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On Two Alternative Axiomatizations of Lattices by McKenzie and Sholander

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Summary. The main result of the article is to prove formally that two sets of axioms, proposed by McKenzie and Sholander [???], axiomatize lattices and distributive lattices, respectively.

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1. Preliminaries: Sholander Axioms

From now on L denotes a non empty lattice structure and v_{64}, v_{65}, v_{66}, v_{67}, v_{103}, v_3, v_{102}, v_{101}, v_{100}, v_2, v_1, v_0 denote elements of L.

Let us consider L. We say that L is satisfying-Sholander-1 if and only if

(Def. 1) for every v_0, v_1, and v_2, $v_0 \sqcap (v_1 \sqcup v_2) = (v_2 \sqcap v_0) \sqcup (v_1 \sqcap v_0)$.

Let us consider v_0. Now we state the propositions:

(1) Suppose L is join-absorbing and for every v_0, v_2, and v_1, $v_0 \sqcap (v_1 \sqcup v_2) = (v_2 \sqcap v_0) \sqcup (v_1 \sqcap v_0)$. Then $v_0 \sqcap v_0 = v_0$.

(2) Suppose L is join-absorbing and for every v_0, v_2, and v_1, $v_0 \sqcap (v_1 \sqcup v_2) = (v_2 \sqcap v_0) \sqcup (v_1 \sqcap v_0)$. Then $v_0 \sqcup v_0 = v_0$. The theorem is a consequence of (1).

Let us consider v_0 and v_1. Now we state the propositions:

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(3) Suppose L is join-absorbing and for every v_0, v_2, and v_1, $v_0 \sqcap (v_1 \sqcup v_2) = (v_2 \sqcap v_0) \sqcup (v_1 \sqcap v_0)$. Then $v_0 \sqcap v_1 = v_1 \sqcap v_0$. The theorem is a consequence of (2).

(4) Suppose L is join-absorbing and for every v_0, v_2, and v_1, $v_0 \sqcap (v_1 \sqcup v_2) = (v_2 \sqcap v_0) \sqcup (v_1 \sqcap v_0)$. Then $v_0 \sqcup v_1 = v_1 \sqcup v_0$. The theorem is a consequence of (1).

Now we state the propositions:

(5) Suppose L is join-absorbing and for every v_0, v_2, and v_1, $v_0 \sqcap (v_1 \sqcup v_2) = (v_2 \sqcap v_0) \sqcup (v_1 \sqcap v_0)$. $(v_0 \sqcap v_1) \sqcap v_2 = v_0 \sqcap (v_1 \sqcap v_2)$. The theorem is a consequence of (1), (2), (4), and (3).

(6) If for every v_1 and v_0, $v_0 \sqcap (v_0 \sqcup v_1) = v_0$, then for every v_0 and v_1, $v_0 \sqcap (v_0 \sqcup v_1) = v_0$.

(7) Suppose L is join-absorbing and for every v_0, v_2, and v_1, $v_0 \sqcap (v_1 \sqcup v_2) = (v_2 \sqcap v_0) \sqcup (v_1 \sqcap v_0)$. $v_0 \sqcup (v_0 \sqcap v_1) = v_0$. The theorem is a consequence of (1), (3), and (4).

Let us consider v_0, v_1, and v_2. Now we state the propositions:

(8) Suppose L is join-absorbing and for every v_0, v_2, and v_1, $v_0 \sqcap (v_1 \sqcup v_2) = (v_2 \sqcap v_0) \sqcup (v_1 \sqcap v_0)$. Then $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$. The theorem is a consequence of (1), (3), (7), (2), (5), and (4).

(9) Suppose L is join-absorbing and for every v_0, v_2, and v_1, $v_0 \sqcap (v_1 \sqcup v_2) = (v_2 \sqcap v_0) \sqcup (v_1 \sqcap v_0)$. Then $v_0 \sqcap (v_1 \sqcup v_2) = (v_0 \sqcap v_1) \sqcup (v_0 \sqcap v_2)$. The theorem is a consequence of (4) and (3).

(10) Suppose L is join-absorbing and for every v_0, v_2, and v_1, $v_0 \sqcap (v_1 \sqcup v_2) = (v_2 \sqcap v_0) \sqcup (v_1 \sqcap v_0)$. Then $v_0 \sqcup (v_1 \sqcap v_2) = (v_0 \sqcup v_1) \sqcap (v_0 \sqcup v_2)$. The theorem is a consequence of (5), (1), (4), (8), (2), and (3).

From now on L denotes a distributive, join-commutative, meet-commutative, non empty lattice structure and v_0, v_1, v_2 denote elements of L.

