

About Supergraphs. Part I

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Summary. Drawing a finite graph is usually done by a finite sequence of the following three operations.

1. Draw a vertex of the graph.
2. Draw an edge between two vertices of the graph.
3. Draw an edge starting from a vertex of the graph and immediately draw a vertex at the other end of it.

By this procedure any finite graph can be constructed. This property of graphs is so obvious that the author of this article has yet to find a reference where it is mentioned explicitly. In introductory books (like [14], [5]) as well as in advanced ones (like [4]), after the initial definition of graphs the examples are usually given by graphical representations rather than explicit set theoretic descriptions, assuming a mutual understanding how the representation is to be translated into a description fitting the definition. However, Mizar does not possess this innate ability of humans to translate pictures into graphs. Therefore, if one wants to create graphs in Mizar without directly providing a set theoretic description (i.e. using the `createGraph` functor), a rigorous approach to the constructing operations is required.

In this paper supergraphs are defined as an inverse mode to subgraphs as given in [10]. The three graph construction operations are defined as modes extending `Supergraph` similar to how the various remove operations were introduced as submodes of `Subgraph` in [10]. Many theorems are proven that describe how graph properties are transferred to special supergraphs. In particular, to prove that disconnected graphs cannot become connected by adding an edge between two vertices that lie in the same component, the theory of replacing a part of a walk with another walk is introduced in the preliminaries.

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1. GENERAL PRELIMINARIES

Let us consider an even integer n and an odd integer m . Now we state the propositions:

- (1) If $n \leq m$, then $n + 1 \leq m$.
- (2) If $m \leq n$, then $m + 1 \leq n$.

Now we state the propositions:

- (3) Let us consider natural numbers i, j . If $i > i - 1 + j$, then $j = 0$.

PROOF: $i - j \geq 1$ by [3, (21), (14)]. \square

- (4) Let us consider finite sequences f, g , and a natural number i . Suppose $i \leq \text{len } f$ and $\text{mid}(f, i, i - 1 + \text{len } g) = g$. Then $i - 1 + \text{len } g \leq \text{len } f$. The theorem is a consequence of (3).

Let us consider a finite sequence p and a natural number n . Now we state the propositions:

- (5) If $n \in \text{dom } p$ and $n + 1 \leq \text{len } p$, then $\text{mid}(p, n, n + 1) = \langle p(n), p(n + 1) \rangle$.
- (6) If $n \in \text{dom } p$ and $n + 2 \leq \text{len } p$, then $\text{mid}(p, n, n + 2) = \langle p(n), p(n + 1), p(n + 2) \rangle$. The theorem is a consequence of (5).

Now we state the proposition:

- (7) Let us consider a non empty set D , finite sequences f, g of elements of D , and a natural number n . Suppose g is a substring of f . Then
 - (i) $\text{len } g = 0$, or
 - (ii) $1 \leq n - 1 + \text{len } g \leq \text{len } f$ and $n \leq n - 1 + \text{len } g$.

The theorem is a consequence of (4).

Let D be a non empty set, f, g be finite sequences of elements of D , and n be a natural number. We say that **g is odd-substring of f and n** if and only if (Def. 1) if $\text{len } g > 0$, then there exists an odd natural number i such that $n \leq i \leq \text{len } f$ and $\text{mid}(f, i, i - 1 + \text{len } g) = g$.

We say that **g is even-substring of f and n** if and only if (Def. 2) if $\text{len } g > 0$, then there exists an even natural number i such that $n \leq i \leq \text{len } f$ and $\text{mid}(f, i, i - 1 + \text{len } g) = g$.

Let us consider a non empty set D , finite sequences f, g of elements of D , and a natural number n . Now we state the propositions:

- (8) If g is odd-substring of f and n , then g is a substring of f .
- (9) If g is even-substring of f and n , then g is a substring of f .

Now we state the propositions:

- (10) Let us consider a non empty set D , finite sequences f, g of elements of D , and natural numbers n, m . Suppose $m \geq n$. Then
- (i) if g is odd-substring of f and m , then g is odd-substring of f and n , and
 - (ii) if g is even-substring of f and m , then g is even-substring of f and n .
- (11) Let us consider a non empty set D , and a finite sequence f of elements of D . If $1 \leq \text{len } f$, then f is odd-substring of f and 0.
- (12) Let us consider a non empty set D , finite sequences f, g of elements of D , and an even element n of \mathbb{N} . If g is odd-substring of f and n , then g is odd-substring of f and $n + 1$.
- (13) Let us consider a non empty set D , finite sequences f, g of elements of D , and an odd element n of \mathbb{N} . If g is even-substring of f and n , then g is even-substring of f and $n + 1$.
- (14) Let us consider a non empty set D , and finite sequences f, g of elements of D . If g is odd-substring of f and 0, then g is odd-substring of f and 1. The theorem is a consequence of (12).

2. GRAPH PRELIMINARIES

Let G be a non-directed-multi graph. Observe that every subgraph of G is non-directed-multi.

Now we state the propositions:

- (15) Every graph is a subgraph of G induced by the vertices of G .
- (16) Let us consider graphs G_1, G_3 , sets V, E , and a subgraph G_2 of G_1 induced by V and E . If $G_2 \approx G_3$, then G_3 is a subgraph of G_1 induced by V and E .
- (17) Let us consider a graph G , a set X , and objects e, y . Suppose e joins a vertex from X and a vertex from $\{y\}$ in G . Then there exists an object x such that
- (i) $x \in X$, and
 - (ii) e joins x and y in G .
- (18) Let us consider a graph G , and a set X . Suppose $X \cap (\text{the vertices of } G) = \emptyset$. Then
- (i) $G.\text{edgesInto}(X) = \emptyset$, and

- (ii) $G.\text{edgesOutOf}(X) = \emptyset$, and
- (iii) $G.\text{edgesInOut}(X) = \emptyset$, and
- (iv) $G.\text{edgesBetween}(X) = \emptyset$.

PROOF: $G.\text{edgesInto}(X) = \emptyset$ by [10, (13)]. $G.\text{edgesOutOf}(X) = \emptyset$ by [10, (13)]. \square

Let us consider a graph G , sets X_1, X_2 , and an object y . Now we state the propositions:

- (19) If X_1 misses X_2 , then $G.\text{edgesBetween}(X_1, \{y\})$ misses $G.\text{edgesBetween}(X_2, \{y\})$.
The theorem is a consequence of (17).

- (20) $G.\text{edgesBetween}(X_1 \cup X_2, \{y\}) = G.\text{edgesBetween}(X_1, \{y\}) \cup G.\text{edgesBetween}(X_2, \{y\})$.
PROOF: Set $E_1 = G.\text{edgesBetween}(X_1, \{y\})$. Set $E_2 = G.\text{edgesBetween}(X_2, \{y\})$.
For every object e such that $e \in G.\text{edgesBetween}(X_1 \cup X_2, \{y\})$ holds $e \in E_1 \cup E_2$ by (17), [10, (17)]. \square

Now we state the proposition:

- (21) Let us consider a trivial graph G . Then there exists a vertex v of G such that
 - (i) the vertices of $G = \{v\}$, and
 - (ii) the source of $G = (\text{the edges of } G) \mapsto v$, and
 - (iii) the target of $G = (\text{the edges of } G) \mapsto v$.

PROOF: Consider v being a vertex of G such that the vertices of $G = \{v\}$. For every object e such that $e \in \text{dom}(\text{the source of } G)$ holds (the source of G)(e) = v by [7, (4)]. For every object e such that $e \in \text{dom}(\text{the target of } G)$ holds (the target of G)(e) = v by [7, (4)]. \square

Let G be a graph. Let us note that every walk of G which is closed, trail-like, and non trivial is also circuit-like and every walk of G which is closed, path-like, and non trivial is also cycle-like.

