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Klein-Beltrami Model. Part II

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Summary. Makarios (with Isabelle/HOL[1]) and John Harrison (with HOL-Light [2]) shown that "the Klein-Beltrami model of the hyperbolic plane satisfy all of Tarski's axioms except his Euclidean axiom" [3, 4, 20, 5].

With the Mizar system [2], [12] we use some ideas are taken from Tim Makarios' MSc thesis [16] for formalized some definitions (like the tangent) and lemmas necessary for the verification of the independence of the parallel postulate.

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1. Beltrami-Cayley-Klein Disk Model

The BK-model yielding a non empty subset of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 1) the interior of the conic for 1, 1, -1, 0, 0 and 0.

Now we state the propositions:

(1) The BK-model misses the absolute.

(2) Let us consider an element P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P =$ the direction of u and $P \in$ the BK-model. Then $u(3) \neq 0$.

Let P be an element of the BK-model. The functor BK-to-REAL2(P) yielding an element of the inside of circle(0,0,1) is defined by

[1] https://www.isa-afp.org/entries/Tarskis_Geometry.html

[2] https://github.com/jrh13/hol-light/blob/master/100/independence.ml

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(Def. 2) there exists a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $u = P$ and $u(3) = 1$ and $it = [u(1), u(2)]$.

Let Q be an element of the inside of circle(0,0,1). The functor REAL2-to-BK(Q) yielding an element of the BK-model is defined by

(Def. 3) there exists an element P of $\mathcal{E}^2_{\mathrm{T}}$ such that $P = Q$ and $it =$ the direction of $[(P)_{\mathbf{1}}, (P)_{\mathbf{2}}, 1]$.

Now we state the propositions:

(3) Let us consider an element P of the BK-model. Then REAL2-to-BK(BK-to-REAL2 P.

PROOF: Consider u being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $u = P$ and $u(3) = 1$ and BK-to-REAL2(P) $= [u(1), u(2)]$. Consider Q being an element of $\mathcal{E}^2_{\mathrm{T}}$ such that $Q =$ BK-to-REAL2(P) and REAL2-to-BK(BK-to-REAL2(P)) = the direction of $[(Q)_{\mathbf{1}}, (Q)_{\mathbf{2}}, 1]$. $[(Q)_{\mathbf{1}}, (Q)_{\mathbf{2}}, 1]$ and u are proportional by [11, (52)], [13, (3)]. □

(4) Let us consider elements P, Q of the BK-model. Then $P = Q$ if and only if BK-to-REAL2(P) = BK-to-REAL2(Q).

(5) Let us consider an element Q of the inside of circle(0,0,1). Then BK-to-REAL2(REA Q.

(6) Let us consider elements P, Q of the BK-model, and elements P_1, P_2, P_3 of the absolute. Suppose $P \neq Q$ and $P_1 \neq P_2$ and P, Q and P_1 are collinear and P, Q and P_2 are collinear and P, Q and P_3 are collinear. Then

(i) $P_3 = P_1$, or

(ii) $P_3 = P_2$.

PROOF: $P_3 = P_1$ or $P_3 = P_2$ by [19, (3)], [**?** , (92)]. □

(7) Let us consider an element P of the BK-model, an element Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a non zero element v of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P \neq Q$ and $Q =$ the direction of v and $v(3) = 1$. Then there exists an element P_1 of the absolute such that P, Q and P_1 are collinear.

PROOF: Consider u being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $u = P$ and $u(3) = 1$ and BK-to-REAL2(P) $= [u(1), u(2)]$. Reconsider $s = [u(1), u(2)]$, $t = [v(1), v(2)]$ as a point of $\mathcal{E}^2_{\mathrm{T}}$. Set $a = 0$. Set $b = 0$. Set $r = 1$. Reconsider $S = s$, $T = t$, $X = [a, b]$ as an element of $\mathcal{R}^2$. Reconsider $w_1 = \frac{-2\cdot|(t-s, s-[a,b])| + \sqrt{\Delta(\sum({}^2(T-S)), 2\cdot|(t-s, s-[a,b])|, \sum({}^2(S-X)) - r^{\mathbf{2}})}}{2\cdot(\sum({}^2(T-S)))}$ as a real number. $s \neq t$ by [1, (77)], [13, (3)]. Consider e_1 being a point of $\mathcal{E}^2_{\mathrm{T}}$ such that $\{e_1\} = \mathrm{HalfLine}(s, t) \cap \mathrm{circle}(a, b, r)$ and $e_1 = (1 - w_1) \cdot s + w_1 \cdot t$. Reconsider $f = [(e_1)_{\mathbf{1}}, (e_1)_{\mathbf{2}}, 1]$ as an element of $\mathcal{E}^3_{\mathrm{T}}$. Reconsider $e_3 = f$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. $1 \cdot e_3 + (-(1 - w_1)) \cdot u + (-w_1) \cdot v = 0_{\mathcal{E}^3_{\mathrm{T}}}$ by [13, (7), (3)], [11, (53), (55)]. □

(8) Let us consider an element P of the BK-model, and a line L of Inc-ProjSp(the real projective plane). Then there exists an element Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that

(i) $P \neq Q$, and

(ii) $Q \in L$.

(9) Let us consider real numbers a, b, c, d, e, and elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $u = [a, b, e]$ and $v = [c, d, 0]$ and $w = [a + c, b + d, e]$. Then $\langle |u, v, w| \rangle = 0$.

(10) Let us consider real numbers a, b, and a non zero real number c. Then $[a, b, c]$ is a non zero element of $\mathcal{E}^3_{\mathrm{T}}$.