Now we state the propositions:

(11) $v_0 \sqcap (v_1 \sqcup v_2) = (v_2 \sqcap v_0) \sqcup (v_1 \sqcap v_0)$.

(12) Let us consider a non empty lattice structure L. Then L is a distributive lattice if and only if L is join-absorbing and satisfying-Sholander-1. The theorem is a consequence of (11), (9), (3), (4), (5), (8), and (7).

Let us observe that every non empty lattice structure which is join-absorbing and satisfying-Sholander-1 is also distributive and lattice-like and every non empty lattice structure which is distributive, join-commutative, and meet-commutative is also satisfying-Sholander-1.

2. Four Axioms for Lattices Proposed by McKenzie

From now on L denotes a non empty lattice structure and v_{103}, v_3, v_{102}, v_{101}, v_{100}, v_2, v_1, v_0 denote elements of L.

Let us consider L. We say that L is satisfying-McKenzie-1 if and only if

(Def. 2) for every v_1, v_2, and v_0, $v_0 \sqcup (v_1 \sqcap (v_0 \sqcap v_2)) = v_0$.

We say that L is satisfying-McKenzie-2 if and only if

(Def. 3) for every v_1, v_2, and v_0, $v_0 \sqcap (v_1 \sqcup (v_0 \sqcup v_2)) = v_0$.

We say that L is satisfying-McKenzie-3 if and only if

(Def. 4) for every v_2, v_1, and v_0, $((v_0 \sqcap v_1) \sqcup (v_1 \sqcap v_2)) \sqcup v_1 = v_1$.

We say that L is satisfying-McKenzie-4 if and only if

(Def. 5) for every v_2, v_1, and v_0, $((v_0 \sqcup v_1) \sqcap (v_1 \sqcup v_2)) \sqcap v_1 = v_1$.

Now we state the propositions:

(13) Suppose L is satisfying-McKenzie-1 and satisfying-McKenzie-2 and for every v_2, v_1, and v_0, $((v_0 \sqcap v_1) \sqcup (v_1 \sqcap v_2)) \sqcup v_1 = v_1$ and for every v_2, v_1, and v_0, $((v_0 \sqcup v_1) \sqcap (v_1 \sqcup v_2)) \sqcap v_1 = v_1$. Then

- (i) for every v_1 and v_0, $v_0 \sqcap (v_0 \sqcup v_1) = v_0$, and
- (ii) for every v_1 and v_0, $v_0 \sqcup (v_0 \sqcap v_1) = v_0$, and
- (iii) L is join-commutative, meet-commutative, meet-associative, and join-associative.

(14) Suppose L is join-commutative, join-associative, meet-commutative, and meet-associative and for every v_1 and v_0, $v_0 \sqcap (v_0 \sqcup v_1) = v_0$ and for every v_1 and v_0, $v_0 \sqcup (v_0 \sqcap v_1) = v_0$. Then

- (i) for every v_1, v_2, and v_0, $v_0 \sqcup (v_1 \sqcap (v_0 \sqcap v_2)) = v_0$, and
- (ii) for every v_1, v_2, and v_0, $v_0 \sqcap (v_1 \sqcup (v_0 \sqcup v_2)) = v_0$, and
- (iii) for every v_2, v_1, and v_0, $((v_0 \sqcap v_1) \sqcup (v_1 \sqcap v_2)) \sqcup v_1 = v_1$, and
- (iv) for every v_2, v_1, and v_0, $((v_0 \sqcup v_1) \sqcap (v_1 \sqcup v_2)) \sqcap v_1 = v_1$.

Let L be a non empty lattice structure. We say that L is satisfying-4-McKenzie-axioms if and only if

(Def. 6) L is satisfying-McKenzie-1, satisfying-McKenzie-2, satisfying-McKenzie-3, and satisfying-McKenzie-4.

One can verify that every non empty lattice structure which is satisfying-4-McKenzie-ax is also satisfying-McKenzie-1, satisfying-McKenzie-2, satisfying-McKenzie-3, and satisfying-McKenzie-4 and every non empty lattice structure which is satisfying-McKenzie- satisfying-McKenzie-2, satisfying-McKenzie-3, and satisfying-McKenzie-4 is also satisfying-4-McKenzie-axioms.

From now on L denotes a non empty lattice structure.

Now we state the proposition:

(15) L is a lattice if and only if L is satisfying-4-McKenzie-axioms. The theorem is a consequence of (14) and (13).

Let us observe that every non empty lattice structure which is lattice-like is also satisfying-4-McKenzie-axioms and every non empty lattice structure which is satisfying-4-McKenzie-axioms is also lattice-like.

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