

# Fubini's Theorem for Non-Negative or Non-Positive Functions

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**Summary.** The goal of this article is to show Fubini's theorem for non-negative or non-positive measurable functions [18], [4], [6], using the Mizar system [3], [17]. We formalized Fubini's theorem in our previous article [13], but in that case we showed the Fubini's theorem for measurable sets and it was not enough as the integral does not appear explicitly.

On the other hand, the theorems obtained in this paper are more general and it can be easily extended to a general integrable function. Furthermore, it also can be easy to extend to functional space  $L^p$  [21]. It should be mentioned also that Hölzl and Heller [19] have introduced the Lebesgue integration theory in Isabelle/HOL and have proved Fubini's theorem there.

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## 1. EXTENDED REAL-VALUED CHARACTERISTIC FUNCTION

Let  $A, X$  be sets and  $e$  be an extended real. The functor  $\chi_{e,A,X}$  yielding a function from  $X$  into  $\overline{\mathbb{R}}$  is defined by

(Def. 1) for every object  $x$  such that  $x \in X$  holds if  $x \in A$ , then  $it(x) = e$  and if  $x \notin A$ , then  $it(x) = 0$ .

Now we state the propositions:

(1) Let us consider a non empty set  $X$ , a set  $A$ , and a real number  $r$ . Then  $r \cdot \chi_{A,X} = \chi_{r,A,X}$ .

- (2) Let us consider a non empty set  $X$ , and a set  $A$ . Then
- (i)  $\chi_{+\infty, A, X} = \bar{\chi}_{A, X}$ , and
  - (ii)  $\chi_{-\infty, A, X} = -\bar{\chi}_{A, X}$ .
- (3) Let us consider sets  $X$ ,  $A$ . Then  $\chi_{A, X}$  is without  $+\infty$  and without  $-\infty$ .
- (4) Let us consider a non empty set  $X$ , a set  $A$ , and a real number  $r$ . Then
- (i)  $\text{rng } \chi_{r, A, X} \subseteq \{0, r\}$ , and
  - (ii)  $\chi_{r, A, X}$  is without  $+\infty$  and without  $-\infty$ .

The theorem is a consequence of (3) and (1).

- (5) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a non empty partial function  $f$  from  $X$  to  $\bar{\mathbb{R}}$ , and a  $\sigma$ -measure  $M$  on  $S$ . Suppose  $f$  is simple function in  $S$ . Then there exists a non empty finite sequence  $E$  of separated subsets of  $S$  and there exists a finite sequence  $a$  of elements of  $\bar{\mathbb{R}}$  and there exists a finite sequence  $r$  of elements of  $\mathbb{R}$  such that  $E$  and  $a$  are representation of  $f$  and for every natural number  $n$ ,  $a(n) = r(n)$  and  $f \upharpoonright E(n) = \chi_{r(n), E(n), X} \upharpoonright E(n)$  and if  $E(n) = \emptyset$ , then  $r(n) = 0$ .

PROOF: Consider  $E$  being a finite sequence of separated subsets of  $S$ ,  $b$  being a finite sequence of elements of  $\bar{\mathbb{R}}$  such that  $E$  and  $b$  are representation of  $f$ . For every natural number  $n$  such that  $E(n) \neq \emptyset$  holds  $b(n) \in \mathbb{R}$  by [8, (3)], [16, (32)]. Define  $\mathcal{Q}[\text{natural number, object}] \equiv$  if  $E(\$_1) \neq \emptyset$ , then  $\$_2 = b(\$_1)$  and if  $E(\$_1) = \emptyset$ , then  $\$_2 = 0$ . For every natural number  $n$  such that  $n \in \text{Seg len } E$  there exists an element  $a$  of  $\bar{\mathbb{R}}$  such that  $\mathcal{Q}[n, a]$ . Consider  $a$  being a finite sequence of elements of  $\bar{\mathbb{R}}$  such that  $\text{dom } a = \text{Seg len } E$  and for every natural number  $n$  such that  $n \in \text{Seg len } E$  holds  $\mathcal{Q}[n, a(n)]$  from [2, Sch. 5]. Define  $\mathcal{R}[\text{natural number, object}] \equiv \$_2 = a(\$_1)$ . For every natural number  $n$  such that  $n \in \text{Seg len } E$  there exists an element  $r$  of  $\mathbb{R}$  such that  $\mathcal{R}[n, r]$ . Consider  $r$  being a finite sequence of elements of  $\mathbb{R}$  such that  $\text{dom } r = \text{Seg len } E$  and for every natural number  $n$  such that  $n \in \text{Seg len } E$  holds  $\mathcal{R}[n, r(n)]$  from [2, Sch. 5]. For every natural number  $n$  such that  $n \in \text{dom } E$  for every object  $x$  such that  $x \in E(n)$  holds  $f(x) = a(n)$ . For every natural number  $n$ ,  $a(n) = r(n)$  and  $f \upharpoonright E(n) = \chi_{r(n), E(n), X} \upharpoonright E(n)$  and if  $E(n) = \emptyset$ , then  $r(n) = 0$  by [8, (3)], [11, (74)], [22, (62)], [8, (49), (47)].  $\square$

Let  $F$  be a finite sequence-like function. Let us observe that  $F$  is disjoint valued if and only if the condition (Def. 2) is satisfied.

- (Def. 2) for every natural numbers  $m, n$  such that  $m, n \in \text{dom } F$  and  $m \neq n$  holds  $F(m)$  misses  $F(n)$ .

Now we state the propositions:

- (6) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , and elements  $E_1, E_2$  of  $S$ . Suppose  $E_1$  misses  $E_2$ . Then  $\langle E_1, E_2 \rangle$  is a finite sequence of separated subsets of  $S$ .
- (7) Let us consider a non empty set  $X$ , subsets  $A_1, A_2$  of  $X$ , and real numbers  $r_1, r_2$ . Then  $\langle \chi_{r_1, A_1, X}, \chi_{r_2, A_2, X} \rangle$  is a summable finite sequence of elements of  $\overline{\mathbb{R}}^X$ . The theorem is a consequence of (4).
- (8) Let us consider a non empty set  $X$ , and a summable finite sequence  $F$  of elements of  $\overline{\mathbb{R}}^X$ . Suppose  $\text{len } F \geq 2$ . Then  $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_2 = F_1 + F_2$ .
- (9) Let us consider a non empty set  $X$ , and a function  $f$  from  $X$  into  $\overline{\mathbb{R}}$ . Then  $f + (X \mapsto 0_{\overline{\mathbb{R}}}) = f$ .
- (10) Let us consider a non empty set  $X$ , and a summable finite sequence  $F$  of elements of  $\overline{\mathbb{R}}^X$ . Then
- (i)  $\text{dom } F = \text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}$ , and
  - (ii) for every natural number  $n$  such that  $n \in \text{dom } F$  holds  $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_n = (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$ , and
  - (iii) for every natural number  $n$  and for every element  $x$  of  $X$  such that  $1 \leq n < \text{len } F$  holds  $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{n+1}(x) = ((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_n(x) + F_{n+1}(x)$ .

