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# About Supergraphs. Part II

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**Summary.** In the previous article [? ] supergraphs and several specializations to formalize the process of drawing graphs were introduced. In this paper another such operation is formalized: Drawing a vertex and then immediately drawing edges connecting this vertex with a subset of the other vertices of the graph. In case the new vertex is joined with all vertices of a given graph $G$, this is known as the join of $G$ and the trivial loopless graph $K_1$. While the join of two graphs is known and found in standard literature (like [? ], [? ] and [? ]), the operation discribed in this article is not.

Alongside the new operation a mode to reverse the directions of a subset of the edges of a graph is introduced. When all edge directions of a graph are reversed, this is commonly known as the converse of a (directed) graph.

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## 1. Reversing Edge Directions

From now on $G$, $G_2$ denote graphs, $V$, $E$ denote sets, and $v$ denotes an object. Let us consider $G$ and $E$.

A reverseEdgeDirections of $G$, $E$ is a graph defined by

(Def. 1) (i) the vertices of *it* = the vertices of $G$ and the edges of *it* = the edges of $G$ and the source of *it* = (the source of $G$)+·(the target of $G$)$\restriction E$ and the target of *it* = (the target of $G$)+·(the source of $G$)$\restriction E$, **if** $E \subseteq$ the edges of $G$,

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(ii) $it \approx G$, **otherwise**.

A reverseEdgeDirections of $G$ is a reverseEdgeDirections of $G$, the edges of $G$. Now we state the propositions:

(1) Let us consider reverseEdgeDirectionss $G_1$, $G_2$ of $G$, $E$. Then $G_1 \approx G_2$.

(2) Let us consider a reverseEdgeDirections $G_1$ of $G$, $E$. If $G_1 \approx G_2$, then $G_2$ is a reverseEdgeDirections of $G$, $E$.

Let us consider $G_2$, $E$, and a reverseEdgeDirections $G_1$ of $G_2$, $E$. Now we state the propositions:

(3) $G_2$ is a reverseEdgeDirections of $G_1$, $E$.

(4) (i) the vertices of $G_1$ = the vertices of $G_2$, and

(ii) the edges of $G_1$ = the edges of $G_2$.

Now we state the propositions:

(5) Let us consider a reverseEdgeDirections $G_1$ of $G_2$. Then $G_2$ is a reverseEdgeDirections of $G_1$. The theorem is a consequence of (4) and (3).

(6) Let us consider a trivial graph $G_2$, a set $E$, and a graph $G_1$. Then $G_1 \approx G_2$ if and only if $G_1$ is a reverseEdgeDirections of $G_2$, $E$.

Let us consider $G_2$, $E$, a reverseEdgeDirections $G_1$ of $G_2$, $E$, and objects $v_1$, $e$, $v_2$. Now we state the propositions:

(7) If $E \subseteq$ the edges of $G_2$ and $e \in E$, then $e$ joins $v_1$ to $v_2$ in $G_2$ iff $e$ joins $v_2$ to $v_1$ in $G_1$. The theorem is a consequence of (3) and (4).

(8) If $E \subseteq$ the edges of $G_2$ and $e \notin E$, then $e$ joins $v_1$ to $v_2$ in $G_2$ iff $e$ joins $v_1$ to $v_2$ in $G_1$. The theorem is a consequence of (3) and (4).

(9) $e$ joins $v_1$ and $v_2$ in $G_2$ if and only if $e$ joins $v_1$ and $v_2$ in $G_1$. The theorem is a consequence of (3).

Now we state the proposition:

(10) Let us consider a reverseEdgeDirections $G_1$ of $G_2$, $E$. Then $v$ is a vertex of $G_1$ if and only if $v$ is a vertex of $G_2$.

Let us consider $G_2$, $E$, $V$, and a reverseEdgeDirections $G_1$ of $G_2$, $E$. Now we state the propositions:

(11) $G_1$.edgesBetween$(V)$ = $G_2$.edgesBetween$(V)$.
PROOF: For every object $e$, $e \in G_1$.edgesBetween$(V)$ iff $e \in G_2$.edgesBetween$(V)$ by (9), [5, (31)]. □

(12) $G_1$.edgesInOut$(V)$ = $G_2$.edgesInOut$(V)$.
PROOF: For every object $e$, $e \in G_1$.edgesInOut$(V)$ iff $e \in G_2$.edgesInOut$(V)$ by (9), [5, (28)]. □

Now we state the proposition:

(13) Let us consider a reverseEdgeDirections $G_1$ of $G_2$, $E$, a vertex $v_1$ of $G_1$, and a vertex $v_2$ of $G_2$. If $v_1 = v_2$, then $v_1$.edgesInOut() = $v_2$.edgesInOut(). The theorem is a consequence of (12).

Let us consider $G_2$, $E$, and a reverseEdgeDirections $G_1$ of $G_2$, $E$. Now we state the propositions:

(14) Every walk of $G_2$ is a walk of $G_1$. The theorem is a consequence of (4) and (9).

(15) Every walk of $G_1$ is a walk of $G_2$. The theorem is a consequence of (3) and (14).

Now we state the propositions:

(16) Let us consider a reverseEdgeDirections $G_1$ of $G_2$, $E$, a walk $W_2$ of $G_2$, and a walk $W_1$ of $G_1$. Suppose $E \subseteq$ the edges of $G_2$ and $W_1 = W_2$ and $W_2$.edges() $\subseteq E$. Then $W_1$ is directed if and only if $W_2$.reverse() is directed.