Let us consider graphs G_1, G_2 , a walk W_1 of G_1 , and a walk W_2 of G_2 . Now we state the propositions:

- (22) If $W_1 = W_2$, then if W_1 is trail-like, then W_2 is trail-like.
- (23) If $W_1 = W_2$, then if W_1 is path-like, then W_2 is path-like. The theorem is a consequence of (22).
- (24) If $W_1 = W_2$, then if W_1 is cycle-like, then W_2 is cycle-like. The theorem is a consequence of (23).
- (25) If $W_1 = W_2$, then if W_1 is vertex-distinct, then W_2 is vertex-distinct.

Now we state the propositions:

- (26) Let us consider a graph G , a walk W of G , and a vertex v of G . If $v \in W.\text{vertices}()$, then $G.\text{walkOf}(v)$ is a substring of W .

(27) Let us consider a graph G , a walk W of G , and an odd element n of \mathbb{N} . Suppose $n + 2 \leq \text{len } W$. Then $G.\text{walkOf}(W(n), W(n + 1), W(n + 2))$ is odd-substring of W and 0. The theorem is a consequence of (6).

Let us consider a graph G , a walk W of G , and objects u, e, v . Now we state the propositions:

(28) If e joins u and v in G and $e \in W.\text{edges}()$, then $G.\text{walkOf}(u, e, v)$ is odd-substring of W and 0 or $G.\text{walkOf}(v, e, u)$ is odd-substring of W and 0. The theorem is a consequence of (27).

(29) If e joins u and v in G and $G.\text{walkOf}(u, e, v)$ is odd-substring of W and 0, then $e \in W.\text{edges}()$ and $u, v \in W.\text{vertices}()$. The theorem is a consequence of (14), (8), and (7).

Let G be a graph and W_1, W_2 be walks of G . The functor $W_1.\text{findFirstVertex}(W_2)$ yielding an odd element of \mathbb{N} is defined by

- (Def. 3) (i) $it \leq \text{len } W_1$ and there exists an even natural number k such that $it = k + 1$ and for every natural number n such that $1 \leq n \leq \text{len } W_2$ holds $W_1(k + n) = W_2(n)$ and for every even natural number l such that for every natural number n such that $1 \leq n \leq \text{len } W_2$ holds $W_1(l + n) = W_2(n)$ holds $k \leq l$, **if** W_2 is odd-substring of W_1 and 0,
- (ii) $it = \text{len } W_1$, **otherwise**.

The functor $W_1.\text{findLastVertex}(W_2)$ yielding an odd element of \mathbb{N} is defined by

- (Def. 4) (i) $it \leq \text{len } W_1$ and there exists an even natural number k such that $it = k + \text{len } W_2$ and for every natural number n such that $1 \leq n \leq \text{len } W_2$ holds $W_1(k + n) = W_2(n)$ and for every even natural number l such that for every natural number n such that $1 \leq n \leq \text{len } W_2$ holds $W_1(l + n) = W_2(n)$ holds $k \leq l$, **if** W_2 is odd-substring of W_1 and 0,
- (ii) $it = \text{len } W_1$, **otherwise**.

Let us consider a graph G and walks W_1, W_2 of G . Now we state the propositions:

- (30) Suppose W_2 is odd-substring of W_1 and 0. Then
- (i) $W_1(W_1.\text{findFirstVertex}(W_2)) = W_2.\text{first}()$, and
 - (ii) $W_1(W_1.\text{findLastVertex}(W_2)) = W_2.\text{last}()$.
- (31) Suppose W_2 is odd-substring of W_1 and 0. Then
- (i) $1 \leq W_1.\text{findFirstVertex}(W_2) \leq \text{len } W_1$, and
 - (ii) $1 \leq W_1.\text{findLastVertex}(W_2) \leq \text{len } W_1$.

Now we state the propositions:

(32) Let us consider a graph G , and a walk W of G . Then

- (i) $1 = W.\text{findFirstVertex}(W)$, and
- (ii) $W.\text{findLastVertex}(W) = \text{len } W$.

The theorem is a consequence of (11).

(33) Let us consider a graph G , and walks W_1, W_2 of G . Suppose W_2 is odd-substring of W_1 and 0. Then $W_1.\text{findFirstVertex}(W_2) \leq W_1.\text{findLastVertex}(W_2)$.

Let G be a graph and W_1, W_2, W_3 be walks of G . The functor $W_1.\text{replaceWith}(W_2, W_3)$ yielding a walk of G is defined by the term

(Def. 5)
$$\begin{cases} ((W_1.\text{cut}(1, W_1.\text{findFirstVertex}(W_2))).\text{append}(W_3)).\text{append}((W_1.\text{cut}(W_1.\text{findLastVertex}(W_2), \text{len } W_1))), & \text{if } W_2 \text{ is odd-substring of } W_1 \\ \mathbf{otherwise.} \end{cases}$$

Let W_1, W_3 be walks of G and e be an object. The functor $W_1.\text{replaceEdgeWith}(e, W_3)$ yielding a walk of G is defined by the term

(Def. 6)
$$\begin{cases} W_1.\text{replaceWith}(G.\text{walkOf}(W_3.\text{first}(), e, W_3.\text{last}()), W_3), & \text{if } e \text{ joins } W_3.\text{first}() \text{ and } W_3.\text{last}() \\ \mathbf{otherwise.} \end{cases}$$

Let W_1, W_2 be walks of G . The functor $W_1.\text{replaceWithEdge}(W_2, e)$ yielding a walk of G is defined by the term

(Def. 7)
$$\begin{cases} W_1.\text{replaceWith}(W_2, G.\text{walkOf}(W_2.\text{first}(), e, W_2.\text{last}())), & \text{if } W_2 \text{ is odd-substring of } W_1 \\ \mathbf{otherwise.} \end{cases}$$

Let us consider a graph G and walks W_1, W_2, W_3 of G . Now we state the propositions:

(34) Suppose W_2 is odd-substring of W_1 and 0 and $W_2.\text{first}() = W_3.\text{first}()$ and $W_2.\text{last}() = W_3.\text{last}()$. Then

- (i) $(W_1.\text{cut}(1, W_1.\text{findFirstVertex}(W_2))).\text{first}() = W_1.\text{first}()$, and
- (ii) $(W_1.\text{cut}(1, W_1.\text{findFirstVertex}(W_2))).\text{last}() = W_3.\text{first}()$, and
- (iii) $((W_1.\text{cut}(1, W_1.\text{findFirstVertex}(W_2))).\text{append}(W_3)).\text{first}() = W_1.\text{first}()$,
and
- (iv) $((W_1.\text{cut}(1, W_1.\text{findFirstVertex}(W_2))).\text{append}(W_3)).\text{last}() = W_3.\text{last}()$,
and
- (v) $(W_1.\text{cut}(W_1.\text{findLastVertex}(W_2), \text{len } W_1)).\text{first}() = W_3.\text{last}()$, and
- (vi) $(W_1.\text{cut}(W_1.\text{findLastVertex}(W_2), \text{len } W_1)).\text{last}() = W_1.\text{last}()$.

The theorem is a consequence of (31) and (30).

(35) (i) $W_1.\text{first}() = (W_1.\text{replaceWith}(W_2, W_3)).\text{first}()$, and
(ii) $W_1.\text{last}() = (W_1.\text{replaceWith}(W_2, W_3)).\text{last}()$.

The theorem is a consequence of (34).

- (36) Suppose W_2 is odd-substring of W_1 and 0 and $W_2.\text{first}() = W_3.\text{first}()$ and $W_2.\text{last}() = W_3.\text{last}()$. Then $(W_1.\text{replaceWith}(W_2, W_3)).\text{vertices}() = ((W_1.\text{cut}(1, W_1.\text{findFirstVertex}(W_2))).\text{vertices}() \cup W_3.\text{vertices}()) \cup (W_1.\text{cut}(W_1.\text{findLastVertex}(W_2), \text{len } W_1)).\text{vertices}())$. The theorem is a consequence of (34).
- (37) Suppose W_2 is odd-substring of W_1 and 0 and $W_2.\text{first}() = W_3.\text{first}()$ and $W_2.\text{last}() = W_3.\text{last}()$. Then $(W_1.\text{replaceWith}(W_2, W_3)).\text{edges}() = ((W_1.\text{cut}(1, W_1.\text{findFirstVertex}(W_2))).\text{edges}() \cup W_3.\text{edges}()) \cup (W_1.\text{cut}(W_1.\text{findLastVertex}(W_2), \text{len } W_1)).\text{edges}())$. The theorem is a consequence of (34).

