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Formalizing Two Generalized Approximation Operators

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Summary. Rough sets, developed by Pawlak [???], are important tool to describe situation of incomplete or partially unknown information. In this article we give the formal characterization of two closely related rough approximations, along the lines proposed in a paper by Gomolińska [???].

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1. Preliminaries

Let R be a non empty set and I be a function from R into 2^R. We say that I is map-reflexive if and only if

(Def. 1) for every element u of R, $u \in I(u)$.

The functor $\mathrm{singleton}_R$ yielding a function from R into 2^R is defined by

(Def. 2) for every element x of R, $it(x) = \{x\}$.

Note that $\mathrm{singleton}_R$ is map-reflexive.

Now we state the proposition:

(1) Let us consider a non empty relational structure R, and a function I from the carrier of R into 2^α. Suppose I is map-reflexive. Then the carrier of $R = \bigcup(I^\circ(\Omega_R))$, where α is the carrier of R.

From now on f, g denote functions and R denotes a non empty, reflexive relational structure.

Now we state the propositions:

(2) $\mathrm{LAp}(R) \mathrel{\dot{\subseteq}} \mathrm{id}_{2^{\alpha}}$, where α is the carrier of R.

PROOF: Set $f = \mathrm{LAp}(R)$. Set $g = \mathrm{id}_{2^{(\text{the carrier of } R)}}$. For every set i such that $i \in \mathrm{dom}\, f$ holds $f(i) \subseteq g(i)$ by [3, (35)]. □

(3) $\mathrm{id}_{2^{\alpha}} \mathrel{\dot{\subseteq}} \mathrm{UAp}(R)$, where α is the carrier of R.

PROOF: Set $f = \mathrm{id}_{2^{(\text{the carrier of } R)}}$. Set $g = \mathrm{UAp}(R)$. For every set i such that $i \in \mathrm{dom}\, f$ holds $f(i) \subseteq g(i)$ by [3, (36)]. □

From now on R denotes a non empty relational structure.

Now we state the propositions:

(4) Let us consider a map f of R, and subsets x, y of R. Then $\mathrm{Flip}\,\mathrm{Flip}\, f = f$.

(5) Let us consider maps f, g of R. Then $\mathrm{Flip}\, f \cdot g = (\mathrm{Flip}\, f) \cdot (\mathrm{Flip}\, g)$.

PROOF: Set $f_2 = \mathrm{Flip}\, f \cdot g$. Set $f_1 = \mathrm{Flip}\, f$. Set $g_1 = \mathrm{Flip}\, g$. For every subset x of R, $f_2(x) = f_1 \cdot g_1(x)$ by [1, (13)]. □

(6) Let us consider a map f of R. Then $f(\emptyset) = \emptyset$ if and only if $(\mathrm{Flip}\, f)((\text{the carrier of } R)) = \text{the carrier of } R$.

2. Generalized Approximation Mappings

Let R be a non empty relational structure. The functor UncertaintyMap R yielding a function from the carrier of R into $2^{(\text{the carrier of } R)}$ is defined by

(Def. 3) for every element x of R, $it(x) = \mathrm{Coim}((\text{the internal relation of } R), x)$.

Now we state the proposition:

(7) Let us consider elements w, u of R. Then $\langle w, u\rangle \in$ the internal relation of R if and only if $w \in (\text{UncertaintyMap}\, R)(u)$.

Let R be a non empty relational structure. The functor $\tau(R)$ yielding a function from the carrier of R into $2^{(\text{the carrier of } R)}$ is defined by

(Def. 4) for every element u of R, $it(u) = (\text{the internal relation of } R)^{\circ} u$.

Now we state the propositions:

(8) Let us consider elements u, w of R. Then $u \in (\text{the internal relation of } R)^{\circ} w$ if and only if $w \in \mathrm{Coim}((\text{the internal relation of } R), u)$.

PROOF: If $u \in (\text{the internal relation of } R)^{\circ} w$, then $w \in \mathrm{Coim}((\text{the internal relation of } R), u)$ by [6, (169)]. Consider t being an object such that $\langle w, t\rangle \in$ the internal relation of R and $t \in \{u\}$. □

(9) Let us consider elements w, u of R. Then $\langle w, u\rangle \in$ the internal relation of R if and only if $u \in (\tau(R))(w)$.

