

On Algebras of Algorithms and Specifications over Uninterpreted Data

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Summary. This paper continues formalization in Mizar [2, 1] of basic notions of the composition-nominative approach to program semantics [13] which was started in [8, 11].

The composition-nominative approach studies mathematical models of computer programs and data on various levels of abstraction and generality and provides tools for reasoning about their properties. Besides formalization of semantics of programs, certain elements of the composition-nominative approach were applied to abstract systems in a mathematical systems theory [4, 6, 7, 5, 3].

In the paper we introduce a definition of the notion of a binominative function over a set D understood as a partial function which maps elements of D to D . The sets of binominative functions and nominative predicates [11] over given sets form the carrier of the generalized Glushkov algorithmic algebra [14]. This algebra can be used to formalize algorithms which operate on various data structures (such as multidimensional arrays, lists, etc.) and reason about their properties.

We formalize the operations of this algebra (also called compositions) which are valid over uninterpreted data and which include among others the sequential composition, the prediction composition, the branching composition, the monotone Floyd-Hoare composition, and the cycle composition. The details on formalization of nominative data and the operations of the algorithmic algebra over them are described in [10, 12, 9].

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1. PRELIMINARIES

From now on x denotes an object and n denotes a natural number.

Let X, Y be sets. Observe that every element of $X \dot{\rightarrow} Y$ is X -defined and every element of $X \dot{\rightarrow} Y$ is Y -valued.

Now we state the proposition:

- (1) Let us consider sets X, Y, Z, T , objects x, y, z , and a function f from $X \times Y \times Z$ into T . Suppose $x \in X$ and $y \in Y$ and $z \in Z$ and $T \neq \emptyset$. Then $f(x, y, z) \in T$.

One can verify that there exists a set which and has not non empty elements.

Let A, B, C be sets. The functor $\cdot(A, B, C)$ yielding a function from $(A \dot{\rightarrow} B) \times (B \dot{\rightarrow} C)$ into $A \dot{\rightarrow} C$ is defined by

- (Def. 1) for every partial function f from A to B and for every partial function g from B to C , $it(f, g) = g \cdot f$.

From now on D denotes a non empty set and p, q denote partial predicates of D .

Now we state the propositions:

- (2) If q is total, then $\text{dom } p \subseteq \text{dom}(p \vee q)$.
 (3) If q is total, then $\text{dom } p \subseteq \text{dom}(p \wedge q)$.
 (4) If q is total, then $\text{dom } p \subseteq \text{dom}(p \Rightarrow q)$.

2. OPERATIONS IN ALGEBRAS OF ALGORITHMS AND SPECIFICATIONS OVER UNINTERPRETED DATA

From now on D denotes a set.

Let us consider D . The functor $\text{FPrg}(D)$ yielding a set is defined by the term

- (Def. 2) $D \dot{\rightarrow} D$.

Let us note that $\text{FPrg}(D)$ is non empty and functional.

A binominative function of D is a partial function from D to D . Now we state the proposition:

- (5) Let us consider a non empty set D , and a binominative function f of D . Then $\text{id}_{\text{field } f}$ is a binominative function of D .

In the sequel p, q denote partial predicates of D and f, g denote binominative functions of D .

Let us consider D and p . Let F be a function from $\text{Pr}(D)$ into $\text{Pr}(D)$. One can verify that $F(p)$ is function-like and relation-like.

Let p be an element of $\text{Pr}(D)$. One can verify that $F(p)$ is function-like and relation-like.

Let us consider p and q . Let F be a function from $\text{Pr}(D) \times \text{Pr}(D)$ into $\text{Pr}(D)$. Let us note that $F(p, q)$ is function-like and relation-like.

Let p, q be elements of $\text{Pr}(D)$. One can verify that $F(p, q)$ is function-like and relation-like.

Let x be an element of $\text{Pr}(D) \times \text{Pr}(D)$. Let us note that $F(x)$ is function-like and relation-like.

Let us consider f . Let F be a function from $\text{FPrg}(D)$ into $\text{FPrg}(D)$. Observe that $F(f)$ is function-like and relation-like.

Let us consider p and g . Let F be a function from $\text{Pr}(D) \times \text{FPrg}(D) \times \text{FPrg}(D)$ into $\text{FPrg}(D)$. One can verify that $F(p, f, g)$ is function-like and relation-like and $F(\langle p, f, g \rangle)$ is function-like and relation-like.

Let us consider q . Let F be a function from $\text{Pr}(D) \times \text{FPrg}(D) \times \text{Pr}(D)$ into $\text{Pr}(D)$. One can verify that $F(p, f, q)$ is function-like and relation-like and $F(\langle p, f, q \rangle)$ is function-like and relation-like.