(11) Let us consider elements u, v of $\mathcal{E}^3_{\mathrm{T}}$, and real numbers a, b, c, d, e. Suppose $u = [a, b, c]$ and $v = [d, e, 0]$ and u and v are proportional. Then $c = 0$.

(12) Let us consider elements P, Q, R of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and non zero elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P =$ the direction of u and $Q =$ the direction of v and $R =$ the direction of w and $(u)_{\mathbf{3}} \neq 0$ and $(v)_{\mathbf{3}} = 0$ and $w = [(u)_{\mathbf{1}} + (v)_{\mathbf{1}}, (u)_{\mathbf{2}} + (v)_{\mathbf{2}}, (u)_{\mathbf{3}}]$. Then

(i) $R \neq P$, and

(ii) $R \neq Q$.

(13) Let us consider a line L of Inc-ProjSp(the real projective plane), and elements P, Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. If $P \neq Q$ and P, $Q \in L$, then $L = \mathrm{Line}(P, Q)$.

(14) Let us consider a line L of Inc-ProjSp(the real projective plane), elements P, Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$. Suppose P, $Q \in L$ and $P =$ the direction of u and $Q =$ the direction of v and $(u)_{\mathbf{3}} \neq 0$ and $(v)_{\mathbf{3}} = 0$. Then

(i) $P \neq Q$, and

(ii) the direction of $[(u)_{\mathbf{1}} + (v)_{\mathbf{1}}, (u)_{\mathbf{2}} + (v)_{\mathbf{2}}, (u)_{\mathbf{3}}] \in L$.

PROOF: $P \neq Q$ by [15, (22)], [13, (3)], (11). Reconsider $w = [(u)_{\mathbf{1}} + (v)_{\mathbf{1}}, (u)_{\mathbf{2}} + (v)_{\mathbf{2}}, (u)_{\mathbf{3}}]$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. $\langle |u, v, w| \rangle = 0$. □

(15) Let us consider elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $(v)_{\mathbf{3}} = 0$ and $w = [(u)_{\mathbf{1}} + (v)_{\mathbf{1}}, (u)_{\mathbf{2}} + (v)_{\mathbf{2}}, (u)_{\mathbf{3}}]$. Then $\langle |u, v, w| \rangle = 0$.

(16) Let us consider a line L of Inc-ProjSp(the real projective plane), an element P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P =$ the direction of u and $P \in L$ and $u(3) \neq 0$. Then there exists

an element Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ and there exists a non zero element v of $\mathcal{E}^3_{\mathrm{T}}$ such that $Q =$ the direction of v and $Q \in L$ and $P \neq Q$ and $v(3) \neq 0$. The theorem is a consequence of (15).

(17) Let us consider an element P of the BK-model, and a line L of Inc-ProjSp(the real projective plane). Suppose $P \in L$. Then there exists an element Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that

(i) $P \neq Q$, and

(ii) $Q \in L$, and

(iii) for every non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that $Q =$ the direction of u holds $u(3) \neq 0$.

The theorem is a consequence of (16).

(18) Let us consider non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$, and a non zero real number k. Suppose $u = k \cdot v$. Then the direction of $u =$ the direction of v.

(19) Let us consider an element P of the BK-model, and an element Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P \neq Q$. Then there exists an element P_1 of the absolute such that P, Q and P_1 are collinear.

PROOF: Reconsider $L = \mathrm{Line}(P, Q)$ as a line of Inc-ProjSp(the real projective plane). Consider R being an element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that $P \neq R$ and $R \in L$ and for every non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that $R =$ the direction of u holds $u(3) \neq 0$. Consider u being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $u = P$ and $u(3) = 1$ and BK-to-REAL2$(P) = [u(1), u(2)]$. Consider v' being an element of $\mathcal{E}^3_{\mathrm{T}}$ such that v' is not zero and the direction of $v' = R$. Reconsider $k = \frac{1}{(v')_{\mathbf{3}}}$ as a non zero real number. $k \cdot v'$ is not zero by [13, (4), (7)], [1, (78)]. Reconsider $v = k \cdot v'$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. the direction of $v = R$ and $v(3) = 1$ by (18), [13, (3), (7)], [1, (78)]. Reconsider $s = [u(1), u(2)]$, $t = [v(1), v(2)]$ as a point of $\mathcal{E}^2_{\mathrm{T}}$. Set $a = 0$. Set $b = 0$. Set $r = 1$. Reconsider $S = s$, $T = t$, $X = [a, b]$ as an element of $\mathcal{R}^2$. Reconsider $w_1 = \frac{-2 \cdot |(t-s, s-[a,b])| + \sqrt{\Delta(\sum({}^2(T-S)), 2 \cdot |(t-s, s-[a,b])|, \sum({}^2(S-X)) - r^{\mathbf{2}})}}{2 \cdot (\sum({}^2(T-S)))}$ as a real number. $s \neq t$ by [1, (77)], [13, (3)]. Consider e_1 being a point of $\mathcal{E}^2_{\mathrm{T}}$ such that $\{e_1\} = \mathrm{HalfLine}(s, t) \cap \mathrm{circle}(a, b, r)$ and $e_1 = (1 - w_1) \cdot s + w_1 \cdot t$. Reconsider $f = [(e_1)_{\mathbf{1}}, (e_1)_{\mathbf{2}}, 1]$ as an element of $\mathcal{E}^3_{\mathrm{T}}$. Reconsider $e_3 = f$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. $1 \cdot e_3 + (-(1 - w_1)) \cdot u + (-w_1) \cdot v = 0_{\mathcal{E}^3_{\mathrm{T}}}$ by [13, (7), (3)], [11, (53), (55)]. □

(20) Let us consider elements P, Q of the BK-model. Suppose $P \neq Q$. Then there exist elements P_1, P_2 of the absolute such that

(i) $P_1 \neq P_2$, and

(ii) P, Q and P_1 are collinear, and

(iii) P, Q and P_2 are collinear.

