rangle\rangle$. The theorem is a consequence of (26) and (30).

(32) There exists d_1 and there exists d_2 and there exists d_3 such that $A = \langle\langle d_1, d_2, d_3\rangle\rangle$. The theorem is a consequence of (29).

(33) Let us consider a finite sequence p of elements of $\mathbb{R}^1$. If $\operatorname{len} p = 3$, then $\mathrm{ColVec2Mx}(\mathrm{M2F}(p)) = p$. The theorem is a consequence of (30).

(34) Let us consider a square matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, a square matrix M_1 over $\mathbb{R}$ of dimension 3, an element v of $\mathcal{E}^3_{\mathrm{T}}$, a finite sequence u_1 of elements of $\mathbb{R}_{\mathrm{F}}$, a finite sequence u_2 of elements of $\mathbb{R}$, and a finite sequence p of elements of $\mathbb{R}^1$. Suppose $p = M \cdot u_1$ and $v = \mathrm{M2F}(p)$ and $\operatorname{len} u_1 = 3$ and $u_1 = u_2$ and $M_1 = M$. Then $v = M_1 \cdot u_2$.

(35) Let us consider a square matrix N over $\mathbb{R}$ of dimension 3, and a finite sequence u_1 of elements of $\mathbb{R}$. If $u_1 = 0_{\mathcal{E}^3_{\mathrm{T}}}$, then $N \cdot u_1 = 0_{\mathcal{E}^3_{\mathrm{T}}}$.

(36) Let us consider a square matrix N over $\mathbb{R}$ of dimension 3, a finite sequence u_1 of elements of $\mathbb{R}$, and an element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose N is invertible and $u = u_1$ and u is not zero. Then $N \cdot u_1 \neq 0_{\mathcal{E}^3_{\mathrm{T}}}$. The theorem is a consequence of (35).

(37) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, a square matrix N_2 over $\mathbb{R}$ of dimension 3, elements P, Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$, and finite sequences v_1, u_2 of elements of $\mathbb{R}$. Suppose $P =$ the direction of u and $Q =$ the direction of v and $u = u_2$ and $v = v_1$ and $N = N_2$ and $N_2 \cdot u_2 = v_1$. Then (the homography of $N)(P) = Q$. The theorem is a consequence of (34).

(38) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, a square matrix N_2 over $\mathbb{R}$ of dimension 3, elements P, Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$, finite sequences v_1, u_2 of elements of $\mathbb{R}$, and a non zero real number a. Suppose $P =$ the direction of u and $Q =$ the direction of v and $u = u_2$ and $v = v_1$ and $N = N_2$ and $N_2 \cdot u_2 = a \cdot v_1$. Then (the homography of $N)(P) = Q$. The theorem is a consequence of (34) and (36).

Let us consider a finite sequence p of elements of $\mathbb{R}$ and a square matrix M over $\mathbb{R}$ of dimension 3. Now we state the propositions:

(39) If $\operatorname{len} p = 3$, then $|(a \cdot p, M \cdot (b \cdot p))| = a \cdot b \cdot |(p, M \cdot p)|$.

(40) If $\operatorname{len} p = 3$, then SumAll QuadraticForm$(a\cdot p, M, b\cdot p) = a\cdot b\cdot($SumAll QuadraticForn
The theorem is a consequence of (39).

Now we state the propositions:

(41) Let us consider real numbers a, b. Then $[a, b, 1]$ is not zero.

(42) Let us consider an element P of $\mathcal{E}^2_{\mathrm{T}}$, an element Q of $\mathcal{E}^2_{\mathrm{T}}$, and a real number r. Then $P \in \mathrm{Sphere}(Q, r)$ if and only if $P \in \mathrm{circle}(Q(1), Q(2), r)$.

In the sequel u, v denote non zero elements of $\mathcal{E}^3_{\mathrm{T}}$.

Now we state the proposition:

(43) If the direction of u = the direction of v and $u(3) = v(3)$ and $v(3) \neq 0$, then $u = v$.

