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Pythagorean Tuning: Pentatonic and Heptatonic Scale

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Summary. In this article, using the Mizar system [3], [4], we define a structure [1], [6] in order to build a Pythagorean pentatonic scale and a Pythagorean heptatonic scale[1] [5], [7].

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1. Preliminaries

Now we state the proposition:

(1) Let us consider an object r. Then $r \in \mathbb{R}_{+\cup\{0\}} \setminus \{0\}$ if and only if r is a positive real number.

Note that there exists a rational number which is positive.

The functor $\mathbb{Q}_+$ yielding a non empty subset of $\mathbb{R}_{+\cup\{0\}}$ is defined by the term

(Def. 1) the set of all r where r is a positive rational number.

Now we state the propositions:

(2) Let us consider an object r. Then r is an element of $\mathbb{Q}_+$ if and only if r is a positive rational number.

(3) $\mathbb{Q}_{+\cup\{0\}} \subseteq \mathbb{Q}$.

[1] `https://en.wikipedia.org/wiki/Pythagorean_tuning`

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The functor $\mathbb{R}_+$ yielding a non empty subset of $\mathbb{R}_{+\cup\{0\}}$ is defined by the term

(Def. 2) $\mathbb{R}_{+\cup\{0\}} \setminus \{0\}$.

Now we state the propositions:

(4) $\mathbb{N}_+ \subseteq \mathbb{Q}_+$.

(5) $\mathbb{N}_+ \subseteq \mathbb{R}_+$. The theorem is a consequence of (1).

(6) $\mathbb{Q}_+ \subseteq \mathbb{R}_+$. The theorem is a consequence of (2) and (1).

2. Real Frequency

We consider structures of music which extend 1-sorted structures and are systems

$$\langle \text{a carrier}, \text{an equidistance}, \text{a Ratio} \rangle$$

where the carrier is a set, the equidistance is a relation between (the carrier) $\times$ (the carrier) and (the carrier) $\times$ (the carrier), the Ratio is a function from (the carrier) $\times$ (the carrier) into the carrier.

Let S be a structure of music and a, b, c, d be elements of S. We say that $\overline{ab} \cong \overline{cd}$ if and only if

(Def. 3) $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the equidistance of S.

Let x, y be elements of $\mathbb{R}_+$. The functor $\mathbb{R}$-ratio(x, y) yielding an element of $\mathbb{R}_+$ is defined by

(Def. 4) there exist positive real numbers r, s such that $x = r$ and $s = y$ and $it = \frac{s}{r}$.

Now we state the proposition:

(7) Let us consider elements a, b, c, d of $\mathbb{R}_+$. Then $\mathbb{R}$-ratio$(a, b) = \mathbb{R}$-ratio(c, d) if and only if $\mathbb{R}$-ratio$(b, a) = \mathbb{R}$-ratio(d, c).

The functor $\mathbb{R}$-ratio yielding a function from $\mathbb{R}_+ \times \mathbb{R}_+$ into $\mathbb{R}_+$ is defined by

(Def. 5) for every element x of $\mathbb{R}_+ \times \mathbb{R}_+$, there exist elements y, z of $\mathbb{R}_+$ such that $x = \langle y, z\rangle$ and $it(x) = \mathbb{R}$-ratio(y, z).

The functor eq-$\mathbb{R}$-ratio yielding a relation between $\mathbb{R}_+ \times \mathbb{R}_+$ and $\mathbb{R}_+ \times \mathbb{R}_+$ is defined by

(Def. 6) for every elements x, y of $\mathbb{R}_+ \times \mathbb{R}_+$, $\langle x, y\rangle \in it$ iff there exist elements a, b, c, d of $\mathbb{R}_+$ such that $x = \langle a, b\rangle$ and $y = \langle c, d\rangle$ and $\mathbb{R}$-ratio$(a, b) = \mathbb{R}$-ratio(c, d).

The functor $\mathbb{R}$-music yielding a structure of music is defined by the term

(Def. 7) $\langle \mathbb{R}_+, \text{eq-}\mathbb{R}\text{-ratio}, \mathbb{R}\text{-ratio} \rangle$.

Now we state the propositions:

(8) (i) $\mathbb{R}$-music is not empty, and

(ii) the carrier of $\mathbb{R}$-music $\subseteq \mathbb{R}_+$, and

(iii) for every elements f_1, f_2, f_3, f_4 of $\mathbb{R}$-music, $\overline{f_1 f_2} \cong \overline{f_3 f_4}$ iff (the Ratio of $\mathbb{R}$-music)(f_1, f_2) = (the Ratio of $\mathbb{R}$-music)(f_3, f_4).

(9) Let us consider elements f_1, f_2, f_3 of $\mathbb{R}$-music. Suppose (the Ratio of $\mathbb{R}$-music)(f_1, f_2) = (the Ratio of $\mathbb{R}$-music)(f_1, f_3). Then $f_2 = f_3$.

(10) $\mathbb{N}_+ \subseteq$ the carrier of $\mathbb{R}$-music.

(11) Let us consider an element fr of $\mathbb{R}$-music, and a non zero natural number n. Then there exists an element h of $\mathbb{R}$-music such that $\langle fr, h\rangle \in [\langle 1, n\rangle]_\alpha$, where α is the equidistance of $\mathbb{R}$-music. The theorem is a consequence of (1) and (8).

(12) Let us consider elements f_1, f_2, f_3 of $\mathbb{R}$-music. Suppose (the Ratio of $\mathbb{R}$-music)(f_1, f_1) = (the Ratio of $\mathbb{R}$-music)(f_2, f_3). Then $f_2 = f_3$.

(13) Let us consider elements f_1, f_2, f_8, f_6, f_9, f_7 of $\mathbb{R}$-music, positive real numbers r_1, r_2, and non zero natural numbers n, m. Suppose $f_8 = n \cdot r_1$ and $f_6 = m \cdot r_1$ and $f_9 = n \cdot r_2$ and $f_7 = m \cdot r_2$. Then $\overline{f_8 f_6} \cong \overline{f_9 f_7}$. The theorem is a consequence of (8).

(14) Let us consider elements f_1, f_2, f_3, f_4 of $\mathbb{R}$-music. Then (the Ratio of $\mathbb{R}$-music)(f_1, f_2) = (the Ratio of $\mathbb{R}$-music)(f_3, f_4) if and only if (the Ratio of $\mathbb{R}$-music)(f_2, f_1) = (the Ratio of $\mathbb{R}$-music)(f_4, f_3). The theorem is a consequence of (7).

3. Rational Frequency

Let x, y be elements of $\mathbb{Q}_+$. The functor $\mathbb{Q}$-ratio(x, y) yielding an element of $\mathbb{Q}_+$ is defined by

(Def. 8) there exist positive rational numbers r, s such that $x = r$ and $s = y$ and $it = \frac{s}{r}$.

Now we state the proposition:

(15) Let us consider elements a, b, c, d of $\mathbb{Q}_+$. Then $\mathbb{Q}$-ratio(a, b) = $\mathbb{Q}$-ratio(c, d) if and only if $\mathbb{Q}$-ratio(b, a) = $\mathbb{Q}$-ratio(d, c).

The functor $\mathbb{Q}$-ratio yielding a function from $\mathbb{Q}_+ \times \mathbb{Q}_+$ into $\mathbb{Q}_+$ is defined by

(Def. 9) for every element x of $\mathbb{Q}_+ \times \mathbb{Q}_+$, there exist elements y, z of $\mathbb{Q}_+$ such that $x = \langle y, z\rangle$ and $it(x) = \mathbb{Q}$-ratio(y, z).

The functor eq-$\mathbb{Q}$-ratio yielding a relation between $\mathbb{Q}_+ \times \mathbb{Q}_+$ and $\mathbb{Q}_+ \times \mathbb{Q}_+$ is defined by

(Def. 10) for every elements x, y of $\mathbb{Q}_+ \times \mathbb{Q}_+$, $\langle x, y\rangle \in it$ iff there exist elements a, b, c, d of $\mathbb{Q}_+$ such that $x = \langle a, b\rangle$ and $y = \langle c, d\rangle$ and $\mathbb{Q}$-ratio$(a, b) =$ $\mathbb{Q}$-ratio(c, d).