Now we state the propositions:

- (38) Let us consider a graph G , walks W_1, W_3 of G , and an object e . Suppose e joins $W_3.\text{first}()$ and $W_3.\text{last}()$ in G and $G.\text{walkOf}(W_3.\text{first}(), e, W_3.\text{last}())$ is odd-substring of W_1 and 0. Then $e \in (W_1.\text{replaceEdgeWith}(e, W_3)).\text{edges}()$ if and only if $e \in (W_1.\text{cut}(1, W_1.\text{findFirstVertex}(G.\text{walkOf}(W_3.\text{first}(), e, W_3.\text{last}())))).\text{edges}()$ or $e \in W_3.\text{edges}()$ or $e \in (W_1.\text{cut}(W_1.\text{findLastVertex}(G.\text{walkOf}(W_3.\text{first}(), e, W_3.\text{last}()), \text{len } W_1)).\text{edges}())$. The theorem is a consequence of (37).
- (39) Let us consider a graph G , walks W_1, W_3 of G , and an object e . Suppose e joins $W_3.\text{first}()$ and $W_3.\text{last}()$ in G and $e \notin W_3.\text{edges}()$ and $G.\text{walkOf}(W_3.\text{first}(), e, W_3.\text{last}())$ is odd-substring of W_1 and 0 and for every even natural numbers n, m such that $n, m \in \text{dom } W_1$ and $W_1(n) = e$ and $W_1(m) = e$ holds $n = m$. Then $e \notin (W_1.\text{replaceEdgeWith}(e, W_3)).\text{edges}()$.

PROOF: Set $W_2 = G.\text{walkOf}(W_3.\text{first}(), e, W_3.\text{last}())$. W_2 is odd-substring of W_1 and 1. Define $\mathcal{P}[\text{natural number}] \equiv \1 is odd and $1 \leq \$1 \leq \text{len } W_1$ and $\text{mid}(W_1, \$1, \$1 - 1 + \text{len } W_2) = W_2$. Consider i being a natural number such that $\mathcal{P}[i]$ and for every natural number n such that $\mathcal{P}[n]$ holds $i \leq n$ from [3, Sch. 5]. Set $j = i - 1 + \text{len } W_2$. W_2 is a substring of W_1 . $1 \leq j \leq \text{len } W_1$ and $i \leq j$. Set $n_1 = i + 1$. Reconsider $k = i - 1$ as an even natural number. For every natural number n such that $1 \leq n \leq \text{len } W_2$ holds $W_1(k + n) = W_2(n)$ by [12, (118)]. For every even natural number l such that for every natural number n such that $1 \leq n \leq \text{len } W_2$ holds $W_1(l + n) = W_2(n)$ holds $k \leq l$ by [11, (12)], (1), [13, (25)], (6). $i \leq \text{len } W_1$ and there exists an even natural number k such that $i = k + 1$ and for every natural number n such that $1 \leq n \leq \text{len } W_2$ holds $W_1(k + n) = W_2(n)$ and for every even natural number l such that for every natural number n such that $1 \leq n \leq \text{len } W_2$ holds $W_1(l + n) = W_2(n)$ holds $k \leq l$. $W_1.\text{findFirstVertex}(W_2) < n_1$. $n_1 \in \text{dom } W_1$ by [13, (25)]. $e \notin (W_1.\text{cut}(1, W_1.\text{findFirstVertex}(W_2))).\text{edges}()$ by [8, (99)], [1, (4)], (31), [8, (36)]. $e \notin (W_1.\text{cut}(W_1.\text{findLastVertex}(W_2), \text{len } W_1)).\text{edges}()$ by [8, (99)], [1, (4)], (31), [8, (36)]. \square

- (40) Let us consider a graph G , a trail T_1 of G , a walk W_3 of G , and an object

e . Suppose e joins $W_3.first()$ and $W_3.last()$ in G and $e \notin W_3.edges()$ and $G.walkOf(W_3.first(), e, W_3.last())$ is odd-substring of T_1 and 0. Then $e \notin (T_1.replaceEdgeWith(e, W_3)).edges()$.

PROOF: For every even natural numbers n, m such that $n, m \in \text{dom } T_1$ and $T_1(n) = e$ and $T_1(m) = e$ holds $n = m$ by [13, (25)], [8, (138)]. \square

- (41) Let us consider a graph G , and walks W_1, W_2 of G . Suppose $W_1.first() = W_2.first()$ and $W_1.last() = W_2.last()$. Then $W_1.replaceWith(W_1, W_2) = W_2$. The theorem is a consequence of (11), (32), and (31).
- (42) Let us consider a graph G , walks W_1, W_3 of G , and an object e . Suppose e joins $W_3.first()$ and $W_3.last()$ in G and $G.walkOf(W_3.first(), e, W_3.last())$ is odd-substring of W_1 and 0. Then there exists a walk W_2 of G such that $W_1.replaceEdgeWith(e, W_3) = W_1.replaceWith(W_2, W_3)$.
- (43) Let us consider a graph G , walks W_1, W_2 of G , and an object e . Suppose W_2 is odd-substring of W_1 and 0 and e joins $W_2.first()$ and $W_2.last()$ in G . Then there exists a walk W_3 of G such that $W_1.replaceWithEdge(W_2, e) = W_1.replaceWith(W_2, W_3)$.
- (44) Let us consider a graph G , walks W_1, W_3 of G , and an object e . Then
- (i) $W_1.first() = (W_1.replaceEdgeWith(e, W_3)).first()$, and
 - (ii) $W_1.last() = (W_1.replaceEdgeWith(e, W_3)).last()$.

The theorem is a consequence of (42) and (35).

- (45) Let us consider a graph G , walks W_1, W_2 of G , and an object e . Then
- (i) $W_1.first() = (W_1.replaceWithEdge(W_2, e)).first()$, and
 - (ii) $W_1.last() = (W_1.replaceWithEdge(W_2, e)).last()$.

The theorem is a consequence of (43) and (35).

- (46) Let us consider a graph G , walks W_1, W_2, W_3 of G , and objects u, v . Then W_1 is walk from u to v if and only if $W_1.replaceWith(W_2, W_3)$ is walk from u to v . The theorem is a consequence of (35).
- (47) Let us consider a graph G , walks W_1, W_3 of G , and objects e, u, v . Then W_1 is walk from u to v if and only if $W_1.replaceEdgeWith(e, W_3)$ is walk from u to v . The theorem is a consequence of (42) and (46).
- (48) Let us consider a graph G , walks W_1, W_2 of G , and objects e, u, v . Then W_1 is walk from u to v if and only if $W_1.replaceWithEdge(W_2, e)$ is walk from u to v . The theorem is a consequence of (43) and (46).
- (49) Let us consider a graph G , and vertices v_1, v_2 of G . Suppose v_1 is isolated and $v_1 \neq v_2$. Then $v_2 \notin G.reachableFrom(v_1)$.
- (50) Let us consider a graph G , and vertices v_1, v_2 of G . If $v_1 \in G.reachableFrom(v_2)$, then $v_2 \in G.reachableFrom(v_1)$.

(51) Let us consider a graph G , and a vertex v of G . If v is isolated, then $\{v\} = G.\text{reachableFrom}(v)$.

PROOF: For every object x , $x \in \{v\}$ iff $x \in G.\text{reachableFrom}(v)$ by [9, (9)], (49). \square

(52) Let us consider a graph G , a vertex v of G , and a subgraph C of G induced by $\{v\}$. If v is isolated, then C is a component of G . The theorem is a consequence of (51).