PROOF: Consider u being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $u = P$ and $u(3) = 1$ and BK-to-REAL2$(P) = [u(1), u(2)]$. Consider v being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $v = Q$ and $v(3) = 1$ and BK-to-REAL2$(Q) = [v(1), v(2)]$. Reconsider $s = [u(1), u(2)]$, $t = [v(1), v(2)]$ as a point of $\mathcal{E}^2_{\mathrm{T}}$. Set $a = 0$. Set $b = 0$. Set $r = 1$. Reconsider $S = s$, $T = t$, $X = [a, b]$ as an element of $\mathcal{R}^2$. Reconsider $w_1 = \frac{-2\cdot|(t-s,s-[a,b])|+\sqrt{\Delta(\sum(^2(T-S)),2\cdot|(t-s,s-[a,b])|,\sum(^2(S-X))-r^{\mathbf{2}})}}{2\cdot(\sum(^2(T-S)))}$ as a real number. Consider e_1 being a point of $\mathcal{E}^2_{\mathrm{T}}$ such that $\{e_1\} =$ HalfLine$(s, t) \cap$ circle(a, b, r) and $e_1 = (1 - w_1) \cdot s + w_1 \cdot t$. Reconsider $w_2 = \frac{-2\cdot|(s-t,t-[a,b])|+\sqrt{\Delta(\sum(^2(S-T)),2\cdot|(s-t,t-[a,b])|,\sum(^2(T-X))-r^{\mathbf{2}})}}{2\cdot(\sum(^2(S-T)))}$ as a real number. Consider e_2 being a point of $\mathcal{E}^2_{\mathrm{T}}$ such that $\{e_2\} =$ HalfLine$(t, s) \cap$ circle(a, b, r) and $e_2 = (1 - w_2) \cdot t + w_2 \cdot s$. Reconsider $f = [(e_1)_{\mathbf{1}}, (e_1)_{\mathbf{2}}, 1]$ as an element of $\mathcal{E}^3_{\mathrm{T}}$. Reconsider $e_3 = f$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. Reconsider $P_1 =$ the direction of e_3 as a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. $1 \cdot e_3 + (-(1 - w_1)) \cdot u + (-w_1) \cdot v = 0_{\mathcal{E}^3_{\mathrm{T}}}$ by [13, (7), (3)], [11, (53), (55)]. Reconsider $g = [(e_2)_{\mathbf{1}}, (e_2)_{\mathbf{2}}, 1]$ as an element of $\mathcal{E}^3_{\mathrm{T}}$. Reconsider $e_4 = g$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. Reconsider $P_2 =$ the direction of e_4 as a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. $1 \cdot e_4 + (-(1 - w_2)) \cdot v + (-w_2) \cdot u = 0_{\mathcal{E}^3_{\mathrm{T}}}$ by [13, (7), (3)], [11, (53), (55)]. $P_1 \neq P_2$ by [15, (22), (1)], [13, (8)], [1, (78)]. □

(21) Let us consider elements P, Q, R of the real projective plane, non zero elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$, and real numbers a, b, c, d. Suppose $P \in$ the BK-model and $Q \in$ the absolute and $P =$ the direction of u and $Q =$ the direction of v and $R =$ the direction of w and $u = [a, b, 1]$ and $v = [c, d, 1]$ and $w = [\frac{a+c}{2}, \frac{b+d}{2}, 1]$. Then

(i) $R \in$ the BK-model, and

(ii) $R \neq P$, and

(iii) P, R and Q are collinear.

PROOF: Reconsider $P_6 = P$ as an element of the BK-model. Consider u_2 being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $u_2 = P_6$ and $u_2(3) = 1$ and BK-to-REAL2$(P_6) = [u_2(1), u_2(2)]$. Consider p being a point of $\mathcal{E}^2_{\mathrm{T}}$ such that $[v(1), v(2)] = p$ and $|p - [0, 0]| = 1$. Reconsider $R_1 = [w(1), w(2)]$ as an element of $\mathcal{E}^2_{\mathrm{T}}$. $|R_1 - [0, 0]|^{\mathbf{2}} < 1$ by [11, (52), (62)], [17, (29)], [**?** , (17)]. Consider P_1 being an element of $\mathcal{E}^2_{\mathrm{T}}$ such that $P_1 = R_1$

and REAL2-to-BK(R_1) = the direction of $[(P_1)_{\mathbf{1}}, (P_1)_{\mathbf{2}}, 1]$. $P \neq R$ by [13, (2)], [? , (43)], (1). $\square$

(22) Let us consider elements P, Q of the real projective plane. Suppose $P \in$ the absolute and $Q \in$ the BK-model. Then there exists an element R of the real projective plane such that

(i) $R \in$ the BK-model, and

(ii) $Q \neq R$, and

(iii) R, Q and P are collinear.

The theorem is a consequence of (21).