The functor $\mathbb{Q}$-music yielding a structure of music is defined by the term

(Def. 11) $\langle\mathbb{Q}_+, \text{eq-}\mathbb{Q}\text{-ratio}, \mathbb{Q}\text{-ratio}\rangle$.

Now we state the propositions:

(16) (i) $\mathbb{Q}$-music is not empty, and

(ii) the carrier of $\mathbb{Q}$-music $\subseteq \mathbb{R}_+$, and

(iii) for every elements f_1, f_2, f_3, f_4 of $\mathbb{Q}$-music, $\overline{f_1 f_2} \cong \overline{f_3 f_4}$ iff (the Ratio of $\mathbb{Q}$-music)$(f_1, f_2) =$ (the Ratio of $\mathbb{Q}$-music)(f_3, f_4).

The theorem is a consequence of (6).

(17) Let us consider elements f_1, f_2, f_3 of $\mathbb{Q}$-music. Suppose (the Ratio of $\mathbb{Q}$-music)$(f_1, f_2) =$ (the Ratio of $\mathbb{Q}$-music)(f_1, f_3). Then $f_2 = f_3$.

(18) $\mathbb{N}_+ \subseteq$ the carrier of $\mathbb{Q}$-music.

(19) Let us consider an element fr of $\mathbb{Q}$-music, and a non zero natural number n. Then there exists an element h of $\mathbb{Q}$-music such that $\langle fr, h\rangle \in [\langle 1, n\rangle]_\alpha$, where α is the equidistance of $\mathbb{Q}$-music. The theorem is a consequence of (2) and (16).

(20) Let us consider elements f_1, f_2, f_3 of $\mathbb{Q}$-music. Suppose (the Ratio of $\mathbb{Q}$-music)$(f_1, f_1) =$ (the Ratio of $\mathbb{Q}$-music)(f_2, f_3). Then $f_2 = f_3$.

(21) Let us consider an element fr of $\mathbb{Q}$-music. Then there exists a positive real number r such that

(i) $fr = r$, and

(ii) for every non zero natural number n, $n \cdot r$ is an element of $\mathbb{Q}$-music.

The theorem is a consequence of (2).

(22) Let us consider elements f_1, f_2, f_8, f_6, f_9, f_7 of $\mathbb{Q}$-music, positive rational numbers r_1, r_2, and non zero natural numbers n, m. Suppose $f_8 = n \cdot r_1$ and $f_6 = m \cdot r_1$ and $f_9 = n \cdot r_2$ and $f_7 = m \cdot r_2$. Then $\overline{f_8 f_6} \cong \overline{f_9 f_7}$. The theorem is a consequence of (16).

(23) Let us consider elements f_1, f_2, f_3, f_4 of $\mathbb{Q}$-music. Then (the Ratio of $\mathbb{Q}$-music)$(f_1, f_2) =$ (the Ratio of $\mathbb{Q}$-music)(f_3, f_4) if and only if (the Ratio of $\mathbb{Q}$-music)$(f_2, f_1) =$ (the Ratio of $\mathbb{Q}$-music)(f_4, f_3). The theorem is a consequence of (15).

4. Musical Structure and Some Axioms

Let S be a structure of music. We say that S is satisfying real if and only if

(Def. 12) the carrier of $S \subseteq \mathbb{R}_+$.

We say that S is equidistant-ratio equivalent if and only if

(Def. 13) for every elements f_1, f_2, f_3, f_4 of S, $\overline{f_1 f_2} \cong \overline{f_3 f_4}$ iff (the Ratio of S)(f_1, f_2) = (the Ratio of S)(f_3, f_4).

We say that S is satisfying interval if and only if

(Def. 14) for every elements f_1, f_2, f_3 of S such that (the Ratio of S)(f_1, f_2) = (the Ratio of S)(f_1, f_3) holds $f_2 = f_3$.

We say that S is unison-ratio stable if and only if

(Def. 15) for every elements f_1, f_2, f_3 of S such that (the Ratio of S)(f_1, f_1) = (the Ratio of S)(f_2, f_3) holds $f_2 = f_3$.

We say that S is ratio symmetric if and only if

(Def. 16) for every elements f_1, f_2, f_3, f_4 of S, (the Ratio of S)(f_1, f_2) = (the Ratio of S)(f_3, f_4) iff (the Ratio of S)(f_2, f_1) = (the Ratio of S)(f_4, f_3).

We say that S is natural if and only if

(Def. 17) $\mathbb{N}_+ \subseteq$ the carrier of S.

We say that S is harmonic closed if and only if

(Def. 18) for every element fr of S and for every non zero natural number n, there exists an element h of S such that $\langle fr, h\rangle \in [\langle 1, n\rangle]_\alpha$, where α is the equidistance of S.

Note that there exists a structure of music which is harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, and non empty.

Let us note that the functor $\mathbb{R}$-music yields a harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Observe that the functor $\mathbb{Q}$-music yields a harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Now we state the propositions:

(24) Let us consider a natural structure of music S. Then every non zero natural number is an element of S.

(25) Let us consider an equidistant-ratio equivalent structure of music M, and elements a, b of M. Then $\overline{ab} \cong \overline{ab}$.

(26) Let us consider an equidistant-ratio equivalent structure of music M, and elements a, b, c, d of M. Then $\overline{ab} \cong \overline{cd}$ if and only if $\overline{cd} \cong \overline{ab}$.

(27) Let us consider an equidistant-ratio equivalent structure of music M, and elements a, b, c, d, e, f of M. Suppose $\overline{ab} \cong \overline{cd}$ and $\overline{cd} \cong \overline{ef}$. Then $\overline{ab} \cong \overline{ef}$.

(28) Let us consider a satisfying interval, equidistant-ratio equivalent structure of music S, and elements a, b, c of S. Then $\overline{ab} \cong \overline{ac}$ if and only if $b = c$. The theorem is a consequence of (25).

From now on M denotes an equidistant-ratio equivalent structure of music and a, b, c, d, e, f denote elements of M.

Now we state the propositions:

(29) $\overline{aa} \cong \overline{aa}$.

(30) The equidistance of M is reflexive in (the carrier of M) $\times$ (the carrier of M). The theorem is a consequence of (25).

(31) Suppose M is not empty. Then

(i) the equidistance of M is reflexive, and

(ii) field(the equidistance of M) = (the carrier of M) $\times$ (the carrier of M).

The theorem is a consequence of (30).

(32) The equidistance of M is symmetric in (the carrier of M) $\times$ (the carrier of M). The theorem is a consequence of (26).

(33) The equidistance of M is transitive in (the carrier of M) $\times$ (the carrier of M). The theorem is a consequence of (27).

(34) The equidistance of M is an equivalence relation of (the carrier of M) $\times$ (the carrier of M). The theorem is a consequence of (30), (32), and (33).

(35) Let us consider a ratio symmetric, equidistant-ratio equivalent structure of music M, and elements a, b, c, d of M. Then $\overline{ab} \cong \overline{cd}$ if and only if $\overline{ba} \cong \overline{dc}$.

(36) Let us consider a unison-ratio stable, equidistant-ratio equivalent structure of music S, and elements a, b, c of S. If $\overline{aa} \cong \overline{bc}$, then $b = c$.

Let S be a natural, satisfying interval, harmonic closed, equidistant-ratio equivalent structure of music, fr be an element of S, and n be a non zero natural number. The n-harmonic of fr in S yielding an element of S is defined by

(Def. 19) $\langle fr, it\rangle \in [\langle 1, n\rangle]_{\alpha}$, where α is the equidistance of S.

We say that S is harmonic linear if and only if

(Def. 20) for every element fr of S and for every non zero natural number n, there exists a positive real number f such that $fr = f$ and the n-harmonic of fr in $S = n \cdot f$.

Now we state the propositions:

(37) $\mathbb{R}$-music is harmonic linear. The theorem is a consequence of (1) and (24).

(38) $\mathbb{Q}$-music is harmonic linear. The theorem is a consequence of (2) and (24).

One can check that there exists a harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music which is harmonic linear.