PROOF: For every natural number  $n$  and for every element  $x$  of  $X$  such that  $1 \leq n < \text{len } F$  holds  $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{n+1}(x) = ((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_n(x) + F_{n+1}(x)$  by [1, (13)], [20, (25)].  $\square$

- (11) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a function  $f$  from  $X$  into  $\overline{\mathbb{R}}$ , a finite sequence  $E$  of separated subsets of  $S$ , and a summable finite sequence  $F$  of elements of  $\overline{\mathbb{R}}^X$ . Suppose  $\text{dom } E = \text{dom } F$  and  $\text{dom } f = \bigcup \text{rng } E$  and for every natural number  $n$  such that  $n \in \text{dom } F$  there exists a real number  $r$  such that  $F_n = r \cdot \chi_{E(n), X}$  and  $f = ((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{\text{len } F}$ . Then
- (i) for every element  $x$  of  $X$  and for every natural numbers  $m, n$  such that  $m, n \in \text{dom } F$  and  $x \in E(m)$  and  $m \neq n$  holds  $F_n(x) = 0$ , and
  - (ii) for every element  $x$  of  $X$  and for every natural numbers  $m, n$  such that  $m, n \in \text{dom } F$  and  $x \in E(m)$  and  $n < m$  holds  $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_n(x) = 0$ , and
  - (iii) for every element  $x$  of  $X$  and for every natural numbers  $m, n$  such that  $m, n \in \text{dom } F$  and  $x \in E(m)$  and  $n \geq m$  holds  $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_n(x) = f(x)$ , and
  - (iv) for every element  $x$  of  $X$  and for every natural number  $m$  such that  $m \in \text{dom } F$  and  $x \in E(m)$  holds  $F_m(x) = f(x)$ , and

(v)  $f$  is simple function in  $S$ .

PROOF: For every element  $x$  of  $X$  and for every natural numbers  $m, n$  such that  $m, n \in \text{dom } F$  and  $x \in E(m)$  and  $m \neq n$  holds  $F_n(x) = 0$ . For every element  $x$  of  $X$  and for every natural numbers  $m, n$  such that  $m, n \in \text{dom } F$  and  $x \in E(m)$  and  $n < m$  holds  $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_n(x) = 0$  by [20, (25)], (10), [1, (13)]. For every element  $x$  of  $X$  and for every natural numbers  $m, n$  such that  $m, n \in \text{dom } F$  and  $x \in E(m)$  and  $n \geq m$  holds  $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_n(x) = f(x)$  by [20, (25)], (10), [1, (13)]. For every element  $x$  of  $X$  and for every natural number  $m$  such that  $m \in \text{dom } F$  and  $x \in E(m)$  holds  $F_m(x) = f(x)$  by [20, (25)], (10), [1, (13), (19)]. For every element  $x$  of  $X$  such that  $x \in \text{dom } f$  holds  $|f(x)| < +\infty$  by [15, (41)]. For every natural number  $n$  and for every elements  $x, y$  of  $X$  such that  $n \in \text{dom } E$  and  $x, y \in E(n)$  holds  $f(x) = f(y)$ .  $\square$

- (12) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , and an element  $E$  of  $S$ . Then  $\chi_{E,X}$  is simple function in  $S$ .

PROOF: Reconsider  $E_2 = X \setminus E$  as an element of  $S$ . Reconsider  $E_3 = \langle E, E_2 \rangle$  as a finite sequence of separated subsets of  $S$ .  $1 \cdot \chi_{E,X} = \chi_{1,E_3}$  and  $0 \cdot \chi_{E_2,X} = \chi_{0,E_3}$ . Reconsider  $F = \langle 1 \cdot \chi_{E,X}, 0 \cdot \chi_{E_2,X} \rangle$  as a summable finite sequence of elements of  $\overline{\mathbb{R}}^X$ . For every natural number  $n$  such that  $n \in \text{dom } F$  there exists a real number  $r$  such that  $F_n = r \cdot \chi_{E_3(n),X}$  by [2, (44)].  $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{\text{len } F} = F_1 + F_2$ .  $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{\text{len } F} = \chi_{E,X}$ .  $\square$

- (13) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , elements  $A, B$  of  $S$ , and an extended real  $e$ . Then  $\chi_{e,A,X}$  is measurable on  $B$ . The theorem is a consequence of (2) and (1).
- (14) Let us consider a set  $X$ , subsets  $A_1, A_2$  of  $X$ , and an extended real  $e$ . Then  $\chi_{e,A_1,X} \upharpoonright A_2 = \chi_{e,A_1 \cap A_2,X} \upharpoonright A_2$ .
- (15) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , elements  $A, B, C$  of  $S$ , and an extended real  $e$ . If  $C \subseteq B$ , then  $\chi_{e,A,X} \upharpoonright B$  is measurable on  $C$ . The theorem is a consequence of (13).
- (16) Let us consider a set  $X$ , subsets  $A_1, A_2$  of  $X$ , an extended real  $e$ , and an object  $x$ . If  $A_1$  misses  $A_2$ , then  $(\chi_{e,A_1,X} \upharpoonright A_2)(x) = 0$ .
- (17) Let us consider a set  $X$ , a subset  $A$  of  $X$ , and an extended real  $e$ . Then
- (i) if  $e \geq 0$ , then  $\chi_{e,A,X}$  is non-negative, and
  - (ii) if  $e \leq 0$ , then  $\chi_{e,A,X}$  is non-positive.
- (18) Let us consider sets  $A, X$ , and a subset  $B$  of  $X$ . Then  $\text{dom}(\chi_{A,X} \upharpoonright B) = B$ .

## 2. SOME PROPERTIES OF INTEGRATION

Now we state the propositions:

- (19) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , and a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . If  $f$  is empty, then  $f$  is simple function in  $S$ .

PROOF: Reconsider  $E_4 = \emptyset$  as an element of  $S$ . Reconsider  $F = \langle E_4 \rangle$  as a finite sequence of separated subsets of  $S$ . For every natural number  $n$  and for every elements  $x, y$  of  $X$  such that  $n \in \text{dom } F$  and  $x, y \in F(n)$  holds  $f(x) = f(y)$  by [2, (2), (38), (40)].  $\square$

- (20) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , and elements  $E_1, E_2$  of  $S$ . Then  $\int \chi_{E_1, X} \upharpoonright E_2 \, dM = M(E_1 \cap E_2)$ .

Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , elements  $E_1, E_2$  of  $S$ , and partial functions  $f, g$  from  $X$  to  $\overline{\mathbb{R}}$ . Now we state the propositions:

- (21) Suppose  $E_1 = \text{dom } f$  and  $f$  is non-negative and  $f$  is measurable on  $E_1$  and  $E_2 = \text{dom } g$  and  $g$  is non-negative and  $g$  is measurable on  $E_2$ . Then  $\int f + g \, dM = \int f \upharpoonright \text{dom}(f + g) \, dM + \int g \upharpoonright \text{dom}(f + g) \, dM$ .
- (22) Suppose  $E_1 = \text{dom } f$  and  $f$  is non-positive and  $f$  is measurable on  $E_1$  and  $E_2 = \text{dom } g$  and  $g$  is non-positive and  $g$  is measurable on  $E_2$ . Then  $\int f + g \, dM = \int f \upharpoonright \text{dom}(f + g) \, dM + \int g \upharpoonright \text{dom}(f + g) \, dM$ . The theorem is a consequence of (21).
- (23) Suppose  $E_1 = \text{dom } f$  and  $f$  is non-negative and  $f$  is measurable on  $E_1$  and  $E_2 = \text{dom } g$  and  $g$  is non-positive and  $g$  is measurable on  $E_2$ . Then
- (i)  $\int f - g \, dM = \int f \upharpoonright \text{dom}(f - g) \, dM - \int g \upharpoonright \text{dom}(f - g) \, dM$ , and
  - (ii)  $\int g - f \, dM = \int g \upharpoonright \text{dom}(g - f) \, dM - \int f \upharpoonright \text{dom}(g - f) \, dM$ .

The theorem is a consequence of (21).

