

Introduction to Stochastic Finance: Random Variables and Arbitrage Theory

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Summary. Using the Mizar system [1], [6], we start to show, that the Call-Option, the Put-Option and the Straddle (more generally defined as in the literature) are random variables ([5], p. 15), see (Def.1) and (Def.2). Next we construct and prove the simple random variables ([2], p. 14) in (Def.8).

In the third Section, we introduce the definition of arbitrage opportunity, see (Def.12). Next we show, that this definition can be characterized in a different way (Lemma 1.3. in [5], p. 5), see (17). In our formalization for Lemma 1.3 we make the assumption that φ is a sequence of real numbers (there are only finitely many values of interest, the values of φ in R^d). For the definition of almost sure with probability 1 see p. 6 in [2]. Last we introduce the risk-neutral probability (Definition 1.4, p. 6 in [5]), here see (Def.16). We give an example in real world: Suppose you have some assets like bonds (riskless assets). Then we can fix our price for these bonds with x for today and $x \cdot (1+r)$ for tomorrow, r is the interest rate. So we simply assume, that in every possible market evolution of tomorrow we have a determined value. Then every probability measure of Ω_{fut1} is a risk-neutral measure, see (21). This example shows the existence of some risk-neutral measure. If you find more than one of them, you can determine - with an additional condition to the probability measures - whether a market model is arbitrage free or not (see Theorem 1.6. in [5], p. 6.)

A short graph for (21):

Suppose we have a portfolio with many (in this example infinitely many) assets. For asset d we have the price $\pi(d)$ for today, and the price $\pi(d) \cdot (1+r)$ for tomorrow with some interest rate $r > 0$.

Let G be a sequence of random variables on Ω_{fut1} , Borel sets. So you have many functions $f_k : \{1, 2, 3, 4\} \rightarrow R$ with $G(k) = f_k$ and f_k is a random variable of Ω_{fut1} , Borel sets. For every f_k we have $f_k(w) = \pi(k) \cdot (1+r)$ for $w \in \{1, 2, 3, 4\}$.

*Today**Tomorrow*

only one scenario

$$\begin{cases} w_{21} = \{1, 2\}, \\ w_{22} = \{3, 4\}, \end{cases}$$

for all $d \in \mathbb{N}$ holds $\pi(d)$

$$\begin{cases} f_d(w) = G(d)(w) = \pi(d) \cdot (1 + r), \\ w \in w_{21} \text{ or } w \in w_{22}, \\ r > 0 \text{ is the interest rate.} \end{cases}$$

Here, every probability measure of Ω_{fut1} is a risk-neutral measure.

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1. PUT-OPTION, CALL-OPTION AND STRADDLE ARE RANDOM VARIABLES

From now on Ω denotes a non empty set and F denotes a σ -field of subsets of Ω .

Now we state the propositions:

- (1) $]0, +\infty[$ is an element of the Borel sets.
- (2) Let us consider a random variable R of F and the Borel sets, an element K of \mathbb{R} , and a function g from Ω into \mathbb{R} . Suppose $g = \chi_{(R - (\Omega \rightarrow K))^{-1}([0, +\infty]), \Omega}$. Then $\text{Call-Option}(R, K) = g \cdot (R - (\Omega \rightarrow K))$.
- (3) Let us consider a random variable R of F and the Borel sets, and a real number K . Then $(\Omega \rightarrow K) - R$ is a random variable of F and the Borel sets.
- (4) Let us consider an element A of F . Then $\chi_{A, \Omega}$ is a random variable of F and the Borel sets.
- (5) $\chi_{\Omega, \Omega}$ is random variable on F and the Borel sets. The theorem is a consequence of (4).
- (6) Let us consider random variables f, R of F and the Borel sets, and a real number K . Then $(f - R)^{-1}([0, +\infty[)$ is an element of F . The theorem is a consequence of (1).

Let us consider Ω and F . Let R be a random variable of F and the Borel sets and K be a real number. Let us note that the functor $\text{Call-Option}(R, K)$ yields a random variable of F and the Borel sets. The functor **Put-Option(R, K)** yielding a function from Ω into \mathbb{R} is defined by

- (Def. 1) for every element w of Ω , if $((\Omega \rightarrow K) - R)(w) \geq 0$, then $it(w) = ((\Omega \rightarrow K) - R)(w)$ and if $((\Omega \rightarrow K) - R)(w) < 0$, then $it(w) = 0$.