(53) Let us consider a non trivial graph G_1 , a vertex v of G_1 , and a subgraph G_2 of G_1 with vertex v removed. Suppose v is isolated. Then

(i) $G_1.\text{componentSet}() = G_2.\text{componentSet}() \cup \{\{v\}\}$, and

(ii) $G_1.\text{numComponents}() = G_2.\text{numComponents}() + 1$.

PROOF: For every object V , $V \in G_1.\text{componentSet}()$ iff $V \in G_2.\text{componentSet}() \cup \{\{v\}\}$ by (51), [8, (127), (135), (171)]. $\{v\} \notin G_2.\text{componentSet}()$ by [10, (47)]. \square

Let G be a graph. Let us observe that every vertex of G which is isolated is also non cut-vertex.

Now we state the propositions:

(54) Let us consider a graph G_1 , a subgraph G_2 of G_1 , a walk W_1 of G_1 , and a walk W_2 of G_2 . If $W_1 = W_2$, then W_1 is cycle-like iff W_2 is cycle-like.

(55) Let us consider a connected graph G_1 , and a component G_2 of G_1 . Then $G_1 \approx G_2$.

Observe that every graph which is complete is also connected and there exists a graph which is non non-directed-multi, non non-multi, non loopless, non directed-simple, non simple, non acyclic, and non finite.

From now on G denotes a graph.

Let us consider G . The functor $\text{Endvertices}(G)$ yielding a subset of the vertices of G is defined by

(Def. 8) for every object v , $v \in \text{it}$ iff there exists a vertex w of G such that $v = w$ and w is endvertex.

Now we state the proposition:

(56) Let us consider a vertex v of G . Then $v \in \text{Endvertices}(G)$ if and only if v is endvertex.

3. SUPERGRAPHS

Let us consider G .

$\text{A Supergraph of } G$ is a graph defined by

(Def. 9) the vertices of $G \subseteq$ the vertices of it and the edges of $G \subseteq$ the edges of it and for every set e such that $e \in$ the edges of G holds (the source of $G)(e) =$ (the source of $it)(e)$ and (the target of $G)(e) =$ (the target of $it)(e)$.

Let us consider graphs G_1, G_2 . Now we state the propositions:

- (57) G_2 is a subgraph of G_1 if and only if G_1 is a Supergraph of G_2 .
- (58) G_2 is subgraph of G_1 and Supergraph of G_1 if and only if $G_1 \approx G_2$. The theorem is a consequence of (57).
- (59) G_1 is a Supergraph of G_2 and G_2 is a Supergraph of G_1 if and only if $G_1 \approx G_2$. The theorem is a consequence of (57).
- (60) G_1 is a Supergraph of G_2 if and only if $G_2 \subseteq G_1$. The theorem is a consequence of (57).

Now we state the propositions:

- (61) G is a Supergraph of G .
- (62) Let us consider a graph G_3 , and a Supergraph G_2 of G_3 . Then every Supergraph of G_2 is a Supergraph of G_3 . The theorem is a consequence of (57).

In the sequel G_2 denotes a graph and G_1 denotes a Supergraph of G_2 .

Now we state the proposition:

- (63) Let us consider graphs G_1, G_2 . Suppose the vertices of $G_2 \subseteq$ the vertices of G_1 and the source of $G_2 \subseteq$ the source of G_1 and the target of $G_2 \subseteq$ the target of G_1 . Then G_1 is a Supergraph of G_2 .

Let us consider G_2 and G_1 . Now we state the propositions:

- (64) (i) the source of $G_2 \subseteq$ the source of G_1 , and
(ii) the target of $G_2 \subseteq$ the target of G_1 .
- (65) Suppose the vertices of $G_2 =$ the vertices of G_1 and the edges of $G_2 =$ the edges of G_1 . Then $G_1 \approx G_2$. The theorem is a consequence of (64).

Now we state the propositions:

- (66) Let us consider graphs G_1, G_2 . Suppose the vertices of $G_2 =$ the vertices of G_1 and the edges of $G_2 =$ the edges of G_1 and the source of $G_2 \subseteq$ the source of G_1 and the target of $G_2 \subseteq$ the target of G_1 . Then $G_1 \approx G_2$. The theorem is a consequence of (63) and (65).
- (67) Let us consider a set x . Then
(i) if $x \in$ the vertices of G_2 , then $x \in$ the vertices of G_1 , and
(ii) if $x \in$ the edges of G_2 , then $x \in$ the edges of G_1 .

The theorem is a consequence of (57).

Let us consider G_2 and G_1 . Now we state the propositions:

- (68) Every vertex of G_2 is a vertex of G_1 .
- (69) (i) the source of $G_2 = (\text{the source of } G_1) \upharpoonright (\text{the edges of } G_2)$, and
 (ii) the target of $G_2 = (\text{the target of } G_1) \upharpoonright (\text{the edges of } G_2)$.
- The theorem is a consequence of (57).

Now we state the proposition:

- (70) Let us consider sets x, y , and an object e . Then
- (i) if e joins x and y in G_2 , then e joins x and y in G_1 , and
- (ii) if e joins x to y in G_2 , then e joins x to y in G_1 , and
- (iii) if e joins a vertex from x and a vertex from y in G_2 , then e joins a vertex from x and a vertex from y in G_1 , and
- (iv) if e joins a vertex from x to a vertex from y in G_2 , then e joins a vertex from x to a vertex from y in G_1 .

The theorem is a consequence of (57).

Let us consider G_2, G_1 , and objects e, v_1, v_2 . Now we state the propositions:

- (71) If e joins v_1 to v_2 in G_1 , then e joins v_1 to v_2 in G_2 or $e \notin$ the edges of G_2 .
- (72) If e joins v_1 and v_2 in G_1 , then e joins v_1 and v_2 in G_2 or $e \notin$ the edges of G_2 . The theorem is a consequence of (71).

Let G be a finite graph. Observe that there exists a Supergraph of G which is finite.

Now we state the propositions:

- (73) (i) $G_2.\text{order}() \subseteq G_1.\text{order}()$, and
 (ii) $G_2.\text{size}() \subseteq G_1.\text{size}()$.
- (74) Let us consider a finite graph G_2 , and a finite Supergraph G_1 of G_2 . Then
- (i) $G_2.\text{order}() \leq G_1.\text{order}()$, and
 (ii) $G_2.\text{size}() \leq G_1.\text{size}()$.

The theorem is a consequence of (57).

- (75) Every walk of G_2 is a walk of G_1 . The theorem is a consequence of (57).
- (76) Let us consider a walk W_2 of G_2 , and a walk W_1 of G_1 . Suppose $W_1 = W_2$. Then
- (i) W_1 is closed iff W_2 is closed, and
- (ii) W_1 is directed iff W_2 is directed, and
- (iii) W_1 is trivial iff W_2 is trivial, and
- (iv) W_1 is trail-like iff W_2 is trail-like, and

- (v) W_1 is path-like iff W_2 is path-like, and
- (vi) W_1 is vertex-distinct iff W_2 is vertex-distinct, and
- (vii) W_1 is cycle-like iff W_2 is cycle-like.

The theorem is a consequence of (57) and (54).

Let G be a non trivial graph. Note that every Supergraph of G is non trivial and every Supergraph of G is non non-directed-multi and every Supergraph of G is non non-multi and every Supergraph of G is non loopless and every Supergraph of G is non directed-simple and every Supergraph of G is non simple and every Supergraph of G is non acyclic and every Supergraph of G is non finite.

In the sequel V denotes a set.

Let us consider G and V .

An addVertices of G and V is a Supergraph of G defined by

- (Def. 10) the vertices of $it = (the\ vertices\ of\ G) \cup V$ and the edges of $it =$ the edges of G and the source of $it =$ the source of G and the target of $it =$ the target of G .