One can check that the functor $\mathbb{R}$-music yields a harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Let us note that the functor $\mathbb{Q}$-music yields a harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music.

Let M be a harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music. We say that M is harmonic stable if and only if

(Def. 21) for every elements f_1, f_2 of M and for every non zero natural numbers n, m, $\overline{\text{the } n\text{-harmonic of } f_1 \text{ in } M \text{ the } m\text{-harmonic of } f_1 \text{ in } M} \cong \overline{\text{the } n\text{-harmonic of } f_2 \text{ in } M \text{ the } m\text{-harmonic of } f_2 \text{ in } M}$.

Now we state the propositions:

(39) $\mathbb{R}$-music is harmonic stable. The theorem is a consequence of (1) and (13).

(40) $\mathbb{Q}$-music is harmonic stable. The theorem is a consequence of (2) and (22).

Observe that there exists a harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music which is harmonic stable.

One can verify that the functor $\mathbb{R}$-music yields a harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Observe that the functor $\mathbb{Q}$-music yields a harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music.

Let M be a harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music and *fr* be an element of M. The functors: the set of unison of *fr* in M, the set of octave of *fr* in M, the set of fifth of *fr* in M, the set of fourth of *fr* in M, and the set of major sixth of *fr* in M yielding subsets of (the carrier of M) $\times$ (the carrier of M) are defined by terms

(Def. 22) $[\langle$the 1-harmonic of *fr* in M, the 1-harmonic of *fr* in $M\rangle]_\alpha$, where α is the equidistance of M,

(Def. 23) $[\langle$the 1-harmonic of *fr* in M, the 2-harmonic of *fr* in $M\rangle]_\alpha$, where α is the equidistance of M,

(Def. 24) $[\langle$the 2-harmonic of *fr* in M, the 3-harmonic of *fr* in $M\rangle]_\alpha$, where α is the equidistance of M,

(Def. 25) $[\langle$the 3-harmonic of *fr* in M, the 4-harmonic of *fr* in $M\rangle]_\alpha$, where α is the equidistance of M,

(Def. 26) $[\langle$the 3-harmonic of *fr* in M, the 5-harmonic of *fr* in $M\rangle]_\alpha$, where α is the equidistance of M,

respectively. The functors: the set of major third of *fr* in M, the set of minor third of *fr* in M, the set of minor sixth of *fr* in M, the set of major tone of *fr* in M, and the set of minor tone of *fr* in M yielding subsets of (the carrier of M) $\times$ (the carrier of M) are defined by terms

(Def. 27) $[\langle$the 4-harmonic of *fr* in M, the 5-harmonic of *fr* in $M\rangle]_\alpha$, where α is the equidistance of M,

(Def. 28) $[\langle$the 5-harmonic of *fr* in M, the 6-harmonic of *fr* in $M\rangle]_\alpha$, where α is the equidistance of M,

(Def. 29) $[\langle$the 5-harmonic of *fr* in M, the 8-harmonic of *fr* in $M\rangle]_\alpha$, where α is the equidistance of M,

(Def. 30) $[\langle$the 8-harmonic of *fr* in M, the 9-harmonic of *fr* in $M\rangle]_\alpha$, where α is the equidistance of M,

(Def. 31) $[\langle$the 9-harmonic of *fr* in M, the 10-harmonic of *fr* in $M\rangle]_\alpha$, where α is the equidistance of M,

respectively. The functors: the set of unison of M, the set of octave of M, the set of fifth of M, the set of fourth of M, and the set of major sixth of M yielding subsets of (the carrier of M) $\times$ (the carrier of M) are defined by terms

(Def. 32) $[\langle 1, 1\rangle]_\alpha$, where α is the equidistance of M,

(Def. 33) $[\langle 1, 2\rangle]_\alpha$, where α is the equidistance of M,

(Def. 34) $[\langle 2, 3\rangle]_\alpha$, where α is the equidistance of M,

(Def. 35) $[\langle 3, 4\rangle]_\alpha$, where α is the equidistance of M,

(Def. 36) $[\langle 3, 5\rangle]_\alpha$, where α is the equidistance of M,

respectively. The functors: the set of major third of M, the set of minor third of M, the set of minor sixth of M, the set of major tone of M, and the set of minor tone of M yielding subsets of (the carrier of M) $\times$ (the carrier of M) are defined by terms

(Def. 37) $[\langle 4, 5\rangle]_\alpha$, where α is the equidistance of M, (Def. 38) $[\langle 5, 6\rangle]_\alpha$, where α is the equidistance of M,

(Def. 39) $[\langle 5, 8\rangle]_\alpha$, where α is the equidistance of M,

(Def. 40) $[\langle 8, 9\rangle]_\alpha$, where α is the equidistance of M,

(Def. 41) $[\langle 9, 10\rangle]_\alpha$, where α is the equidistance of M,

respectively. Let S be a harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music. We say that S is fifth constructible if and only if

(Def. 42) for every element fr of S, there exists an element q of S such that $\langle fr, q\rangle \in$ the set of fifth of S.

Now we state the propositions:

(41) Let us consider an element fr of $\mathbb{R}$-music. Then there exist positive real numbers f, q_1 such that

(i) $f = fr$, and

(ii) $q_1 = \frac{(3 \textbf{ qua } \text{real number})}{2} \cdot f$, and

(iii) $\langle f, q_1\rangle \in$ the set of fifth of $\mathbb{R}$-music.

The theorem is a consequence of (1) and (24).

(42) $\mathbb{R}$-music is fifth constructible. The theorem is a consequence of (41) and (1).

(43) Let us consider an element fr of $\mathbb{Q}$-music. Then there exist positive rational numbers f, q_1 such that

(i) $f = fr$, and

(ii) $q_1 = \frac{(3 \textbf{ qua } \text{rational number})}{2} \cdot f$, and

(iii) $\langle f, q_1\rangle \in$ the set of fifth of $\mathbb{Q}$-music.

The theorem is a consequence of (2) and (24).

(44) $\mathbb{Q}$-music is fifth constructible. The theorem is a consequence of (43) and (2).

Let us observe that there exists a harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music which is fifth constructible.

Let us note that the functor $\mathbb{R}$-music yields a fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Let us note that the functor $\mathbb{Q}$-music yields a fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music.

Let M be a fifth constructible, harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music and fr be an element of M. The fifth of fr in M yielding an element of M is defined by

(Def. 43) $\langle fr, it \rangle \in$ the set of fifth of M.

Now we state the propositions:

(45) Let us consider a fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music M, and an element fr of M. Then the set of fifth of fr in $M =$ the set of fifth of M. The theorem is a consequence of (24) and (27).

(46) Let us consider an element fr of $\mathbb{R}$-music. Then there exists a positive real number f such that

(i) $fr = f$, and

(ii) the fifth of fr in $\mathbb{R}$-music $= \frac{(3 \textbf{ qua } \text{real number})}{2} \cdot f$.

The theorem is a consequence of (1) and (41).

(47) Let us consider an element fr of $\mathbb{Q}$-music. Then there exists a positive rational number f such that

(i) $fr = f$, and

(ii) the fifth of fr in $\mathbb{Q}$-music $= \frac{(3 \textbf{ qua } \text{rational number})}{2} \cdot f$.

The theorem is a consequence of (2) and (43).

Let M be a fifth constructible, harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music. We say that M is classical fifth if and only if

(Def. 44) for every element fr of M, there exists a positive real number f such that $fr = f$ and the fifth of fr in $M = \frac{(3 \textbf{ qua } \text{real number})}{2} \cdot f$.

One can verify that there exists a fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music which is classical fifth.

One can verify that the functor $\mathbb{R}$-music yields a classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music.

One can check that the functor $\mathbb{Q}$-music yields a classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music.

5. Harmonic

Now we state the propositions:

(48) Let us consider a harmonic closed, natural, unison-ratio stable, satisfying interval, equidistant-ratio equivalent structure of music M, and an element fr of M. Then the 1-harmonic of fr in $M = fr$. The theorem is a consequence of (36).

(49) Let us consider a harmonic stable, harmonic closed, natural, unison-ratio stable, satisfying interval, equidistant-ratio equivalent structure of music M, and elements a, b of M. Then $\overline{aa} \cong \overline{bb}$. The theorem is a consequence of (48).