The functor Dir101 yielding a point of the projective space over $\mathcal{E}^3_\mathrm{T}$ is defined by the term

(Def. 1) the direction of $[1, 0, 1]$.

The functor Dirm101 yielding a point of the projective space over $\mathcal{E}^3_\mathrm{T}$ is defined by the term

(Def. 2) the direction of $[-1, 0, 1]$.

The functor Dir011 yielding a point of the projective space over $\mathcal{E}^3_\mathrm{T}$ is defined by the term

(Def. 3) the direction of $[0, 1, 1]$.

Now we state the propositions:

(44) (i) Dir101, Dirm101 and Dir011 are not collinear, and

(ii) Dir101, Dirm101 and Dir010 are not collinear, and

(iii) Dir101, Dir011 and Dir010 are not collinear, and

(iv) Dirm101, Dir011 and Dir010 are not collinear.

PROOF: Dir101, Dirm101 and Dir011 are not collinear by [14, (2), (4)], [2, (78)], (1). Dir101, Dirm101 and Dir010 are not collinear by [14, (2), (4)], [2, (78)], (1). Dir101, Dir011 and Dir010 are not collinear by [14, (2), (4)], [2, (78)], (1). Dirm101, Dir011 and Dir010 are not collinear by [14, (2), (4)], [2, (78)], (1). □

(45) symmetric3$(1, 1, 1, 0, 0, 0) = I^{3\times 3}_{\mathbb{R}_\mathrm{F}}$.

(46) Let us consider elements r, a, b, c, d, e, f, g, h, i of $\mathbb{R}_\mathrm{F}$, and a square matrix M over $\mathbb{R}_\mathrm{F}$ of dimension 3. Suppose $M = \langle\langle a, b, c\rangle, \langle d, e, f\rangle, \langle g, h, i\rangle\rangle$. Then $r \cdot M = \langle\langle r \cdot a, r \cdot b, r \cdot c\rangle, \langle r \cdot d, r \cdot e, r \cdot f\rangle, \langle r \cdot g, r \cdot h, r \cdot i\rangle\rangle$.

(47) Let us consider a real number a, and an element r of $\mathbb{R}_\mathrm{F}$. Suppose $r = a \cdot a$. Then (symmetric3$(a, a, -a, 0, 0, 0)) \cdot ($symmetric3$(a, a, -a, 0, 0, 0)) = r \cdot (I^{3\times 3}_{\mathbb{R}_\mathrm{F}})$. The theorem is a consequence of (46).

Let us consider a non zero real number a. Now we state the propositions:

(48) (symmetric3$(a, a, -a, 0, 0, 0)) \cdot ($symmetric3$(\frac{1}{a}, \frac{1}{a}, -\frac{1}{a}, 0, 0, 0)) = I^{3\times 3}_{\mathbb{R}_\mathrm{F}}$.

(49) (symmetric3$(\frac{1}{a}, \frac{1}{a}, -\frac{1}{a}, 0, 0, 0)) \cdot ($symmetric3$(a, a, -a, 0, 0, 0)) = I^{3\times 3}_{\mathbb{R}_\mathrm{F}}$. The theorem is a consequence of (48).

Now we state the propositions:

(50) (symmetric3$(1, 1, -1, 0, 0, 0)) \cdot ($symmetric3$(1, 1, -1, 0, 0, 0)) = I^{3\times 3}_{\mathbb{R}_\mathrm{F}}$. The theorem is a consequence of (48).

(51) Let us consider a non zero real number a, and a square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3. If $N = \text{symmetric3}(a, a, -a, 0, 0, 0)$, then N is invertible. The theorem is a consequence of (48) and (49).

(52) (i) $\text{symmetric3}(1, 1, -1, 0, 0, 0)$ is an invertible square matrix over $\mathbb{R}_{\rm F}$ of dimension 3, and

(ii) $(\text{symmetric3}(1, 1, -1, 0, 0, 0))^{\smile} = \text{symmetric3}(1, 1, -1, 0, 0, 0)$.

The theorem is a consequence of (50).