Now we state the proposition:

- (24) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , an element  $E$  of  $S$ , a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ , and a real number  $r$ . Suppose  $E = \text{dom } f$  and  $f$  is non-positive or non-negative and  $f$  is measurable on  $E$ . Then  $\int r \cdot f \, dM = r \cdot (\int f \, dM)$ .

## 3. SECTIONS OF PARTIAL FUNCTION

Now we state the proposition:

(25) Let us consider non empty sets  $X, Y$ , an element  $A$  of  $2^{X \times Y}$ , and sets  $x, y$ . Suppose  $x \in X$  and  $y \in Y$ . Then

(i)  $\langle x, y \rangle \in A$  iff  $x \in Y\text{section}(A, y)$ , and

(ii)  $\langle x, y \rangle \in A$  iff  $y \in X\text{section}(A, x)$ .

Let  $X_1, X_2$  be non empty sets,  $Y$  be a set,  $f$  be a partial function from  $X_1 \times X_2$  to  $Y$ , and  $x$  be an element of  $X_1$ . The functor  $\text{ProjPMap1}(f, x)$  yielding a partial function from  $X_2$  to  $Y$  is defined by

(Def. 3)  $\text{dom } it = X\text{section}(\text{dom } f, x)$  and for every element  $y$  of  $X_2$  such that  $\langle x, y \rangle \in \text{dom } f$  holds  $it(y) = f(x, y)$ .

Let  $y$  be an element of  $X_2$ . The functor  $\text{ProjPMap2}(f, y)$  yielding a partial function from  $X_1$  to  $Y$  is defined by

(Def. 4)  $\text{dom } it = Y\text{section}(\text{dom } f, y)$  and for every element  $x$  of  $X_1$  such that  $\langle x, y \rangle \in \text{dom } f$  holds  $it(x) = f(x, y)$ .

Now we state the propositions:

(26) Let us consider non empty sets  $X_1, X_2$ , a set  $Y$ , a partial function  $f$  from  $X_1 \times X_2$  to  $Y$ , an element  $x$  of  $X_1$ , and an element  $y$  of  $X_2$ . Then

(i) if  $x \in \text{dom } \text{ProjPMap2}(f, y)$ , then  $(\text{ProjPMap2}(f, y))(x) = f(x, y)$ , and

(ii) if  $y \in \text{dom } \text{ProjPMap1}(f, x)$ , then  $(\text{ProjPMap1}(f, x))(y) = f(x, y)$ .

(27) Let us consider non empty sets  $X_1, X_2, Y$ , a function  $f$  from  $X_1 \times X_2$  into  $Y$ , an element  $x$  of  $X_1$ , and an element  $y$  of  $X_2$ . Then

(i)  $\text{ProjPMap1}(f, x) = \text{curry}(f, x)$ , and

(ii)  $\text{ProjPMap2}(f, y) = \text{curry}'(f, y)$ .

The theorem is a consequence of (26).

(28) Let us consider non empty sets  $X, Y, Z$ , a partial function  $f$  from  $X \times Y$  to  $Z$ , an element  $x$  of  $X$ , an element  $y$  of  $Y$ , and a set  $A$ . Then

(i)  $X\text{section}(f^{-1}(A), x) = (\text{ProjPMap1}(f, x))^{-1}(A)$ , and

(ii)  $Y\text{section}(f^{-1}(A), y) = (\text{ProjPMap2}(f, y))^{-1}(A)$ .

(29) Let us consider non empty sets  $X_1, X_2$ , an element  $x$  of  $X_1$ , an element  $y$  of  $X_2$ , a real number  $r$ , and a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ . Then

(i)  $\text{ProjPMap1}(r \cdot f, x) = r \cdot \text{ProjPMap1}(f, x)$ , and

(ii)  $\text{ProjPMap2}(r \cdot f, y) = r \cdot \text{ProjPMap2}(f, y)$ .

(30) Let us consider non empty sets  $X_1, X_2$ , a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ , an element  $x$  of  $X_1$ , and an element  $y$  of  $X_2$ . Suppose for every element  $z$  of  $X_1 \times X_2$  such that  $z \in \text{dom } f$  holds  $f(z) = 0$ . Then

- (i)  $(\text{ProjPMap2}(f, y))(x) = 0$ , and
  - (ii)  $(\text{ProjPMap1}(f, x))(y) = 0$ .
- (31) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , an element  $x$  of  $X_1$ , an element  $y$  of  $X_2$ , and a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ . Suppose  $f$  is simple function in  $\sigma(\text{MeasRect}(S_1, S_2))$ . Then

- (i)  $\text{ProjPMap1}(f, x)$  is simple function in  $S_2$ , and
- (ii)  $\text{ProjPMap2}(f, y)$  is simple function in  $S_1$ .

PROOF: Consider  $F$  being a finite sequence of separated subsets of  $\sigma(\text{MeasRect}(S_1, S_2))$  such that  $\text{dom } f = \bigcup \text{rng } F$  and for every natural number  $n$  and for every elements  $z_1, z_2$  of  $X_1 \times X_2$  such that  $n \in \text{dom } F$  and  $z_1, z_2 \in F(n)$  holds  $f(z_1) = f(z_2)$ . Define  $\mathcal{F}_\infty(\text{natural number}) = \text{MeasurableXsection}(F(\$1), x)$ . Consider  $F_1$  being a finite sequence of elements of  $S_2$  such that  $\text{len } F_1 = \text{len } F$  and for every natural number  $n$  such that  $n \in \text{dom } F_1$  holds  $F_1(n) = F(n)$  from [7, Sch. 1]. Reconsider  $F_2 = F$  as a finite sequence of elements of  $2^{X_1 \times X_2}$ . Reconsider  $F_3 = F_1$  as a finite sequence of elements of  $2^{X_2}$ . For every natural number  $n$  such that  $n \in \text{dom } F_3$  holds  $F_3(n) = \text{Xsection}(F_2(n), x)$ . For every natural number  $n$  and for every elements  $y_1, y_2$  of  $X_2$  such that  $n \in \text{dom } F_1$  and  $y_1, y_2 \in F_1(n)$  holds  $(\text{ProjPMap1}(f, x))(y_1) = (\text{ProjPMap1}(f, x))(y_2)$  by [8, (3)], [13, (20)], (26). Define  $\mathcal{G}(\text{natural number}) = \text{MeasurableYsection}(F(\$1), y)$ . Consider  $G$  being a finite sequence of elements of  $S_1$  such that  $\text{len } G = \text{len } F$  and for every natural number  $n$  such that  $n \in \text{dom } G$  holds  $G(n) = \mathcal{G}(n)$  from [7, Sch. 1]. Reconsider  $G_1 = G$  as a finite sequence of elements of  $2^{X_1}$ . For every natural number  $n$  such that  $n \in \text{dom } G_1$  holds  $G_1(n) = \text{Ysection}(F_2(n), y)$ . For every natural number  $n$  and for every elements  $x_1, x_2$  of  $X_1$  such that  $n \in \text{dom } G$  and  $x_1, x_2 \in G(n)$  holds  $(\text{ProjPMap2}(f, y))(x_1) = (\text{ProjPMap2}(f, y))(x_2)$  by [8, (3)], [13, (21)], (26).  $\square$

Let us consider non empty sets  $X_1, X_2$ , an element  $x$  of  $X_1$ , an element  $y$  of  $X_2$ , and a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ . Now we state the propositions:

- (32) If  $f$  is non-negative, then  $\text{ProjPMap1}(f, x)$  is non-negative and  $\text{ProjPMap2}(f, y)$  is non-negative.

