FORMALIZED MATHEMATICS
Vol. 26, No. 1, Pages 1–10, 2018
DOI: 10.2478/forma-2018-0001

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Introduction to Stochastic Finance: Random Variables and Arbitrage Theory

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Summary. Using the Mizar system [1], [6], we start to show, that the Call-Option, the Put-Option and the Straddle (more generally defined as in the literature) are random variables ([5], p. 15). Next we construct and prove the simple random variables ([2], p. 14).

In the second part, we introduce the definition of arbitrage opportunity. Next we show, that this definition can be characterized in a different way (Lemma 1.3. in [5], p. 5). In our formalization for Lemma 1.3 we make the assumption that φ is a sequence of real numbers (there are only finitley many valued of interest, the values of φ in R^d). For the definition of almost sure with probability 1 see p. 6 in [2]. Last we introduce the risk-neutral probability (Definition 1.4, p. 6 in [5]). We give an example in real world: Suppose you have some assets like bonds (riskless assets). Then we can fix our price for these bonds with x for today and $x(1+r)$ for tomorrow, r is the interest rate. So we simply assume, that in every possible market evolution of tomorrow we have a determinated value. Then every probability measure of Ω_{fut1} is a risk-neutral measure (see FINANCE7???:13). This example shows the existence of some risk-neutral measure. If you find more than one of them, you can determine - with an additional conidition to the probability measures - whether a market model is arbitrage free or not (see Theorem 1.6. in [5], p. 6.)

A short graph for FINANCE7???:13:

Suppose we have a portfolio with many (in this example infinitely many) assets. For asset d we have the price $jpi.d$ for today, and the price $jpi.d(1+r)$ for tomorrow with some interest rate $r > 0$.

Let G be a sequence of random variables on Ω_{fut1}, Borel sets. So you have many functions $f_k : \{1, 2, 3, 4\} \to R$ with $G.k = f_k$ and f_k is a random variable of Ω_{fut1}, Borel sets. For every f_k we have $f_k(w) = jpi.k(1+r)$ for $w \in \{1, 2, 3, 4\}$.

ISSN 1426–2630(Print), 1898-9934(Online)

$$\begin{array}{lll} & \textit{Today} & \textit{Tomorrow} \\ & \text{only one scenario} & \left\{ \begin{array}{l} \\ w_{21} = \{1,2\} \\ w_{22} = \{3,4\} \end{array} \right. \end{array}$$

$$\begin{array}{lll} & \textit{Today} & \textit{Tomorrow} \\ \text{for all } d \in \mathbb{N} \text{ holds}: & jpi.d & \left\{ \begin{array}{l} f_d(w) = (G.d)(w) = jpi.d(1+r), \\ w \in w_{w21} \text{ or } w \in w_{22}, \\ r > 0 \text{ is the interest rate.} \end{array} \right. \end{array}$$

Here, every probability measure of Ω_{fut1} is a risk neutral measure.

MSC: 03B35

Keywords:

MML identifier: `FINANCE6`, version: `8.1.07 5.47.1318`

1. Put-Option, Call-Option and Straddle are Random Variables

From now on Ω denotes a non empty set and F denotes a σ-field of subsets of Ω.

Let t be a real number and n be a natural number. The functor $\boxed{\mathrm{Conv}(n,t)}$ yielding an element of $\mathbb{R}$ is defined by the term

(Def. 1) $\begin{cases} t, & \textbf{if } n = 1, \\ 0, & \textbf{otherwise}. \end{cases}$

The functor $\boxed{\mathrm{Conv2}(n,t)}$ yielding an element of $\mathbb{R}$ is defined by the term

(Def. 2) $\begin{cases} 1, & \textbf{if } n = 0, \\ \mathrm{Conv}(n,t), & \textbf{otherwise}. \end{cases}$

The functor $\boxed{\mathrm{Conv4}(n,t)}$ yielding an element of $\mathbb{R}$ is defined by the term

(Def. 3) $\begin{cases} -1, & \textbf{if } n = 0, \\ \mathrm{Conv}(n,t), & \textbf{otherwise}. \end{cases}$

Now we state the propositions:

(1) $]0, +\infty[$ is an element of the Borel sets.