(23) Let us consider a line L of Inc-ProjSp(the real projective plane), points p, q of Inc-ProjSp(the real projective plane), and elements P, Q of the real projective plane. Suppose $p = P$ and $q = Q$ and $P \in$ the BK-model and $Q \in$ the absolute and q lies on L and p lies on L. Then there exist points p_1, p_2 of Inc-ProjSp(the real projective plane) and there exist elements P_1, P_2 of the real projective plane such that $p_1 = P_1$ and $p_2 = P_2$ and $P_1 \neq P_2$ and $P_1, P_2 \in$ the absolute and p_1 lies on L and p_2 lies on L. The theorem is a consequence of (1), (22), and (20).

(24) Let us consider an element P of the BK-model, and an element Q of the absolute. Then there exists an element R of the absolute such that

(i) $Q \neq R$, and

(ii) Q, P and R are collinear.

The theorem is a consequence of (1) and (23).

(25) Let us consider an element P of the BK-model, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P =$ the direction of u and $u(3) = 1$. Then $u(1)^{\mathbf{2}} + u(2)^{\mathbf{2}} < 1$.

(26) Let us consider elements P_1, P_2 of the absolute, an element Q of the BK-model, and non zero elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Suppose the direction of $u = P_1$ and the direction of $v = P_2$ and the direction of $w = Q$ and $u(3) = 1$ and $v(3) = 1$ and $w(3) = 1$ and $v(1) = -u(1)$ and $v(2) = -u(2)$ and P_1, Q and P_2 are collinear. Then there exists a real number a such that

(i) $-1 < a < 1$, and

(ii) $w(1) = a \cdot (u(1))$, and

(iii) $w(2) = a \cdot (u(2))$.

The theorem is a consequence of (25).

2. Tangent

Let P be an element of the absolute. The functor PoleInfty(P) yielding an element of the real projective plane is defined by

(Def. 4) there exists a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that $P =$ the direction of u and $u(3) = 1$ and $u(1)^{\mathbf{2}} + u(2)^{\mathbf{2}} = 1$ and *it* = the direction of $[-u(2), u(1), 0]$.

Now we state the propositions:

(27) Let us consider an element P of the absolute. Then $P \neq$ PoleInfty(P).

(28) Let us consider elements P_1, P_2 of the absolute. Suppose PoleInfty(P_1) = PoleInfty(P_2). Then

(i) $P_1 = P_2$, or

(ii) there exists a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that $P_1 =$ the direction of u and $P_2 =$ the direction of $[-(u)_{\mathbf{1}}, -(u)_{\mathbf{2}}, 1]$ and $(u)_{\mathbf{3}} = 1$.

PROOF: Consider u_1 being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that $P_1 =$ the direction of u_1 and $u_1(3) = 1$ and $u_1(1)^{\mathbf{2}} + u_1(2)^{\mathbf{2}} = 1$ and PoleInfty(P_1) = the direction of $[-u_1(2), u_1(1), 0]$. Consider u_2 being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that $P_2 =$ the direction of u_2 and $u_2(3) = 1$ and $u_2(1)^{\mathbf{2}} + u_2(2)^{\mathbf{2}} = 1$ and PoleInfty(P_2) = the direction of $[-u_2(2), u_2(1), 0]$. Reconsider $w_1 = [-u_1(2), u_1(1), 0]$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. Reconsider $w_2 = [-u_2(2), u_2(1), 0]$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. Consider a being a real number such that $a \neq 0$ and $w_1 = a \cdot w_2$. If $a = 1$, then $P_1 = P_2$ by [13, (3)]. If $a = -1$, then there exists a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that $P_1 =$ the direction of u and $P_2 =$ the direction of $[-(u)_{\mathbf{1}}, -(u)_{\mathbf{2}}, 1]$ and $(u)_{\mathbf{3}} = 1$ by [13, (3)]. □

Let P be an element of the absolute. The functor tangent(P) yielding a line of the real projective plane is defined by

(Def. 5) there exists an element p of the real projective plane such that $p = P$ and *it* = Line(p, PoleInfty(P)).

Let us consider an element P of the absolute. Now we state the propositions:

(29) $P \in$ tangent(P).

(30) tangent(P) $\cap$ (the absolute) = $\{P\}$.

PROOF: $\{P\} \subseteq$ tangent(P) $\cap$ (the absolute). tangent(P) $\cap$ (the absolute) $\subseteq \{P\}$ by [19, (11)], [15, (26), (22), (1)]. □

Now we state the propositions:

(31) Let us consider elements P_1, P_2 of the absolute. If tangent(P_1) = tangent(P_2), then $P_1 = P_2$. The theorem is a consequence of (30).

(32) Let us consider elements P, Q of the absolute. Then there exists an element R of the real projective plane such that

(i) $R \in \text{tangent}(P)$, and

(ii) $R \in \text{tangent}(Q)$.

(33) Let us consider elements P_1, P_2 of the absolute. Suppose $P_1 \neq P_2$. Then there exists an element P of the real projective plane such that $\text{tangent}(P_1) \cap \text{tangent}(P_2) = \{P\}$. The theorem is a consequence of (31).

(34) Let us consider a square matrix M over $\mathbb{R}$ of dimension 3, an element P of the absolute, an element Q of the real projective plane, non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$, and finite sequences f_3, f_7 of elements of $\mathbb{R}$. Suppose $M = \text{symmetric3}(1, 1, -1, 0, 0, 0)$ and $P =$ the direction of u and $Q =$ the direction of v and $u = f_3$ and $v = f_7$ and $Q \in \text{tangent}(P)$. Then $\text{SumAll QuadraticForm}(f_7, M, f_3) = 0$.