PROOF: For every object  $q$  such that  $q \in \text{dom } \text{ProjPMap1}(f, x)$  holds  $0 \leq (\text{ProjPMap1}(f, x))(q)$  by (26), [5, (51)]. For every object  $p$  such that  $p \in \text{dom } \text{ProjPMap2}(f, y)$  holds  $0 \leq (\text{ProjPMap2}(f, y))(p)$  by (26), [5, (51)].  $\square$

- (33) If  $f$  is non-positive, then  $\text{ProjPMap1}(f, x)$  is non-positive and  $\text{ProjPMap2}(f, y)$  is non-positive.

PROOF: For every set  $q$  such that  $q \in \text{dom ProjPMap1}(f, x)$  holds  $0 \geq (\text{ProjPMap1}(f, x))(q)$  by (26), [14, (8)]. For every set  $p$  such that  $p \in \text{dom ProjPMap2}(f, y)$  holds  $0 \geq (\text{ProjPMap2}(f, y))(p)$  by (26), [14, (8)].  
□

Now we state the propositions:

(34) Let us consider non empty sets  $X_1, X_2$ , an element  $x$  of  $X_1$ , an element  $y$  of  $X_2$ , a subset  $A$  of  $X_1 \times X_2$ , and a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ . Then

- (i)  $\text{ProjPMap1}(f \upharpoonright A, x) = \text{ProjPMap1}(f, x) \upharpoonright \text{Xsection}(A, x)$ , and
- (ii)  $\text{ProjPMap2}(f \upharpoonright A, y) = \text{ProjPMap2}(f, y) \upharpoonright \text{Ysection}(A, y)$ .

The theorem is a consequence of (25).

(35) Let us consider non empty sets  $X_1, X_2$ , a subset  $A$  of  $X_1 \times X_2$ , an element  $x$  of  $X_1$ , and an element  $y$  of  $X_2$ . Then

- (i)  $\text{ProjPMap1}(\overline{\chi}_{A, X_1 \times X_2}, x) = \overline{\chi}_{\text{Xsection}(A, x), X_2}$ , and
- (ii)  $\text{ProjPMap2}(\overline{\chi}_{A, X_1 \times X_2}, y) = \overline{\chi}_{\text{Ysection}(A, y), X_1}$ .

The theorem is a consequence of (27) and (25).

(36) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , partial functions  $f, g$  from  $X$  to  $\overline{\mathbb{R}}$ , and an element  $E$  of  $S$ . Suppose  $f \upharpoonright E = g \upharpoonright E$  and  $E \subseteq \text{dom } f$  and  $E \subseteq \text{dom } g$  and  $f$  is measurable on  $E$ . Then  $g$  is measurable on  $E$ .

(37) Let us consider non empty sets  $X_1, X_2$ , a subset  $A$  of  $X_1 \times X_2$ , a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ , an element  $x$  of  $X_1$ , an element  $y$  of  $X_2$ , and a sequence  $F$  of partial functions from  $X_1 \times X_2$  into  $\overline{\mathbb{R}}$ . Suppose  $A \subseteq \text{dom } f$  and for every natural number  $n$ ,  $\text{dom}(F(n)) = A$  and for every element  $z$  of  $X_1 \times X_2$  such that  $z \in A$  holds  $F \# z$  is convergent and  $\lim(F \# z) = f(z)$ . Then

- (i) there exists a sequence  $F_5$  of partial functions from  $X_1$  into  $\overline{\mathbb{R}}$  with the same dom such that for every natural number  $n$ ,  $F_5(n) = \text{ProjPMap2}(F(n), y)$  and for every element  $x$  of  $X_1$  such that  $x \in \text{Ysection}(A, y)$  holds  $F_5 \# x$  is convergent and  $(\text{ProjPMap2}(f, y))(x) = \lim(F_5 \# x)$ , and
- (ii) there exists a sequence  $F_6$  of partial functions from  $X_2$  into  $\overline{\mathbb{R}}$  with the same dom such that for every natural number  $n$ ,  $F_6(n) = \text{ProjPMap1}(F(n), x)$  and for every element  $y$  of  $X_2$  such that  $y \in \text{Xsection}(A, x)$  holds  $F_6 \# y$  is convergent and  $(\text{ProjPMap1}(f, x))(y) = \lim(F_6 \# y)$ .

PROOF: Define  $\mathcal{R}[\text{element of } \mathbb{N}, \text{object}] \equiv \$2 = \text{ProjPMap2}(F(\$1), y)$ . For every element  $n$  of  $\mathbb{N}$ , there exists an element  $f$  of  $X_1 \rightarrow \overline{\mathbb{R}}$  such that  $\mathcal{R}[n, f]$  by [10, (45)]. There exists a sequence  $F_5$  of partial functions from



$X_1$  into  $\overline{\mathbb{R}}$  with the same dom such that for every natural number  $n$ ,  $F_5(n) = \text{ProjPMap2}(F(n), y)$  and for every element  $x$  of  $X_1$  such that  $x \in \text{Ysection}(A, y)$  holds  $F_5 \# x$  is convergent and  $(\text{ProjPMap2}(f, y))(x) = \lim(F_5 \# x)$  by (25), [9, (63)]. Define  $\mathcal{Q}[\text{element of } \mathbb{N}, \text{object}] \equiv \mathcal{S}_2 = \text{ProjPMap1}(F(\mathcal{S}_1))$ . For every element  $n$  of  $\mathbb{N}$ , there exists an element  $f$  of  $X_2 \rightarrow \overline{\mathbb{R}}$  such that  $\mathcal{Q}[n, f]$  by [10, (45)]. Consider  $F_6$  being a sequence of  $X_2 \rightarrow \overline{\mathbb{R}}$  such that for every element  $n$  of  $\mathbb{N}$ ,  $\mathcal{Q}[n, F_6(n)]$  from [9, Sch. 3]. For every natural number  $n$ ,  $\text{dom}(F_6(n)) = \text{Xsection}(A, x)$ . For every natural numbers  $m, n$ ,  $\text{dom}(F_6(m)) = \text{dom}(F_6(n))$ . For every natural number  $n$ ,  $F_6(n) = \text{ProjPMap1}(F(n), x)$ . For every element  $y$  of  $X_2$  such that  $y \in \text{Xsection}(A, x)$  holds  $F_6 \# y$  is convergent and  $(\text{ProjPMap1}(f, x))(y) = \lim(F_6 \# y)$  by (25), [9, (63)].  $\square$

- (38) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , an element  $A$  of  $S_1$ , an element  $B$  of  $S_2$ , and an element  $x$  of  $X_1$ . Then  $M_2(B \cap \text{MeasurableXsection}(E, x)) \cdot (\chi_{A, X_1}(x)) = \int \text{ProjPMap1}(\chi_{A \times B, X_1 \times X_2} \upharpoonright E, x) dM_2$ .

PROOF: Set  $C_1 = \chi_{A \times B, X_1 \times X_2} \upharpoonright E$ .  $\text{ProjPMap1}(\chi_{A \times B, X_1 \times X_2}, x) = \text{curry}(\chi_{A \times B, X_1 \times X_2})$ .  $\text{ProjPMap1}(C_1, x) = \text{ProjPMap1}(\chi_{A \times B, X_1 \times X_2}, x) \upharpoonright \text{Xsection}(E, x)$ . For every element  $y$  of  $X_2$ ,  $(\text{ProjPMap1}(C_1, x))(y) = (\chi_{A, X_1} \upharpoonright \text{MeasurableYsection}(E, y))(x) \cdot (\chi_{B, X_2}(y))$  by (25), (26), [8, (49)], [12, (2)].  $\square$

- (39) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , an element  $A$  of  $S_1$ , an element  $B$  of  $S_2$ , and an element  $y$  of  $X_2$ . Then  $M_1(A \cap \text{MeasurableYsection}(E, y)) \cdot (\chi_{B, X_2}(y)) = \int \text{ProjPMap2}(\chi_{A \times B, X_1 \times X_2} \upharpoonright E, y) dM_1$ .