Proof: For every odd element $n$ of $\mathbb{N}$ such that $n < \operatorname{len} W_1$ holds $W_1(n+1)$ joins $W_1(n)$ to $W_1(n+2)$ in $G_1$ by [4, (1)], [6, (12)], [1, (13)], [8, (3)]. □

(17) Let us consider a reverseEdgeDirections $G_1$ of $G_2$, a walk $W_2$ of $G_2$, and a walk $W_1$ of $G_1$. Suppose $W_1 = W_2$. Then $W_1$ is directed if and only if $W_2$.reverse() is directed. The theorem is a consequence of (16).

(18) Let us consider a reverseEdgeDirections $G_1$ of $G_2$, $E$, a walk $W_2$ of $G_2$, and a walk $W_1$ of $G_1$. If $W_1 = W_2$, then $W_1$ is chordal iff $W_2$ is chordal. The theorem is a consequence of (3).

(19) Let us consider a reverseEdgeDirections $G_1$ of $G_2$, $E$, and objects $v_1$, $v_2$. Then there exists a walk $W_1$ of $G_1$ such that $W_1$ is walk from $v_1$ to $v_2$ if and only if there exists a walk $W_2$ of $G_2$ such that $W_2$ is walk from $v_1$ to $v_2$. The theorem is a consequence of (15) and (14).

(20) Let us consider a reverseEdgeDirections $G_1$ of $G_2$, $E$, a vertex $v_1$ of $G_1$, and a vertex $v_2$ of $G_2$. If $v_1 = v_2$, then $G_1$.reachableFrom($v_1$) = $G_2$.reachableFrom($v_2$). The theorem is a consequence of (19).

(21) Let us consider a reverseEdgeDirections $G_1$ of $G_2$, $E$. Then

(i) $G_1$.componentSet() = $G_2$.componentSet(), and

(ii) $G_1$.numComponents() = $G_2$.numComponents().

The theorem is a consequence of (10) and (20).

Let $G$ be a trivial graph and $E$ be a set. Observe that every reverseEdgeDirections of $G$, $E$ is trivial.

Let $G$ be a non trivial graph. Let us observe that every reverseEdgeDirections of $G$, $E$ is non trivial.

Now we state the propositions:

(22) Let us consider a reverseEdgeDirections $G_1$ of $G_2$, $E$, a set $v$, and a subgraph $G_3$ of $G_1$ with vertex $v$ removed. Then every subgraph of $G_2$ with vertex $v$ removed is a reverseEdgeDirections of $G_3$, $E \setminus G_1$.edgesInOut($\{v\}$). The theorem is a consequence of (11), (2), (3), and (6).

(23) Let us consider a reverseEdgeDirections $G_1$ of $G_2$, $E$, a vertex $v_1$ of $G_1$, and a vertex $v_2$ of $G_2$. Suppose $v_1 = v_2$. Then

(i) $v_1$ is isolated iff $v_2$ is isolated, and

(ii) $v_1$ is endvertex iff $v_2$ is endvertex, and

(iii) $v_1$ is cut-vertex iff $v_2$ is cut-vertex.

The theorem is a consequence of (3).

Let us consider $G_2$, $E$, and a reverseEdgeDirections $G_1$ of $G_2$, $E$. Now we state the propositions:

(24) (i) $G_1$.order() $= G_2$.order(), and

(ii) $G_1$.size() $= G_2$.size().

The theorem is a consequence of (4).

(25) Suppose $E \subseteq$ the edges of $G_2$ and $G_2$ is non-directed-multi and for every objects $e_1$, $e_2$, $v_1$, $v_2$ such that $e_1$ joins $v_1$ and $v_2$ in $G_2$ and $e_2$ joins $v_1$ and $v_2$ in $G_2$ holds $e_1$, $e_2 \in E$ or $e_1 \notin E$ and $e_2 \notin E$. Then $G_1$ is non-directed-multi.

PROOF: For every objects $e_1$, $e_2$, $v_1$, $v_2$ such that $e_1$ joins $v_1$ to $v_2$ in $G_1$ and $e_2$ joins $v_1$ to $v_2$ in $G_1$ holds $e_1 = e_2$ by [5, (16)], (9), (7), (8). □

Let $G$ be a non-directed-multi graph. Let us note that every reverseEdgeDirections of $G$ is non-directed-multi.

Let $G$ be a non non-directed-multi graph. Observe that every reverseEdgeDirections of $G$ is non non-directed-multi.

Let $G$ be a non-multi graph and $E$ be a set. One can verify that every reverseEdgeDirections of $G$, $E$ is non-multi.

Let $G$ be a non non-multi graph. Let us note that every reverseEdgeDirections of $G$, $E$ is non non-multi.

Let $G$ be a loopless graph. One can check that every reverseEdgeDirections of $G$, $E$ is loopless.

Let $G$ be a non loopless graph. One can check that every reverseEdgeDirections of $G$, $E$ is non loopless.

Let $G$ be a connected graph. Let us observe that every reverseEdgeDirections of $G$, $E$ is connected.

Let $G$ be a non connected graph. Observe that every reverseEdgeDirections of $G$, $E$ is non connected.

Let $G$ be an acyclic graph. Note that every reverseEdgeDirections of $G$, $E$ is acyclic.

Let $G$ be a non acyclic graph. One can verify that every reverseEdgeDirections of $G$, $E$ is non acyclic.

Let $G$ be a complete graph. Observe that every reverseEdgeDirections of $G$, $E$ is complete.

Let $G$ be a non complete graph. Observe that every reverseEdgeDirections of $G$, $E$ is non complete.

Let $G$ be a chordal graph. Note that every reverseEdgeDirections of $G$, $E$ is chordal.

Let $G$ be a finite graph. Let us note that every reverseEdgeDirections of $G$, $E$ is finite.

Let $G$ be a non finite graph. One can verify that every reverseEdgeDirections of $G$, $E$ is non finite.