Proof: Consider p being an element of the real projective plane such that $p = P$ and $\text{tangent}(P) = \text{Line}(p, \text{PoleInfty}(P))$. Consider w being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that $P =$ the direction of w and $w(3) = 1$ and $w(1)^{\mathbf{2}} + w(2)^{\mathbf{2}} = 1$ and $\text{PoleInfty}(P) =$ the direction of $[-w(2), w(1), 0]$. Consider a_1 being a real number such that $a_1 \neq 0$ and $w = a_1 \cdot u$. $w(1) = a_1 \cdot ((u)_{\mathbf{1}})$ and $w(2) = a_1 \cdot ((u)_{\mathbf{2}})$ and $w(3) = a_1 \cdot ((u)_{\mathbf{3}})$ by [7, (44)]. $\text{len}\, f_3 = \text{width}\, M$ and $\text{len}\, f_7 = \text{len}\, M$ and $\text{len}\, f_3 = \text{len}\, M$ and $\text{len}\, f_7 = \text{width}\, M$ and $\text{len}\, f_3 > 0$ and $\text{len}\, f_7 > 0$ by [21, (153)]. □

(35) Let us consider elements P, Q, R of the absolute, and points P_1, P_2, P_3, P_4 of the real projective plane. Suppose P, Q, R are mutually different and $P_1 = P$ and $P_2 = Q$ and $P_3 = R$ and $P_4 \in \text{tangent}(P)$ and $P_4 \in \text{tangent}(Q)$. Then

(i) P_1, P_2 and P_3 are not collinear, and

(ii) P_1, P_2 and P_4 are not collinear, and

(iii) P_1, P_3 and P_4 are not collinear, and

(iv) P_2, P_3 and P_4 are not collinear.

Proof: $P_4 \notin$ the absolute. Consider p being an element of the real projective plane such that $p = P$ and $\text{tangent}(P) = \text{Line}(p, \text{PoleInfty}(P))$. Consider q being an element of the real projective plane such that $q = Q$ and $\text{tangent}(Q) = \text{Line}(q, \text{PoleInfty}(Q))$. P_1, P_2 and P_4 are not collinear by [19, (2), (4), (7)]. P_1, P_3 and P_4 are not collinear by [19, (4), (6), (11)], (30). P_2, P_3 and P_4 are not collinear by [19, (4), (6), (11)], (30). □

(36) Let us consider elements P, Q of the absolute, an element R of the real projective plane, and non zero elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P =$ the direction of u and $Q =$ the direction of v and $R =$ the direction of w

and $R \in \text{tangent}(P)$ and $R \in \text{tangent}(Q)$ and $u(3) = 1$ and $v(3) = 1$ and $w(3) = 0$. Then

(i) $P = Q$, or

(ii) $u(1) = -v(1)$ and $u(2) = -v(2)$.

The theorem is a consequence of (34).

(37) Let us consider an element P of the absolute, an element R of the real projective plane, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $R \in \text{tangent}(P)$ and $R =$ the direction of u and $u(3) = 0$. Then $R = \text{PoleInfty}(P)$. The theorem is a consequence of (34).

(38) Let us consider a non zero real number a, and an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose $N = \text{symmetric3}(a, a, -a, 0, 0, 0)$. Then (the homography of $N)^\circ$(the absolute) = the absolute.

Proof: (The homography of $N)^\circ$(the absolute) $\subseteq$ the absolute by [**?** , (89)], [6, (3)], [13, (3)], [10, (8)]. The absolute $\subseteq$ (the homography of $N)^\circ$(the absolute) by [**?** , (89)], [1, (78)], [13, (4), (3)]. □

(39) Let us consider a non zero element r_1 of $\mathbb{R}_{\mathrm{F}}$, and invertible square matrices M, O over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose $O = \text{symmetric3}(1, 1, -1, 0, 0, 0)$ and $M = r_1 \cdot O$. Then (the homography of $M)^\circ$(the absolute) = the absolute.

Proof: $r_1 \neq 0$ by [18, (34)]. □

(40) Let us consider an element P of the absolute. Then tangent(P) misses the BK-model. The theorem is a consequence of (29), (23), and (30).

(41) Let us consider elements P, P_3, P_4 of the real projective plane, elements P_1, P_2 of the absolute, and an element Q of the real projective plane. Suppose $P_1 \neq P_2$ and $P_3 = P_1$ and $P_4 = P_2$ and $P \in$ the BK-model and P, P_3 and P_4 are collinear and $Q \in \text{tangent}(P_1)$ and $Q \in \text{tangent}(P_2)$. Then there exists an element R of the real projective plane such that

(i) $R \in$ the absolute, and

(ii) P, Q and R are collinear.

The theorem is a consequence of (40), (7), (37), (28), and (26).

(42) Let us consider elements P, R, S of the real projective plane, and an element Q of the absolute. Suppose $P \in$ the BK-model and $R \in \text{tangent}(Q)$ and P, S and R are collinear and $R \neq S$. Then $Q \neq S$. The theorem is a consequence of (29), (23), and (30).

3. Subgroup of K-Isometry

Let h be an element of EnsHomography3. We say that h is K-isometry if and only if

(Def. 6) there exists an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3 such that $h =$ the homography of N and (the homography of N)$^{\circ}$(the absolute) = the absolute.