(2) Let us consider a random variable R of F and the Borel sets, and a real number K. Then $\chi_{(R-(\Omega \longmapsto K))^{-1}([0,+\infty[),\Omega}$ is a function from Ω into $\mathbb{R}$.

(3) Let us consider a random variable R of F and the Borel sets, an element K of $\mathbb{R}$, and a function g_2 from Ω into $\mathbb{R}$. Suppose $g_2 = \chi_{(R-(\Omega \longmapsto K))^{-1}([0,+\infty[),\Omega}$. Then the Call-Option on R and $K = g_2 \cdot (R - (\Omega \longmapsto K))$.

Let us consider Ω and F. Let R be a random variable of F and the Borel sets and K be an element of $\mathbb{R}$. One can check that the Call-Option on R and

K yields a random variable of F and the Borel sets. Let R be a function from Ω into $\mathbb{R}$ and w be an element of Ω. The functor $\boxed{\text{SetForPut-Option}(R, w)}$ yielding an element of $\mathbb{R}$ is defined by the term

(Def. 4) $\begin{cases} R(w), & \textbf{if } R(w) \geqslant 0, \\ 0, & \textbf{otherwise}. \end{cases}$

Let us consider F. Let R be a random variable of F and the Borel sets and K be a real number. The functor $\boxed{\text{Put-Option}(R, K)}$ yielding a function from Ω into $\mathbb{R}$ is defined by

(Def. 5) for every element w of Ω, if $((\Omega \longmapsto K) - R)(w) \geqslant 0$, then $it(w) = ((\Omega \longmapsto K) - R)(w)$ and if $((\Omega \longmapsto K) - R)(w) < 0$, then $it(w) = 0$.

Let us consider a random variable R of F and the Borel sets and a real number K. Now we state the propositions:

(4) $(\Omega \longmapsto K) - R$ is a random variable of F and the Borel sets.

(5) $\chi_{((\Omega \longmapsto K) - R)^{-1}([0, +\infty[), \Omega}$ is a function from Ω into $\mathbb{R}$.

Now we state the proposition:

(6) Let us consider a random variable R of F and the Borel sets, a real number K, and a function g_2 from Ω into $\mathbb{R}$. Suppose $g_2 = \chi_{((\Omega \longmapsto K) - R)^{-1}([0, +\infty[), \Omega}$. Then $\text{Put-Option}(R, K) = g_2 \cdot ((\Omega \longmapsto K) - R)$.

Let us consider Ω and F. Let R be a random variable of F and the Borel sets and K be a real number. Note that the functor Put-Option(R, K) yields a random variable of F and the Borel sets.

2. Simple Random Variables

Now we state the propositions:

(7) Let us consider an element A of F. Then $\chi_{A, \Omega}$ is a random variable of F and the Borel sets.

(8) $\chi_{\Omega, \Omega}$ is random variable on F and the Borel sets.

PROOF: Set $g_2 = \chi_{\Omega, \Omega}$. For every set x such that $x \in$ the Borel sets holds $g_2{}^{-1}(x) \in F$ by [9, (5), (22), (4)]. □

Let us consider Ω and F. Let R be a random variable of F and the Borel sets and K be an element of $\mathbb{R}$. The functor $\boxed{\text{Straddle}(R, K)}$ yielding a random variable of F and the Borel sets is defined by the term

(Def. 6) Put-Option(R, K) + (the Call-Option on R and K).

Now we state the proposition:

(9) Let us consider a random variable R of F and the Borel sets, an element K of $\mathbb{R}$, and an element w of Ω. Then $(\text{Straddle}(R, K))(w) = |(R - (\Omega \longmapsto K))(w)|$.