(53) Let us consider an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3, a square matrix N_1 over $\mathbb{R}_{\rm F}$ of dimension 3, and square matrices M, N_2 over $\mathbb{R}$ of dimension 3. Suppose $M = \text{symmetric3}(1, 1, -1, 0, 0, 0)$ and $N_1 = M$ and $N_2 = (\mathbb{R}_{\rm F} \to \mathbb{R})N^{\smile}$. Then $N^{\rm T} \cdot N_1 \cdot N = ((\mathbb{R}_{\rm F} \to \mathbb{R})((\mathbb{R} \to \mathbb{R}_{\rm F})N_2{}^{\rm T})^{\smile}) \cdot M \cdot ((\mathbb{R}_{\rm F} \to \mathbb{R})((\mathbb{R} \to \mathbb{R}_{\rm F})N_2)^{\smile})$.

PROOF: $(\mathbb{R}_{\rm F} \to \mathbb{R})((\mathbb{R} \to \mathbb{R}_{\rm F})N_2{}^{\rm T})^{\smile} = N^{\rm T}$ by [25, (13), (16)]. □

(54) Let us consider a natural number n, an element a of $\mathbb{R}_{\rm F}$, a real number r, a square matrix A over $\mathbb{R}_{\rm F}$ of dimension n, and a square matrix r_1 over $\mathbb{R}$ of dimension n. If $a = r$ and $A = r_1$, then $a \cdot A = r \cdot r_1$.

(55) Let us consider a natural number n, an element a of $\mathbb{R}_{\rm F}$, and square matrices A, B over $\mathbb{R}_{\rm F}$ of dimension n. If $n > 0$, then $(a \cdot A) \cdot B = a \cdot (A \cdot B)$. The theorem is a consequence of (54).

(56) $\text{symmetric3}(a, a, -a, 0, 0, 0) = a \cdot (\text{symmetric3}(1, 1, -1, 0, 0, 0))$. The theorem is a consequence of (46).

(57) If $M = \text{symmetric3}(a, a, -a, 0, 0, 0)$, then $M \cdot M \cdot M = a \cdot a \cdot a \cdot (\text{symmetric3}(1, 1, -1, 0,$
The theorem is a consequence of (47), (55), and (56).

Let us consider a natural number n, a real number a, a square matrix M over $\mathbb{R}$ of dimension n, and a finite sequence x of elements of $\mathbb{R}$. Now we state the propositions:

(58) If $n > 0$ and $\operatorname{len} x = n$, then $M \cdot (a \cdot x) = (a \cdot M) \cdot x$.

(59) If $n > 0$ and $\operatorname{len} x = n$, then $a \cdot (M \cdot x) = (a \cdot M) \cdot x$. The theorem is a consequence of (58).

Now we state the propositions:

(60) Let us consider a natural number n, and a square matrix N over $\mathbb{R}$ of dimension n. Suppose N is invertible. Then

(i) $N^{\rm T}$ is invertible, and

(ii) $\operatorname{Inv} N^{\rm T} = (\operatorname{Inv} N)^{\rm T}$.

(61) Let us consider a non zero real number r, and matrices N, O, M over $\mathbb{R}$ of dimension 3×3. Suppose N is invertible and $M = r \cdot O$ and $M = N^{\rm T} \cdot O \cdot N$. Then $(\operatorname{Inv} N)^{\rm T} \cdot O \cdot (\operatorname{Inv} N) = (\frac{1}{r}) \cdot O$. The theorem is a consequence of (60).

(62) Let us consider a real number r, square matrices M, N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, and square matrices M_1, N_2 over $\mathbb{R}$ of dimension 3. Suppose $M_1 = M$ and $N_2 = N$ and N is symmetric and $M_1 = r \cdot N_2$. Then M is symmetric.