Now we state the propositions:

(26) Let us consider a reverseEdgeDirections $G_1$ of $G_2$. Then

(i) the source of $G_1$ = the target of $G_2$, and

(ii) the target of $G_1$ = the source of $G_2$.

(27) Let us consider a reverseEdgeDirections $G_1$ of $G_2$, and objects $v_1$, $e$, $v_2$. Then $e$ joins $v_1$ to $v_2$ in $G_2$ if and only if $e$ joins $v_2$ to $v_1$ in $G_1$. The theorem is a consequence of (26).

## 2. Adding a Vertex and Several Edges to a Graph

Let us consider $G$, $v$, and $V$.

A addAdjVertexToAll of $G$, $v$, $V$ is a reverseEdgeDirections of $G$ defined by

(Def. 2) (i) the vertices of *it* = (the vertices of $G$) $\cup \{v\}$ and the edges of *it* = (the edges of $G$) $\cup$ ($V \longmapsto$ (the edges of $G$)) and the source of *it* = (the source of $G$)$+\cdot$(($V \longmapsto$ (the edges of $G$)) $\longmapsto v$) and the target of *it* = (the target of $G$)$+\cdot\pi_1(V \boxtimes \{$the edges of $G\})$, **if** $V \subseteq$ the vertices of $G$ and $v \notin$ the vertices of $G$,

(ii) *it* $\approx G$, **otherwise**.

A addAdjVertexFromAll of $G$, $v$, $V$ is a reverseEdgeDirections of $G$ defined by

(Def. 3) (i) the vertices of *it* = (the vertices of $G$) $\cup \{v\}$ and the edges of *it* = (the edges of $G$) $\cup$ ($V \longmapsto$ (the edges of $G$)) and the source of *it* = (the source of $G$)$+\cdot\pi_1(V \boxtimes \{$the edges of $G\})$ and the target of $it$ = (the target of $G$)$+\cdot$(($V \longmapsto$ (the edges of $G$)) $\longmapsto v$), **if** $V \subseteq$ the vertices of $G$ and $v \notin$ the vertices of $G$,

(ii) $it \approx G$, **otherwise**.

A addAdjVertexToAll of $G$, $v$ is a addAdjVertexToAll of $G$, $v$, the vertices of $G$.

A addAdjVertexFromAll of $G$, $v$ is a addAdjVertexFromAll of $G$, $v$, the vertices of $G$. Now we state the propositions:

(28) Let us consider addAdjVertexToAlls $G_1$, $G_2$ of $G$, $v$, $V$. Then $G_1 \approx G_2$.

(29) Let us consider addAdjVertexFromAlls $G_1$, $G_2$ of $G$, $v$, $V$. Then $G_1 \approx G_2$.

(30) Let us consider a addAdjVertexToAll $G_1$ of $G$, $v$, $V$. If $G_1 \approx G_2$, then $G_2$ is a addAdjVertexToAll of $G$, $v$, $V$.

(31) Let us consider a addAdjVertexFromAll $G_1$ of $G$, $v$, $V$. If $G_1 \approx G_2$, then $G_2$ is a addAdjVertexFromAll of $G$, $v$, $V$.

(32) Let us consider a addAdjVertexToAll $G_1$ of $G$, $v$, $V$, and a addAdjVertexFromAll $G_2$ of $G$, $v$, $V$. Then

(i) the vertices of $G_1$ = the vertices of $G_2$, and

(ii) the edges of $G_1$ = the edges of $G_2$.

(33) Let us consider a addAdjVertexToAll $G_1$ of $G_2$, $v$, $V$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$. Then $G_1$.edgesOutOf($\{v\}$) = $V \longmapsto$ (the edges of $G_2$).

PROOF: For every object $e$, $e \in G_1$.edgesOutOf($\{v\}$) iff $e \in V \longmapsto$ (the edges of $G_2$) by [2, (5)], [7, (13)], [3, (13)], [7, (7)]. □

(34) Let us consider a addAdjVertexFromAll $G_1$ of $G_2$, $v$, $V$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$. Then $G_1$.edgesInto($\{v\}$) = $V \longmapsto$ (the edges of $G_2$).

PROOF: For every object $e$, $e \in G_1$.edgesInto($\{v\}$) iff $e \in V \longmapsto$ (the edges of $G_2$) by [2, (5)], [7, (13)], [3, (13)], [7, (7)]. □

(35) Let us consider a addAdjVertexToAll $G_1$ of $G$, $v$, $V$, and a addAdjVertexFromAll $G_2$ of $G$, $v$, $V$. Suppose $V \subseteq$ the vertices of $G$ and $v \notin$ the vertices of $G$. Then

(i) $G_2$ is a reverseEdgeDirections of $G_1$, $G_1$.edgesOutOf($\{v\}$), and

(ii) $G_1$ is a reverseEdgeDirections of $G_2$, $G_2$.edgesInto($\{v\}$).

The theorem is a consequence of (33) and (34).

(36) Let us consider a addAdjVertexToAll $G_1$ of $G$, $v$, $V$, a addAdjVertexFromAll $G_2$ of $G$, $v$, $V$, and objects $v_1$, $e$, $v_2$. Then $e$ joins $v_1$ and $v_2$ in $G_1$ if

and only if $e$ joins $v_1$ and $v_2$ in $G_2$. The theorem is a consequence of (35) and (9).

(37) Let us consider a addAdjVertexToAll $G_1$ of $G$, $v$, $V$, a addAdjVertexFromAll $G_2$ of $G$, $v$, $V$, and an object $w$. Then $w$ is a vertex of $G_1$ if and only if $w$ is a vertex of $G_2$.