(50) Let us consider a harmonic stable, harmonic linear, harmonic closed, natural, unison-ratio stable, satisfying interval, equidistant-ratio equivalent structure of music M, and an element fr of M. Then the set of octave of fr in $M =$ the set of octave of M. The theorem is a consequence of (48), (27), and (24).

(51) Let us consider a fifth constructible, harmonic closed, natural, satisfying interval, equidistant-ratio equivalent, non empty structure of music M, and an element fr of M. Then there exists a sequence s_{11} of M such that

(i) $s_{11}(0) = fr$, and

(ii) for every natural number n, $\langle s_{11}(n), s_{11}(n+1)\rangle \in$ the set of fifth of M.

Proof: Define $\mathcal{P}[\text{set}, \text{set}, \text{set}] \equiv$ there exist positive real numbers x, y such that $\langle \$_2, \$_3\rangle \in$ the set of fifth of M. For every natural number n and for every element x of M, there exists an element y of M such that $\mathcal{P}[n, x, y]$. Consider s_{11} being a sequence of M such that $s_{11}(0) = fr$ and for every natural number n, $\mathcal{P}[n, s_{11}(n), s_{11}(n+1)]$. □

Let M be a structure of music and a, b, c be elements of M. We say that b is between a and c if and only if

(Def. 45) there exist positive real numbers r_1, r_2, r_3 such that $a = r_1$ and $b = r_2$ and $c = r_3$ and $r_1 \leqslant r_2 < r_3$.

Let S be a harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music. We say that S is octave constructible if and only if

(Def. 46) for every element fr of S, there exists an element o of S such that $\langle fr, o\rangle \in$ the set of octave of S.

Now we state the propositions:

(52) Let us consider an element fr of $\mathbb{R}$-music. Then there exist positive real numbers f, q_1 such that

(i) $f = fr$, and

(ii) $q_1 = 2 \cdot f$, and

(iii) $\langle f, q_1 \rangle \in$ the set of octave of $\mathbb{R}$-music.

The theorem is a consequence of (1) and (24).

(53) $\mathbb{R}$-music is octave constructible. The theorem is a consequence of (52) and (1).

(54) Let us consider an element fr of $\mathbb{Q}$-music. Then there exist positive rational numbers f, q_1 such that

(i) $f = fr$, and

(ii) $q_1 = 2 \cdot f$, and

(iii) $\langle f, q_1 \rangle \in$ the set of octave of $\mathbb{Q}$-music.

The theorem is a consequence of (2) and (24).

(55) $\mathbb{Q}$-music is octave constructible. The theorem is a consequence of (54) and (2).

Let us note that there exists a classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music which is octave constructible.

Let us observe that the functor $\mathbb{R}$-music yields an octave constructible, classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Let us note that the functor $\mathbb{Q}$-music yields an octave constructible, classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music.

Let M be an octave constructible, harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music and fr be an element of M. The octave of fr in M yielding an element of M is defined by

(Def. 47) $\langle fr, it \rangle \in$ the set of octave of M.

Let M be a satisfying real, non empty structure of music and r be an element of M. The functor ${}^{@}r$ yielding a positive real number is defined by the term

(Def. 48) r.

Let M be an octave constructible, harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music. We say that M is classical octave if and only if

(Def. 49) for every element *fr* of M, there exists a positive real number f such that $fr = f$ and the octave of *fr* in $M = 2 \cdot f$.

Now we state the propositions:

(56) $\mathbb{R}$-music is classical octave. The theorem is a consequence of (52) and (1).

(57) $\mathbb{Q}$-music is classical octave. The theorem is a consequence of (54) and (2).

One can verify that there exists an octave constructible, classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music which is classical octave.

Observe that the functor $\mathbb{R}$-music yields a classical octave, octave constructible, classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Observe that the functor $\mathbb{Q}$-music yields a classical octave, octave constructible, classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music.

Let M be an octave constructible, harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music. We say that M is octave descending constructible if and only if

(Def. 50) for every element *fr* of M, there exists an element o of M such that $\langle o, fr \rangle \in$ the set of octave of M.

Now we state the propositions:

(58) Let us consider an element *fr* of $\mathbb{R}$-music. Then there exist positive real numbers f, q_1 such that

(i) $f = fr$, and

(ii) $q_1 = \frac{(1 \textbf{ qua } \text{real number})}{2} \cdot f$, and

(iii) $\langle q_1, f \rangle \in$ the set of octave of $\mathbb{R}$-music.

The theorem is a consequence of (1), (24), and (35).

(59) $\mathbb{R}$-music is octave descending constructible. The theorem is a consequence of (58) and (1).

(60) Let us consider an element fr of $\mathbb{Q}$-music. Then there exist positive rational numbers f, q_1 such that

  (i) $f = fr$, and

  (ii) $q_1 = \frac{(1 \textbf{ qua } \text{rational number})}{2} \cdot f$, and

  (iii) $\langle q_1, f \rangle \in$ the set of octave of $\mathbb{Q}$-music.

The theorem is a consequence of (2), (24), and (35).

(61) $\mathbb{Q}$-music is octave descending constructible. The theorem is a consequence of (60) and (2).

One can verify that there exists a classical octave, octave constructible, classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music which is octave descending constructible.

One can verify that the functor $\mathbb{R}$-music yields an octave descending constructible, classical octave, octave constructible, classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Note that the functor $\mathbb{Q}$-music yields an octave descending constructible, classical octave, octave constructible, classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music.

Let M be an octave descending constructible, octave constructible, fifth constructible, harmonic closed, natural, ratio symmetric, satisfying interval, equidistant-ratio equivalent structure of music and fr be an element of M. The octave descending of fr in M yielding an element of M is defined by

(Def. 51) $\langle it, fr \rangle \in$ the set of octave of M.

Now we state the propositions:

(62) Let us consider an octave descending constructible, classical octave, octave constructible, fifth constructible, harmonic closed, natural, ratio symmetric, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music M, and an element fr of M. Then there exists a positive real number r such that

  (i) $fr = r$, and

  (ii) the octave descending of fr in $M = \frac{r}{2}$.

The theorem is a consequence of (1).

(63) Let us consider classical octave, octave constructible, classical fifth, fifth constructible, harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structures of music M_1, M_2, an element f_1 of M_1, and an element f_2 of M_2. Suppose $f_1 = f_2$. Then

(i) the fifth of f_1 in M_1 = the fifth of f_2 in M_2, and

(ii) the octave of f_1 in M_1 = the octave of f_2 in M_2.

(64) Let us consider octave descending constructible, classical octave, octave constructible, fifth constructible, harmonic closed, natural, ratio symmetric, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structures of music M_1, M_2, an element fr_1 of M_1, and an element fr_2 of M_2. Suppose $fr_1 = fr_2$. Then the octave descending of fr_1 in M_1 = the octave descending of fr_2 in M_2. The theorem is a consequence of (62).

Let M be an octave descending constructible, octave constructible, fifth constructible, harmonic closed, natural, ratio symmetric, satisfying interval, equidistant-ratio equivalent structure of music and f_{10}, fr be elements of M. The reduct fifth of the fr with fundamental frequency f_{10} in M yielding an element of M is defined by the term

$$\text{(Def. 52)}\quad \begin{cases} \text{the fifth of } fr \text{ in } M, \textbf{ if } \text{the fifth of } fr \text{ in } M \text{ is between } f_{10} \text{ and the} \\ \quad \text{octave of } f_{10} \text{ in } M, \\ \text{the octave descending of (the fifth of } fr \text{ in } M\text{) in } M, \textbf{ otherwise}. \end{cases}$$

Now we state the propositions:

(65) Let us consider octave descending constructible, classical octave, octave constructible, classical fifth, fifth constructible, harmonic closed, natural, ratio symmetric, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structures of music M_1, M_2, elements fr_1, f_{11} of M_1, and elements fr_2, f_{12} of M_2. Suppose $fr_1 = fr_2$ and $f_{11} = f_{12}$. Then the reduct fifth of the fr_1 with fundamental frequency f_{11} in M_1 = the reduct fifth of the fr_2 with fundamental frequency f_{12} in M_2. The theorem is a consequence of (63) and (64).