Let us consider a real number r and square matrices O, M over $\mathbb{R}$ of dimension 3. Now we state the propositions:

(63) Suppose $O = \mathrm{symmetric3}(1, 1, -1, 0, 0, 0)$ and $M = r \cdot O$. Then

(i) $O \cdot M = r \cdot (1_{\mathbb{R}}\, \mathrm{matrix}(3))$, and

(ii) $M \cdot O = r \cdot (1_{\mathbb{R}}\, \mathrm{matrix}(3))$.

The theorem is a consequence of (50).

(64) If $O = \mathrm{symmetric3}(1, 1, -1, 0, 0, 0)$ and $M = r \cdot O$, then $M^{\mathrm{T}} \cdot O^{\mathrm{T}} \cdot O \cdot (M^{\mathrm{T}} \cdot O) = r^{\mathbf{2}} \cdot O$.

PROOF: Reconsider $M_1 = M$ as a square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. M_1 is symmetric. $r \cdot (1_{\mathbb{R}}\, \mathrm{matrix}(3)) \cdot O \cdot (r \cdot (1_{\mathbb{R}}\, \mathrm{matrix}(3))) = r^{\mathbf{2}} \cdot O$ by [10, (16)], [11, (1)], (46), [10, (19)]. □

Now we state the propositions:

(65) Let us consider square matrices O, N over $\mathbb{R}$ of dimension 3. Then $N^{\mathrm{T}} \cdot O^{\mathrm{T}} \cdot O \cdot (N^{\mathrm{T}} \cdot O) = (O^{\mathrm{T}} \cdot (N \cdot O \cdot (N^{\mathrm{T}}))) \cdot O$.

(66) Let us consider square matrices N_2, M_1 over $\mathbb{R}$ of dimension 3, and finite sequences p_1, p_2, p_3 of elements of $\mathbb{R}$. Suppose $p_1 = \langle 1, 0, 0\rangle$ and $p_2 = \langle 0, 1, 0\rangle$ and $p_3 = \langle 0, 0, 1\rangle$ and $N_2 \cdot p_1 = M_1 \cdot p_1$ and $N_2 \cdot p_2 = M_1 \cdot p_2$ and $N_2 \cdot p_3 = M_1 \cdot p_3$. Then $N_2 = M_1$.

(67) Let us consider a non zero real number a, and an element u of $\mathcal{E}^3_{\mathrm{T}}$. If $a \cdot u = 0_{\mathcal{E}^3_{\mathrm{T}}}$, then u is zero.

(68) Let us consider non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$, and real numbers a, b. Suppose ($a \neq 0$ or $b \neq 0$) and $a \cdot u + b \cdot v = 0_{\mathcal{E}^3_{\mathrm{T}}}$. Then u and v are proportional.

PROOF: Reconsider $a_1 = a \cdot u$, $b_1 = b \cdot v$ as an element of $\mathcal{E}^3_{\mathrm{T}}$. Consider c being a real number such that $c \neq 0$ and $a_1 = c \cdot b_1$. $a \neq 0$ and $b \neq 0$ by [12, (22)], [17, (3), (1)], (67). □

(69) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, and points P, Q, R of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P \neq Q$ and (the homography of N)(P) $= Q$ and (the homography of N)(Q) $= P$ and P, Q and R are collinear. Then (the homography of N)((the homography of N)(R)) $= R$.

PROOF: Consider u_1, v_1 being elements of $\mathcal{E}^3_{\mathrm{T}}$, u_4 being a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$, p_1 being a finite sequence of elements of $\mathbb{R}^1$ such that $P =$ the direction of u_1 and u_1 is not zero and $u_1 = u_4$ and $p_1 = N \cdot u_4$