Proof: If $\langle w, u \rangle \in$ the internal relation of R, then $u \in (\tau(R))(w)$ by [6, (169)]. $w \in \mathrm{Coim}((\text{the internal relation of } R), u)$. Consider x being an object such that $\langle w, x \rangle \in$ the internal relation of R and $x \in \{u\}$. $\square$

Let R be a non empty relational structure and f be a function from the carrier of R into $2^{(\text{the carrier of } R)}$. The functor $ff_0(f)$ yielding a map of R is defined by

(Def. 5) for every subset x of R, $it(x) = \{u$, where u is an element of $R : f(u)$ meets $x\}$.

The functors: $f_0(R)$ and $ff_1(R)$ yielding maps of R are defined by terms

(Def. 6) $ff_0(\tau(R))$,

(Def. 7) $ff_0(\mathrm{UncertaintyMap}\, R)$,

respectively. Now we state the propositions:

(10) If the internal relation of R is symmetric, then $\mathrm{UncertaintyMap}\, R = \tau(R)$.

Proof: Set $f = \mathrm{UncertaintyMap}\, R$. Set $g = \tau(R)$. For every element x of R, $f(x) = g(x)$ by [6, (169)], [4, (20)]. $\square$

(11) If the internal relation of R is symmetric, then $f_0(R) = ff_1(R)$. The theorem is a consequence of (10).

(12) the internal relation of R is symmetric if and only if for every elements u, v of R such that $u \in (\tau(R))(v)$ holds $v \in (\tau(R))(u)$. The theorem is a consequence of (10), (7), and (9).

(13) $f_0(R) = \mathrm{UAp}(R)$.

(14) $\mathrm{Flip}\, f_0(R) = \mathrm{LAp}(R)$. The theorem is a consequence of (13).

(15) Let us consider an approximation space R, and a subset x of R. Then $(f_0(R))(x)$ is exact. The theorem is a consequence of (13).

(16) If the internal relation of R is total and reflexive, then $\mathrm{id}_{2^\alpha} \mathrel{\dot{\subseteq}} f_0(R)$, where α is the carrier of R.

Proof: Set $f = \mathrm{id}_{2^{(\text{the carrier of } R)}}$. Set $g = f_0(R)$. For every set i such that $i \in \mathrm{dom}\, f$ holds $f(i) \subseteq g(i)$ by [2, (1)], (9). $\square$

(17) If R is reflexive, then $\mathrm{Flip}\, f_0(R) \mathrel{\dot{\subseteq}} \mathrm{id}_{2^\alpha}$, where α is the carrier of R. The theorem is a consequence of (14) and (2).

(18) If the internal relation of R is total and reflexive, then $\mathrm{id}_{2^\alpha} \mathrel{\dot{\subseteq}} ff_1(R)$, where α is the carrier of R.

Proof: Set $f = \mathrm{id}_{2^{(\text{the carrier of } R)}}$. Set $g = ff_1(R)$. For every set i such that $i \in \mathrm{dom}\, f$ holds $f(i) \subseteq g(i)$ by [2, (1)], (7). $\square$

In the sequel f denotes a function from the carrier of R into $2^{(\text{the carrier of } R)}$. Now we state the proposition:

(19) $(ff_0(f))(\emptyset) = \emptyset$.

Let us consider R and f. Note that $ff_0(f)$ preserves empty set.

Now we state the propositions:

(20) $(f_0(R))(\emptyset) = \emptyset$.

(21) $(ff_1(R))(\emptyset) = \emptyset$.

Let R be a non empty, reflexive relational structure. One can verify that the internal relation of R is total and reflexive.

Now we state the propositions:

(22) If f is map-reflexive, then $(ff_0(f))(($the carrier of $R)) =$ the carrier of R.

(23) Suppose the internal relation of R is reflexive and total. Then $(f_0(R))(($the carrier of $R)) =$ the carrier of R.

PROOF: The carrier of $R \subseteq \{u$, where u is an element of $R : (\tau(R))(u)$ meets $\Omega_R\}$ by [2, (1)], [6, (169)]. □

(24) Suppose the internal relation of R is reflexive and total. Then $(ff_1(R))(($the carrier of $R)) =$ the carrier of R.