Now we state the proposition:

- (7) Let us consider a random variable R of F and the Borel sets, a real number K , and a function g from Ω into \mathbb{R} . Suppose $g = \chi_{((\Omega \mapsto K) - R)^{-1}([0, +\infty])}$. Then $\text{Put-Option}(R, K) = g \cdot ((\Omega \mapsto K) - R)$.

Let us consider Ω and F . Let R be a random variable of F and the Borel sets and K be a real number. Note that the functor $\text{Put-Option}(R, K)$ yields a random variable of F and the Borel sets.

2. SIMPLE RANDOM VARIABLES

Let us consider Ω and F . Let R be a random variable of F and the Borel sets and K be a real number. The functor **Straddle(R, K)** yielding a random variable of F and the Borel sets is defined by the term

- (Def. 2) $\text{Put-Option}(R, K) + \text{Call-Option}(R, K)$.

Now we state the proposition:

- (8) Let us consider a random variable R of F and the Borel sets, a real number K , and an element w of Ω . Then $(\text{Straddle}(R, K))(w) = |(R - (\Omega \mapsto K))(w)|$.

Let us consider Ω and F . The functors: **the set of constants F** and **the set of χ_F** yielding sets are defined by terms

- (Def. 3) $\{f, \text{ where } f \text{ is a function from } \Omega \text{ into } \mathbb{R} : f \text{ is random variable on } F \text{ and the Borel sets and constant}\}$,

- (Def. 4) $\{\chi_{A, \Omega}, \text{ where } A \text{ is an element of } F : \chi_{A, \Omega} \text{ is random variable on } F \text{ and the Borel sets}\}$,

respectively. Let X be a set. We say that **X is F -random membered** if and only if

- (Def. 5) for every object x such that $x \in X$ there exists a function f from Ω into \mathbb{R} such that $f = x$ and f is random variable on F and the Borel sets.

Observe that the set of constants F is non empty and the set of χ_F is non empty and the set of constants F is F -random membered and the set of χ_F is F -random membered and there exists a set which is F -random membered and non empty.

Let D be an F -random membered, non empty set, C_1 be a sequence of D , and n be a natural number. **The change type of C_1 and n** yielding a random variable of F and the Borel sets is defined by the term

- (Def. 6) $C_1(n)$.

Let C_2 be a sequence of D and w be an element of Ω . **The change all types of C_2 and w** yielding a function from \mathbb{N} into \mathbb{R} is defined by

(Def. 7) for every natural number n , $it(n) = (\text{the change type of } C_2 \text{ and } n)(w)$.

Let D_1, D_2 be F -random membered, non empty sets, C_1 be a sequence of D_1 , C_2 be a sequence of D_2 , and n be a natural number. **The simple \mathcal{RV} of C_1, C_2 and n** yielding a function from Ω into \mathbb{R} is defined by

(Def. 8) for every element w of Ω , $it(w) = (\sum_{\alpha=0}^{\kappa} ((\text{the change all types of } C_2 \text{ and } w) \cdot (\text{the change all types of } C_1 \text{ and } w))(\alpha))_{\kappa \in \mathbb{N}}(n)$.

Observe that the simple \mathcal{RV} of C_1, C_2 and n yields a random variable of F and the Borel sets.

3. ARBITRAGE THEORY: DEFINITION AND ALTERNATIVE REPRESENTATION

From now on φ denotes a sequence of real numbers and π denotes a price function.

Let us consider Ω and F . Let q be a natural number and G be a sequence of the set of random variables on F and the Borel sets. **The change element to functions G and q** yielding a real-valued random variable on F is defined by the term

(Def. 9) $G(q)$.