Let us consider Ω and F. The functor set-of-constant-RV(F) yielding a set is defined by the term

(Def. 7) $\{f$, where f is a function from Ω into $\mathbb{R}$: there exists an element K of $\mathbb{R}$ such that f is random variable on F and the Borel sets and $f = \Omega \longmapsto K\}$.

One can check that set-of-constant-RV(F) is non empty.

The functor set-of-chi-RV(F) yielding a set is defined by the term

(Def. 8) $\{f$, where f is a function from Ω into $\mathbb{R}$: there exists an element A of F such that $\chi_{A,\Omega}$ is random variable on F and the Borel sets and $\chi_{A,\Omega} = f\}$.

One can verify that set-of-chi-RV(F) is non empty.

Let C_1 be a sequence of set-of-chi-RV(F) and n be a natural number. The functor Conv-chi-RV(C_1, n) yielding a random variable of F and the Borel sets is defined by the term

(Def. 9) $C_1(n)$.

Let C_2 be a sequence of set-of-constant-RV(F). The functor Conv-constant-RV(C_2, n) yielding a random variable of F and the Borel sets is defined by the term

(Def. 10) $C_2(n)$.

Let w be an element of Ω. The functor Conv2-constant-RV(C_2, w) yielding a function from $\mathbb{N}$ into $\mathbb{R}$ is defined by

(Def. 11) for every natural number n, $it(n) = ($Conv-constant-RV$(C_2, n))(w)$.

Let C_1 be a sequence of set-of-chi-RV(F). The functor Conv2-chi-RV(C_1, w) yielding a function from $\mathbb{N}$ into $\mathbb{R}$ is defined by

(Def. 12) for every natural number n, $it(n) = ($Conv-chi-RV$(C_1, n))(w)$.

Let C_2 be a sequence of set-of-constant-RV(F) and n be a natural number. The functor simple-RV-help(C_1, C_2, n) yielding a function from Ω into $\mathbb{R}$ is defined by

(Def. 13) for every element w of Ω, $it(w) = (\sum_{\alpha=0}^{\kappa}($Conv2-constant-RV$(C_2, w) \cdot$ Conv2-chi-RV$(C_1, w))(\alpha))_{\kappa \in \mathbb{N}}(n)$.

A simple-RV of C_2, C_1, n is a random variable of F and the Borel sets defined by

(Def. 14) for every element w of Ω, $it(w) = ($simple-RV-help$(C_1, C_2, n))(w)$.

3. Arbitrage Theory: Definition and Alternative Representation

From now on φ denotes a sequence of real numbers and jpi denotes a price function.

Let us consider Ω and F. Let q be a natural number and G be a sequence of the set of random variables on F and the Borel sets. The functor

Change-Element-to-Func(G, q) yielding a real-valued random variable on F is defined by the term

(Def. 15) $G(q)$.

Let us consider a sequence G of the set of random variables on F and the Borel sets and a natural number n. Now we state the propositions:

(10) (The $\mathcal{RV}$-portfolio value for future extension of φ, F, G and $n)^{-1}([0, +\infty[) \in F$. The theorem is a consequence of (1).

(11) (The $\mathcal{RV}$-portfolio value for future extension of φ, F, G and $n)^{-1}(]0, +\infty[) \in F$. The theorem is a consequence of (1).

Let us consider φ, Ω, and F. Let G be a sequence of the set of random variables on F and the Borel sets and n be a natural number. The functors: ArbitrageElSigma1$(\varphi, \Omega, F, G, n)$ and ArbitrageElSigma2$(\varphi, \Omega, F, G, n)$ yielding elements of F are defined by terms

(Def. 16) (the $\mathcal{RV}$-portfolio value for future extension of φ, F, G and $n)^{-1}([0, +\infty[)$,

(Def. 17) (the $\mathcal{RV}$-portfolio value for future extension of φ, F, G and $n)^{-1}(]0, +\infty[)$,

respectively. Let P_1 be a probability on F and *jpi* be a price function. We say that there exists an arbitrage opportunity w.r.t. P_1, G, *jpi* and n if and only if

(Def. 18) there exists a sequence φ of real numbers such that the buy portfolio extension of φ, *jpi*, and $n \leqslant 0$ and $P_1(\text{ArbitrageElSigma1}(\varphi, \Omega, F, G, n)) = 1$ and $P_1(\text{ArbitrageElSigma2}(\varphi, \Omega, F, G, n)) > 0$.