PROOF: Set  $C_1 = \chi_{A \times B, X_1 \times X_2} \upharpoonright E$ .  $\text{ProjPMap2}(\chi_{A \times B, X_1 \times X_2}, y) = \text{curry}'(\chi_{A \times B, X_1 \times X_2})$ .  $\text{ProjPMap2}(C_1, y) = \text{ProjPMap2}(\chi_{A \times B, X_1 \times X_2}, y) \upharpoonright \text{Ysection}(E, y)$ . For every element  $x$  of  $X_1$ ,  $(\text{ProjPMap2}(C_1, y))(x) = (\chi_{B, X_2} \upharpoonright \text{MeasurableXsection}(E, x))(y) \cdot (\chi_{A, X_1}(x))$  by (25), (26), [8, (49)], [12, (2)].  $\square$

- (40) Let us consider non empty sets  $X_1, X_2$ , an element  $x$  of  $X_1$ , an element  $y$  of  $X_2$ , a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ , and an extended real  $e$ . Then

- (i)  $\langle x, y \rangle \in \text{dom } f$  and  $f(x, y) = e$  iff  $y \in \text{dom ProjPMap1}(f, x)$  and  $(\text{ProjPMap1}(f, x))(y) = e$ , and
- (ii)  $\langle x, y \rangle \in \text{dom } f$  and  $f(x, y) = e$  iff  $x \in \text{dom ProjPMap2}(f, y)$  and  $(\text{ProjPMap2}(f, y))(x) = e$ .

The theorem is a consequence of (25) and (26).

- (41) Let us consider non empty sets  $X_1, X_2$ , sets  $A, Z$ , a partial function  $f$  from  $X_1 \times X_2$  to  $Z$ , and an element  $x$  of  $X_1$ . Then  $X\text{section}(f^{-1}(A), x) = (\text{ProjPMap1}(f, x))^{-1}(A)$ .
- (42) Let us consider non empty sets  $X_1, X_2$ , sets  $A, Z$ , a partial function  $f$  from  $X_1 \times X_2$  to  $Z$ , and an element  $y$  of  $X_2$ . Then  $Y\text{section}(f^{-1}(A), y) = (\text{ProjPMap2}(f, y))^{-1}(A)$ .
- (43) Let us consider non empty sets  $X_1, X_2$ , subsets  $A, B$  of  $X_1 \times X_2$ , and a set  $p$ . Then
- (i)  $X\text{section}(A \setminus B, p) = X\text{section}(A, p) \setminus X\text{section}(B, p)$ , and
  - (ii)  $Y\text{section}(A \setminus B, p) = Y\text{section}(A, p) \setminus Y\text{section}(B, p)$ .
- (44) Let us consider non empty sets  $X_1, X_2$ , an element  $x$  of  $X_1$ , an element  $y$  of  $X_2$ , and partial functions  $f_1, f_2$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ . Then
- (i)  $\text{ProjPMap1}(f_1 + f_2, x) = \text{ProjPMap1}(f_1, x) + \text{ProjPMap1}(f_2, x)$ , and
  - (ii)  $\text{ProjPMap1}(f_1 - f_2, x) = \text{ProjPMap1}(f_1, x) - \text{ProjPMap1}(f_2, x)$ , and
  - (iii)  $\text{ProjPMap2}(f_1 + f_2, y) = \text{ProjPMap2}(f_1, y) + \text{ProjPMap2}(f_2, y)$ , and
  - (iv)  $\text{ProjPMap2}(f_1 - f_2, y) = \text{ProjPMap2}(f_1, y) - \text{ProjPMap2}(f_2, y)$ .

The theorem is a consequence of (42), (41), (43), (26), and (40).

- (45) Let us consider non empty sets  $X_1, X_2$ , a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ , and an element  $x$  of  $X_1$ . Then
- (i)  $\text{ProjPMap1}(\max_+(f), x) = \max_+(\text{ProjPMap1}(f, x))$ , and
  - (ii)  $\text{ProjPMap1}(\max_-(f), x) = \max_-(\text{ProjPMap1}(f, x))$ .
- (46) Let us consider non empty sets  $X_1, X_2$ , a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ , and an element  $y$  of  $X_2$ . Then
- (i)  $\text{ProjPMap2}(\max_+(f), y) = \max_+(\text{ProjPMap2}(f, y))$ , and
  - (ii)  $\text{ProjPMap2}(\max_-(f), y) = \max_-(\text{ProjPMap2}(f, y))$ .
- (47) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ , an element  $x$  of  $X_1$ , an element  $y$  of  $X_2$ , and an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ . Suppose  $E \subseteq \text{dom } f$  and  $f$  is measurable on  $E$ . Then
- (i)  $\text{ProjPMap1}(f, x)$  is measurable on  $\text{MeasurableXsection}(E, x)$ , and
  - (ii)  $\text{ProjPMap2}(f, y)$  is measurable on  $\text{MeasurableYsection}(E, y)$ .

The theorem is a consequence of (45) and (46).

Let  $X_1, X_2, Y$  be non empty sets,  $F$  be a sequence of partial functions from  $X_1 \times X_2$  into  $Y$ , and  $x$  be an element of  $X_1$ . The functor  $\text{ProjPMap1}(F, x)$  yielding a sequence of partial functions from  $X_2$  into  $Y$  is defined by

(Def. 5) for every natural number  $n$ ,  $it(n) = \text{ProjPMap1}(F(n), x)$ .

Let  $y$  be an element of  $X_2$ . The functor  $\text{ProjPMap2}(F, y)$  yielding a sequence of partial functions from  $X_1$  into  $Y$  is defined by

(Def. 6) for every natural number  $n$ ,  $it(n) = \text{ProjPMap2}(F(n), y)$ .

Now we state the proposition:

(48) Let us consider non empty sets  $X_1, X_2$ , a subset  $E$  of  $X_1 \times X_2$ , an element  $x$  of  $X_1$ , and an element  $y$  of  $X_2$ . Then

(i)  $\text{ProjPMap1}(\chi_{E, X_1 \times X_2}, x) = \chi_{X_{\text{section}(E, x)}, X_2}$ , and

(ii)  $\text{ProjPMap2}(\chi_{E, X_1 \times X_2}, y) = \chi_{Y_{\text{section}(E, y)}, X_1}$ .

The theorem is a consequence of (25) and (27).

Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , an element  $E$  of  $S$ , and an extended real  $e$ . Now we state the propositions:

(49)  $\int \chi_{e, E, X} dM = e \cdot (M(E))$ . The theorem is a consequence of (2), (12), and (1).

(50)  $\int \chi_{e, E, X} \upharpoonright E dM = e \cdot (M(E))$ . The theorem is a consequence of (15), (2), (13), (49), (16), (1), and (12).

Now we state the propositions:

(51) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , elements  $E_1, E_2$  of  $S$ , and an extended real  $e$ . Then  $\int \chi_{e, E_1, X} \upharpoonright E_2 dM = e \cdot (M(E_1 \cap E_2))$ . The theorem is a consequence of (14), (17), (13), (16), (15), and (50).

(52) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , an element  $x$  of  $X_1$ , and an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ . Suppose  $M_2$  is  $\sigma$ -finite. Then

(i)  $(Y\text{vol}(E, M_2))(x) = \int \text{ProjPMap1}(\chi_{E, X_1 \times X_2}, x) dM_2$ , and

(ii)  $(Y\text{vol}(E, M_2))(x) = \int^+ \text{ProjPMap1}(\chi_{E, X_1 \times X_2}, x) dM_2$ , and

(iii)  $(Y\text{vol}(E, M_2))(x) = \int' \text{ProjPMap1}(\chi_{E, X_1 \times X_2}, x) dM_2$ .