Let us consider φ . Let n be a natural number. The functors: **the first \mathcal{AO} -set of φ, Ω, F** and **the second \mathcal{AO} -set of φ, Ω, F, G and n** yielding elements of F are defined by terms

(Def. 10) $(\text{the } \mathcal{RV}\text{-portfolio value for future extension of } \varphi, F, G \text{ and } n)^{-1}([0, +\infty[)$,

(Def. 11) $(\text{the } \mathcal{RV}\text{-portfolio value for future extension of } \varphi, F, G \text{ and } n)^{-1}(]0, +\infty[)$,

respectively. Let P be a probability on F and π be a price function. We say that **there exists an \mathcal{AO} w.r.t. P, G, π and n** if and only if

(Def. 12) there exists a sequence φ of real numbers such that the buy portfolio extension of φ, π , and $n \leq 0$ and $P(\text{the first } \mathcal{AO}\text{-set of } \varphi, \Omega, F, G \text{ and } n) = 1$ and $P(\text{the second } \mathcal{AO}\text{-set of } \varphi, \Omega, F, G \text{ and } n) > 0$.

Let r be a real number. **The first \mathcal{RV} of r** yielding an element of the set of random variables on Ω_{now} and the Borel sets is defined by the term

(Def. 13) $\{1, 2, 3, 4\} \mapsto r$.

Let π be a price function and d be a natural number. **The first \mathcal{RV} of π, r and d** yielding an element of the set of random variables on Ω_{fut1} and the Borel sets is defined by the term

(Def. 14) the first \mathcal{RV} of $\pi(d) \cdot (1 + r)$.

Now we state the propositions:

(9) There exists a sequence G of the set of random variables on Ω_{now} and the Borel sets such that

- (i) $G(0) = \{1, 2, 3, 4\} \mapsto 1$, and
- (ii) $G(1) = \{1, 2, 3, 4\} \mapsto 5$, and
- (iii) for every natural number k such that $k > 1$ holds $G(k) = \{1, 2, 3, 4\} \mapsto 0$.

PROOF: Define $\mathcal{U}(\text{natural number}) = (\$1 = 0 \rightarrow \text{the first } \mathcal{RV} \text{ of } 1, (\$1 = 1 \rightarrow \text{the first } \mathcal{RV} \text{ of } 5, \text{the first } \mathcal{RV} \text{ of } 0))$. Consider f being a sequence of the set of random variables on Ω_{now} and the Borel sets such that for every element d of \mathbb{N} , $f(d) = \mathcal{U}(d)$ from [3, Sch. 4]. $f(0) = (0 = 0 \rightarrow \text{the first } \mathcal{RV} \text{ of } 1, (0 = 1 \rightarrow \text{the first } \mathcal{RV} \text{ of } 5, \text{the first } \mathcal{RV} \text{ of } 0))$. $f(1) = (1 = 0 \rightarrow \text{the first } \mathcal{RV} \text{ of } 1, (1 = 1 \rightarrow \text{the first } \mathcal{RV} \text{ of } 5, \text{the first } \mathcal{RV} \text{ of } 0))$. For every natural number k such that $k > 1$ holds $f(k) = \{1, 2, 3, 4\} \mapsto 0$. \square

- (10) Let us consider a probability P on Ω_{now} , and a sequence G of the set of random variables on Ω_{now} and the Borel sets. Suppose $G(0) = \{1, 2, 3, 4\} \mapsto 1$ and $G(1) = \{1, 2, 3, 4\} \mapsto 5$ and for every natural number k such that $k > 1$ holds $G(k) = \{1, 2, 3, 4\} \mapsto 0$. Then there exists a price function π such that there exists an \mathcal{AO} w.r.t. P, G, π and 1.

PROOF: Set $\Omega = \{1, 2, 3, 4\}$. Set $F = \Omega_{now}$. $P(\Omega) = 1$ and $P(\emptyset) = 0$ by [9, (30), (4), (31)]. Define $\mathcal{U}(\text{element of } \mathbb{N}) = (\$1 = 0 \rightarrow 1, (\$1 = 1 \rightarrow 1, 0)) (\in \mathbb{R})$. Consider f being a function from \mathbb{N} into \mathbb{R} such that for every element d of \mathbb{N} , $f(d) = \mathcal{U}(d)$ from [3, Sch. 4]. f is a price function. Reconsider $\pi = f$ as a price function. Define $\mathcal{U}(\text{element of } \mathbb{N}) = (\$1 = 0 \rightarrow -1, (\$1 = 1 \rightarrow 1, 0)) (\in \mathbb{R})$. Consider φ being a sequence of real numbers such that for every element k of \mathbb{N} , $\varphi(k) = \mathcal{U}(k)$ from [3, Sch. 4]. $P(\text{the first } \mathcal{AO}\text{-set of } \varphi, \Omega, F, G \text{ and } 1) = 1$ and $P(\text{the second } \mathcal{AO}\text{-set of } \varphi, \Omega, F, G \text{ and } 1) > 0$ and the buy portfolio extension of φ, π , and $1 \leq 0$ by [8, (9)], [10, (7)]. \square