Let r be a real number. The functor RVfirst(r) yielding an element of the set of random variables on Ω_{now} and the Borel sets is defined by the term

(Def. 19) $\{1, 2, 3, 4\} \longmapsto r$.

Let *jpi* be a price function and d be a natural number. The functor RVfourth(jpi, r, d) yielding an element of the set of random variables on Ω_{fut1} and the Borel sets is defined by the term

(Def. 20) $\{1, 2, 3, 4\} \longmapsto jpi(d) \cdot (1 + r)$.

Let n be a natural number. The functor RVswitchsecond(n) yielding an element of the set of random variables on Ω_{now} and the Borel sets is defined by the term

(Def. 21) $\begin{cases} \text{RVfirst}(5), & \textbf{if } n = 1, \\ \text{RVfirst}(0), & \textbf{otherwise}. \end{cases}$

The functor RVswitchfirst(n) yielding an element of the set of random variables on Ω_{now} and the Borel sets is defined by the term

(Def. 22) $\begin{cases} \text{RVfirst}(1), & \textbf{if } n = 0, \\ \text{RVswitchsecond}(n), & \textbf{otherwise}. \end{cases}$

Now we state the propositions:

(12) There exists a sequence G of the set of random variables on Ω_{now} and the Borel sets such that

(i) $G(0) = \{1,2,3,4\} \longmapsto 1$, and

(ii) $G(1) = \{1,2,3,4\} \longmapsto 5$, and

(iii) for every natural number k such that $k > 1$ holds $G(k) = \{1,2,3,4\} \longmapsto 0$.

PROOF: Define $\mathcal{U}(\text{natural number}) = \text{RVswitchfirst}(\$_1)$. Consider f being a sequence of the set of random variables on Ω_{now} and the Borel sets such that for every element d of $\mathbb{N}$, $f(d) = \mathcal{U}(d)$ from [3, Sch. 4]. $f(0) = \text{RVswitchfirst}(0)$. $f(1) = \text{RVswitchfirst}(1)$. For every natural number k such that $k > 1$ holds $f(k) = \{1,2,3,4\} \longmapsto 0$. $\square$

(13) Let us consider a probability P_1 on Ω_{now}, and a sequence G of the set of random variables on Ω_{now} and the Borel sets. Suppose $G(0) = \{1,2,3,4\} \longmapsto 1$ and $G(1) = \{1,2,3,4\} \longmapsto 5$ and for every natural number k such that $k > 1$ holds $G(k) = \{1,2,3,4\} \longmapsto 0$. Then there exists a price function jpi such that there exists an arbitrage opportunity w.r.t. P_1, G, jpi and 1.

PROOF: Set $\Omega = \{1,2,3,4\}$. Set $F = \Omega_{now}$. $P_1(\Omega) = 1$ and $P_1(\emptyset) = 0$ by [9, (30), (4), (31)]. Define $\mathcal{U}(\text{natural number}) = \text{Conv2}(\$_1, 1)$. Consider f being a sequence of real numbers such that for every element d of $\mathbb{N}$, $f(d) = \mathcal{U}(d)$ from [3, Sch. 4]. f is a price function. Consider jpi being a price function such that $jpi = f$. Define $\mathcal{U}(\text{natural number}) = \text{Conv4}(\$_1, 1)$. Consider φ being a sequence of real numbers such that for every element k of $\mathbb{N}$, $\varphi(k) = \mathcal{U}(k)$ from [3, Sch. 4]. $P_1(\text{ArbitrageElSigma1}(\varphi, \Omega, F, G, 1)) = 1$ and $P_1(\text{ArbitrageElSigma2}(\varphi, \Omega, F, G, 1)) > 0$ and the buy portfolio extension of φ, jpi, and $1 \leqslant 0$ by [8, (9)], [10, (7)]. $\square$

(14) Let us consider a natural number n, a real number r, and a sequence G of the set of random variables on F and the Borel sets. Then $\{w$, where w is an element of Ω : the portfolio value for future extension of n, φ, F, G and $w \geqslant 0\}$ = (the $\mathcal{RV}$-portfolio value for future extension of φ, F, G and $n)^{-1}([0, +\infty[)$. The theorem is a consequence of (1).