(38) Let us consider a addAdjVertexToAll $G_1$ of $G_2$, $v$, $V$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$. Let us consider objects $e_1$, $u$. Then

(i) $e_1$ does not join $u$ to $v$ in $G_1$, and

(ii) if $u \notin V$, then $e_1$ does not join $v$ to $u$ in $G_1$, and

(iii) for every object $e_2$ such that $e_1$ joins $v$ to $u$ in $G_1$ and $e_2$ joins $v$ to $u$ in $G_1$ holds $e_1 = e_2$.

Proof: $e_1$ does not join $u$ to $v$ in $G_1$ by [2, (5)], [3, (13)]. If $u \notin V$, then $e_1$ does not join $v$ to $u$ in $G_1$ by [3, (13), (11)], [2, (5)]. $e_1 \notin$ the edges of $G_2$ and $e_2 \notin$ the edges of $G_2$ by [2, (5)]. Consider $x_1$, $y_1$ being objects such that $x_1 \in V$ and $y_1 \in \{$the edges of $G_2\}$ and $e_1 = \langle x_1, y_1 \rangle$. Consider $x_2$, $y_2$ being objects such that $x_2 \in V$ and $y_2 \in \{$the edges of $G_2\}$ and $e_2 = \langle x_2, y_2 \rangle$. □

(39) Let us consider a addAdjVertexFromAll $G_1$ of $G_2$, $v$, $V$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$. Let us consider objects $e_1$, $u$. Then

(i) $e_1$ does not join $v$ to $u$ in $G_1$, and

(ii) if $u \notin V$, then $e_1$ does not join $u$ to $v$ in $G_1$, and

(iii) for every object $e_2$ such that $e_1$ joins $u$ to $v$ in $G_1$ and $e_2$ joins $u$ to $v$ in $G_1$ holds $e_1 = e_2$.

Proof: $e_1$ does not join $v$ to $u$ in $G_1$ by [2, (5)], [3, (13)]. If $u \notin V$, then $e_1$ does not join $u$ to $v$ in $G_1$ by [3, (13), (11)], [2, (5)]. $e_1 \notin$ the edges of $G_2$ and $e_2 \notin$ the edges of $G_2$ by [2, (5)]. Consider $x_1$, $y_1$ being objects such that $x_1 \in V$ and $y_1 \in \{$the edges of $G_2\}$ and $e_1 = \langle x_1, y_1 \rangle$. Consider $x_2$, $y_2$ being objects such that $x_2 \in V$ and $y_2 \in \{$the edges of $G_2\}$ and $e_2 = \langle x_2, y_2 \rangle$. □

(40) Let us consider a addAdjVertexToAll $G_1$ of $G_2$, $v$, $V$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$. Let us consider objects $e$, $v_1$, $v_2$. Suppose $v_1 \neq v$. If $e$ joins $v_1$ to $v_2$ in $G_1$, then $e$ joins $v_1$ to $v_2$ in $G_2$.

Proof: $e \in$ the edges of $G_2$ by [7, (13)], [3, (13)], [7, (7)]. □

(41) Let us consider a addAdjVertexFromAll $G_1$ of $G_2$, $v$, $V$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$. Let us consider objects $e$, $v_1$, $v_2$. Suppose $v_2 \neq v$. If $e$ joins $v_1$ to $v_2$ in $G_1$, then $e$ joins $v_1$ to $v_2$ in $G_2$.

PROOF: $e \in$ the edges of $G_2$ by [7, (13)], [3, (13)], [7, (7)]. □

(42) Let us consider a addAdjVertexToAll $G_1$ of $G_2$, $v$, $V$, and an object $v_1$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$ and $v_1 \in V$. Then $\langle v_1$, the edges of $G_2 \rangle$ joins $v$ to $v_1$ in $G_1$.

(43) Let us consider a addAdjVertexFromAll $G_1$ of $G_2$, $v$, $V$, and an object $v_1$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$ and $v_1 \in V$. Then $\langle v_1$, the edges of $G_2 \rangle$ joins $v_1$ to $v$ in $G_1$.

Let us consider $G$, $v$, $V$, a addAdjVertexToAll $G_1$ of $G$, $v$, $V$, and a addAdjVertexFromAll $G_2$ of $G$, $v$, $V$. Now we state the propositions:

(44) Every walk of $G_1$ is a walk of $G_2$. The theorem is a consequence of (35) and (14).

(45) Every walk of $G_2$ is a walk of $G_1$. The theorem is a consequence of (35) and (14).

Let us consider $G$, $v$, and $V$.

A addAdjVertexAll of $G$, $v$, $V$ is a reverseEdgeDirections of $G$ defined by

(Def. 4) (i) the vertices of *it* = (the vertices of $G) \cup \{v\}$ and for every object $e$, $e$ does not join $v$ and $v$ in *it* and for every object $v_1$, if $v_1 \notin V$, then $e$ does not join $v_1$ and $v$ in *it* and for every object $v_2$ such that $v_1 \neq v$ and $v_2 \neq v$ and $e$ joins $v_1$ to $v_2$ in *it* holds $e$ joins $v_1$ to $v_2$ in $G$ and there exists a set $E$ such that $\overline{\overline{V}} = \overline{\overline{E}}$ and $E$ misses the edges of $G$ and the edges of *it* = (the edges of $G) \cup E$ and for every object $v_1$ such that $v_1 \in V$ there exists an object $e_1$ such that $e_1 \in E$ and $e_1$ joins $v_1$ and $v$ in *it* and for every object $e_2$ such that $e_2$ joins $v_1$ and $v$ in *it* holds $e_1 = e_2$, **if** $V \subseteq$ the vertices of $G$ and $v \notin$ the vertices of $G$,

(ii) *it* $\approx G$, **otherwise**.