The theorem is a consequence of (48), (12), and (27).

(53) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , an element  $y$  of  $X_2$ , and an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ . Suppose  $M_1$  is  $\sigma$ -finite. Then

(i)  $(X\text{vol}(E, M_1))(y) = \int \text{ProjPMap2}(\chi_{E, X_1 \times X_2}, y) dM_1$ , and

(ii)  $(X\text{vol}(E, M_1))(y) = \int^+ \text{ProjPMap2}(\chi_{E, X_1 \times X_2}, y) dM_1$ , and

(iii)  $(X\text{vol}(E, M_1))(y) = \int' \text{ProjPMap2}(\chi_{E, X_1 \times X_2}, y) dM_1$ .

The theorem is a consequence of (48), (12), and (27).

- (54) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , an element  $E$  of  $S$ , and a real number  $r$ . Then  $\int r \cdot \chi_{E,X} dM = r \cdot (\int \chi_{E,X} dM)$ . The theorem is a consequence of (12).
- (55) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , an element  $y$  of  $X_2$ , an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ , and a real number  $r$ . Suppose  $M_1$  is  $\sigma$ -finite. Then
- (i)  $(r \cdot \text{Xvol}(E, M_1))(y) = \int \text{ProjPMap2}(\chi_{r,E,X_1 \times X_2}, y) dM_1$ , and
  - (ii) if  $r \geq 0$ , then  $(r \cdot \text{Xvol}(E, M_1))(y) = \int^+ \text{ProjPMap2}(\chi_{r,E,X_1 \times X_2}, y) dM_1$ .

PROOF: Set  $p_2 = \text{ProjPMap2}(\chi_{E,X_1 \times X_2}, y)$ .  $\chi_{r,E,X_1 \times X_2} = r \cdot \chi_{E,X_1 \times X_2}$ .  $\text{ProjPMap2}(\chi_{r,E,X_1 \times X_2}, y) = r \cdot p_2$ .  $p_2$  is non-negative.  $\chi_{E,X_1 \times X_2}$  is simple function in  $\sigma(\text{MeasRect}(S_1, S_2))$ .  $\int \text{ProjPMap2}(\chi_{r,E,X_1 \times X_2}, y) dM_1 = r \cdot (\int^+ p_2 dM_1)$ . If  $r \geq 0$ , then  $(r \cdot \text{Xvol}(E, M_1))(y) = \int^+ \text{ProjPMap2}(\chi_{r,E,X_1 \times X_2}, y) dM_1$  by [14, (20)], (31), [14, (39), (89)].  $\square$

- (56) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , an element  $x$  of  $X_1$ , an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ , and a real number  $r$ . Suppose  $M_2$  is  $\sigma$ -finite. Then
- (i)  $(r \cdot \text{Yvol}(E, M_2))(x) = \int \text{ProjPMap1}(\chi_{r,E,X_1 \times X_2}, x) dM_2$ , and
  - (ii) if  $r \geq 0$ , then  $(r \cdot \text{Yvol}(E, M_2))(x) = \int^+ \text{ProjPMap1}(\chi_{r,E,X_1 \times X_2}, x) dM_2$ .

PROOF: Set  $p_2 = \text{ProjPMap1}(\chi_{E,X_1 \times X_2}, x)$ .  $\chi_{r,E,X_1 \times X_2} = r \cdot \chi_{E,X_1 \times X_2}$ .  $\text{ProjPMap1}(\chi_{r,E,X_1 \times X_2}, x) = r \cdot p_2$ .  $p_2$  is non-negative.  $\chi_{E,X_1 \times X_2}$  is simple function in  $\sigma(\text{MeasRect}(S_1, S_2))$ .  $\int \text{ProjPMap1}(\chi_{r,E,X_1 \times X_2}, x) dM_2 = r \cdot (\int^+ p_2 dM_2)$ . If  $r \geq 0$ , then  $(r \cdot \text{Yvol}(E, M_2))(x) = \int^+ \text{ProjPMap1}(\chi_{r,E,X_1 \times X_2}, x) dM_2$  by [14, (20)], (31), [14, (39), (89)].  $\square$

- (57) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , and a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Suppose  $\text{dom } f \in S$  and for every element  $x$  of  $X$  such that  $x \in \text{dom } f$  holds  $0 = f(x)$ . Then
- (i) for every element  $E$  of  $S$  such that  $E \subseteq \text{dom } f$  holds  $f$  is measurable on  $E$ , and
  - (ii)  $\int f dM = 0$ .

The theorem is a consequence of (15) and (50).

- (58) Let us consider non empty sets  $X_1, X_2, Y$ , a sequence  $F$  of partial functions from  $X_1 \times X_2$  into  $Y$ , an element  $x$  of  $X_1$ , and an element  $y$  of  $X_2$ . Suppose  $F$  has the same dom. Then
- (i)  $\text{ProjPMap1}(F, x)$  has the same dom, and
  - (ii)  $\text{ProjPMap2}(F, y)$  has the same dom.

4. FUBINI'S THEOREM FOR NON-NEGATIVE OR NON-POSITIVE FUNCTIONS

Let  $X_1, X_2$  be non empty sets,  $S_1$  be a  $\sigma$ -field of subsets of  $X_1$ ,  $M_1$  be a  $\sigma$ -measure on  $S_1$ , and  $f$  be a partial function from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ . The functor **Integral1**( $M_1, f$ ) yielding a function from  $X_2$  into  $\overline{\mathbb{R}}$  is defined by

(Def. 7) for every element  $y$  of  $X_2$ ,  $it(y) = \int \text{ProjPMap2}(f, y) dM_1$ .

Let  $S_2$  be a  $\sigma$ -field of subsets of  $X_2$  and  $M_2$  be a  $\sigma$ -measure on  $S_2$ . The functor **Integral2**( $M_2, f$ ) yielding a function from  $X_1$  into  $\overline{\mathbb{R}}$  is defined by

(Def. 8) for every element  $x$  of  $X_1$ ,  $it(x) = \int \text{ProjPMap1}(f, x) dM_2$ .

Now we state the propositions:

(59) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ , an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ , and an element  $V$  of  $S_2$ . Suppose  $M_1$  is  $\sigma$ -finite and  $f$  is non-negative or non-positive and  $E = \text{dom } f$  and  $f$  is measurable on  $E$ . Then **Integral1**( $M_1, f$ ) is measurable on  $V$ .

(60) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ , an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ , and an element  $U$  of  $S_1$ . Suppose  $M_2$  is  $\sigma$ -finite and  $f$  is non-negative or non-positive and  $E = \text{dom } f$  and  $f$  is measurable on  $E$ . Then **Integral2**( $M_2, f$ ) is measurable on  $U$ .

(61) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , an element  $y$  of  $X_2$ , and an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ . Suppose  $M_1$  is  $\sigma$ -finite. Then  $(\text{Xvol}(E, M_1))(y) = \int \chi_{\text{MeasurableYsection}(E, y), X_1} dM_1$ .

(62) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , an element  $x$  of  $X_1$ , and an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ . Suppose  $M_2$  is  $\sigma$ -finite. Then  $(\text{Yvol}(E, M_2))(x) = \int \chi_{\text{MeasurableXsection}(E, x), X_2} dM_2$ .