Now we state the propositions:

- (77) Let us consider addVerticess G_1, G_2 of G and V . Then $G_1 \approx G_2$.
- (78) Let us consider an addVertices G_1 of G_2 and V . Then $G_1 \approx G_2$ if and only if $V \subseteq$ the vertices of G_2 .
- (79) Let us consider graphs G_1, G_2 , and a set V . Suppose $G_1 \approx G_2$ and $V \subseteq$ the vertices of G_2 . Then G_1 is an addVertices of G_2 and V . The theorem is a consequence of (59).
- (80) Let us consider an addVertices G_1 of G and V . If $G_1 \approx G_2$, then G_2 is an addVertices of G and V . The theorem is a consequence of (58) and (62).
- (81) Let us consider an addVertices G_1 of G_2 and V . Then $G_1.edgesBetween(the\ vertices\ of\ G_2) =$ the edges of G_1 .
 PROOF: Set $E_1 =$ the edges of G_1 . Set $V_2 =$ the vertices of G_2 . For every object $e, e \in E_1$ iff $e \in G_1.edgesInto(V_2) \cap G_1.edgesOutOf(V_2)$ by [6, (5)].
 \square
- (82) Let us consider a graph G_3 , sets V_1, V_2 , and an addVertices G_2 of G_3 and V_2 . Then every addVertices of G_2 and V_1 is an addVertices of G_3 and $V_1 \cup V_2$. The theorem is a consequence of (62).
- (83) Let us consider a graph G_3 , sets V_1, V_2 , and an addVertices G_1 of G_3 and $V_1 \cup V_2$. Then there exists an addVertices G_2 of G_3 and V_2 such that G_1 is an addVertices of G_2 and V_1 .
- (84) Let us consider an addVertices G_1 of G_2 and V . Then G_2 is a subgraph of G_1 induced by the vertices of G_2 . The theorem is a consequence of (57)

and (81).

(85) Let us consider an addVertices G_1 of G_2 and V , and objects x, y, e . Then e joins x to y in G_1 if and only if e joins x to y in G_2 .

(86) Let us consider an addVertices G_1 of G_2 and V , and an object v . If $v \in V$, then v is a vertex of G_1 .

(87) Let us consider an addVertices G_1 of G_2 and V , and objects x, y, e . Then e joins x and y in G_1 if and only if e joins x and y in G_2 . The theorem is a consequence of (85).

(88) Let us consider an addVertices G_1 of G_2 and V , and a vertex v of G_1 . Suppose $v \in V \setminus$ (the vertices of G_2). Then v is isolated and non cut-vertex. PROOF: $v.edgesInOut() = \emptyset$ by [10, (61)], [6, (5)]. \square

(89) Let us consider an addVertices G_1 of G_2 and V . Suppose $V \setminus$ (the vertices of G_2) $\neq \emptyset$. Then G_1 is non trivial, non connected, non tree-like, and non complete.

PROOF: Consider v_1 being an object such that $v_1 \in V \setminus$ (the vertices of G_2). $\bar{\alpha} \neq 1$, where α is the vertices of G_1 by [2, (69)]. v_1 is isolated. \square

Let G be a non-directed-multi graph and V be a set. Note that every addVertices of G and V is non-directed-multi.

Let G be a non-multi graph. One can verify that every addVertices of G and V is non-multi.

Let G be a loopless graph. Observe that every addVertices of G and V is loopless.

Let G be a directed-simple graph. Let us note that every addVertices of G and V is directed-simple and every addVertices of G and V is directed-simple.

Let us consider G_2, V , an addVertices G_1 of G_2 and V , and a walk W of G_1 . Now we state the propositions:

- (90) (i) $W.vertices()$ misses $V \setminus$ (the vertices of G_2), or
(ii) W is trivial.

The theorem is a consequence of (85).

(91) If $W.vertices()$ misses $V \setminus$ (the vertices of G_2), then W is a walk of G_2 .

The theorem is a consequence of (57).

Let G be an acyclic graph and V be a set. Note that every addVertices of G and V is acyclic.

Now we state the proposition:

(92) Let us consider an addVertices G_1 of G_2 and V . Then G_2 is chordal if and only if G_1 is chordal.

PROOF: If G_2 is chordal, then G_1 is chordal by (90), (91), (76), [8, (114)]. G_2 is a subgraph of G_1 induced by the vertices of G_2 . \square

Let G be a chordal graph and V be a set. Observe that every addVertices of G and V is chordal.

From now on v denotes an object.

Let us consider G and v .

An addVertex of G and v is an addVertices of G and $\{v\}$.

Let us consider G_2 , v , and an addVertex G_1 of G_2 and v . Now we state the propositions:

(93) $G_1 \approx G_2$ if and only if $v \in$ the vertices of G_2 .

(94) v is a vertex of G_1 . The theorem is a consequence of (86).

Let us consider G . Let us observe that every addVertex of G and the vertices of G is non trivial, non connected, and non complete and there exists a graph which is non trivial, non connected, and non complete.

Let G be a non connected graph and V be a set. One can check that every addVertices of G and V is non connected.

Now we state the propositions:

(95) Let us consider an addVertices G_1 of G_2 and V . Then

(i) $G_1.size() = G_2.size()$, and

(ii) $G_1.order() = G_2.order() + \overline{V \setminus \alpha}$,

where α is the vertices of G_2 .

(96) Let us consider a finite graph G_2 , a finite set V , and an addVertices G_1 of G_2 and V . Then $G_1.order() = G_2.order() + \overline{V \setminus \alpha}$, where α is the vertices of G_2 .

(97) Let us consider a graph G_2 , an object v , and an addVertex G_1 of G_2 and v . Suppose $v \notin$ the vertices of G_2 . Then $G_1.order() = G_2.order() + 1$. The theorem is a consequence of (95).

(98) Let us consider a finite graph G_2 , an object v , and an addVertex G_1 of G_2 and v . Suppose $v \notin$ the vertices of G_2 . Then $G_1.order() = G_2.order() + 1$. The theorem is a consequence of (96).

Let G be a finite graph and V be a finite set. One can check that every addVertices of G and V is finite.

Let v be an object. One can check that every addVertex of G and v is finite.

Let G be a graph and V be a non finite set. One can check that every addVertices of G and V is non finite.

Let us consider G . Let v_1, e, v_2 be objects.

An addEdge of G, v_1, e and v_2 is a Supergraph of G defined by

(Def. 11) (i) the vertices of $it =$ the vertices of G and the edges of $it =$ (the edges of G) $\cup \{e\}$ and the source of $it =$ (the source of G) $\cdot (e \mapsto v_1)$ and

the target of $it = (\text{the target of } G) + \cdot (e \mapsto v_2)$, **if** $v_1, v_2 \in$ the vertices of G and $e \notin$ the edges of G ,

(ii) $it \approx G$, **otherwise**.

Now we state the propositions:

(99) Let us consider objects v_1, e, v_2 , and addEdges G_1, G_2 of G, v_1, e and v_2 . Then $G_1 \approx G_2$.

(100) Let us consider vertices v_1, v_2 of G_2 , an object e , and an addEdge G_1 of G_2, v_1, e and v_2 . Then $G_1 \approx G_2$ if and only if $e \in$ the edges of G_2 .

(101) Let us consider objects v_1, e, v_2 , and an addEdge G_1 of G, v_1, e and v_2 . If $G_1 \approx G_2$, then G_2 is an addEdge of G, v_1, e and v_2 . The theorem is a consequence of (58) and (62).

Let us consider G_2 , vertices v_1, v_2 of G_2 , an object e , and an addEdge G_1 of G_2, v_1, e and v_2 . Now we state the propositions:

(102) The vertices of $G_1 =$ the vertices of G_2 .

(103) $G_1.\text{edgesBetween}(\text{the vertices of } G_2) =$ the edges of G_1 . The theorem is a consequence of (102).

(104) Every vertex of G_1 is a vertex of G_2 .

(105) If $e \notin$ the edges of G_2 , then e joins v_1 to v_2 in G_1 .