A addAdjVertexAll of $G$, $v$ is a addAdjVertexAll of $G$, $v$, the vertices of $G$.

One can verify that a addAdjVertexToAll of $G$, $v$, $V$ is a addAdjVertexAll of $G$, $v$, $V$.

Observe that a addAdjVertexFromAll of $G$, $v$, $V$ is a addAdjVertexAll of $G$, $v$, $V$. Now we state the propositions:

(46) Let us consider a addAdjVertexAll $G_1$ of $G_2$, $v$, $\emptyset$. Then the edges of $G_2$ = the edges of $G_1$.

(47) Let us consider a non empty set $V$, and a addAdjVertexAll $G_1$ of $G_2$, $v$, $V$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$. Then the edges of $G_1 \neq \emptyset$.

(48) Let us consider a addAdjVertexAll $G_1$ of $G$, $v$, $V$. If $G_1 \approx G_2$, then $G_2$ is a addAdjVertexAll of $G$, $v$, $V$.

(49) Let us consider a addAdjVertexAll $G_1$ of $G_2$, $v$, $V$, and objects $v_1$, $e$, $v_2$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$ and $v_1 \neq v$ and $v_2 \neq v$ and $e$ joins $v_1$ and $v_2$ in $G_1$. Then $e$ joins $v_1$ and $v_2$ in $G_2$.

(50) Let us consider a addAdjVertexAll $G_1$ of $G_2$, $v$, $V$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$. Then $v$ is a vertex of $G_1$.

(51) Let us consider a addAdjVertexAll $G_1$ of $G_2$, $v$, $V$, a set $E$, and objects $v_1$, $e$, $v_2$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$ and the edges of $G_1 =$ (the edges of $G_2$) $\cup E$ and $E$ misses the edges of $G_2$ and $e$ joins $v_1$ and $v_2$ in $G_1$ and $e \notin$ the edges of $G_2$. Then

(i) $e \in E$, and

(ii) $v_1 = v$ and $v_2 \in V$ or $v_2 = v$ and $v_1 \in V$.

(52) Let us consider a addAdjVertexAll $G_1$ of $G_2$, $v$, $V$, and a set $E$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$ and the edges of $G_1 =$ (the edges of $G_2$) $\cup E$ and $E$ misses the edges of $G_2$. Then there exist functions $f$, $g$ from $E$ into $V \cup \{v\}$ such that

(i) the source of $G_1 =$ (the source of $G_2$)$+\cdot f$, and

(ii) the target of $G_1 =$ (the target of $G_2$)$+\cdot g$, and

(iii) for every object $e$ such that $e \in E$ holds $e$ joins $f(e)$ to $g(e)$ in $G_1$ and ($f(e) = v$ iff $g(e) \neq v$).

Proof: Consider $E_1$ being a set such that $\overline{\overline{V}} = \overline{\overline{E_1}}$ and $E_1$ misses the edges of $G_2$ and the edges of $G_1 =$ (the edges of $G_2$)$\cup E_1$ and for every object $v_1$ such that $v_1 \in V$ there exists an object $e_1$ such that $e_1 \in E_1$ and $e_1$ joins $v_1$ and $v$ in $G_1$ and for every object $e_2$ such that $e_2$ joins $v_1$ and $v$ in $G_1$ holds $e_1 = e_2$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an object $v_2$ such that $\$_1$ joins $\$_2$ to $v_2$ in $G_1$. For every object $e$ such that $e \in E$ there exists an object $v_1$ such that $v_1 \in V \cup \{v\}$ and $\mathcal{P}[e, v_1]$. Consider $f$ being a function from $E$ into $V \cup \{v\}$ such that for every object $e$ such that $e \in E$ holds $\mathcal{P}[e, f(e)]$ from [2, Sch. 1]. Define $\mathcal{Q}[\text{object}, \text{object}] \equiv \$_1$ joins $f(\$_1)$ to $\$_2$ in $G_1$. For every object $e$ such that $e \in E$ there exists an object $v_2$ such that $v_2 \in V \cup \{v\}$ and $\mathcal{Q}[e, v_2]$ by [5, (16)], (51). Consider $g$ being a function from $E$ into $V \cup \{v\}$ such that for every object $e$ such that $e \in E$ holds $\mathcal{Q}[e, g(e)]$ from [2, Sch. 1]. For every object $e$ such that $e \in \text{dom}$(the source of $G_1$) holds (the source of $G_1$)$(e) =$ ((the source of $G_2$)$+\cdot f$)$(e)$ by [3, (11), (13)].

For every object $e$ such that $e \in \mathrm{dom}$(the target of $G_1$) holds (the target of $G_1$)($e$) = ((the target of $G_2$)$+\cdot g$)($e$) by [3, (11), (13)]. □

(53) Let us consider a addAdjVertexAll $G_1$ of $G_2$, $v$, $V$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$. Then the edges of $G_2 = G_1$.edgesBetween(the vertic of $G_2$).

PROOF: Set $B = G_1$.edgesBetween(the vertices of $G_2$). For every object $e$, $e \in$ the edges of $G_2$ iff $e \in B$ by [2, (5)], [5, (31)], (49). □

(54) Let us consider a graph $G_2$, sets $v$, $V$, and a addAdjVertexAll $G_1$ of $G_2$, $v$, $V$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$. Then $G_2$ is a subgraph of $G_1$ with vertex $v$ removed. The theorem is a consequence of (53).

(55) Every addAdjVertexAll of $G_2$, $v$, $\emptyset$ is a addVertex of $G_2$, $v$. The theorem is a consequence of (46).