Now we state the proposition:

(43) Let us consider an element h of EnsHomography3. Suppose $h =$ the homography of $I^{3\times 3}_{\mathbb{R}_{\rm F}}$. Then h is K-isometry.
PROOF: h is K-isometry by [9, (14)], [6, (108)]. □

The set of K-isometries yielding a non empty subset of EnsHomography3 is defined by the term

(Def. 7) $\{h$, where h is an element of EnsHomography3 : h is K-isometry$\}$.

The subgroup of K-isometries yielding a strict subgroup of GroupHomography3 is defined by

(Def. 8) the carrier of *it* = the set of K-isometries.

Now we state the propositions:

(44) Let us consider an element h of the set of K-isometries, and an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3. Suppose $h =$ the homography of N. Then (the homography of N)$^{\circ}$(the absolute) = the absolute.

(45) (i) the homography of $I^{3\times 3}_{\mathbb{R}_{\rm F}} = \mathbf{1}_{\rm GroupHomography3}$, and

(ii) the homography of $I^{3\times 3}_{\mathbb{R}_{\rm F}} = \mathbf{1}_{\alpha}$,
where α is the subgroup of K-isometries.

(46) Let us consider invertible square matrices N_1, N_2 over $\mathbb{R}_{\rm F}$ of dimension 3, and elements h_1, h_2 of the subgroup of K-isometries. Suppose $h_1 =$ the homography of N_1 and $h_2 =$ the homography of N_2. Then

(i) $h_1 \cdot h_2$ is an element of the subgroup of K-isometries, and

(ii) $h_1 \cdot h_2 =$ the homography of $N_1 \cdot N_2$.

(47) Let us consider an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3, and an element h of the subgroup of K-isometries. Suppose $h =$ the homography of N. Then

(i) $h^{-1} =$ the homography of $N^{\smile}$, and

(ii) the homography of $N^{\smile}$ is an element of the subgroup of K-isometries.

The theorem is a consequence of (45).

(48) Let us consider an element s of the projective space over $\mathcal{E}^3_{\rm T}$, and elements p, q, r of the absolute. Suppose p, q, r are mutually different and $s \in$ tangent$(p) \cap$ tangent(q). Then there exists an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3 such that

(i) (the homography of N)$^{\circ}$(the absolute) = the absolute, and

(ii) (the homography of N)(Dir101) $= p$, and

(iii) (the homography of N)(Dirm101) $= q$, and

(iv) (the homography of N)(Dir011) $= r$, and

(v) (the homography of N)(Dir010) $= s$.

PROOF: Reconsider $P_1 = p$, $P_2 = q$, $P_3 = r$, $P_4 = s$ as a point of the real projective plane. P_1, P_2 and P_3 are not collinear and P_1, P_2 and P_4 are not collinear and P_1, P_3 and P_4 are not collinear and P_2, P_3 and P_4 are not collinear. Consider N being an invertible square matrix over $\mathbb{R}_{\rm F}$ of dimension 3 such that (the homography of N)(Dir101) $= P_1$ and (the homography of N)(Dirm101) $= P_2$ and (the homography of N)(Dir011) $= P_3$ and (the homography of N)(Dir010) $= P_4$. Consider n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9 being elements of $\mathbb{R}_{\rm F}$ such that $N = \langle\langle n_1, n_2, n_3\rangle, \langle n_4, n_5, n_6\rangle, \langle n_7, n_8, n_9\rangle\rangle$. Reconsider $b = -1$ as an element of $\mathbb{R}_{\rm F}$. Reconsider $a = 1$ as an element of $\mathbb{R}_{\rm F}$. Reconsider $a = 1$, $b = -1$ as a non zero element of $\mathbb{R}_{\rm F}$. Reconsider $N_1 = \langle\langle a, 0, 0\rangle, \langle 0, a, 0\rangle, \langle 0, 0, b\rangle\rangle$ as an invertible square matrix over $\mathbb{R}_{\rm F}$ of dimension 3. Reconsider $M = N^{\rm T} \cdot N_1 \cdot N$ as an invertible square matrix over $\mathbb{R}_{\rm F}$ of dimension 3. Consider v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9 being elements of $\mathbb{R}_{\rm F}$ such that $M = \langle\langle v_1, v_2, v_3\rangle, \langle v_4, v_5, v_6\rangle, \langle v_7, v_8, v_9\rangle\rangle$. Reconsider $r_1 = v_1$, $r_2 = v_2$, $r_3 = v_3$, $r_4 = v_5$, $r_5 = v_6$, $r_6 = v_9$ as a real number. Consider Q being a point of the projective space over $\mathcal{E}^3_{\rm T}$ such that Dir101 $= Q$ and for every element u of $\mathcal{E}^3_{\rm T}$ such that u is not zero and $Q =$ the direction of u holds qfconic$(r_1, r_4, r_6, 2 \cdot r_2, 2 \cdot r_3, 2 \cdot r_5, u) = 0$. Consider Q being a point of the projective space over $\mathcal{E}^3_{\rm T}$ such that Dirm101 $= Q$ and for every element u of $\mathcal{E}^3_{\rm T}$ such that u is not zero and $Q =$ the direction of u holds qfconic$(r_1, r_4, r_6, 2 \cdot r_2, 2 \cdot r_3, 2 \cdot r_5, u) = 0$. Consider Q being a point of the projective space over $\mathcal{E}^3_{\rm T}$ such that Dir011 $= Q$ and for every element u of $\mathcal{E}^3_{\rm T}$ such that u is not zero and $Q =$ the direction of u holds qfconic$(r_1, r_4, r_6, 2 \cdot r_2, 2 \cdot r_3, 2 \cdot r_5, u) = 0$. $r_3 = 0$ and $r_1 = -r_6$ and $r_2 = 0$ and $r_5 = 0$ and $r_1 = r_4$ by [15, (26)], [11, (24)], [15, (22), (1)]. $r_1 \neq 0$ by [**?**, (22)], [18, (34)]. (The homography of M)$^\circ$(the absolute) = the absolute. □