(66) Let us consider a classical fifth, fifth constructible, harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music M, and an element fr of M. Then there exist positive real numbers r, s such that

(i) $r = fr$, and

(ii) $s = \frac{(3 \textbf{ qua } \text{real number})}{2} \cdot r$, and

(iii) the fifth of fr in $M = s$.

(67) Let us consider an octave descending constructible, classical octave, octave constructible, classical fifth, fifth constructible, harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music M, and elements f_{10}, fr of M. Suppose fr is between f_{10} and the octave of f_{10} in M. Then there exist positive real numbers r_1, r_2, r_3 such that

(i) $f_{10} = r_1$, and

(ii) $fr = r_2$, and

(iii) the octave of f_{10} in $M = 2 \cdot r_1$, and

(iv) $r_1 \leqslant r_2 \leqslant 2 \cdot r_1$.

(68) Let us consider an octave descending constructible, classical octave, octave constructible, classical fifth, fifth constructible, harmonic closed, natural, ratio symmetric, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music M, and elements f_{10}, fr of M. Suppose fr is between f_{10} and the octave of f_{10} in M. Then the reduct fifth of the fr with fundamental frequency f_{10} in M is between f_{10} and the octave of f_{10} in M. The theorem is a consequence of (67) and (62).

A space of music is an octave descending constructible, classical octave, octave constructible, classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Now we state the propositions:

(69) $\mathbb{R}$-music is a space of music.

(70) $\mathbb{Q}$-music is a space of music.

6. Spiral of Fifths

Now we state the proposition:

(71) Let us consider an octave descending constructible, octave constructible, fifth constructible, harmonic closed, natural, ratio symmetric, satisfying interval, equidistant-ratio equivalent, non empty structure of music M, and elements f_{10}, fr of M. Then there exists a sequence s_{11} of M such that

(i) $s_{11}(0) = fr$, and

(ii) for every natural number n, $s_{11}(n + 1)$ = the reduct fifth of the $s_{11}(n)$ with fundamental frequency f_{10} in M.

Proof: Define $\mathcal{P}[\text{set}, \text{set}, \text{set}] \equiv$ there exist elements x, y of M such that $x = \$_2$ and $y = \$_3$ and $y =$ the reduct fifth of the x with fundamental frequency f_{10} in M. For every natural number n and for every element x of M, there exists an element y of M such that $\mathcal{P}[n, x, y]$. Consider s_{11} being a sequence of M such that $s_{11}(0) = \mathit{fr}$ and for every natural number n, $\mathcal{P}[n, s_{11}(n), s_{11}(n+1)]$. □

Let M be an octave descending constructible, octave constructible, fifth constructible, harmonic closed, natural, ratio symmetric, satisfying interval, equidistant-ratio equivalent, non empty structure of music and f_{10}, fr be elements of M. The spiral of fifths of fr with fundamental frequency f_{10} in M yielding a sequence of M is defined by

(Def. 53) $\mathit{it}(0) = \mathit{fr}$ and for every natural number n, $\mathit{it}(n+1) =$ the reduct fifth of the $\mathit{it}(n)$ with fundamental frequency f_{10} in M.

From now on M denotes an octave descending constructible, classical octave, octave constructible, classical fifth, fifth constructible, harmonic closed, natural, ratio symmetric, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music and f_{10}, fr denote elements of M.

Now we state the propositions:

(72) Suppose fr is between f_{10} and the octave of f_{10} in M. Let us consider a natural number n. Then (the spiral of fifths of fr with fundamental frequency f_{10} in M)(n) is between f_{10} and the octave of f_{10} in M.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ (the spiral of fifths of fr with fundamental frequency f_{10} in M)$(\$_1)$ is between f_{10} and the octave of f_{10} in M. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$ from [2, Sch. 2]. □

(73) (The spiral of fifths of f_{10} with fundamental frequency f_{10} in M)$(1) = \frac{(3 \textbf{ qua } \text{real number})}{2} \cdot ({}^{@}f_{10})$. The theorem is a consequence of (66).

(74) (The spiral of fifths of f_{10} with fundamental frequency f_{10} in M)$(2) = \frac{(9 \textbf{ qua } \text{real number})}{8} \cdot ({}^{@}f_{10})$. The theorem is a consequence of (73), (66), and (62).

(75) (The spiral of fifths of f_{10} with fundamental frequency f_{10} in M)$(3) = \frac{(27 \textbf{ qua } \text{real number})}{16} \cdot ({}^{@}f_{10})$. The theorem is a consequence of (74) and (66).

(76) (The spiral of fifths of f_{10} with fundamental frequency f_{10} in M)$(4) = \frac{(81 \textbf{ qua } \text{real number})}{64} \cdot ({}^{@}f_{10})$. The theorem is a consequence of (75), (66), and (62).

(77) (The spiral of fifths of f_{10} with fundamental frequency f_{10} in M)$(5) = \frac{(243 \textbf{ qua } \text{real number})}{128} \cdot ({}^{@}f_{10})$. The theorem is a consequence of (76) and (66).

(78) $\frac{{}^{@}(\text{the spiral of fifths of } \mathit{fr} \text{ with fundamental frequency } \mathit{fr} \text{ in } M)(2)}{{}^{@}\mathit{fr}} =$

$\frac{3 \cdot 3 \textbf{ qua } \text{real number}}{2 \cdot 2 \cdot 2}$. The theorem is a consequence of (74).

(79) $\frac{^@(\text{the spiral of fifths of } fr \text{ with fundamental frequency } fr \text{ in } M)(4)}{^@(\text{the spiral of fifths of } fr \text{ with fundamental frequency } fr \text{ in } M)(2)} = \frac{3 \cdot 3 \textbf{ qua } \text{real number}}{2 \cdot 2 \cdot 2}$. The theorem is a consequence of (74) and (76).

(80) $\frac{^@(\text{the spiral of fifths of } fr \text{ with fundamental frequency } fr \text{ in } M)(1)}{^@(\text{the spiral of fifths of } fr \text{ with fundamental frequency } fr \text{ in } M)(4)} = \frac{32 \textbf{ qua } \text{real number}}{27}$. The theorem is a consequence of (73) and (76).

(81) $\frac{^@(\text{the spiral of fifths of } fr \text{ with fundamental frequency } fr \text{ in } M)(3)}{^@(\text{the spiral of fifths of } fr \text{ with fundamental frequency } fr \text{ in } M)(1)} = \frac{9 \textbf{ qua } \text{real number}}{8}$. The theorem is a consequence of (73) and (75).

(82) $\frac{^@(\text{the octave of } fr \text{ in } M)}{^@(\text{the spiral of fifths of } fr \text{ with fundamental frequency } fr \text{ in } M)(3)} = \frac{32 \textbf{ qua } \text{real number}}{27}$. The theorem is a consequence of (75).

Let M be a space of music and s_{10} be an element of (the carrier of $M)^2$. We say that s_{10} is monotonic if and only if

(Def. 54) there exists an element fr of M and there exist positive real numbers r_1, r_2 such that $s_{10}(1) = fr$ and $s_{10}(1) = r_1$ and $s_{10}(2) = r_2$ and $r_1 < r_2$ and $s_{10}(2) =$ the octave of fr in M.

Let s_{10} be an element of (the carrier of $M)^3$. We say that s_{10} is ditonic if and only if

(Def. 55) there exists an element fr of M and there exist positive real numbers r_1, r_2, r_3 such that $s_{10}(1) = fr$ and $s_{10}(1) = r_1$ and $s_{10}(2) = r_2$ and $s_{10}(3) = r_3$ and $r_1 < r_2 < r_3$ and $s_{10}(3) =$ the octave of fr in M.

Let s_{10} be an element of (the carrier of $M)^4$. We say that s_{10} is tritonic if and only if

(Def. 56) there exists an element fr of M and there exist positive real numbers r_1, r_2, r_3, r_4 such that $s_{10}(1) = fr$ and $s_{10}(1) = r_1$ and $s_{10}(2) = r_2$ and $s_{10}(3) = r_3$ and $s_{10}(4) = r_4$ and $r_1 < r_2 < r_3$ and $r_3 < r_4$ and $s_{10}(4) =$ the octave of fr in M.