(56) Let us consider an object $v_1$, and a addAdjVertexAll $G_1$ of $G_2$, $v$, $\{v_1\}$. Suppose $v_1 \in$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$. Then there exists an object $e$ such that

(i) $e \notin$ the edges of $G_2$, and

(ii) $G_1$ is addAdjVertex of $G_2$, $v$, $e$, $v_1$ or addAdjVertex of $G_2$, $v_1$, $e$, $v$.

The theorem is a consequence of (52).

(57) Let us consider a subset $W$ of $V$, and a addAdjVertexAll $G_1$ of $G_2$, $v$, $V$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$. Then there exists a function $f$ from $W$ into $G_1$.edgesBetween($W, \{v\}$) such that

(i) $f$ is one-to-one and onto, and

(ii) for every object $w$ such that $w \in W$ holds $f(w)$ joins $w$ and $v$ in $G_1$.

PROOF: Consider $E$ being a set such that $\overline{\overline{V}} = \overline{\overline{E}}$ and $E$ misses the edges of $G_2$ and the edges of $G_1$ = (the edges of $G_2$) $\cup E$ and for every object $v_1$ such that $v_1 \in V$ there exists an object $e_1$ such that $e_1 \in E$ and $e_1$ joins $v_1$ and $v$ in $G_1$ and for every object $e_2$ such that $e_2$ joins $v_1$ and $v$ in $G_1$ holds $e_1 = e_2$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv \$_2$ joins $\$_1$ and $v$ in $G_1$. For every object $w$ such that $w \in W$ there exists an object $e$ such that $e \in G_1$.edgesBetween($W, \{v\}$) and $\mathcal{P}[w, e]$ by [5, (17)]. Consider $f$ being a function from $W$ into $G_1$.edgesBetween($W, \{v\}$) such that for every object $w$ such that $w \in W$ holds $\mathcal{P}[w, f(w)]$ from [2, Sch. 1]. For every objects $w_1$, $w_2$ such that $w_1, w_2 \in W$ and $f(w_1) = f(w_2)$ holds $w_1 = w_2$ by [5, (15)]. For every object $e$ such that $e \in G_1$.edgesBetween($W, \{v\}$) holds $e \in \mathrm{rng}\, f$ by [**?** , (17)], [2, (4)]. □

(58) Let us consider a addAdjVertexAll $G_1$ of $G_2$, $v$, $V$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$ and $E$ misses the edges of $G_2$ and the edges of $G_1 =$ (the edges of $G_2$)$\cup E$. Then $E = G_1$.edgesBetween($V, \{v\}$).
Proof: Consider $E_1$ being a set such that $\overline{\overline{V}} = \overline{\overline{E_1}}$ and $E_1$ misses the edges of $G_2$ and the edges of $G_1 =$ (the edges of $G_2$)$\cup E_1$ and for every object $v_1$ such that $v_1 \in V$ there exists an object $e_1$ such that $e_1 \in E_1$ and $e_1$ joins $v_1$ and $v$ in $G_1$ and for every object $e_2$ such that $e_2$ joins $v_1$ and $v$ in $G_1$ holds $e_1 = e_2$. For every object $e$, $e \in E$ iff $e \in G_1$.edgesBetween($V, \{v\}$) by (51), [5, (17)]. □

(59) Let us consider a addAdjVertexAll $G_1$ of $G_2$, $v$, $V$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$. Then

(i) $G_1$.edgesBetween($V, \{v\}$) misses the edges of $G_2$, and

(ii) the edges of $G_1 =$ (the edges of $G_2$) $\cup$ $G_1$.edgesBetween($V, \{v\}$).

Proof: $G_1$.edgesBetween($V, \{v\}$) $\cap$ (the edges of $G_2$) $= \emptyset$ by [**?** , (17), (72)], [5, (13)]. For every object $e$ such that $e \in$ the edges of $G_1$ holds $e \in$ (the edges of $G_2$) $\cup$ $G_1$.edgesBetween($V, \{v\}$) by (51), [5, (17)]. □

(60) Let us consider a graph $G_3$, an object $v$, sets $V_1$, $V_2$, a addAdjVertexAll $G_1$ of $G_3$, $v$, $V_1 \cup V_2$, and a subgraph $G_2$ of $G_1$ with edges $G_1$.edgesBetween($V_2, \{v\}$) removed. Suppose $V_1 \cup V_2 \subseteq$ the vertices of $G_3$ and $v \notin$ the vertices of $G_3$ and $V_1$ misses $V_2$. Then $G_2$ is a addAdjVertexAll of $G_3$, $v$, $V_1$.
Proof: Consider $E$ being a set such that $\overline{\overline{V_1 \cup V_2}} = \overline{\overline{E}}$ and $E$ misses the edges of $G_3$ and the edges of $G_1 =$ (the edges of $G_3$) $\cup E$ and for every object $v_1$ such that $v_1 \in V_1 \cup V_2$ there exists an object $e_1$ such that $e_1 \in E$ and $e_1$ joins $v_1$ and $v$ in $G_1$ and for every object $e_2$ such that $e_2$ joins $v_1$ and $v$ in $G_1$ holds $e_1 = e_2$. $E = G_1$.edgesBetween($V_1 \cup V_2, \{v\}$). For every object $e$ such that $e \in$ the edges of $G_3$ holds $e \in$ (the edges of $G_3$)$\setminus G_1$.edgesBetween($V_2, \{v\}$) by [**?** , (17)]. $G_2$ is a reverseEdgeDirections of $G_3$. □