- (11) Let us consider a natural number n , a real number r , and a sequence G of the set of random variables on F and the Borel sets. Then $\{w, \text{ where } w \text{ is an element of } \Omega : \text{the portfolio value for future extension of } n, \varphi, F, G \text{ and } w \geq 0\} = (\text{the } \mathcal{RV}\text{-portfolio value for future extension of } \varphi, F, G \text{ and } n)^{-1}([0, +\infty])$. The theorem is a consequence of (1).

Let us consider natural numbers d, d_1 , a real number r , and a sequence G of the set of random variables on F and the Borel sets. Now we state the propositions:

- (12) Suppose $d_1 = d - 1$. Then $\{w, \text{ where } w \text{ is an element of } \Omega : \text{the portfolio value for future of } d, \varphi, F, G \text{ and } w \geq (1 + r) \cdot (\text{the buy portfolio of } \varphi, \pi, \text{ and } d)\} = ((\text{the } \mathcal{RV}\text{-portfolio value for future of } \varphi, F, G \text{ and } d_1) - (\Omega \mapsto (1 + r) \cdot (\text{the buy portfolio of } \varphi, \pi, \text{ and } d)))^{-1}([0, +\infty])$.

PROOF: Set $S_1 = \{w, \text{ where } w \text{ is an element of } \Omega : \text{ the portfolio value for future of } d, \varphi, F, G \text{ and } w \geq (1+r) \cdot (\text{the buy portfolio of } \varphi, \pi, \text{ and } d)\}$. Set $S_2 = ((\text{the } \mathcal{RV}\text{-portfolio value for future of } \varphi, F, G \text{ and } d_1) - (\Omega \mapsto (1+r) \cdot (\text{the buy portfolio of } \varphi, \pi, \text{ and } d)))^{-1}([0, +\infty[)$. For every object $x, x \in S_1$ iff $x \in S_2$ by (1), [10, (7)]. \square

- (13) $((\text{The } \mathcal{RV}\text{-portfolio value for future of } \varphi, F, G \text{ and } d_1) - (\Omega \mapsto (1+r) \cdot (\text{the buy portfolio of } \varphi, \pi, \text{ and } d)))^{-1}([0, +\infty[)$ is an event of F .

Now we state the proposition:

- (14) Let us consider a natural number d , a real number r , and a sequence G of the set of random variables on F and the Borel sets. Then $\{w, \text{ where } w \text{ is an element of } \Omega : \text{ the portfolio value for future extension of } d, \varphi, F, G \text{ and } w > 0\} = (\text{the } \mathcal{RV}\text{-portfolio value for future extension of } \varphi, F, G \text{ and } d)^{-1}(]0, +\infty[)$. The theorem is a consequence of (1).

Let us consider natural numbers d, d_1 , a real number r , and a sequence G of the set of random variables on F and the Borel sets. Now we state the propositions:

- (15) Suppose $d_1 = d - 1$. Then $\{w, \text{ where } w \text{ is an element of } \Omega : \text{ the portfolio value for future of } d, \varphi, F, G \text{ and } w > (1+r) \cdot (\text{the buy portfolio of } \varphi, \pi, \text{ and } d)\} = ((\text{the } \mathcal{RV}\text{-portfolio value for future of } \varphi, F, G \text{ and } d_1) - (\Omega \mapsto (1+r) \cdot (\text{the buy portfolio of } \varphi, \pi, \text{ and } d)))^{-1}(]0, +\infty[)$.