(63) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ , an element  $x$  of  $X_1$ , and an element  $y$  of  $X_2$ . Then

(i)  $\text{ProjPMap1}(\chi_{E, X_1 \times X_2}, x) = \chi_{\text{MeasurableXsection}(E, x), X_2}$ , and

(ii)  $\text{ProjPMap2}(\chi_{E, X_1 \times X_2}, y) = \chi_{\text{MeasurableYsection}(E, y), X_1}$ .

(64) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , and an element

$E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ . Suppose  $M_1$  is  $\sigma$ -finite. Then  $X\text{vol}(E, M_1) = \text{Integral1}(M_1, \chi_{E, X_1 \times X_2})$ . The theorem is a consequence of (61) and (63).

- (65) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , and an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ . Suppose  $M_2$  is  $\sigma$ -finite. Then  $Y\text{vol}(E, M_2) = \text{Integral2}(M_2, \chi_{E, X_1 \times X_2})$ . The theorem is a consequence of (62) and (63).

Let  $X_1, X_2$  be non empty sets,  $S_1$  be a  $\sigma$ -field of subsets of  $X_1$ ,  $S_2$  be a  $\sigma$ -field of subsets of  $X_2$ ,  $M_1$  be a  $\sigma$ -measure on  $S_1$ , and  $M_2$  be a  $\sigma$ -measure on  $S_2$ . The functor  $\text{ProdMeas}(M_1, M_2)$  yielding a  $\sigma$ -measure on  $\sigma(\text{MeasRect}(S_1, S_2))$  is defined by the term

(Def. 9)  $\text{Prod } \sigma\text{-Meas}(M_1, M_2)$ .

Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ , and elements  $E_1, E_2$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ . Now we state the propositions:

- (66) Suppose  $E_1 = \text{dom } f$  and  $f$  is non-negative and  $f$  is measurable on  $E_1$ . Then

- (i)  $\text{Integral1}(M_1, f)$  is non-negative, and
- (ii)  $\text{Integral1}(M_1, f|_{E_2})$  is non-negative, and
- (iii)  $\text{Integral2}(M_2, f)$  is non-negative, and
- (iv)  $\text{Integral2}(M_2, f|_{E_2})$  is non-negative.

The theorem is a consequence of (47) and (32).

- (67) Suppose  $E_1 = \text{dom } f$  and  $f$  is non-positive and  $f$  is measurable on  $E_1$ . Then

- (i)  $\text{Integral1}(M_1, f)$  is non-positive, and
- (ii)  $\text{Integral1}(M_1, f|_{E_2})$  is non-positive, and
- (iii)  $\text{Integral2}(M_2, f)$  is non-positive, and
- (iv)  $\text{Integral2}(M_2, f|_{E_2})$  is non-positive.

The theorem is a consequence of (47) and (33).

Now we state the propositions:

- (68) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ , elements  $E_1, E_2$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ , and an element  $V$  of  $S_2$ . Suppose  $M_1$  is  $\sigma$ -finite and  $f$  is non-negative or non-positive and  $E_1 = \text{dom } f$  and  $f$  is measurable on  $E_1$ . Then  $\text{Integral1}(M_1, f|_{E_2})$  is measurable on  $V$ . The theorem is a consequence of (59).

- (69) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ , elements  $E_1, E_2$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ , and an element  $U$  of  $S_1$ . Suppose  $M_2$  is  $\sigma$ -finite and  $f$  is non-negative or non-positive and  $E_1 = \text{dom } f$  and  $f$  is measurable on  $E_1$ . Then  $\text{Integral2}(M_2, f \upharpoonright E_2)$  is measurable on  $U$ . The theorem is a consequence of (60).
- (70) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ , an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ , and an element  $y$  of  $X_2$ . Suppose  $E = \text{dom } f$  and  $f$  is non-negative or non-positive and  $f$  is measurable on  $E$  and for every element  $x$  of  $X_1$  such that  $x \in \text{dom ProjPMap2}(f, y)$  holds  $(\text{ProjPMap2}(f, y))(x) = 0$ . Then  $(\text{Integral1}(M_1, f))(y) = 0$ . The theorem is a consequence of (47), (32), and (33).
- (71) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ , an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ , and an element  $x$  of  $X_1$ . Suppose  $E = \text{dom } f$  and  $f$  is non-negative or non-positive and  $f$  is measurable on  $E$  and for every element  $y$  of  $X_2$  such that  $y \in \text{dom ProjPMap1}(f, x)$  holds  $(\text{ProjPMap1}(f, x))(y) = 0$ . Then  $(\text{Integral2}(M_2, f))(x) = 0$ . The theorem is a consequence of (47), (32), and (33).
- (72) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , elements  $E, E_1, E_2$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ , and a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ . Suppose  $E = \text{dom } f$  and  $f$  is non-negative or non-positive and  $f$  is measurable on  $E$  and  $E_1$  misses  $E_2$ . Then
- (i)  $\text{Integral1}(M_1, f \upharpoonright (E_1 \cup E_2)) = \text{Integral1}(M_1, f \upharpoonright E_1) + \text{Integral1}(M_1, f \upharpoonright E_2)$ ,  
and
  - (ii)  $\text{Integral2}(M_2, f \upharpoonright (E_1 \cup E_2)) = \text{Integral2}(M_2, f \upharpoonright E_1) + \text{Integral2}(M_2, f \upharpoonright E_2)$ .
- (73) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ , and an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ . Suppose  $E = \text{dom } f$  and  $f$  is measurable on  $E$ . Then
- (i)  $\text{Integral1}(M_1, -f) = -\text{Integral1}(M_1, f)$ , and
  - (ii)  $\text{Integral2}(M_2, -f) = -\text{Integral2}(M_2, f)$ .
- The theorem is a consequence of (29) and (47).

Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , partial functions  $f, g$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ , and elements  $E_1, E_2$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ . Now we state the propositions:

- (74) Suppose  $E_1 = \text{dom } f$  and  $f$  is non-negative and  $f$  is measurable on  $E_1$  and  $E_2 = \text{dom } g$  and  $g$  is non-negative and  $g$  is measurable on  $E_2$ . Then
- (i)  $\text{Integral1}(M_1, f+g) = \text{Integral1}(M_1, f \upharpoonright \text{dom}(f+g)) + \text{Integral1}(M_1, g \upharpoonright \text{dom}(f+g))$ , and
  - (ii)  $\text{Integral2}(M_2, f+g) = \text{Integral2}(M_2, f \upharpoonright \text{dom}(f+g)) + \text{Integral2}(M_2, g \upharpoonright \text{dom}(f+g))$ .

PROOF: Set  $f_1 = f \upharpoonright (A \cap B)$ . Set  $g_1 = g \upharpoonright (A \cap B)$ .  $\text{Integral1}(M_1, f_1)$  is non-negative and  $\text{Integral1}(M_1, g_1)$  is non-negative and  $\text{Integral2}(M_2, f_1)$  is non-negative and  $\text{Integral2}(M_2, g_1)$  is non-negative. For every element  $y$  of  $X_2$ ,  $(\text{Integral1}(M_1, f_1) + \text{Integral1}(M_1, g_1))(y) = (\text{Integral1}(M_1, f+g))(y)$  by (47), (32), [14, (88)], (44). For every element  $x$  of  $X_1$ ,  $(\text{Integral2}(M_2, f_1) + \text{Integral2}(M_2, g_1))(x) = (\text{Integral2}(M_2, f+g))(x)$  by (47), (32), [14, (88)], (44).  $\square$

- (75) Suppose  $E_1 = \text{dom } f$  and  $f$  is non-positive and  $f$  is measurable on  $E_1$  and  $E_2 = \text{dom } g$  and  $g$  is non-positive and  $g$  is measurable on  $E_2$ . Then
- (i)  $\text{Integral1}(M_1, f+g) = \text{Integral1}(M_1, f \upharpoonright \text{dom}(f+g)) + \text{Integral1}(M_1, g \upharpoonright \text{dom}(f+g))$ , and
  - (ii)  $\text{Integral2}(M_2, f+g) = \text{Integral2}(M_2, f \upharpoonright \text{dom}(f+g)) + \text{Integral2}(M_2, g \upharpoonright \text{dom}(f+g))$ .