(61) Let us consider a graph $G_3$, an object $v$, a set $V$, a vertex $v_1$ of $G_3$, and a addAdjVertexAll $G_1$ of $G_3$, $v$, $V \cup \{v_1\}$. Suppose $V \subseteq$ the vertices of $G_3$ and $v \notin$ the vertices of $G_3$ and $v_1 \notin V$. Then there exists a addAdjVertexAll $G_2$ of $G_3$, $v$, $V$ and there exists an object $e$ such that $e \notin$ the edges of $G_3$ and $G_1$ is addEdge of $G_2$, $v$, $e$, $v_1$ or addEdge of $G_2$, $v_1$, $e$, $v$.
Proof: Reconsider $W = \{v_1\}$ as a subset of $V \cup \{v_1\}$. Consider $f$ being a function from $W$ into $G_1$.edgesBetween($W, \{v\}$) such that $f$ is one-to-one and onto and for every object $w$ such that $w \in W$ holds $f(w)$ joins $w$ and $v$ in $G_1$. $f(v_1) \notin$ the edges of $G_3$ by [**?** , (72)], [5, (13)]. $v$ is a vertex of $G_1$. □

(62) Let us consider a graph $G_3$, an object $v$, a set $V$, a vertex $v_1$ of $G_3$, an object $e$, and a addAdjVertexAll $G_2$ of $G_3$, $v$, $V$. Suppose $V \subseteq$ the vertices of $G_3$ and $v \notin$ the vertices of $G_3$ and $v_1 \notin V$ and $e \notin$ the edges of $G_2$. Let us consider a graph $G_1$. Suppose $G_1$ is addEdge of $G_2$, $v_1$, $e$, $v$ or addEdge of $G_2$, $v$, $e$, $v_1$. Then $G_1$ is a addAdjVertexAll of $G_3$, $v$, $V \cup \{v_1\}$.

PROOF: Consider $E$ being a set such that $\overline{\overline{V}} = \overline{\overline{E}}$ and $E$ misses the edges of $G_3$ and the edges of $G_2 =$ (the edges of $G_3) \cup E$ and for every object $v_1$ such that $v_1 \in V$ there exists an object $e_1$ such that $e_1 \in E$ and $e_1$ joins $v_1$ and $v$ in $G_2$ and for every object $e_2$ such that $e_2$ joins $v_1$ and $v$ in $G_2$ holds $e_1 = e_2$. Consider $f$ being a function such that $f$ is one-to-one and $\operatorname{dom} f = E$ and $\operatorname{rng} f = V$. Set $f_1 = f {+\cdot} (e \dot{\longmapsto} v_1)$. $\operatorname{rng} f \cap \operatorname{rng}(e \dot{\longmapsto} v_1) = \emptyset$. For every object $w$ such that $w \in \operatorname{rng} f \cup \operatorname{rng}(e \dot{\longmapsto} v_1)$ holds $w \in \operatorname{rng} f_1$ by [3, (83), (18)]. $v$ is a vertex of $G_2$ and $v_1$ is a vertex of $G_3$. □

Let us consider $G_2$, $v$, $V$, a addAdjVertexAll $G_1$ of $G_2$, $v$, $V$, and a walk $W$ of $G_1$. Now we state the propositions:

(63) Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$. Then

(i) if $W$.edges() $\subseteq$ the edges of $G_2$ and $W$ is not trivial, then $v \notin$ $W$.vertices(), and

(ii) if $v \notin W$.vertices(), then $W$.edges() $\subseteq$ the edges of $G_2$.

PROOF: Consider $E$ being a set such that $\overline{\overline{V}} = \overline{\overline{E}}$ and $E$ misses the edges of $G_2$ and the edges of $G_1 =$ (the edges of $G_2) \cup E$ and for every object $v_1$ such that $v_1 \in V$ there exists an object $e_1$ such that $e_1 \in E$ and $e_1$ joins $v_1$ and $v$ in $G_1$ and for every object $e_2$ such that $e_2$ joins $v_1$ and $v$ in $G_1$ holds $e_1 = e_2$. For every object $e$ such that $e \in W$.edges() holds $e \in$ the edges of $G_2$ by [4, (100), (87), (1)], (49). □

(64) Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$ and ($W$.edges() $\subseteq$ the edges of $G_2$ and $W$ is not trivial or $v \notin W$.vertices()). Then $W$ is a walk of $G_2$. The theorem is a consequence of (63).

(65) If $W$.vertices() $\subseteq$ the vertices of $G_2$, then $W$.edges() $\subseteq$ the edges of $G_2$. The theorem is a consequence of (63).

Now we state the propositions:

(66) Let us consider addAdjVertexAlls $G_1$, $G_2$ of $G$, $v$, $V$. Then

(i) the vertices of $G_1 =$ the vertices of $G_2$, and

(ii) every vertex of $G_1$ is a vertex of $G_2$.

PROOF: The vertices of $G_1 =$ the vertices of $G_2$ by [5, (85)]. □

(67) Let us consider addAdjVertexAlls $G_1$, $G_2$ of $G$, $v$, $V$, and objects $v_1$, $e_1$, $v_2$. Suppose $e_1$ joins $v_1$ and $v_2$ in $G_1$. Then there exists an object $e_2$ such that $e_2$ joins $v_1$ and $v_2$ in $G_2$.

(68) Let us consider addAdjVertexAlls $G_1$, $G_2$ of $G$, $v$, $V$. Then there exists a function $f$ from the edges of $G_1$ into the edges of $G_2$ such that

(i) $f\restriction$(the edges of $G$) $= \mathrm{id}_\alpha$, and

(ii) $f$ is one-to-one and onto, and

(iii) for every objects $v_1$, $e$, $v_2$ such that $e$ joins $v_1$ and $v_2$ in $G_1$ holds $f(e)$ joins $v_1$ and $v_2$ in $G_2$,

where $\alpha$ is the edges of $G$. The theorem is a consequence of (67), (47), and (51).