Let D be a set. We introduce the notation $\text{id}_{\text{PP}}(D)$ as a synonym of id_D .

Let us observe that the functor $\text{id}_{\text{PP}}(D)$ yields a binominative function of D . Let D be a non empty set and d be an element of D . The functor $\text{id}_{\text{PP}}(d)$ yielding an element of D is defined by the term

(Def. 3) $(\text{id}_{\text{PP}}(D))(d)$.

Let us consider D . The functor $\bullet(D)$ yielding a function from $\text{FPrg}(D) \times \text{FPrg}(D)$ into $\text{FPrg}(D)$ is defined by the term

(Def. 4) $\cdot(D, D, D)$.

Let us consider D, f , and g . The functor $f \bullet g$ yielding a binominative function of D is defined by the term

(Def. 5) $(\bullet(D))(f, g)$.

Let us consider D . The functor $\cdot(D)$ yielding a function from $\text{FPrg}(D) \times \text{Pr}(D)$ into $\text{Pr}(D)$ is defined by the term

(Def. 6) $\cdot(D, D, \text{Boolean})$.

Let us consider D, f , and p . The functor $f \cdot p$ yielding a partial predicate of D is defined by the term

(Def. 7) $(\cdot(D))(f, p)$.

Let F be a function from $\text{Pr}(D) \times \text{FPrg}(D) \times \text{FPrg}(D)$ into $\text{FPrg}(D)$, p be a partial predicate of D , and f, g be binominative functions of D . One can verify that $F(p, f, g)$ is function-like and relation-like.

Now we state the proposition:

- (6) If $x \in \text{dom}(f \cdot p)$, then $x \in \text{dom}(p \cdot f)$ and $((p \cdot f)(x) = \text{true}$ or $(p \cdot f)(x) = \text{false}$).

The scheme *PredToNomPredEx* deals with a set \mathcal{D} and a set D_1 and a unary predicate \mathcal{P} and states that

- (Sch. 1) There exists a partial predicate p of \mathcal{D} such that $\text{dom } p = D_1$ and for every object d such that $d \in \text{dom } p$ holds if $\mathcal{P}[d]$, then $p(d) = \text{true}$ and if $\mathcal{P}[d]$, then $p(d) = \text{false}$

provided

- $D_1 \subseteq \mathcal{D}$.

The scheme *PredToNomPredUnique* deals with a set \mathcal{D} and a set D_1 and a unary predicate \mathcal{P} and states that

- (Sch. 2) For every partial predicates p, q of \mathcal{D} such that $\text{dom } p = D_1$ and for every object d such that $d \in \text{dom } p$ holds if $\mathcal{P}[d]$, then $p(d) = \text{true}$ and if $\mathcal{P}[d]$, then $p(d) = \text{false}$ and $\text{dom } q = D_1$ and for every object d such that $d \in \text{dom } q$ holds if $\mathcal{P}[d]$, then $q(d) = \text{true}$ and if $\mathcal{P}[d]$, then $q(d) = \text{false}$ holds $p = q$.

Let us consider D . The functor **isEmpty(D)** yielding a partial predicate of D is defined by

- (Def. 8) $\text{dom } it = D$ and for every object d such that $d \in \text{dom } it$ holds if $d = \emptyset$, then $it(d) = \text{true}$ and if $d \neq \emptyset$, then $it(d) = \text{false}$.

Let D be a set with non non empty elements. The functor **Empty $_D$** yielding a binominative function of D is defined by the term

- (Def. 9) $D \mapsto \emptyset$.

Let us consider D . The functor **\perp_D** yielding a binominative function of D is defined by the term

- (Def. 10) \emptyset .

In the sequel D denotes a non empty set, p, q denote partial predicates of D , and f, g, h denote binominative functions of D .

Let us consider D . The functor **IF(D)** yielding a function from $\text{Pr}(D) \times \text{FPrg}(D) \times \text{FPrg}(D)$ into $\text{FPrg}(D)$ is defined by

- (Def. 11) for every partial predicate p of D and for every binominative functions f, g of D , $\text{dom } it(p, f, g) = \{d, \text{ where } d \text{ is an element of } D : d \in \text{dom } p \text{ and } p(d) = \text{true} \text{ and } d \in \text{dom } f \text{ or } d \in \text{dom } p \text{ and } p(d) = \text{false} \text{ and } d \in \text{dom } g\}$ and for every object d , if $d \in \text{dom } p$ and $p(d) = \text{true}$ and $d \in \text{dom } f$, then $it(p, f, g)(d) = f(d)$ and if $d \in \text{dom } p$ and $p(d) = \text{false}$ and $d \in \text{dom } g$, then $it(p, f, g)(d) = g(d)$.