Let us consider natural numbers d, d_2, a real number r, and a sequence G of the set of random variables on F and the Borel sets. Now we state the propositions:

(15) Suppose $d_2 = d - 1$. Then $\{w$, where w is an element of Ω : the portfolio value for future of d, φ, F, G and $w \geqslant (1 + r) \cdot$ (the buy portfolio of φ, jpi, and $d)\}$ = ((the $\mathcal{RV}$-portfolio value for future of φ, F, G and $d_2) - (\Omega \longmapsto (1 + r) \cdot$ (the buy portfolio of φ, jpi, and $d)))^{-1}([0, +\infty[)$.

PROOF: Set $S_1 = \{w$, where w is an element of Ω : the portfolio value for future of d, φ, F, G and $w \geqslant (1+r) \cdot$ (the buy portfolio of φ, jpi, and d)$\}$. Set $S_2 =$ ((the $\mathcal{RV}$-portfolio value for future of φ, F, G and d_2) $-$ ($\Omega \longmapsto (1+r) \cdot$ (the buy portfolio of φ, jpi, and d)))$^{-1}$($[0, +\infty[$). For every object x, $x \in S_1$ iff $x \in S_2$ by (1), [10, (7)]. $\square$

(16) ((The $\mathcal{RV}$-portfolio value for future of φ, F, G and d_2) $-$ ($\Omega \longmapsto (1+r) \cdot$ (the buy portfolio of φ, jpi, and d)))$^{-1}$($[0, +\infty[$) is an event of F.

Now we state the proposition:

(17) Let us consider a natural number d, a real number r, and a sequence G of the set of random variables on F and the Borel sets. Then $\{w$, where w is an element of Ω : the portfolio value for future extension of d, φ, F, G and $w > 0\}$ = (the $\mathcal{RV}$-portfolio value for future extension of φ, F, G and d)$^{-1}$($]0, +\infty[$). The theorem is a consequence of (1).

Let us consider natural numbers d, d_2, a real number r, and a sequence G of the set of random variables on F and the Borel sets. Now we state the propositions:

(18) Suppose $d_2 = d-1$. Then $\{w$, where w is an element of Ω : the portfolio value for future of d, φ, F, G and $w > (1+r) \cdot$ (the buy portfolio of φ, jpi, and d)$\}$ = ((the $\mathcal{RV}$-portfolio value for future of φ, F, G and d_2) $-$ ($\Omega \longmapsto (1+r) \cdot$ (the buy portfolio of φ, jpi, and d)))$^{-1}$($]0, +\infty[$).
PROOF: Set $S_1 = \{w$, where w is an element of Ω : the portfolio value for future of d, φ, F, G and $w > (1+r) \cdot$ (the buy portfolio of φ, jpi, and d)$\}$. Set $S_2 =$ ((the $\mathcal{RV}$-portfolio value for future of φ, F, G and d_2) $-$ ($\Omega \longmapsto (1+r) \cdot$ (the buy portfolio of φ, jpi, and d)))$^{-1}$($]0, +\infty[$). For every object x, $x \in S_1$ iff $x \in S_2$ by (1), [10, (7)]. $\square$

(19) ((The $\mathcal{RV}$-portfolio value for future of φ, F, G and d_2) $-$ ($\Omega \longmapsto (1+r) \cdot$ (the buy portfolio of φ, jpi, and d)))$^{-1}$($]0, +\infty[$) is an event of F.