Let s_{10} be an element of (the carrier of $M)^5$. We say that s_{10} is tetratonic if and only if

(Def. 57) there exists an element fr of M and there exist positive real numbers r_1, r_2, r_3, r_4, r_5 such that $s_{10}(1) = fr$ and $s_{10}(1) = r_1$ and $s_{10}(2) = r_2$ and $s_{10}(3) = r_3$ and $s_{10}(4) = r_4$ and $s_{10}(5) = r_5$ and $r_1 < r_2 < r_3$ and $r_3 < r_4 < r_5$ and $s_{10}(5) =$ the octave of fr in M.

Let n be a natural number and s_{10} be an element of (the carrier of $M)^n$. We say that s_{10} is pentatonic if and only if

(Def. 58) $n = 6$ and there exists an element fr of M and there exist positive real numbers r_1, r_2, r_3, r_4, r_5, r_6 such that $s_{10}(1) = fr$ and $s_{10}(1) = r_1$ and $s_{10}(2) = r_2$ and $s_{10}(3) = r_3$ and $s_{10}(4) = r_4$ and $s_{10}(5) = r_5$ and $s_{10}(6) =$

r_6 and $r_1 < r_2 < r_3$ and $r_3 < r_4 < r_5$ and $r_5 < r_6$ and $s_{10}(6) =$ the octave of *fr* in M.

Let s_{10} be an element of (the carrier of $M)^7$. We say that s_{10} is hexatonic if and only if

(Def. 59) there exists an element *fr* of M and there exist positive real numbers r_1, r_2, r_3, r_4, r_5, r_6, r_7 such that $s_{10}(1) =$ *fr* and $s_{10}(1) = r_1$ and $s_{10}(2) = r_2$ and $s_{10}(3) = r_3$ and $s_{10}(4) = r_4$ and $s_{10}(5) = r_5$ and $s_{10}(6) = r_6$ and $s_{10}(7) = r_7$ and $r_1 < r_2 < r_3$ and $r_3 < r_4 < r_5$ and $r_5 < r_6 < r_7$ and $s_{10}(7) =$ the octave of *fr* in M.

Let n be a natural number and s_{10} be an element of (the carrier of $M)^n$. We say that s_{10} is heptatonic if and only if

(Def. 60) $n = 8$ and there exists an element *fr* of M and there exist positive real numbers r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8 such that $s_{10}(1) =$ *fr* and $s_{10}(1) = r_1$ and $s_{10}(2) = r_2$ and $s_{10}(3) = r_3$ and $s_{10}(4) = r_4$ and $s_{10}(5) = r_5$ and $s_{10}(6) = r_6$ and $s_{10}(7) = r_7$ and $s_{10}(8) = r_8$ and $r_1 < r_2 < r_3$ and $r_3 < r_4 < r_5$ and $r_5 < r_6 < r_7$ and $r_7 < r_8$ and $s_{10}(8) =$ the octave of *fr* in M.

Let s_{10} be an element of (the carrier of $M)^9$. We say that s_{10} is octatonic if and only if

(Def. 61) there exists an element *fr* of M and there exist positive real numbers r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9 such that $s_{10}(1) =$ *fr* and $s_{10}(1) = r_1$ and $s_{10}(2) = r_2$ and $s_{10}(3) = r_3$ and $s_{10}(4) = r_4$ and $s_{10}(5) = r_5$ and $s_{10}(6) = r_6$ and $s_{10}(7) = r_7$ and $s_{10}(8) = r_8$ and $s_{10}(9) = r_9$ and $r_1 < r_2 < r_3$ and $r_3 < r_4 < r_5$ and $r_5 < r_6 < r_7$ and $r_7 < r_8 < r_9$ and $s_{10}(9) =$ the octave of *fr* in M.

7. Pentatonic Pythagorean Scale

Let M be a space of music and *fr* be an element of M. The pentatonic pythagorean scale of *fr* in M yielding an element of (the carrier of $M)^6$ is defined by

(Def. 62) $it(1) =$ *fr* and $it(2) =$ (the spiral of fifths of *fr* with fundamental frequency *fr* in $M)(2)$ and $it(3) =$ (the spiral of fifths of *fr* with fundamental frequency *fr* in $M)(4)$ and $it(4) =$ (the spiral of fifths of *fr* with fundamental frequency *fr* in $M)(1)$ and $it(5) =$ (the spiral of fifths of *fr* with fundamental frequency *fr* in $M)(3)$ and $it(6) =$ the octave of *fr* in M.

From now on M denotes a space of music and f_{10}, *fr*, f_1, f_2 denote elements of M.

Now we state the proposition:

(83) The pentatonic pythagorean scale of fr in M is pentatonic. The theorem is a consequence of (74), (76), (73), and (75).

Let M be a space of music and f_1, f_2 be elements of M. The interval between f_1 and f_2 yielding a positive real number is defined by

(Def. 63) there exist positive real numbers r_1, r_2 such that $r_1 = f_1$ and $r_2 = f_2$ and $it = \frac{r_2}{r_1}$.

The pythagorean tone yielding a positive real number is defined by the term

(Def. 64) $\frac{(9 \textbf{ qua } \text{real number})}{8}$.

The pythagorean semiditone yielding a positive real number is defined by the term

(Def. 65) $\frac{(32 \textbf{ qua } \text{real number})}{27}$.

The pythagorean major third yielding a positive real number is defined by the term

(Def. 66) (the pythagorean tone) $\cdot$ (the pythagorean tone).

The pythagorean pure major third yielding a positive real number is defined by the term

(Def. 67) $\frac{(5 \textbf{ qua } \text{real number})}{4}$.

The syntonic comma yielding a positive real number is defined by the term

(Def. 68) $\frac{\text{the pythagorean major third}}{\text{the pythagorean pure major third}}$.

Now we state the propositions:

(84) The syntonic comma $= \frac{(81 \textbf{ qua } \text{real number})}{80}$.

(85) The pythagorean tone $<$ the pythagorean semiditone.

(86) (The pythagorean tone) $\cdot$ (the pythagorean tone) $\cdot$ (the pythagorean semiditone) $\cdot$ (the pythagorean tone) $\cdot$ (the pythagorean semiditone) $= 2$.

Let M be a space of music and fr be an element of M. The functors: the first degree of pentatonic scale of fr in M, the second degree of pentatonic scale of fr in M, the third degree of pentatonic scale of fr in M, the fourth degree of pentatonic scale of fr in M, and the fifth degree of pentatonic scale of fr in M yielding elements of M are defined by terms

(Def. 69) (the pentatonic pythagorean scale of fr in M)(1),

(Def. 70) (the pentatonic pythagorean scale of fr in M)(2),

(Def. 71) (the pentatonic pythagorean scale of fr in M)(3),

(Def. 72) (the pentatonic pythagorean scale of fr in M)(4),

(Def. 73) (the pentatonic pythagorean scale of fr in M)(5),

respectively. The octave of pentatonic scale of fr in M yielding an element of M is defined by the term

(Def. 74) the octave of fr in M.

Now we state the propositions:

(87) There exist elements r_1, r_2 of $\mathbb{R}_+$ such that the interval between f_1 and $f_2 = \mathbb{R}\text{-ratio}(r_1, r_2)$.

(88) Let us consider positive real numbers r_1, r_2, r_3, r_4, r_5, r_6. Suppose (the pentatonic pythagorean scale of *fr* in $M)(1) = r_1$ and (the pentatonic pythagorean scale of *fr* in $M)(2) = r_2$ and (the pentatonic pythagorean scale of *fr* in $M)(3) = r_3$ and (the pentatonic pythagorean scale of *fr* in $M)(4) = r_4$ and (the pentatonic pythagorean scale of *fr* in $M)(5) = r_5$ and (the pentatonic pythagorean scale of *fr* in $M)(6) = r_6$. Then

(i) $\frac{r_2}{r_1} = \frac{(9 \textbf{ qua } \text{real number})}{8}$, and

(ii) $\frac{r_3}{r_2} = \frac{(9 \textbf{ qua } \text{real number})}{8}$, and

(iii) $\frac{r_4}{r_3} = \frac{(32 \textbf{ qua } \text{real number})}{27}$, and

(iv) $\frac{r_5}{r_4} = \frac{(9 \textbf{ qua } \text{real number})}{8}$, and

(v) $\frac{r_6}{r_5} = \frac{(32 \textbf{ qua } \text{real number})}{27}$.