PROOF: Set $S_1 = \{w, \text{ where } w \text{ is an element of } \Omega : \text{ the portfolio value for future of } d, \varphi, F, G \text{ and } w > (1+r) \cdot (\text{the buy portfolio of } \varphi, \pi, \text{ and } d)\}$. Set $S_2 = ((\text{the } \mathcal{RV}\text{-portfolio value for future of } \varphi, F, G \text{ and } d_1) - (\Omega \mapsto (1+r) \cdot (\text{the buy portfolio of } \varphi, \pi, \text{ and } d)))^{-1}(]0, +\infty[)$. For every object $x, x \in S_1$ iff $x \in S_2$ by (1), [10, (7)]. \square

- (16) $((\text{The } \mathcal{RV}\text{-portfolio value for future of } \varphi, F, G \text{ and } d_1) - (\Omega \mapsto (1+r) \cdot (\text{the buy portfolio of } \varphi, \pi, \text{ and } d)))^{-1}(]0, +\infty[)$ is an event of F .

Now we state the proposition:

- (17) Let us consider a price function π , and natural numbers d, d_1 . Suppose $d > 0$ and $d_1 = d - 1$. Let us consider a probability P on F , and a real number r . Suppose $r > -1$. Let us consider a sequence G of the set of random variables on F and the Borel sets. Suppose the element of F , the Borel sets, G , and $0 = \Omega \mapsto 1 + r$. Then there exists an \mathcal{AO} w.r.t. P, G, π and d if and only if there exists a sequence φ_1 of real numbers such that $P(((\text{the } \mathcal{RV}\text{-portfolio value for future of } \varphi_1, F, G \text{ and } d_1) - (\Omega \mapsto (1+r) \cdot (\text{the buy portfolio of } \varphi_1, \pi, \text{ and } d)))^{-1}([0, +\infty[)) = 1$ and $P(((\text{the } \mathcal{RV}\text{-portfolio value for future of } \varphi_1, F, G \text{ and } d_1) - (\Omega \mapsto (1+r) \cdot (\text{the buy portfolio of } \varphi_1, \pi, \text{ and } d)))^{-1}(]0, +\infty[)) > 0$.

PROOF: If there exists an \mathcal{AO} w.r.t. P, G, π and d , then there exists

a sequence φ_1 of real numbers such that $P(((\text{the } \mathcal{RV}\text{-portfolio value for future of } \varphi_1, F, G \text{ and } d_1) - (\Omega \mapsto (1+r) \cdot (\text{the buy portfolio of } \varphi_1, \pi, \text{ and } d)))^{-1}([0, +\infty[)) = 1$ and $P(((\text{the } \mathcal{RV}\text{-portfolio value for future of } \varphi_1, F, G \text{ and } d_1) - (\Omega \mapsto (1+r) \cdot (\text{the buy portfolio of } \varphi_1, \pi, \text{ and } d)))^{-1}(]0, +\infty[)) > 0$ by (11), (1), [7, (14)], (12). Define \mathcal{U} (natural number) = $(\$_1 = 0 \rightarrow -(\text{the buy portfolio of } \varphi_1, \pi, \text{ and } d), \varphi_1(\$_1)) \in \mathbb{R}$. Consider φ being a sequence of real numbers such that for every element n of \mathbb{N} , $\varphi(n) = \mathcal{U}(n)$ from [3, Sch. 4]. For every natural number n , if $n = 0$, then $\varphi(n) = -(\text{the buy portfolio of } \varphi_1, \pi, \text{ and } d)$ and if $n > 0$, then $\varphi(n) = \varphi_1(n)$. The buy portfolio extension of φ, π , and $d = 0$ by (1), [7, (11)]. $P(\text{the first } \mathcal{AO}\text{-set of } \varphi, \Omega, F, G \text{ and } d) = 1$ by (1), [7, (12)], [10, (7)]. $P(\text{the second } \mathcal{AO}\text{-set of } \varphi, \Omega, F, G \text{ and } d) > 0$ by (1), [7, (12)], [10, (7)]. \square

4. RISK-NEUTRAL PROBABILITY MEASURE

Let us consider Ω and F . Let R be a real-valued random variable on F and r be a real number. **The r -discounted value of R** yielding a real-valued random variable on F is defined by the term

(Def. 15) $R \cdot \left(\frac{1}{1+r}\right)$.