The theorem is a consequence of (73) and (74).

- (76) Suppose  $E_1 = \text{dom } f$  and  $f$  is non-negative and  $f$  is measurable on  $E_1$  and  $E_2 = \text{dom } g$  and  $g$  is non-positive and  $g$  is measurable on  $E_2$ . Then
- (i)  $\text{Integral1}(M_1, f-g) = \text{Integral1}(M_1, f \upharpoonright \text{dom}(f-g)) - \text{Integral1}(M_1, g \upharpoonright \text{dom}(f-g))$ , and
  - (ii)  $\text{Integral1}(M_1, g-f) = \text{Integral1}(M_1, g \upharpoonright \text{dom}(g-f)) - \text{Integral1}(M_1, f \upharpoonright \text{dom}(g-f))$ , and
  - (iii)  $\text{Integral2}(M_2, f-g) = \text{Integral2}(M_2, f \upharpoonright \text{dom}(f-g)) - \text{Integral2}(M_2, g \upharpoonright \text{dom}(f-g))$ , and
  - (iv)  $\text{Integral2}(M_2, g-f) = \text{Integral2}(M_2, g \upharpoonright \text{dom}(g-f)) - \text{Integral2}(M_2, f \upharpoonright \text{dom}(g-f))$ .

The theorem is a consequence of (74) and (73).

Now we state the propositions:



(77) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , and an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ . Suppose  $M_1$  is  $\sigma$ -finite and  $M_2$  is  $\sigma$ -finite. Then

- (i)  $\int Y\text{vol}(E, M_2) dM_1 = \int \chi_{E, X_1 \times X_2} d\text{ProdMeas}(M_1, M_2)$ , and
- (ii)  $\int X\text{vol}(E, M_1) dM_2 = \int \chi_{E, X_1 \times X_2} d\text{ProdMeas}(M_1, M_2)$ .

(78) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ , a partial function  $f$  from  $X_1 \times X_2$  to  $\mathbb{R}$ , and a real number  $r$ . Suppose  $E = \text{dom } f$  and  $f$  is non-negative or non-positive and  $f$  is measurable on  $E$ . Then

- (i)  $\text{Integral1}(M_1, r \cdot f) = r \cdot \text{Integral1}(M_1, f)$ , and
- (ii)  $\text{Integral2}(M_2, r \cdot f) = r \cdot \text{Integral2}(M_2, f)$ .

The theorem is a consequence of (32), (33), (29), and (47).

Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , and an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ . Now we state the propositions:

- (79) (i)  $\text{Integral1}(M_1, \chi_{E, X_1 \times X_2} \upharpoonright E) = \text{Integral1}(M_1, \chi_{E, X_1 \times X_2})$ , and  
(ii)  $\text{Integral2}(M_2, \chi_{E, X_1 \times X_2} \upharpoonright E) = \text{Integral2}(M_2, \chi_{E, X_1 \times X_2})$ .

The theorem is a consequence of (34) and (48).

- (80) (i)  $\text{Integral1}(M_1, \bar{\chi}_{E, X_1 \times X_2} \upharpoonright E) = \text{Integral1}(M_1, \bar{\chi}_{E, X_1 \times X_2})$ , and  
(ii)  $\text{Integral2}(M_2, \bar{\chi}_{E, X_1 \times X_2} \upharpoonright E) = \text{Integral2}(M_2, \bar{\chi}_{E, X_1 \times X_2})$ .

The theorem is a consequence of (34), (35), (2), (50), and (49).

Now we state the propositions:

(81) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ , and an extended real  $e$ . Then

- (i)  $\text{Integral1}(M_1, \chi_{e, E, X_1 \times X_2} \upharpoonright E) = \text{Integral1}(M_1, \chi_{e, E, X_1 \times X_2})$ , and
- (ii)  $\text{Integral2}(M_2, \chi_{e, E, X_1 \times X_2} \upharpoonright E) = \text{Integral2}(M_2, \chi_{e, E, X_1 \times X_2})$ .

The theorem is a consequence of (1), (78), (79), (2), and (80).

(82) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , and an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ . Suppose  $M_1$  is  $\sigma$ -finite and  $M_2$  is  $\sigma$ -finite. Then

- (i)  $\int \chi_{E, X_1 \times X_2} d\text{ProdMeas}(M_1, M_2) = \int \text{Integral1}(M_1, \chi_{E, X_1 \times X_2}) dM_2$ ,  
and

$$(ii) \int \chi_{E, X_1 \times X_2} \upharpoonright E \, d \text{ProdMeas}(M_1, M_2) = \int \text{Integral1}(M_1, \chi_{E, X_1 \times X_2} \upharpoonright E) \, dM_2,$$

and

$$(iii) \int \chi_{E, X_1 \times X_2} \, d \text{ProdMeas}(M_1, M_2) = \int \text{Integral2}(M_2, \chi_{E, X_1 \times X_2}) \, dM_1,$$

and

$$(iv) \int \chi_{E, X_1 \times X_2} \upharpoonright E \, d \text{ProdMeas}(M_1, M_2) = \int \text{Integral2}(M_2, \chi_{E, X_1 \times X_2} \upharpoonright E) \, dM_1.$$

The theorem is a consequence of (64), (77), (79), and (65).

- (83) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ , and a real number  $r$ . Suppose  $M_1$  is  $\sigma$ -finite and  $M_2$  is  $\sigma$ -finite. Then

$$(i) \int \chi_{r, E, X_1 \times X_2} \, d \text{ProdMeas}(M_1, M_2) = \int \text{Integral1}(M_1, \chi_{r, E, X_1 \times X_2}) \, dM_2,$$

and

$$(ii) \int \chi_{r, E, X_1 \times X_2} \upharpoonright E \, d \text{ProdMeas}(M_1, M_2) = \int \text{Integral1}(M_1, \chi_{r, E, X_1 \times X_2} \upharpoonright E) \, dM_2,$$

and

$$(iii) \int \chi_{r, E, X_1 \times X_2} \, d \text{ProdMeas}(M_1, M_2) = \int \text{Integral2}(M_2, \chi_{r, E, X_1 \times X_2}) \, dM_1,$$

and

$$(iv) \int \chi_{r, E, X_1 \times X_2} \upharpoonright E \, d \text{ProdMeas}(M_1, M_2) = \int \text{Integral2}(M_2, \chi_{r, E, X_1 \times X_2} \upharpoonright E) \, dM_1.$$

The theorem is a consequence of (1), (12), (64), (82), (78), (81), and (65).

- (84) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , an element  $A$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ , and a partial function  $f$  from  $X_1 \times X_2$  to  $\mathbb{R}$ . Suppose  $M_1$  is  $\sigma$ -finite and  $M_2$  is  $\sigma$ -finite and  $f$  is non-negative or non-positive and  $A = \text{dom } f$  and  $f$  is measurable on  $A$ . Then

$$(i) \int f \, d \text{ProdMeas}(M_1, M_2) = \int \text{Integral1}(M_1, f) \, dM_2, \text{ and}$$

$$(ii) \int f \, d \text{ProdMeas}(M_1, M_2) = \int \text{Integral2}(M_2, f) \, dM_1.$$

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