The theorem is a consequence of (83), (78), (79), (80), (81), and (82).

(89) There exist positive real numbers r_1, r_2, r_3, r_4, r_5, r_6 such that

(i) (the pentatonic pythagorean scale of *fr* in $M)(1) = r_1$, and

(ii) (the pentatonic pythagorean scale of *fr* in $M)(2) = r_2$, and

(iii) (the pentatonic pythagorean scale of *fr* in $M)(3) = r_3$, and

(iv) (the pentatonic pythagorean scale of *fr* in $M)(4) = r_4$, and

(v) (the pentatonic pythagorean scale of *fr* in $M)(5) = r_5$, and

(vi) (the pentatonic pythagorean scale of *fr* in $M)(6) = r_6$, and

(vii) $\frac{r_2}{r_1} = \frac{(9 \textbf{ qua } \text{real number})}{8}$, and

(viii) $\frac{r_3}{r_2} = \frac{(9 \textbf{ qua } \text{real number})}{8}$, and

(ix) $\frac{r_4}{r_3} = \frac{(32 \textbf{ qua } \text{real number})}{27}$, and

(x) $\frac{r_5}{r_4} = \frac{(9 \textbf{ qua } \text{real number})}{8}$, and

(xi) $\frac{r_6}{r_5} = \frac{(32 \textbf{ qua } \text{real number})}{27}$.

The theorem is a consequence of (1) and (88).

(90) $\frac{(9 \textbf{ qua } \text{real number})}{8} = \frac{(9 \textbf{ qua } \text{rational number})}{8}$.

(91) (i) the interval between the first degree of pentatonic scale of *fr* in M and (the second degree of pentatonic scale of *fr* in $M)$ = the pythagorean tone, and

(ii) the interval between the second degree of pentatonic scale of *fr* in M and (the third degree of pentatonic scale of *fr* in M) = the pythagorean tone, and

(iii) the interval between the third degree of pentatonic scale of *fr* in M and (the fourth degree of pentatonic scale of *fr* in M) = the pythagorean semiditone, and

(iv) the interval between the fourth degree of pentatonic scale of *fr* in M and (the fifth degree of pentatonic scale of *fr* in M) = the pythagorean tone, and

(v) the interval between the fifth degree of pentatonic scale of *fr* in M and (the octave of pentatonic scale of *fr* in M) = the pythagorean semiditone.

The theorem is a consequence of (89).

(92) the fifth of *fr* in M is between *fr* and the octave of *fr* in M.

Let us consider positive real numbers r_1, r_2. Now we state the propositions:

(93) Suppose $f_1 = r_1$ and $f_2 = r_2$ and $r_2 = \frac{(4 \textbf{ qua } \text{real number})}{3} \cdot r_1$. Then

(i) the fifth of f_2 in $M = 2 \cdot r_1$, and

(ii) the fifth of f_2 in M is not between f_1 and the octave of f_1 in M.

(94) Suppose $f_1 = r_1$ and $f_2 = r_2$ and $r_2 = \frac{(4 \textbf{ qua } \text{real number})}{3} \cdot r_1$. Then

(i) if the fifth of f_2 in M is between f_{10} and the octave of f_{10} in M, then the octave descending of (the reduct fifth of the f_2 with fundamental frequency f_{10} in M) in $M = f_1$, and

(ii) if the fifth of f_2 in M is not between f_{10} and the octave of f_{10} in M, then the reduct fifth of the f_2 with fundamental frequency f_{10} in $M = f_1$.

The theorem is a consequence of (62).

(95) Suppose $f_1 = r_1$ and $f_2 = r_2$ and $r_2 = \frac{(4 \textbf{ qua } \text{real number})}{3} \cdot r_1$. Then the reduct fifth of the f_2 with fundamental frequency f_1 in $M = f_1$. The theorem is a consequence of (94) and (93).

8. Heptatonic Pythagorean Scale

Let S be a space of music. We say that S is fourth constructible if and only if

(Def. 75) for every element *fr* of S, there exists an element q of S such that $\langle$*fr*, $q\rangle \in$ the set of fourth of S.

Now we state the propositions:

(96) Let us consider a space of music M. Suppose $M = \mathbb{R}$-music. Let us consider an element *fr* of M. Then there exist positive real numbers f, q_1 such that

(i) $f = fr$, and

(ii) $q_1 = \frac{(4 \textbf{ qua } \text{real number})}{3} \cdot f$, and

(iii) $\langle f, q_1 \rangle \in$ the set of fourth of M.

The theorem is a consequence of (1) and (24).

(97) $\mathbb{R}$-music is fourth constructible. The theorem is a consequence of (96) and (1).

One can verify that there exists a space of music which is fourth constructible.

Let M be a fourth constructible space of music and *fr* be an element of M. The fourth of *fr* in M yielding an element of M is defined by

(Def. 76) $\langle fr, it \rangle \in$ the set of fourth of M.

We say that M is classical fourth if and only if

(Def. 77) for every element *fr* of M, there exists a positive real number f such that $fr = f$ and the fourth of *fr* in $M = \frac{(4 \textbf{ qua } \text{real number})}{3} \cdot f$.

Now we state the proposition:

(98) Let us consider a fourth constructible space of music M. Suppose $M = \mathbb{R}$-music. Let us consider an element *fr* of M. Then there exists a positive real number f such that

(i) $fr = f$, and

(ii) the fourth of *fr* in $M = \frac{(4 \textbf{ qua } \text{real number})}{3} \cdot f$.

The theorem is a consequence of (1) and (96).

Let us note that there exists a fourth constructible space of music which is classical fourth.

Let M be a satisfying real, non empty structure of music. We say that M is euclidean if and only if

(Def. 78) for every elements f_1, f_2 of M, (the Ratio of $M)(f_1, f_2) = \frac{@f_2}{@f_1}$.

One can verify that there exists a satisfying real, non empty structure of music which is euclidean and every satisfying real, non empty structure of music which is euclidean is also satisfying interval and every satisfying real, non empty structure of music which is euclidean is also unison-ratio stable and every satisfying real, non empty structure of music which is euclidean is also ratio symmetric and there exists a classical fourth, fourth constructible space of music which is euclidean.

A heptatonic pythagorean score is a classical fourth, fourth constructible space of music. From now on H denotes a heptatonic pythagorean score and *fr* denotes an element of H.

Let H be a heptatonic pythagorean score and *fr* be an element of H. The heptatonic pythagorean scale of *fr* in H yielding an element of (the carrier of $H)^8$ is defined by

(Def. 79) $it(1)$ = (the spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(1)$ and $it(2)$ = (the spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(3)$ and $it(3)$ = (the spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(5)$ and $it(4)$ = the fourth of *fr* in H and $it(5)$ = (the spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(2)$ and $it(6)$ = (the spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(4)$ and $it(7)$ = (the spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(6)$ and $it(8)$ = the octave of (the spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(1)$ in H.

Now we state the propositions:

(99) the fourth of *fr* in H is between *fr* and the octave of *fr* in H.

(100) Let us consider a natural number n. Then (the spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(n)$ is between *fr* and the octave of *fr* in H.

(101) (The spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(1) =$ *fr*. The theorem is a consequence of (66) and (62).

(102) (The spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(2) = \frac{(3 \textbf{ qua } \text{real number})}{2} \cdot (^{@}fr)$. The theorem is a consequence of (101) and (66).

(103) (The spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(3) = \frac{(9 \textbf{ qua } \text{real number})}{8} \cdot (^{@}fr)$. The theorem is a consequence of (102), (66), and (62).

(104) (The spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(4) = \frac{(27 \textbf{ qua } \text{real number})}{16} \cdot (^{@}fr)$. The theorem is a consequence of (103) and (66).