Let $G$ be a loopless graph. Let us consider $v$ and $V$. One can verify that every addAdjVertexAll of $G$, $v$, $V$ is loopless.

Let $G$ be a non-directed-multi graph. Let us note that every addAdjVertexAll of $G$, $v$, $V$ is non-directed-multi.

Let $G$ be a non-multi graph. One can verify that every addAdjVertexAll of $G$, $v$, $V$ is non-multi.

Let $G$ be a directed-simple graph. Note that every addAdjVertexAll of $G$, $v$, $V$ is directed-simple.

Let $G$ be a simple graph. Let us observe that every addAdjVertexAll of $G$, $v$, $V$ is simple.

Now we state the proposition:

(69) Let us consider a addAdjVertexAll $G_1$ of $G_2$, $v$, $V$, a walk $W$ of $G_1$, and vertices $v_1$, $v_2$ of $G_2$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$ and $W$.first() $= v_1$ and $W$.last() $= v_2$ and $v_2 \notin G_2$.reachableFrom($v_1$). Then $v \in W$.vertices(). The theorem is a consequence of (64).

Let us consider $G_2$, $v$, $V$, and a addAdjVertexAll $G_1$ of $G_2$, $v$, $V$. Now we state the propositions:

(70) Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$ and $G_2$ is acyclic and for every component $G_3$ of $G_2$ and for every vertices $w_1$, $w_2$ of $G_3$ such that $w_1, w_2 \in V$ holds $w_1 = w_2$. Then $G_1$ is acyclic.

Proof: Consider $E$ being a set such that $\overline{\overline{V}} = \overline{\overline{E}}$ and $E$ misses the edges of $G_2$ and the edges of $G_1 =$ (the edges of $G_2$) $\cup E$ and for every object $v_1$ such that $v_1 \in V$ there exists an object $e_1$ such that $e_1 \in E$ and $e_1$ joins $v_1$ and $v$ in $G_1$ and for every object $e_2$ such that $e_2$ joins $v_1$ and $v$ in $G_1$ holds $e_1 = e_2$. There exists no walk $W$ of $G_1$ such that $W$ is cycle-like by (65), [**?** , (57)], [4, (170)], [**?** , (24)]. $\square$

(71) Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$ and ($G_2$ is not acyclic or there exists a component $G_3$ of $G_2$ and there exist vertices $w_1$, $w_2$ of $G_3$ such that $w_1, w_2 \in V$ and $w_1 \neq w_2$). Then $G_1$ is not acyclic.

(72) Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$ and for every component $G_3$ of $G_2$, there exists a vertex $w$ of $G_3$ such that $w \in V$. Then $G_1$ is connected.

PROOF: For every vertex $u$ of $G_1$ such that $u \neq v$ there exists a walk $W_1$ of $G_1$ such that $W_1$ is walk from $u$ to $v$ by [**?** , (75)], [4, (19), (63)]. For every vertices $u$, $w$ of $G_1$, there exists a walk $W_1$ of $G_1$ such that $W_1$ is walk from $u$ to $w$ by [4, (13), (23), (31)]. □

Let $G$ be a connected graph, $v$ be an object, and $V$ be a non empty set. Note that every addAdjVertexAll of $G$, $v$, $V$ is connected.

Let us consider $G_2$, $v$, $V$, and a addAdjVertexAll $G_1$ of $G_2$, $v$, $V$. Now we state the propositions:

(73) Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$ and there exists a component $G_3$ of $G_2$ such that for every vertex $w$ of $G_3$, $w \notin V$. Then $G_1$ is not connected.

PROOF: Consider $G_3$ being a component of $G_2$ such that for every vertex $w$ of $G_3$, $w \notin V$. Set $v_1 =$ the vertex of $G_3$. There exists no walk $W$ of $G_1$ such that $W$ is walk from $v_1$ to $v$ by [4, (160), (127), (125)]. □

(74) Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$ and there exists a component $G_3$ of $G_2$ such that the vertices of $G_3$ misses $V$. Then $G_1$ is not connected. The theorem is a consequence of (73).

Let $G$ be a non connected graph and $v$ be an object. Let us note that every addAdjVertexAll of $G$, $v$, $\emptyset$ is non connected.

Now we state the proposition:

(75) Let us consider a addAdjVertexAll $G_1$ of $G_2$, $v$, $V$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$. Then $G_1$ is complete if and only if $G_2$ is complete and $V =$ the vertices of $G_2$.

PROOF: For every vertices $u$, $v$ of $G_1$ such that $u \neq v$ holds $u$ and $v$ are adjacent by [5, (14)], [**?** , (70)]. □

Let $G$ be a complete graph. Note that every addAdjVertexAll of $G$, the vertices of $G$ is complete.

Now we state the propositions:

(76) Let us consider a addAdjVertexAll $G_1$ of $G_2$, $v$, $V$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$. Then

(i) $G_1$.order() $= G_2$.order() $+ 1$, and

(ii) $G_1$.size() $= G_2$.size() $+ \overline{\overline{V}}$.

(77) Let us consider a finite graph $G_2$, an object $v$, a set $V$, and a addAdjVertexAll $G_1$ of $G_2$, $v$, $V$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$. Then $G_1$.order() $= G_2$.order() $+ 1$.

(78) Let us consider a finite graph $G_2$, an object $v$, a finite set $V$, and a addAdjVertexAll $G_1$ of $G_2$, $v$, $V$. Suppose $V \subseteq$ the vertices of $G_2$ and $v \notin$ the vertices of $G_2$. Then $G_1$.size() $= G_2$.size() $+ \overline{\overline{V}}$.

Let $G$ be a finite graph, $v$ be an object, and $V$ be a set. Note that every addAdjVertexAll of $G$, $v$, $V$ is finite.

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