Let us consider G_2 , vertices v_1, v_2 of G_2 , an object e , an addEdge G_1 of G_2, v_1, e and v_2 , and objects e_1, w_1, w_2 . Now we state the propositions:

(106) Suppose $e \notin$ the edges of G_2 . Then if e_1 joins w_1 and w_2 in G_1 and $e_1 \notin$ the edges of G_2 , then $e_1 = e$.

(107) Suppose $e \notin$ the edges of G_2 . Then suppose e_1 joins w_1 and w_2 in G_1 and $e_1 \notin$ the edges of G_2 . Then

(i) $w_1 = v_1$ and $w_2 = v_2$, or

(ii) $w_1 = v_2$ and $w_2 = v_1$.

The theorem is a consequence of (106) and (105).

Now we state the propositions:

(108) Let us consider vertices v_1, v_2 of G_2 , a set e , and an addEdge G_1 of G_2, v_1, e and v_2 . Suppose $e \notin$ the edges of G_2 . Then G_2 is a subgraph of G_1 with edge e removed. The theorem is a consequence of (57).

(109) Let us consider vertices v_1, v_2 of G_2 , an object e , an addEdge G_1 of G_2, v_1, e and v_2 , and a walk W of G_1 . Suppose if $e \in W.\text{edges}()$, then $e \in$ the edges of G_2 . Then W is a walk of G_2 . The theorem is a consequence of (57).

Let G be a trivial graph and v_1, e, v_2 be objects. Note that every addEdge of G, v_1, e and v_2 is trivial.

Let G be a connected graph. Note that every `addEdge` of G , v_1 , e and v_2 is connected.

Let G be a complete graph. One can check that every `addEdge` of G , v_1 , e and v_2 is complete.

Now we state the propositions:

(110) Let us consider vertices v_1, v_2 of G_2 , an object e , and an `addEdge` G_1 of G_2 , v_1 , e and v_2 . Suppose $e \notin$ the edges of G_2 . Then

(i) $G_1.\text{order}() = G_2.\text{order}()$, and

(ii) $G_1.\text{size}() = G_2.\text{size}() + 1$.

(111) Let us consider a finite graph G_2 , vertices v_1, v_2 of G_2 , an object e , and an `addEdge` G_1 of G_2 , v_1 , e and v_2 . Suppose $e \notin$ the edges of G_2 . Then $G_1.\text{size}() = G_2.\text{size}() + 1$.

Let G be a finite graph and v_1, e, v_2 be objects. Let us note that every `addEdge` of G , v_1 , e and v_2 is finite.

Now we state the propositions:

(112) Let us consider vertices v_1, v_2 of G_2 , an object e , and an `addEdge` G_1 of G_2 , v_1 , e and v_2 . If G_2 is loopless and $v_1 \neq v_2$, then G_1 is loopless. The theorem is a consequence of (105).

(113) Let us consider a vertex v of G_2 , an object e , and an `addEdge` G_1 of G_2 , v , e and v . Suppose G_2 is not loopless or $e \notin$ the edges of G_2 . Then G_1 is not loopless. The theorem is a consequence of (105).

Let us consider G . Let v be a vertex of G . Note that every `addEdge` of G , v , the edges of G and v is non loopless.

Let us consider G_2 , vertices v_1, v_2 of G_2 , an object e , and an `addEdge` G_1 of G_2 , v_1 , e and v_2 . Now we state the propositions:

(114) If G_2 is non-directed-multi and there exists no object e_3 such that e_3 joins v_1 to v_2 in G_2 , then G_1 is non-directed-multi. The theorem is a consequence of (71) and (105).

(115) Suppose $e \notin$ the edges of G_2 and there exists an object e_2 such that e_2 joins v_1 to v_2 in G_2 . Then G_1 is not non-directed-multi. The theorem is a consequence of (105) and (70).

(116) If G_2 is non-multi and v_1 and v_2 are not adjacent, then G_1 is non-multi. The theorem is a consequence of (72) and (105).

(117) If $e \notin$ the edges of G_2 and v_1 and v_2 are adjacent, then G_1 is not non-multi.

PROOF: There exist objects e_1, e_2, u_1, u_2 such that e_1 joins u_1 and u_2 in G_1 and e_2 joins u_1 and u_2 in G_1 and $e_1 \neq e_2$ by (70), (105), [10, (16)]. \square

- (118) If G_2 is acyclic and $v_2 \notin G_2.\text{reachableFrom}(v_1)$, then G_1 is acyclic. The theorem is a consequence of (57), (54), and (105).
- (119) If $e \notin$ the edges of G_2 and $v_2 \in G_2.\text{reachableFrom}(v_1)$, then G_1 is not acyclic. The theorem is a consequence of (75), (105), and (113).
- (120) If G_2 is not connected and $v_2 \in G_2.\text{reachableFrom}(v_1)$, then G_1 is not connected. The theorem is a consequence of (68), (109), (27), (105), (75), (47), and (40).
- (121) Suppose $e \notin$ the edges of G_2 and for every vertices v_3, v_4 of G_2 such that v_3 and v_4 are not adjacent holds $v_3 = v_4$ or $v_1 = v_3$ and $v_2 = v_4$ or $v_1 = v_4$ and $v_2 = v_3$. Then G_1 is complete.
 PROOF: For every vertices u_1, u_2 of G_1 such that $u_1 \neq u_2$ holds u_1 and u_2 are adjacent by (104), (70), (105), [10, (16)]. \square
- (122) If G_2 is not complete and v_1 and v_2 are adjacent, then G_1 is not complete. The theorem is a consequence of (68), (72), and (105).

Let us consider G . Let v_1, e, v_2 be objects.

An **addAdjVertex** of G, v_1, e and v_2 is a Supergraph of G defined by

- (Def. 12) (i) the vertices of $it = (\text{the vertices of } G) \cup \{v_2\}$ and the edges of $it = (\text{the edges of } G) \cup \{e\}$ and the source of $it = (\text{the source of } G) + \cdot(e \dashrightarrow v_1)$ and the target of $it = (\text{the target of } G) + \cdot(e \dashrightarrow v_2)$, **if** $v_1 \in$ the vertices of G and $v_2 \notin$ the vertices of G and $e \notin$ the edges of G ,
- (ii) the vertices of $it = (\text{the vertices of } G) \cup \{v_1\}$ and the edges of $it = (\text{the edges of } G) \cup \{e\}$ and the source of $it = (\text{the source of } G) + \cdot(e \dashrightarrow v_1)$ and the target of $it = (\text{the target of } G) + \cdot(e \dashrightarrow v_2)$, **if** $v_1 \notin$ the vertices of G and $v_2 \in$ the vertices of G and $e \notin$ the edges of G ,
- (iii) $it \approx G$, **otherwise**.

Let v_1 be a vertex of G and e, v_2 be objects.

One can verify that an **addAdjVertex** of G, v_1, e and v_2 can equivalently be formulated as follows:

- (Def. 13) (i) the vertices of $it = (\text{the vertices of } G) \cup \{v_2\}$ and the edges of $it = (\text{the edges of } G) \cup \{e\}$ and the source of $it = (\text{the source of } G) + \cdot(e \dashrightarrow v_1)$ and the target of $it = (\text{the target of } G) + \cdot(e \dashrightarrow v_2)$, **if** $v_2 \notin$ the vertices of G and $e \notin$ the edges of G ,
- (ii) $it \approx G$, **otherwise**.

Let v_1, e be objects and v_2 be a vertex of G .

Note that an **addAdjVertex** of G, v_1, e and v_2 can equivalently be formulated as follows:

- (Def. 14) (i) the vertices of $it = (\text{the vertices of } G) \cup \{v_1\}$ and the edges of $it = (\text{the edges of } G) \cup \{e\}$ and the source of $it = (\text{the source of } G) + \cdot (e \dashrightarrow v_1)$ and the target of $it = (\text{the target of } G) + \cdot (e \dashrightarrow v_2)$, **if** $v_1 \notin \text{the vertices of } G$ and $e \notin \text{the edges of } G$,
- (ii) $it \approx G$, **otherwise**.