and $v_1 = \mathrm{M2F}(p_1)$ and v_1 is not zero and (the homography of N)$(P) =$ the direction of v_1. Consider u_2, v_2 being elements of $\mathcal{E}^3_{\mathrm{T}}$, u_5 being a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$, p_2 being a finite sequence of elements of $\mathbb{R}^1$ such that $Q =$ the direction of u_2 and u_2 is not zero and $u_2 = u_5$ and $p_2 = N \cdot u_5$ and $v_2 = \mathrm{M2F}(p_2)$ and v_2 is not zero and (the homography of N)$(Q) =$ the direction of v_2. Consider u_3 being an element of $\mathcal{E}^3_{\mathrm{T}}$ such that u_3 is not zero and $R =$ the direction of u_3. Consider l_1 being a real number such that $l_1 \neq 0$ and $v_2 = l_1 \cdot u_1$. Consider l_2 being a real number such that $l_2 \neq 0$ and $v_1 = l_2 \cdot u_2$. $\langle|u_1, u_2, u_3|\rangle = 0$. Consider a, b, c being real numbers such that $a \cdot u_1 + b \cdot u_2 + c \cdot u_3 = 0_{\mathcal{E}^3_{\mathrm{T}}}$ and ($a \neq 0$ or $b \neq 0$ or $c \neq 0$). $c \neq 0$ by [12, (22)], [17, (3), (1)], [8, (15)]. (The homography of $N \cdot N$)$(R) = R$. □

(70) Let us consider a natural number n, elements u, v of $\mathcal{E}^n_{\mathrm{T}}$, and real numbers a, b. If $(1-a) \cdot u + a \cdot v = (1-b) \cdot v + b \cdot u$, then $(1-(a+b)) \cdot u = (1-(a+b)) \cdot v$.
PROOF: Reconsider $r_1 = u$, $r_2 = v$ as an element of $\mathcal{R}^n$. $(1-a) \cdot r_1 + a \cdot r_2 - a \cdot r_2 = (1-a) \cdot r_1$ by [8, (42)]. $(1-b) \cdot r_2 - a \cdot r_2 + b \cdot r_1 - b \cdot r_1 = (1-b) \cdot r_2 - a \cdot r_2$ by [8, (42)]. □

(71) The projective space over $\mathcal{E}^3_{\mathrm{T}}$ is proper.

The real projective plane yielding a non empty, proper projective plane defined in terms of collinearity is defined by the term

(Def. 4) the projective space over $\mathcal{E}^3_{\mathrm{T}}$.

From now on P, Q, R denote points of Inc-ProjSp(the real projective plane), L denotes a line of Inc-ProjSp(the real projective plane), and p, q, r denote points of the real projective plane.

Now we state the propositions:

(72) Let us consider an element L of L(the real projective plane). Then there exists p and there exists q such that $p \neq q$ and $L = \mathrm{Line}(p, q)$.

(73) There exists p and there exists q such that $p \neq q$ and $L = \mathrm{Line}(p, q)$.

(74) If $R = r$ and $L = \mathrm{Line}(p, q)$, then R lies on L iff p, q and r are collinear.

(75) Inc-ProjSp(the real projective plane) is an incidence projective plane.
PROOF: Inc-ProjSp(the real projective plane) is 2-dimensional by (73), [19, (3)], (74). □

(76) Let us consider lines L_1, L_2 of the real projective plane. Then L_1 meets L_2. The theorem is a consequence of (75).

In the sequel u, v, w denote non zero elements of $\mathcal{E}^3_{\mathrm{T}}$.

Now we state the propositions:

(77) Suppose $p =$ the direction of u and $q =$ the direction of v and $R =$ the direction of w and $L = \mathrm{Line}(p, q)$. Then R lies on L if and only if

$\langle|u, v, w|\rangle = 0$. The theorem is a consequence of (74).

(78) Let us consider elements p, q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose $p \neq q$ and $p =$ the direction of u and $q =$ the direction of v. Then $u \times v$ is not zero.

Let p, q be points of the real projective plane. Assume $p \neq q$. The functor $\mathrm{L2P}(p, q)$ yielding a point of the real projective plane is defined by

(Def. 5) there exist non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$ such that $p =$ the direction of u and $q =$ the direction of v and *it* = the direction of $u \times v$.

Now we state the propositions:

(79) Let us consider points p, q of the real projective plane. Suppose $p \neq q$. Then

(i) $\mathrm{L2P}(q, p) = \mathrm{L2P}(p, q)$, and

(ii) $p \neq \mathrm{L2P}(p, q)$.