Let d, n be natural numbers, jpi be a price function, and φ be a sequence of real numbers. The functor $\boxed{\mathrm{Helpme}(\varphi, jpi, n, d)}$ yielding an element of $\mathbb{R}$ is defined by the term

$$\text{(Def. 23)} \quad \begin{cases} -(\text{the buy portfolio of } \varphi, jpi, \text{ and } d), & \textbf{if } n = 0, \\ \varphi(n), & \textbf{otherwise}. \end{cases}$$

Now we state the proposition:

(20) Let us consider a price function jpi, and natural numbers d, d_2. Suppose $d > 0$ and $d_2 = d - 1$. Let us consider a probability P_1 on F, and a real number r. Suppose $r > -1$. Let us consider a sequence G of the set of random variables on F and the Borel sets. Suppose the element of F, the Borel sets, G, and $0 = \Omega \longmapsto 1 + r$. Then there exists an arbitrage opportunity w.r.t. P_1, G, jpi and d if and only if there exists a sequence

$my\varphi$ of real numbers such that P_1(((the $\mathcal{RV}$-portfolio value for future of $my\varphi$, F, G and d_2) $-$ ($\Omega \longmapsto (1+r) \cdot$ (the buy portfolio of $my\varphi$, jpi, and d)))$^{-1}$($[0, +\infty[$)) $= 1$ and P_1(((the $\mathcal{RV}$-portfolio value for future of $my\varphi$, F, G and d_2) $-$ ($\Omega \longmapsto (1+r) \cdot$ (the buy portfolio of $my\varphi$, jpi, and d)))$^{-1}$($]0, +\infty[$)) > 0.

PROOF: If there exists an arbitrage opportunity w.r.t. P_1, G, jpi and d, then there exists a sequence $my\varphi$ of real numbers such that P_1(((the $\mathcal{RV}$-portfolio value for future of $my\varphi$, F, G and d_2) $-$ ($\Omega \longmapsto (1+r) \cdot$ (the buy portfolio of $my\varphi$, jpi, and d)))$^{-1}$($[0, +\infty[$)) $= 1$ and P_1(((the $\mathcal{RV}$-portfolio value for future of $my\varphi$, F, G and d_2) $-$ ($\Omega \longmapsto (1+r) \cdot$ (the buy portfolio of $my\varphi$, jpi, and d)))$^{-1}$($]0, +\infty[$)) > 0 by (14), (1), [7, (14)], (15). Define $\mathcal{U}$(natural number) $=$ Helpme$(my\varphi, jpi, \$_1, d)$. Consider φ being a sequence of real numbers such that for every element n of $\mathbb{N}$, $\varphi(n) = \mathcal{U}(n)$ from [3, Sch. 4]. For every natural number n, if $n = 0$, then $\varphi(n) = -$(the buy portfolio of $my\varphi$, jpi, and d) and if $n > 0$, then $\varphi(n) = my\varphi(n)$. The buy portfolio extension of φ, jpi, and $d = 0$ by (1), [7, (11)]. P_1(ArbitrageElSigm
1 by (1), [7, (12)], [10, (7)]. $P_1(\text{ArbitrageElSigma2}(\varphi, \Omega, F, G, d)) > 0$ by (1), [7, (12)], [10, (7)]. $\square$

4. Risk-Neutral Probability Measure

Let us consider Ω and F. Let R be a real-valued random variable on F, d be a natural number, and r be a real number. The functor Real-RV(d, r, R) yielding a real-valued random variable on F is defined by the term

(Def. 24) $R \cdot (\frac{1}{1+r})$.

Let jpi be a price function and G be a sequence of the set of random variables on F and the Borel sets. We say that there exists a risk neutral measure w.r.t. G, jpi and
if and only if

(Def. 25) there exists a probability P_1 on F such that for every natural number d, $jpi(d) = E_{P_1}\{\text{Real-RV}(d, r, \text{Change-Element-to-Func}(G, d))\}$.

From now on P_1 denotes a probability on Ω_{fut1}.