Now we state the propositions:

- (123) Let us consider objects v_1, e, v_2 , and $\text{addAdjVertexs } G_1, G_2$ of G, v_1, e and v_2 . Then $G_1 \approx G_2$.
- (124) Let us consider objects v_1, e, v_2 , and an $\text{addAdjVertex } G_1$ of G, v_1, e and v_2 . If $G_1 \approx G_2$, then G_2 is an addAdjVertex of G, v_1, e and v_2 . The theorem is a consequence of (58) and (62).
- (125) Let us consider a vertex v_1 of G_2 , objects e, v_2 , and an $\text{addAdjVertex } G_1$ of G_2, v_1, e and v_2 . Suppose $e \notin \text{the edges of } G_2$ and $v_2 \notin \text{the vertices of } G_2$. Then there exists an $\text{addVertex } G_3$ of G_2 and v_2 such that G_1 is an addEdge of G_3, v_1, e and v_2 . The theorem is a consequence of (94).
- (126) Let us consider objects v_1, e , a vertex v_2 of G_2 , and an $\text{addAdjVertex } G_1$ of G_2, v_1, e and v_2 . Suppose $e \notin \text{the edges of } G_2$ and $v_1 \notin \text{the vertices of } G_2$. Then there exists an $\text{addVertex } G_3$ of G_2 and v_1 such that G_1 is an addEdge of G_3, v_1, e and v_2 . The theorem is a consequence of (94).
- (127) Let us consider a graph G_3 , a vertex v_1 of G_3 , objects e, v_2 , an $\text{addVertex } G_2$ of G_3 and v_2 , and an $\text{addEdge } G_1$ of G_2, v_1, e and v_2 . Suppose $e \notin \text{the edges of } G_3$ and $v_2 \notin \text{the vertices of } G_3$. Then G_1 is an addAdjVertex of G_3, v_1, e and v_2 . The theorem is a consequence of (62), (68), and (94).
- (128) Let us consider a graph G_3 , objects v_1, e , a vertex v_2 of G_3 , an $\text{addVertex } G_2$ of G_3 and v_1 , and an $\text{addEdge } G_1$ of G_2, v_1, e and v_2 . Suppose $e \notin \text{the edges of } G_3$ and $v_1 \notin \text{the vertices of } G_3$. Then G_1 is an addAdjVertex of G_3, v_1, e and v_2 . The theorem is a consequence of (62), (68), and (94).
- (129) Let us consider a vertex v_1 of G_2 , objects e, v_2 , and an $\text{addAdjVertex } G_1$ of G_2, v_1, e and v_2 . Suppose $v_2 \notin \text{the vertices of } G_2$ and $e \notin \text{the edges of } G_2$. Then v_2 is a vertex of G_1 .
- (130) Let us consider objects v_1, e , a vertex v_2 of G_2 , and an $\text{addAdjVertex } G_1$ of G_2, v_1, e and v_2 . Suppose $v_1 \notin \text{the vertices of } G_2$ and $e \notin \text{the edges of } G_2$. Then v_1 is a vertex of G_1 .
- (131) Let us consider a vertex v_1 of G_2 , objects e, v_2 , and an $\text{addAdjVertex } G_1$ of G_2, v_1, e and v_2 . Suppose $v_2 \notin \text{the vertices of } G_2$ and $e \notin \text{the edges of } G_2$. Then
- (i) e joins v_1 to v_2 in G_1 , and
- (ii) e joins v_1 and v_2 in G_1 .

(132) Let us consider objects v_1, e , a vertex v_2 of G_2 , and an addAdjVertex G_1 of G_2 , v_1, e and v_2 . Suppose $v_1 \notin$ the vertices of G_2 and $e \notin$ the edges of G_2 . Then

- (i) e joins v_1 to v_2 in G_1 , and
- (ii) e joins v_1 and v_2 in G_1 .

(133) Let us consider a vertex v_1 of G_2 , objects e, v_2 , and an addAdjVertex G_1 of G_2 , v_1, e and v_2 . Suppose $v_2 \notin$ the vertices of G_2 and $e \notin$ the edges of G_2 . Let us consider objects e_1, w . If $w \neq v_1$ or $e_1 \neq e$, then e_1 does not join w and v_2 in G_1 . The theorem is a consequence of (72) and (131).

(134) Let us consider objects v_1, e , a vertex v_2 of G_2 , and an addAdjVertex G_1 of G_2 , v_1, e and v_2 . Suppose $v_1 \notin$ the vertices of G_2 and $e \notin$ the edges of G_2 . Let us consider objects e_1, w . If $w \neq v_2$ or $e_1 \neq e$, then e_1 does not join v_1 and w in G_1 . The theorem is a consequence of (72) and (132).

Let us consider G_2 , objects v_1, e, v_2 , and an addAdjVertex G_1 of G_2 , v_1, e and v_2 . Now we state the propositions:

(135) G_1 .edgesBetween(the vertices of G_2) = the edges of G_2 . The theorem is a consequence of (131), (70), and (132).

(136) G_2 is a subgraph of G_1 induced by the vertices of G_2 . The theorem is a consequence of (57), (135), (15), and (16).

Now we state the propositions:

(137) Let us consider a vertex v_1 of G_2 , an object e , a set v_2 , and an addAdjVertex G_1 of G_2 , v_1, e and v_2 . Suppose $e \notin$ the edges of G_2 and $v_2 \notin$ the vertices of G_2 . Then G_2 is a subgraph of G_1 with vertex v_2 removed. The theorem is a consequence of (136).

(138) Let us consider a set v_1 , an object e , a vertex v_2 of G_2 , and an addAdjVertex G_1 of G_2 , v_1, e and v_2 . Suppose $e \notin$ the edges of G_2 and $v_1 \notin$ the vertices of G_2 . Then G_2 is a subgraph of G_1 with vertex v_1 removed. The theorem is a consequence of (136).

(139) Let us consider a non trivial graph G , a vertex v_1 of G , objects e, v_2 , an addAdjVertex G_1 of G , v_1, e and v_2 , a subgraph G_2 of G_1 with vertex v_1 removed, and a subgraph G_3 of G with vertex v_1 removed. Suppose $e \notin$ the edges of G and $v_2 \notin$ the vertices of G . Then G_2 is an addVertex of G_3 and v_2 .

PROOF: v_1 is a vertex of G_1 and $v_1 \neq v_2$. For every object $e_1, e_1 \in G_1$.edgesBetween((the vertices of G_1) \setminus \{v_1\}) iff $e_1 \in G$.edgesBetween((the vertices of G) \setminus \{v_1\}) by [10, (31)], (71), [10, (16), (13)]. For every object e_1 such that $e_1 \in \text{dom}(\text{the source of } G_2)$ holds (the source of G_2)(e_1) = (the source of

$G_3)(e_1)$. For every object e_1 such that $e_1 \in \text{dom}(\text{the target of } G_2)$ holds (the target of $G_2)(e_1) = (\text{the target of } G_3)(e_1)$. \square

- (140) Let us consider a non trivial graph G , objects v_1, e , a vertex v_2 of G , an addAdjVertex G_1 of G , v_1, e and v_2 , a subgraph G_2 of G_1 with vertex v_2 removed, and a subgraph G_3 of G with vertex v_2 removed. Suppose $e \notin$ the edges of G and $v_1 \notin$ the vertices of G . Then G_2 is an addVertex of G_3 and v_1 .