Proof: Consider u_1, v_1 being non zero elements of $\mathcal{E}^3_{\mathrm{T}}$ such that $p =$ the direction of u_1 and $q =$ the direction of v_1 and $\mathrm{L2P}(p, q) =$ the direction of $u_1 \times v_1$. Consider u_2, v_2 being non zero elements of $\mathcal{E}^3_{\mathrm{T}}$ such that $q =$ the direction of u_2 and $p =$ the direction of v_2 and $\mathrm{L2P}(q, p) =$ the direction of $u_2 \times v_2$. Consider a being a real number such that $a \neq 0$ and $u_1 = a \cdot v_2$. Consider b being a real number such that $b \neq 0$ and $v_1 = b \cdot u_2$. $a \cdot v_2 \times b \cdot u_2 = (-a \cdot b) \cdot (u_2 \times v_2)$ by [8, (44)], [14, (8)]. $u_1 \times v_1$ is not zero by [10, (51)], [18, (22)]. $u_2 \times v_2$ is not zero by [10, (51)], [18, (22)]. $p \neq \mathrm{L2P}(p, q)$ by [18, (22), (1)], (21), [10, (44)]. □

(80) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Then dom(the homography of N) = the projective points over $\mathcal{E}^3_{\mathrm{T}}$.

2. Absolute

Let a, b, c, d, e, f be real numbers. The interior of the conic for a, b, c, d, e and f yielding a subset of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 6) $\{P$, where P is a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$: for every element u of $\mathcal{E}^3_{\mathrm{T}}$ such that u is not zero and $P =$ the direction of u holds $\mathrm{qfconic}(a, b, c, d, e, f, u)$ is negative$\}$.

Now we state the proposition:

(81) Let us consider real numbers a, b, c, d, e, f, and non zero elements u_1, u_2 of $\mathcal{E}^3_{\mathrm{T}}$. Suppose the direction of u_1 = the direction of u_2 and $\mathrm{qfconic}(a, b, c, d, e, f, u_1)$ is negative. Then $\mathrm{qfconic}(a, b, c, d, e, f, u_2)$ is negative.

The absolute yielding a non empty subset of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 7) $\operatorname{conic}(1, 1, -1, 0, 0, 0)$.

Now we state the proposition:

(82) Let us consider a square matrix O over $\mathbb{R}$ of dimension 3, an element P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a finite sequence p of elements of $\mathbb{R}$. Suppose $O = \operatorname{symmetric3}(1, 1, -1, 0, 0, 0)$ and $P =$ the direction of u and $u = p$. Then $P \in$ the absolute if and only if $\operatorname{SumAll} \operatorname{QuadraticForm}(p, O, p) = 0$. The theorem is a consequence of (40).

Let us consider an element P of the absolute. Now we state the propositions:

(83) If $P =$ the direction of u, then $u(3) \neq 0$.
PROOF: Consider Q being a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that $P = Q$ and for every element u of $\mathcal{E}^3_{\mathrm{T}}$ such that u is not zero and $Q =$ the direction of u holds $\operatorname{qfconic}(1, 1, -1, 0, 0, 0, u) = 0$. $u(3) \neq 0$ by [7, (1)], [14, (3), (4)]. □

(84) If $P =$ the direction of u and $u(3) = 1$, then $[u(1), u(2)] \in \operatorname{circle}(0, 0, 1)$. The theorem is a consequence of (13).

Now we state the propositions:

(85) Let us consider a point P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P =$ the direction of u and $u(3) = 1$ and $[u(1), u(2)] \in \operatorname{circle}(0, 0, 1)$. Then P is an element of the absolute.

(86) Let us consider a point P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P =$ the direction of u and $u(3) = 1$. Then $[u(1), u(2)] \in \operatorname{circle}(0, 0, 1)$ if and only if P is an element of the absolute.

Let P be an element of the absolute. The absolute to unit circle of P yielding an element of $\operatorname{circle}(0, 0, 1)$ is defined by

(Def. 8) there exists a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that $P =$ the direction of u and $u(3) = 1$ and *it* $= [u(1), u(2)]$.