Let us consider D , p , f , and g . The functor $\mathbf{IF}(p, f, g)$ yielding a binominative function of D is defined by the term

(Def. 12) $(\mathbf{IF}(D))(p, f, g)$.

Now we state the proposition:

(7) Suppose $x \in \text{dom}(\mathbf{IF}(p, f, g))$. Then

- (i) $x \in \text{dom } p$ and $p(x) = \text{true}$ and $x \in \text{dom } f$, or
- (ii) $x \in \text{dom } p$ and $p(x) = \text{false}$ and $x \in \text{dom } g$.

Let us consider D . The functor $\mathbf{FH}(D)$ yielding a function from $\text{Pr}(D) \times \text{FPrg}(D) \times \text{Pr}(D)$ into $\text{Pr}(D)$ is defined by

(Def. 13) for every partial predicates p, q of D and for every binominative function f of D , $\text{dom } it(p, f, q) = \{d, \text{ where } d \text{ is an element of } D : d \in \text{dom } p \text{ and } p(d) = \text{false} \text{ or } d \in \text{dom}(q \cdot f) \text{ and } (q \cdot f)(d) = \text{true} \text{ or } d \in \text{dom } p \text{ and } p(d) = \text{true} \text{ and } d \in \text{dom}(q \cdot f) \text{ and } (q \cdot f)(d) = \text{false}\}$ and for every object d , if $d \in \text{dom } p$ and $p(d) = \text{false}$ or $d \in \text{dom}(q \cdot f)$ and $(q \cdot f)(d) = \text{true}$, then $it(p, f, q)(d) = \text{true}$ and if $d \in \text{dom } p$ and $p(d) = \text{true}$ and $d \in \text{dom}(q \cdot f)$ and $(q \cdot f)(d) = \text{false}$, then $it(p, f, q)(d) = \text{false}$.

Let us consider D , p , q , and f . The functor $\mathbf{FH}(p, f, q)$ yielding a partial predicate of D is defined by the term

(Def. 14) $(\mathbf{FH}(D))(p, f, q)$.

Now we state the proposition:

(8) Suppose $x \in \text{dom}(\mathbf{FH}(p, f, q))$. Then

- (i) $x \in \text{dom } p$ and $p(x) = \text{false}$, or
- (ii) $x \in \text{dom}(q \cdot f)$ and $(q \cdot f)(x) = \text{true}$, or
- (iii) $x \in \text{dom } p$ and $p(x) = \text{true}$ and $x \in \text{dom}(q \cdot f)$ and $(q \cdot f)(x) = \text{false}$.

Let us consider D . The functor $\mathbf{WH}(D)$ yielding a function from $\text{Pr}(D) \times \text{FPrg}(D)$ into $\text{FPrg}(D)$ is defined by

(Def. 15) for every partial predicate p of D and for every binominative function f of D , $\text{dom } it(p, f) = \{d, \text{ where } d \text{ is an element of } D : \text{there exists a natural number } n \text{ such that for every natural number } i \text{ such that } i \leq n-1 \text{ holds } d \in \text{dom}(p \cdot (f^i)) \text{ and } (p \cdot (f^i))(d) = \text{true} \text{ and } d \in \text{dom}(p \cdot (f^n)) \text{ and } (p \cdot (f^n))(d) = \text{false}\}$ and for every object d such that $d \in \text{dom } it(p, f)$ there exists a natural number n such that for every natural number i such that $i \leq n-1$ holds $d \in \text{dom}(p \cdot (f^i))$ and $(p \cdot (f^i))(d) = \text{true}$ and $d \in \text{dom}(p \cdot (f^n))$ and $(p \cdot (f^n))(d) = \text{false}$ and $it(p, f)(d) = (f^n)(d)$.

Let us consider D , p , and f . The functor $\mathbf{WH}(p, f)$ yielding a binominative function of D is defined by the term

(Def. 16) $(\text{WH}(D))(p, f)$.

The functor $\sim(D)$ yielding a function from $\text{Pr}(D)$ into $\text{Pr}(D)$ is defined by

(Def. 17) for every partial predicate p of D , $\text{dom}(it(p)) = \{d, \text{ where } d \text{ is an element of } D : d \notin \text{dom } p\}$ and for every element d of D such that $d \notin \text{dom } p$ holds $it(p)(d) = \text{true}$.

Let D be a non empty set and p be a partial predicate of D . The functor $\sim(p)$ yielding a partial predicate of D is defined by the term

(Def. 18) $(\sim(D))(p)$.

Now we state the propositions:

- (9) Let us consider an element d of D . Then $d \in \text{dom } p$ if and only if $d \notin \text{dom}(\sim(p))$.
- (10) If p is total, then $\text{dom}(\sim(p)) = \emptyset$.

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