PROOF: The carrier of $R \subseteq \{u$, where u is an element of $R :$ (UncertaintyMap $R)(u)$ meets $\Omega_R\}$ by [2, (1)], (7). □

Let us consider elements u, w of R and a subset x of R. Now we state the propositions:

(25) If $f(u) = f(w)$, then $u \in (ff_0(f))(x)$ iff $w \in (ff_0(f))(x)$.

(26) If (UncertaintyMap $R)(u) =$ (UncertaintyMap $R)(w)$, then $u \in (ff_1(R))(x)$ iff $w \in (ff_1(R))(x)$.

(27) If $(\tau(R))(u) = (\tau(R))(w)$, then $u \in (f_0(R))(x)$ iff $w \in (f_0(R))(x)$.

Now we state the proposition:

(28) Let us consider a function f from the carrier of R into 2^α, and a subset x of R. Then (Flip $ff_0(f))(x) = \{w$, where w is an element of $R : f(w) \subseteq x\}$, where α is the carrier of R.

PROOF: (Flip $ff_0(f))(x) \subseteq \{w$, where w is an element of $R : f(w) \subseteq x\}$ by [5, (24)]. Consider w being an element of R such that $y = w$ and $f(w) \subseteq x$. Reconsider $y_1 = y$ as an element of R. $y_1 \notin (ff_0(f))(x^{\rm c})$ by [5, (24)]. □

Let us consider a subset x of R. Now we state the propositions:

(29) (Flip $f_0(R))(x) = \{w$, where w is an element of $R : (\tau(R))(w) \subseteq x\}$.

PROOF: (Flip $f_0(R))(x) \subseteq \{w$, where w is an element of $R : (\tau(R))(w) \subseteq x\}$ by [5, (24)]. Consider w being an element of R such that $y = w$ and $(\tau(R))(w) \subseteq x$. Reconsider $y_1 = y$ as an element of R. $y_1 \notin (f_0(R))(x^{\rm c})$ by [5, (24)]. □

(30) $(\mathrm{Flip}\, ff_1(R))(x) = \{w$, where w is an element of $R : (\mathrm{UncertaintyMap}\, R)(w) \subseteq x\}$.

PROOF: $(\mathrm{Flip}\, ff_1(R))(x) \subseteq \{w$, where w is an element of $R : (\mathrm{UncertaintyMap}\, R)(w$ $x\}$ by [5, (24)]. Consider w being an element of R such that $y = w$ and $(\mathrm{UncertaintyMap}\, R)(w) \subseteq x$. Reconsider $y_1 = y$ as an element of R. $y_1 \notin (ff_1(R))(x^{\mathrm{c}})$ by [5, (24)]. □

Let us consider elements u, w of R and a subset x of R. Now we state the propositions:

(31) If $f(u) = f(w)$, then $u \in (\mathrm{Flip}\, ff_0(f))(x)$ iff $w \in (\mathrm{Flip}\, ff_0(f))(x)$. The theorem is a consequence of (28).

(32) If $(\tau(R))(u) = (\tau(R))(w)$, then $u \in (\mathrm{Flip}\, f_0(R))(x)$ iff $w \in (\mathrm{Flip}\, f_0(R))(x)$. The theorem is a consequence of (29).

(33) If $(\mathrm{UncertaintyMap}\, R)(u) = (\mathrm{UncertaintyMap}\, R)(w)$, then $u \in (\mathrm{Flip}\, ff_1(R))(x)$ iff $w \in (\mathrm{Flip}\, ff_1(R))(x)$. The theorem is a consequence of (30).

Let us consider an element w of R. Now we state the propositions:

(34) If R is reflexive, then $w \in (\mathrm{UncertaintyMap}\, R)(w)$. The theorem is a consequence of (7).

(35) If R is reflexive, then $w \in (\tau(R))(w)$. The theorem is a consequence of (9).

Let R be a reflexive, non empty relational structure. Note that UncertaintyMap R is map-reflexive and $\tau(R)$ is map-reflexive.

Now we state the propositions:

(36) If R is reflexive, then $\mathrm{Flip}\, ff_1(R) \mathrel{\dot{\subseteq}} \mathrm{id}_{2^{\alpha}}$, where α is the carrier of R. The theorem is a consequence of (34) and (30).