Now we state the propositions:

(21) Let us consider a real number r. Suppose $r > 0$. Let us consider a price function jpi, and a natural number d. Then there exists a real-valued random variable f on Ω_{fut1} such that

(i) $f = \{1, 2, 3, 4\} \longmapsto jpi(d) \cdot (1+r)$, and

(ii) f is integrable on P2M P_1, and

(iii) f is simple function in Ω_{fut1}.

PROOF: Set $\Omega_2 = \{1, 2, 3, 4\}$. Define $\mathcal{U}(\text{element of } \Omega_2) = jpi(d) \cdot (1 + r)(\in \mathbb{R})$. Consider f being a function from Ω_2 into $\mathbb{R}$ such that for every element d of Ω_2, $f(d) = \mathcal{U}(d)$ from [3, Sch. 4]. Set $g = \Omega_2 \longmapsto jpi(d) \cdot (1 + r)(\in \mathbb{R})$. $f = g$ by (1), [10, (7)]. f is integrable on P2M P_1 by [7, (9), (3)], [4, (12)], [10, (7)]. □

(22) Let us consider a natural number n, and a real number r. Suppose $r > 0$. Let us consider a price function jpi, a natural number d, and a real-valued random variable R on Ω_{fut1}. Suppose $R = \{1, 2, 3, 4\} \longmapsto jpi(d) \cdot (1 + r)$ and R is integrable on P2M P_1 and R is simple function in Ω_{fut1}. Then $jpi(d) = E_{P_1}\{\text{Real-RV}(d, r, R)\}$.

PROOF: Set $F_2 = \Omega_{fut1}$. $\overline{\mathbb{R}}(R) = R$ and R is non-negative by [10, (7)]. Set $m_3 = jpi(d) \cdot (1 + r)$. for every object x such that $x \in \operatorname{dom} \overline{\mathbb{R}}(R)$ holds $(\overline{\mathbb{R}}(R))(x) = m_3$ and $\operatorname{dom} \overline{\mathbb{R}}(R) \in F_2$ and $0 \leqslant m_3$ by [10, (7)], [9, (5)]. □

(23) Let us consider a real number r. Suppose $r > 0$. Let us consider a price function jpi. Then there exists a sequence G of the set of random variables on Ω_{fut1} and the Borel sets such that for every natural number d, Change-Element-to-Func$(G, d) = \{1, 2, 3, 4\} \longmapsto jpi(d) \cdot (1 + r)$ and Change-Element-to-Func(G, d) is integrable on P2M P_1 and Change-Element-to-Func is simple function in Ω_{fut1}.

PROOF: Define $\mathcal{U}(\text{natural number}) = \text{RVfourth}(jpi, r, \$_1)$. Consider g being a sequence of the set of random variables on Ω_{fut1} and the Borel sets such that for every element d of $\mathbb{N}$, $g(d) = \mathcal{U}(d)$ from [3, Sch. 4]. There exists a real-valued random variable R on Ω_{fut1} such that $R = \{1, 2, 3, 4\} \longmapsto jpi(d) \cdot (1+r)(\in \mathbb{R})$ and R is integrable on P2M P_1 and R is simple function in Ω_{fut1}. □

(24) Let us consider a real number r. Suppose $r > 0$. Let us consider a price function jpi, and a sequence G of the set of random variables on Ω_{fut1} and the Borel sets. Suppose for every natural number d, Change-Element-to-Func$(G, d) = \{1, 2, 3, 4\} \longmapsto jpi(d) \cdot (1 + r)$ and Change-Element-to-Func(G, d) is integrable on P2M P_1 and Change-Element-to-Func(G, d) is simple function in Ω_{fut1}. Then

(i) there exists a risk neutral measure w.r.t. G, jpi and r, and

(ii) for every natural number s, $jpi(s) = E_{P_1}\{\text{Real-RV}(s, r, \text{Change-Element-to-Fun}$

The theorem is a consequence of (22).

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Received March 27, 2018