Let π be a price function and G be a sequence of the set of random variables on F and the Borel sets. We say that **there exists a risk neutral measure w.r.t. G, π and r** if and only if

(Def. 16) there exists a probability P on F such that for every natural number d , $\pi(d) = E_P\{\text{the } r\text{-discounted value of (the change element to functions } G \text{ and } d)\}$.

From now on P denotes a probability on Ω_{fut1} .

Now we state the propositions:

(18) Let us consider a real number r . Suppose $r > 0$. Let us consider a price function π , and a natural number d . Then there exists a real-valued random variable f on Ω_{fut1} such that

- (i) $f = \{1, 2, 3, 4\} \mapsto \pi(d) \cdot (1+r)$, and
- (ii) f is integrable on $P2M(P)$, and
- (iii) f is simple function in Ω_{fut1} .

PROOF: Set $\Omega_2 = \{1, 2, 3, 4\}$. Define \mathcal{U} (element of Ω_2) = $\pi(d) \cdot (1+r) \in \mathbb{R}$. Consider f being a function from Ω_2 into \mathbb{R} such that for every element d of Ω_2 , $f(d) = \mathcal{U}(d)$ from [3, Sch. 4]. Set $g = \Omega_2 \mapsto \pi(d) \cdot (1+r) \in \mathbb{R}$. For

every object x such that $x \in \text{dom } f$ holds $f(x) = g(x)$ by (1), [10, (7)]. f is integrable on $\text{P2M}(P)$ by (4), [7, (9), (3)], [4, (12)]. \square

- (19) Let us consider a natural number n , and a real number r . Suppose $r > 0$. Let us consider a price function π , a natural number d , and a real-valued random variable R on Ω_{fut1} . Suppose $R = \{1, 2, 3, 4\} \mapsto \pi(d) \cdot (1 + r)$ and R is integrable on $\text{P2M}(P)$ and R is simple function in Ω_{fut1} . Then $\pi(d) = E_P\{\text{the } r\text{-discounted value of } R\}$.

PROOF: Set $F = \Omega_{fut1}$. $\overline{\mathbb{R}}(R) = R$ and R is non-negative by [10, (7)]. Set $m = \pi(d) \cdot (1 + r)$. for every object x such that $x \in \text{dom } \overline{\mathbb{R}}(R)$ holds $(\overline{\mathbb{R}}(R))(x) = m$ and $\text{dom } \overline{\mathbb{R}}(R) \in F$ and $0 \leq m$ by [10, (7)], [9, (5)]. \square

- (20) Let us consider a real number r . Suppose $r > 0$. Let us consider a price function π . Then there exists a sequence G of the set of random variables on Ω_{fut1} and the Borel sets such that for every natural number d , $G(d) = \{1, 2, 3, 4\} \mapsto \pi(d) \cdot (1 + r)$ and the change element to functions G and d is integrable on $\text{P2M}(P)$ and the change element to functions G and d is simple function in Ω_{fut1} .

PROOF: Define $\mathcal{U}(\text{natural number}) = \text{the first } \mathcal{RV} \text{ of } \pi, r \text{ and } \$_1$. Consider g being a sequence of the set of random variables on Ω_{fut1} and the Borel sets such that for every element d of \mathbb{N} , $g(d) = \mathcal{U}(d)$ from [3, Sch. 4]. There exists a real-valued random variable R on Ω_{fut1} such that $R = \{1, 2, 3, 4\} \mapsto \pi(d) \cdot (1 + r) (\in \mathbb{R})$ and R is integrable on $\text{P2M}(P)$ and R is simple function in Ω_{fut1} . \square

- (21) Let us consider a real number r . Suppose $r > 0$. Let us consider a price function π , and a sequence G of the set of random variables on Ω_{fut1} and the Borel sets. Suppose for every natural number d , $G(d) = \{1, 2, 3, 4\} \mapsto \pi(d) \cdot (1 + r)$ and the change element to functions G and d is integrable on $\text{P2M}(P)$ and the change element to functions G and d is simple function in Ω_{fut1} . Then

- (i) there exists a risk neutral measure w.r.t. G , π and r , and
- (ii) for every natural number s , $\pi(s) = E_P\{\text{the } r\text{-discounted value of (the change element to functions } G \text{ and } s)\}$.

The theorem is a consequence of (19).

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