PROOF: v_2 is a vertex of G_1 and $v_1 \neq v_2$. For every object $e_1, e_1 \in G_1.\text{edgesBetween}(\text{the vertices of } G_1 \setminus \{v_2\})$ iff $e_1 \in G.\text{edgesBetween}(\text{the vertices of } G) \setminus \{v_2\}$ by [10, (31)], (71), [10, (16)], (13)]. For every object e_1 such that $e_1 \in \text{dom}(\text{the source of } G_2)$ holds (the source of $G_2)(e_1) = (\text{the source of } G_3)(e_1)$. For every object e_1 such that $e_1 \in \text{dom}(\text{the target of } G_2)$ holds (the target of $G_2)(e_1) = (\text{the target of } G_3)(e_1)$. \square

- (141) Let us consider a vertex v_1 of G_2 , objects e, v_2 , an addAdjVertex G_1 of G_2 , v_1, e and v_2 , and a vertex w of G_1 . Suppose $e \notin$ the edges of G_2 and $v_2 \notin$ the vertices of G_2 and $w = v_2$. Then w is endvertex.

PROOF: There exists an object e_1 such that $w.\text{edgesInOut}() = \{e_1\}$ and e_1 does not join w and w in G_1 by [10, (61)], (133), (131). \square

- (142) Let us consider objects v_1, e , a vertex v_2 of G_2 , an addAdjVertex G_1 of G_2 , v_1, e and v_2 , and a vertex w of G_1 . Suppose $e \notin$ the edges of G_2 and $v_1 \notin$ the vertices of G_2 and $w = v_1$. Then w is endvertex.

PROOF: There exists an object e_1 such that $w.\text{edgesInOut}() = \{e_1\}$ and e_1 does not join w and w in G_1 by [10, (61)], (134), (132). \square

- (143) Let us consider a vertex v_1 of G_2 , objects e, v_2 , and an addAdjVertex G_1 of G_2 , v_1, e and v_2 . Suppose $v_2 \notin$ the vertices of G_2 and $e \notin$ the edges of G_2 . Then G_1 is not trivial. The theorem is a consequence of (125) and (89).

- (144) Let us consider objects v_1, e , a vertex v_2 of G_2 , and an addAdjVertex G_1 of G_2 , v_1, e and v_2 . Suppose $v_1 \notin$ the vertices of G_2 and $e \notin$ the edges of G_2 . Then G_1 is not trivial. The theorem is a consequence of (126) and (89).

Let G be a graph and v be a vertex of G . Note that every addAdjVertex of G, v , the edges of G and the vertices of G is non trivial and every addAdjVertex of G , the vertices of G , the edges of G and v is non trivial.

Let G be a trivial graph. One can check that every addAdjVertex of G, v , the edges of G and the vertices of G is complete and every addAdjVertex of G , the vertices of G , the edges of G and v is complete.

Let G be a loopless graph and v_1, e, v_2 be objects. Let us observe that every addAdjVertex of G, v_1, e and v_2 is loopless.

Let G be a non-directed-multi graph. One can verify that every addAdjVertex of G , v_1 , e and v_2 is non-directed-multi.

Let G be a non-multi graph. One can verify that every addAdjVertex of G , v_1 , e and v_2 is non-multi.

Let G be a directed-simple graph. One can verify that every addAdjVertex of G , v_1 , e and v_2 is directed-simple.

Let G be a simple graph. One can verify that every addAdjVertex of G , v_1 , e and v_2 is simple.

Now we state the propositions:

(145) Let us consider a vertex v_1 of G_2 , objects e , v_2 , an addAdjVertex G_1 of G_2 , v_1 , e and v_2 , and a walk W of G_1 . Suppose $e \notin$ the edges of G_2 and $v_2 \notin$ the vertices of G_2 and ($e \notin W.edges()$ and W is not trivial or $v_2 \notin W.vertices()$). Then W is a walk of G_2 . The theorem is a consequence of (125), (68), (94), (108), (90), (91), (137), (129), and (143).

(146) Let us consider objects v_1 , e , a vertex v_2 of G_2 , an addAdjVertex G_1 of G_2 , v_1 , e and v_2 , and a walk W of G_1 . Suppose $e \notin$ the edges of G_2 and $v_1 \notin$ the vertices of G_2 and ($e \notin W.edges()$ and W is not trivial or $v_1 \notin W.vertices()$). Then W is a walk of G_2 . The theorem is a consequence of (126), (68), (94), (108), (90), (91), (138), (130), and (144).

(147) Let us consider objects v_1 , e , v_2 , an addAdjVertex G_1 of G_2 , v_1 , e and v_2 , and a trail T of G_1 . Suppose $e \notin$ the edges of G_2 and $T.first()$, $T.last() \in$ the vertices of G_2 . Then $e \notin T.edges()$. The theorem is a consequence of (129), (141), (145), (130), (142), and (146).

Let G be a connected graph and v_1 , e , v_2 be objects. Observe that every addAdjVertex of G , v_1 , e and v_2 is connected.

Let G be a non connected graph. One can verify that every addAdjVertex of G , v_1 , e and v_2 is non connected.

Let G be an acyclic graph. One can check that every addAdjVertex of G , v_1 , e and v_2 is acyclic.

Let G be a tree-like graph. Let us observe that every addAdjVertex of G , v_1 , e and v_2 is tree-like.

Now we state the propositions:

(148) Let us consider a vertex v_1 of G_2 , objects e , v_2 , and an addAdjVertex G_1 of G_2 , v_1 , e and v_2 . Suppose $e \notin$ the edges of G_2 and $v_2 \notin$ the vertices of G_2 and G_2 is not trivial. Then G_1 is not complete.

PROOF: There exist vertices u , v of G_1 such that $u \neq v$ and u and v are not adjacent by [10, (21)], (68), (129), (133). \square

(149) Let us consider objects v_1 , e , a vertex v_2 of G_2 , and an addAdjVertex G_1 of G_2 , v_1 , e and v_2 . Suppose $e \notin$ the edges of G_2 and $v_1 \notin$ the vertices

of G_2 and G_2 is not trivial. Then G_1 is not complete.

PROOF: There exist vertices u, v of G_1 such that $u \neq v$ and u and v are not adjacent by [10, (21)], (68), (130), (134). \square

Let G be a non complete graph and v_1, e, v_2 be objects. Let us note that every `addAdjVertex` of G, v_1, e and v_2 is non complete.

Let v be a vertex of G . Let us note that every `addAdjVertex` of G, v , the edges of G and the vertices of G is non complete and every `addAdjVertex` of G , the vertices of G , the edges of G and v is non complete.

Now we state the propositions:

(150) Let us consider a vertex v_1 of G_2 , objects e, v_2 , and an `addAdjVertex` G_1 of G_2, v_1, e and v_2 . Suppose $e \notin$ the edges of G_2 and $v_2 \notin$ the vertices of G_2 . Then

(i) $G_1.\text{order}() = G_2.\text{order}() + 1$, and

(ii) $G_1.\text{size}() = G_2.\text{size}() + 1$.

(151) Let us consider objects v_1, e , a vertex v_2 of G_2 , and an `addAdjVertex` G_1 of G_2, v_1, e and v_2 . Suppose $e \notin$ the edges of G_2 and $v_1 \notin$ the vertices of G_2 . Then

(i) $G_1.\text{order}() = G_2.\text{order}() + 1$, and

(ii) $G_1.\text{size}() = G_2.\text{size}() + 1$.

(152) Let us consider a finite graph G_2 , a vertex v_1 of G_2 , objects e, v_2 , and an `addAdjVertex` G_1 of G_2, v_1, e and v_2 . Suppose $e \notin$ the edges of G_2 and $v_2 \notin$ the vertices of G_2 . Then

(i) $G_1.\text{order}() = G_2.\text{order}() + 1$, and

(ii) $G_1.\text{size}() = G_2.\text{size}() + 1$.

(153) Let us consider a finite graph G_2 , objects v_1, e , a vertex v_2 of G_2 , and an `addAdjVertex` G_1 of G_2, v_1, e and v_2 . Suppose $e \notin$ the edges of G_2 and $v_1 \notin$ the vertices of G_2 . Then

(i) $G_1.\text{order}() = G_2.\text{order}() + 1$, and

(ii) $G_1.\text{size}() = G_2.\text{size}() + 1$.

Let G be a finite graph and v_1, e, v_2 be objects. Let us observe that every `addAdjVertex` of G, v_1, e and v_2 is finite.

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