(49) Let us consider elements p_1, q_1, r_1, p_2, q_2, r_2 of the absolute, and elements s_1, s_2 of the real projective plane. Suppose p_1, q_1, r_1 are mutually different and p_2, q_2, r_2 are mutually different and $s_1 \in$ tangent$(p_1) \cap$ tangent(q_1) and $s_2 \in$ tangent$(p_2) \cap$ tangent(q_2). Then there exists an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3 such that

(i) (the homography of N)$^\circ$(the absolute) = the absolute, and

(ii) (the homography of N)$(p_1) = p_2$, and

(iii) (the homography of N)$(q_1) = q_2$, and

(iv) (the homography of N)$(r_1) = r_2$, and

(v) (the homography of N)$(s_1) = s_2$.

The theorem is a consequence of (48) and (47).

(50) Let us consider elements p_1, q_1, r_1, p_2, q_2, r_2 of the absolute. Suppose p_1, q_1, r_1 are mutually different and p_2, q_2, r_2 are mutually different. Then there exists an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that

(i) (the homography of N)$^\circ$(the absolute) = the absolute, and

(ii) (the homography of N)$(p_1) = p_2$, and

(iii) (the homography of N)$(q_1) = q_2$, and

(iv) (the homography of N)$(r_1) = r_2$.

The theorem is a consequence of (33), (48), and (47).

(51) Let us consider a collinearity space C, and elements p, q, r, s of C. If $\mathrm{Line}(p, q) = \mathrm{Line}(r, s)$, then r, s and p are collinear.

(52) Let us consider a collinearity space C, and elements p, q of C. Then $\mathrm{Line}(p, q) = \mathrm{Line}(q, p)$.

PROOF: $\mathrm{Line}(p, q) \subseteq \mathrm{Line}(q, p)$ by [19, (4)]. $\mathrm{Line}(q, p) \subseteq \mathrm{Line}(p, q)$ by [19, (4)]. $\square$

(53) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, and elements p, q, r, s of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose Line((the homography of N)(p), (the homography of N)(q)) = Line((the homography of N)(r), (the homogr
of N)(s)). Then

(i) p, q and r are collinear, and

(ii) p, q and s are collinear, and

(iii) r, s and p are collinear, and

(iv) r, s and q are collinear.

The theorem is a consequence of (51) and (52).

Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 and elements p, q, r, s, t, u, n_1, n_2, n_3, n_4 of the real projective plane. Now we state the propositions:

(54) Suppose $p \neq q$ and $r \neq s$ and $n_1 \neq n_2$ and $n_3 \neq n_4$ and p, q and t are collinear and r, s and t are collinear and $n_1 =$ (the homography of N)(p) and $n_2 =$ (the homography of N)(q) and $n_3 =$ (the homography of N)(r) and $n_4 =$ (the homography of N)(s) and n_1, n_2 and u are collinear and n_3, n_4 and u are collinear. Then

(i) $u =$ (the homography of N)(t), or

(ii) Line(n_1, n_2) = Line(n_3, n_4).

(55) Suppose $p \neq q$ and $r \neq s$ and $n_1 \neq n_2$ and $n_3 \neq n_4$ and p, q and t are collinear and r, s and t are collinear and $n_1 =$ (the homography of N)(p) and $n_2 =$ (the homography of N)(q) and $n_3 =$ (the homography of N)(r) and $n_4 =$ (the homography of N)(s) and n_1, n_2 and u are collinear and n_3, n_4 and u are collinear and p, q and r are not collinear. Then $u =$ (the homography of N)(t). The theorem is a consequence of (54) and (53).

Now we state the propositions:

(56) Let us consider elements p, q of the absolute, and elements a, b of the BK-model. Then there exists an invertible square matrix N over $\mathbb{R}_\mathrm{F}$ of dimension 3 such that

(i) (the homography of N)$^\circ$(the absolute) = the absolute, and

(ii) (the homography of N)(a) = b, and

(iii) (the homography of N)(p) = q.

Proof: Consider p' being an element of the absolute such that $p \neq p'$ and p, a and p' are collinear. Consider q' being an element of the absolute such that $q \neq q'$ and q, b and q' are collinear. Consider t being an element of the real projective plane such that tangent(p)$\cap$tangent(p') = $\{t\}$. Consider u being an element of the real projective plane such that tangent(q) $\cap$ tangent(q') = $\{u\}$. Reconsider $a' = a$ as an element of the real projective plane. Consider R_1 being an element of the real projective plane such that $R_1 \in$ the absolute and a', t and R_1 are collinear. Reconsider $b' = b$ as an element of the real projective plane. Consider R_2 being an element of the real projective plane such that $R_2 \in$ the absolute and b', u and R_2 are collinear. p, p', R_1 are mutually different by [19, (11), (4)], [14, (1)], (30). Consider N being an invertible square matrix over $\mathbb{R}_\mathrm{F}$ of dimension 3 such that (the homography of N)$^\circ$(the absolute) = the absolute and (the homography of N)(p) = q and (the homography of N)(p') = q' and (the homography of N)(R_1) = R_2 and (the homography of N)(t) = u. Reconsider $p_5 = p$, $p_6 = p'$, $p_7 = R_1$, $p_8 = t$, $p_9 = a$, $n_1 = q$, $n_2 = q'$, $n_3 = R_2$, $n_4 = u$, $n_5 = b$ as an element of the real projective plane. $n_5 =$ (the homography of N)(p_9). $\square$