Now we state the proposition:

(87) The carrier of $\operatorname{TopUnitCircle} 2 = \operatorname{circle}(0, 0, 1)$.
PROOF: The carrier of $\operatorname{TopUnitCircle} 2 \subseteq \operatorname{circle}(0, 0, 1)$ by [16, (9)], [12, (54)], [15, (52)]. $\operatorname{circle}(0, 0, 1) \subseteq$ the carrier of $\operatorname{TopUnitCircle} 2$ by [15, (52)], [12, (54)], [16, (9)]. □

Let u be a non zero element of $\mathcal{E}^2_{\mathrm{T}}$. Assume $u \in \operatorname{circle}(0, 0, 1)$. The unit circle to absolut yielding an element of the absolute is defined by the term

(Def. 9) the direction of $[u(1), u(2), 1]$.

Now we state the proposition:

(88) Let us consider an element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $\mathrm{qfconic}(1,1,-1,0,0,0,u) = 0$ and $u(3) = 1$. Then $[u(1), u(2)] \in \mathrm{Sphere}(0_{\mathcal{E}^2_{\mathrm{T}}}, 1)$. The theorem is a consequence of (13).

Let us consider an element P of the absolute. Now we state the propositions:

(89) There exists u such that

(i) $u(1)^{\mathbf{2}} + u(2)^{\mathbf{2}} = 1$, and

(ii) $u(3) = 1$, and

(iii) $P =$ the direction of u.

The theorem is a consequence of (83), (84), and (14).

(90) There exists an element Q of the absolute such that $P \neq Q$.

Proof: Consider Q being a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that $P = Q$ and for every element u of $\mathcal{E}^3_{\mathrm{T}}$ such that u is not zero and $Q =$ the direction of u holds $\mathrm{qfconic}(1,1,-1,0,0,0,u) = 0$. Consider u being an element of $\mathcal{E}^3_{\mathrm{T}}$ such that u is not zero and the direction of $u = P$. $u(3) \neq 0$. $[u(1), u(2), -u(3)]$ is not zero by [14, (4)], [2, (78)], (83). Reconsider $v = [u(1), u(2), -u(3)]$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. Reconsider $R =$ the direction of v as an element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. $R \neq P$ by [18, (22), (1)], [14, (2)], [8, (44)]. For every element w of $\mathcal{E}^3_{\mathrm{T}}$ such that w is not zero and $R =$ the direction of w holds $\mathrm{qfconic}(1,1,-1,0,0,0,w) = 0$ by [18, (22), (1)], [8, (44)], [14, (2)]. □

Now we state the propositions:

(91) Let us consider real numbers a, b. Suppose $a^{\mathbf{2}} + b^{\mathbf{2}} = 1$. Then $[-b, a, 0]$ is not zero.

(92) Let us consider elements P, Q, R of the absolute. If P, Q, R are mutually different, then P, Q and R are not collinear.