(37) $(f_0(R)) \cdot (f_0(R)) = f_0(R)$ if and only if $(\mathrm{Flip}\, f_0(R)) \cdot (\mathrm{Flip}\, f_0(R)) = \mathrm{Flip}\, f_0(R)$. The theorem is a consequence of (5).

(38) If R is reflexive, then $\bigcup((\mathrm{UncertaintyMap}\, R)^{\circ}(\Omega_R))$ = the carrier of R. The theorem is a consequence of (34).

Let R be a non empty relational structure. Let us note that $f_0(R)$ is $\subseteq$-monotone and $ff_1(R)$ is $\subseteq$-monotone.

Now we state the propositions:

(39) Let us consider a map f of R. Suppose f is $\subseteq$-monotone. Then Flip f is $\subseteq$-monotone.

PROOF: Set $g = \mathrm{Flip}\, f$. For every subsets A, B of R such that $A \subseteq B$ holds $g(A) \subseteq g(B)$ by [5, (12)]. □

(40) Flip $f_0(R)$ is $\subseteq$-monotone.

(41) Flip $ff_1(R)$ is $\subseteq$-monotone.

(42) Let us consider a function f from the carrier of R into 2^{α}, and subsets x, y of R. Then $(ff_0(f))(x \cup y) = (ff_0(f))(x) \cup (ff_0(f))(y)$, where α is the carrier of R.

Let us consider subsets x, y of R. Now we state the propositions:

(43) $(f_0(R))(x \cup y) = (f_0(R))(x) \cup (f_0(R))(y)$. The theorem is a consequence of (42).

(44) $(ff_1(R))(x \cup y) = (ff_1(R))(x) \cup (ff_1(R))(y)$. The theorem is a consequence of (42).

Now we state the proposition:

(45) Let us consider a function f from the carrier of R into 2^{α}, and subsets x, y of R. Then $(\mathrm{Flip}\, ff_0(f))(x) \cup (\mathrm{Flip}\, ff_0(f))(y) \subseteq (\mathrm{Flip}\, ff_0(f))(x \cup y)$, where α is the carrier of R. The theorem is a consequence of (28).

Let us consider subsets x, y of R. Now we state the propositions:

(46) $(\mathrm{Flip}\, f_0(R))(x) \cup (\mathrm{Flip}\, f_0(R))(y) \subseteq (\mathrm{Flip}\, f_0(R))(x \cup y)$. The theorem is a consequence of (45).

(47) $(\mathrm{Flip}\, ff_1(R))(x) \cup (\mathrm{Flip}\, ff_1(R))(y) \subseteq (\mathrm{Flip}\, ff_1(R))(x \cup y)$. The theorem is a consequence of (45).

Now we state the proposition:

(48) Let us consider a function f from the carrier of R into 2^{α}, and subsets x, y of R. Then $(ff_0(f))(x \cap y) \subseteq (ff_0(f))(x) \cap (ff_0(f))(y)$, where α is the carrier of R.

Let us consider subsets x, y of R. Now we state the propositions:

(49) $(f_0(R))(x \cap y) \subseteq (f_0(R))(x) \cap (f_0(R))(y)$. The theorem is a consequence of (48).

(50) $(ff_1(R))(x \cap y) \subseteq (ff_1(R))(x) \cap (ff_1(R))(y)$. The theorem is a consequence of (48).

Now we state the proposition:

(51) Let us consider a function f from the carrier of R into 2^{α}, and subsets x, y of R. Then $(\mathrm{Flip}\, ff_0(f))(x) \cap (\mathrm{Flip}\, ff_0(f))(y) = (\mathrm{Flip}\, ff_0(f))(x \cap y)$, where α is the carrier of R.

Let us consider subsets x, y of R. Now we state the propositions:

(52) $(\mathrm{Flip}\, f_0(R))(x) \cap (\mathrm{Flip}\, f_0(R))(y) = (\mathrm{Flip}\, f_0(R))(x \cap y)$. The theorem is a consequence of (51).

(53) $(\mathrm{Flip}\, ff_1(R))(x) \cap (\mathrm{Flip}\, ff_1(R))(y) = (\mathrm{Flip}\, ff_1(R))(x \cap y)$. The theorem is a consequence of (51).

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