(57) Let us consider elements p, q, r, s of the absolute. Suppose p, q, r are mutually different and q, p, s are mutually different. Then there exists an invertible square matrix N over $\mathbb{R}_\mathrm{F}$ of dimension 3 such that

(i) (the homography of N)$^\circ$(the absolute) = the absolute, and

(ii) (the homography of N)(p) $= q$, and

(iii) (the homography of N)(q) $= p$, and

(iv) (the homography of N)(r) $= s$, and

(v) for every element t of the real projective plane such that $t \in$ tangent(p)$\cap$ tangent(q) holds (the homography of N)(t) $= t$.

The theorem is a consequence of (33), (48), and (47).

Let us consider elements P, Q of the BK-model. Now we state the propositions:

(58) Suppose $P \neq Q$. Then there exist elements P_1, P_2, P_3, P_4 of the absolute and there exists an element P_5 of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that $P_1 \neq P_2$ and P, Q and P_1 are collinear and P, Q and P_2 are collinear and P, P_5 and P_3 are collinear and Q, P_5 and P_4 are collinear and P_1, P_2, P_3 are mutually different and P_1, P_2, P_4 are mutually different and $P_5 \in$ tangent(P_1) $\cap$ tangent(P_2). The theorem is a consequence of (20), (32), (41), (30), (42), (29), (40), and (7).

(59) Suppose $P \neq Q$. Then there exists an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that

(i) (the homography of N)$^\circ$(the absolute) = the absolute, and

(ii) (the homography of N)(P) $= Q$, and

(iii) (the homography of N)(Q) $= P$, and

(iv) there exist elements P_1, P_2 of the absolute such that $P_1 \neq P_2$ and P, Q and P_1 are collinear and P, Q and P_2 are collinear and (the homography of N)(P_1) $= P_2$ and (the homography of N)(P_2) $= P_1$.

Proof: Consider P_1, P_2, P_3, P_4 being elements of the absolute, P_5 being an element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that $P_1 \neq P_2$ and P, Q and P_1 are collinear and P, Q and P_2 are collinear and P, P_5 and P_3 are collinear and Q, P_5 and P_4 are collinear and P_1, P_2, P_3 are mutually different and P_1, P_2, P_4 are mutually different and $P_5 \in$ tangent(P_1)$\cap$tangent(P_2). Consider N_1 being an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that (the homography of N_1)$^\circ$(the absolute) = the absolute and (the homography of N_1)(Dir101) $= P_1$ and (the homography of N_1)(Dirm101) $= P_2$ and (the homography of N_1)(Dir011) $= P_3$ and (the homography of N_1)(Dir010) $= P_5$. Consider N_2 being an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that (the homography of N_2)$^\circ$(the absolute) = the absolute and (the homography of N_2)(Dir101) $= P_2$ and (the homography of N_2)(Dirm101) $= P_1$ and (the homography of N_2)(Dir011) $= P_4$ and (the homography of N_2)(Dir010) $= P_5$. Reconsider $N = N_2 \cdot (N_1{}^{\smile})$ as an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Reconsider $h_1 =$ the homography of N_1 as an element of EnsHomography3. Reconsider $h_5 = h_1$ as an element of the subgroup of K-isometries. Reconsider $h_2 =$ the homography of N_2 as an element of EnsHomography3. Reconsider $h_6 = h_2$ as an element of the subgroup of K-isometries. Reconsider $h_3 =$ the homography of $N_1{}^{\smile}$ as an element of EnsHomography3. ${h_5}^{-1} = h_3$. Reconsider $h_7 = h_3$ as an element of the subgroup of K-isometries. Reconsider $h_4 = h_6 \cdot h_7$ as an element of the subgroup of K-isometries. Consider h being an element of EnsHomography3 such that $h_4 = h$ and h is K-isometry. (the homography of $N)(P) = Q$ and (the homography of $N)(Q) = P$ by [9, (15)], [8, (102), (57)], [14, (2), (1)]. □

4. Main Lemmas

Now we state the propositions:

(60) Let us consider elements P, Q of the BK-model. Then there exists an element h of the subgroup of K-isometries and there exists an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that $h =$ the homography of N and (the homography of $N)(P) = Q$ and (the homography of $N)(Q) = P$. The theorem is a consequence of (43) and (59).

(61) Let us consider elements P, Q, R, S, T, U of the BK-model. Suppose there exist elements h_1, h_2 of the subgroup of K-isometries and there exist invertible square matrices N_1, N_2 over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that $h_1 =$ the homography of N_1 and $h_2 =$ the homography of N_2 and (the homography of $N_1)(P) = R$ and (the homography of $N_1)(Q) = S$ and (the homography of $N_2)(R) = T$ and (the homography of $N_2)(S) = U$. Then there exists an element h_3 of the subgroup of K-isometries and there exists an invertible square matrix N_3 over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that $h_3 =$ the homography of N_3 and (the homography of $N_3)(P) = T$ and (the homography of $N_3)(Q) = U$. The theorem is a consequence of (46).

(62) Let us consider elements P, Q, R of the BK-model, an element h of the subgroup of K-isometries, and an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose $h =$ the homography of N and (the homography of $N)(P) = R$ and (the homography of $N)(Q) = R$. Then $P = Q$.

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