Proof: Consider u_{12} being an element of $\mathcal{E}^3_{\mathrm{T}}$ such that u_{12} is not zero and $P =$ the direction of u_{12}. Consider u_{16} being an element of $\mathcal{E}^3_{\mathrm{T}}$ such that u_{16} is not zero and $Q =$ the direction of u_{16}. Consider u_{20} being an element of $\mathcal{E}^3_{\mathrm{T}}$ such that u_{20} is not zero and $R =$ the direction of u_{20}. Reconsider $u_{13} = (u_{12})_{\mathbf{1}}$, $u_{14} = (u_{12})_{\mathbf{2}}$, $u_{15} = (u_{12})_{\mathbf{3}}$, $u_{17} = (u_{16})_{\mathbf{1}}$, $u_{18} = (u_{16})_{\mathbf{2}}$, $u_{19} = (u_{16})_{\mathbf{3}}$, $u_{21} = (u_{20})_{\mathbf{1}}$, $u_{22} = (u_{20})_{\mathbf{2}}$, $u_{23} = (u_{20})_{\mathbf{3}}$ as a real number. $u_{12}(3) \neq 0$ and $u_{16}(3) \neq 0$ and $u_{20}(3) \neq 0$. Reconsider $v_5 = \frac{u_{13}}{u_{15}}$, $v_6 = \frac{u_{14}}{u_{15}}$, $v_8 = \frac{u_{17}}{u_{19}}$, $v_9 = \frac{u_{18}}{u_{19}}$, $v_{11} = \frac{u_{21}}{u_{23}}$, $v_{12} = \frac{u_{22}}{u_{23}}$ as a real number. Reconsider $v_4 = [v_5, v_6, 1]$, $v_7 = [v_8, v_9, 1]$, $v_{10} = [v_{11}, v_{12}, 1]$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. $P =$ the direction of v_4 and $Q =$ the direction of v_7 and $R =$ the direction of v_{10} by [14, (8), (3)], [18, (1), (22)]. Consider t_1, t_2, t_3 being elements of $\mathcal{E}^3_{\mathrm{T}}$ such that $P =$ the direction of t_1 and $Q =$ the direction of t_2 and $R =$ the direction of t_3 and t_1 is not zero and

t_2 is not zero and t_3 is not zero and there exist real numbers a_1, b_1, c_1 such that $a_1 \cdot t_1 + b_1 \cdot t_2 + c_1 \cdot t_3 = 0_{\mathcal{E}^3_{\mathrm{T}}}$ and ($a_1 \neq 0$ or $b_1 \neq 0$ or $c_1 \neq 0$). Consider a_1, b_1, c_1 being real numbers such that $a_1 \cdot t_1 + b_1 \cdot t_2 + c_1 \cdot t_3 = 0_{\mathcal{E}^3_{\mathrm{T}}}$ and $a_1 \neq 0$ or $b_1 \neq 0$ or $c_1 \neq 0$. Consider l_1 being a real number such that $l_1 \neq 0$ and $t_1 = l_1 \cdot v_4$. Consider l_2 being a real number such that $l_2 \neq 0$ and $t_2 = l_2 \cdot v_7$. Consider l_3 being a real number such that $l_3 \neq 0$ and $t_3 = l_3 \cdot v_{10}$. Reconsider $A = [(v_4)_{\mathbf{1}}, (v_4)_{\mathbf{2}}]$, $B = [(v_7)_{\mathbf{1}}, (v_7)_{\mathbf{2}}]$, $C = [(v_{10})_{\mathbf{1}}, (v_{10})_{\mathbf{2}}]$ as an element of $\mathcal{E}^2_{\mathrm{T}}$. $A \neq B$ by [2, (77)], [14, (3)]. $A \neq C$ by [2, (77)], [14, (3)]. $B \neq C$ by [2, (77)], [14, (3)]. $A \in \mathrm{Sphere}(0_{\mathcal{E}^2_{\mathrm{T}}}, 1)$. $\mathrm{qfconic}(1, 1, -1, 0, 0, 0, v_7) = 0$. $B \in \mathrm{Sphere}(0_{\mathcal{E}^2_{\mathrm{T}}}, 1)$. $C \in \mathrm{Sphere}(0_{\mathcal{E}^2_{\mathrm{T}}}, 1)$. □

(93) Let us consider a non zero real number r, and invertible square matrices O, N, M over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose $O = \mathrm{symmetric3}(1, 1, -1, 0, 0, 0)$ and $M = \mathrm{symmetric3}(r, r, -r, 0, 0, 0)$ and $M = N^{\mathrm{T}} \cdot O \cdot N$ and (the homography of M)$^{\circ}$(the absolute) = the absolute. Then (the homography of N)$^{\circ}$(the absolute) = the absolute.

Proof: (The homography of N)$^{\circ}$(the absolute) $\subseteq$ the absolute by [11, (13)], [12, (24), (22)], [20, (50)]. The absolute $\subseteq$ (the homography of N)$^{\circ}$(the absolute) by [11, (15)], [12, (24), (22)], [20, (50)]. □

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