(105) (The spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(5) = \frac{(81 \textbf{ qua } \text{real number})}{64} \cdot (^{@}fr)$. The theorem is a consequence of (104), (66), and (62).

(106) (The spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(6) = \frac{(243 \textbf{ qua } \text{real number})}{128} \cdot (^{@}fr)$. The theorem is a consequence of

(105) and (66).

(107) (i) (the heptatonic pythagorean scale of fr in $H)(1) = 1 \cdot ({}^{@}\mathit{fr})$, and

(ii) (the heptatonic pythagorean scale of fr in $H)(2) = \frac{(9\ \textbf{qua}\ \text{real number})}{8} \cdot ({}^{@}\mathit{fr})$, and

(iii) (the heptatonic pythagorean scale of fr in $H)(3) = \frac{(81\ \textbf{qua}\ \text{real number})}{64} \cdot ({}^{@}\mathit{fr})$, and

(iv) (the heptatonic pythagorean scale of fr in $H)(4) = \frac{(4\ \textbf{qua}\ \text{real number})}{3} \cdot ({}^{@}\mathit{fr})$, and

(v) (the heptatonic pythagorean scale of fr in $H)(5) = \frac{(3\ \textbf{qua}\ \text{real number})}{2} \cdot ({}^{@}\mathit{fr})$, and

(vi) (the heptatonic pythagorean scale of fr in $H)(6) = \frac{(27\ \textbf{qua}\ \text{real number})}{16} \cdot ({}^{@}\mathit{fr})$, and

(vii) (the heptatonic pythagorean scale of fr in $H)(7) = \frac{(243\ \textbf{qua}\ \text{real number})}{128} \cdot ({}^{@}\mathit{fr})$, and

(viii) (the heptatonic pythagorean scale of fr in $H)(8) = 2 \cdot ({}^{@}\mathit{fr})$.

The theorem is a consequence of (101), (103), (105), (102), (104), and (106).

(108) The heptatonic pythagorean scale of fr in H is heptatonic. The theorem is a consequence of (107).

The pythagorean semitone yielding a positive real number is defined by the term

(Def. 80) $\frac{(256\ \textbf{qua}\ \text{real number})}{243}$.

Now we state the propositions:

(109) $\frac{\text{the pythagorean tone}}{2} <$ the pythagorean semitone.

(110) (The pythagorean tone) $\cdot$ (the pythagorean tone) $\cdot$ (the pythagorean semitone)$\cdot$(the pythagorean tone)$\cdot$(the pythagorean tone)$\cdot$(the pythagorean tone) $\cdot$ (the pythagorean semitone) $= 2$.

Let H be a heptatonic pythagorean score and fr be an element of H. The functors: the first degree of heptatonic scale of fr in H, the second degree of heptatonic scale of fr in H, the third degree of heptatonic scale of fr in H, the fourth degree of heptatonic scale of fr in H, and the fifth degree of heptatonic scale of fr in H yielding elements of H are defined by terms

(Def. 81) (the heptatonic pythagorean scale of fr in $H)(1)$,

(Def. 82) (the heptatonic pythagorean scale of fr in $H)(2)$,

(Def. 83) (the heptatonic pythagorean scale of fr in $H)(3)$,

(Def. 84) (the heptatonic pythagorean scale of fr in $H)(4)$, (Def. 85) (the heptatonic pythagorean scale of *fr* in H)(5),

respectively. The functors: the sixth degree of heptatonic scale of *fr* in H, the seventh degree of heptatonic scale of *fr* in H, and the eight degree of heptatonic scale of *fr* in H yielding elements of H are defined by terms

(Def. 86) (the heptatonic pythagorean scale of *fr* in H)(6),

(Def. 87) (the heptatonic pythagorean scale of *fr* in H)(7),

(Def. 88) the octave of *fr* in H,

respectively. Now we state the proposition:

(111) (i) the interval between the first degree of heptatonic scale of *fr* in H and (the second degree of heptatonic scale of *fr* in H) = the pythagorean tone, and

(ii) the interval between the second degree of heptatonic scale of *fr* in H and (the third degree of heptatonic scale of *fr* in H) = the pythagorean tone, and

(iii) the interval between the third degree of heptatonic scale of *fr* in H and (the fourth degree of heptatonic scale of *fr* in H) = the pythagorean semitone, and

(iv) the interval between the fourth degree of heptatonic scale of *fr* in H and (the fifth degree of heptatonic scale of *fr* in H) = the pythagorean tone, and

(v) the interval between the fifth degree of heptatonic scale of *fr* in H and (the sixth degree of heptatonic scale of *fr* in H) = the pythagorean tone, and

(vi) the interval between the sixth degree of heptatonic scale of *fr* in H and (the seventh degree of heptatonic scale of *fr* in H) = the pythagorean tone, and

(vii) the interval between the seventh degree of heptatonic scale of *fr* in H and (the eight degree of heptatonic scale of *fr* in H) = the pythagorean semitone.

The theorem is a consequence of (107).

From now on H denotes a heptatonic pythagorean score and *fr* denotes an element of H.

Let M be a space of music, n be a natural number, and s_{10} be an element of (the carrier of $M)^n$. Assume s_{10} is heptatonic. We say that s_{10} is perfect fifth if and only if

(Def. 89) $\langle s_{10}(1), s_{10}(5)\rangle$, $\langle s_{10}(2), s_{10}(6)\rangle$, $\langle s_{10}(3), s_{10}(7)\rangle$, $\langle s_{10}(4), s_{10}(8)\rangle \in$ the set of fifth of M.

Now we state the proposition:

(112) Let us consider an euclidean heptatonic pythagorean score H, and an element *fr* of H. Then the heptatonic pythagorean scale of *fr* in H is perfect fifth. The theorem is a consequence of (108), (107), and (24).

Let H be a heptatonic pythagorean score and *fr* be an element of H. The heptatonic pythagorean scale ascending of *fr* in H yielding an element of (the carrier of $H)^8$ is defined by the term

(Def. 90) the heptatonic pythagorean scale of (the octave of *fr* in H) in H.

Now we state the propositions:

(113) (i) (the heptatonic pythagorean scale ascending of *fr* in $H)(1) = 2 \cdot ({}^{@}fr)$, and

(ii) (the heptatonic pythagorean scale ascending of *fr* in $H)(2) = \frac{9 \textbf{ qua } \text{real number}}{4} \cdot ({}^{@}fr)$, and

(iii) (the heptatonic pythagorean scale ascending of *fr* in $H)(3) = \frac{81 \textbf{ qua } \text{real number}}{32} \cdot ({}^{@}fr)$, and

(iv) (the heptatonic pythagorean scale ascending of *fr* in $H)(4) = \frac{8 \textbf{ qua } \text{real number}}{3} \cdot ({}^{@}fr)$, and

(v) (the heptatonic pythagorean scale ascending of *fr* in $H)(5) = (3 \textbf{ qua } \text{real number}) \cdot ({}^{@}fr)$, and

(vi) (the heptatonic pythagorean scale ascending of *fr* in $H)(6) = \frac{27 \textbf{ qua } \text{real number}}{8} \cdot ({}^{@}fr)$, and

(vii) (the heptatonic pythagorean scale ascending of *fr* in $H)(7) = \frac{243 \textbf{ qua } \text{real number}}{64} \cdot ({}^{@}fr)$, and

(viii) (the heptatonic pythagorean scale ascending of *fr* in $H)(8) = 4 \cdot ({}^{@}fr)$.

The theorem is a consequence of (107).

(114) (The heptatonic pythagorean scale of *fr* in $H)(8)$ = (the heptatonic pythagorean scale ascending of *fr* in $H)(1)$. The theorem is a consequence of (107) and (113).

(115) (i) the interval between the fifth degree of heptatonic scale of *fr* in H and (the second degree of heptatonic scale of (the octave of *fr* in H) in $H) = \frac{(3 \textbf{ qua } \text{real number})}{2}$, and

(ii) the interval between the sixth degree of heptatonic scale of *fr* in H and (the third degree of heptatonic scale of (the octave of *fr* in H) in $H) = \frac{(3 \textbf{ qua } \text{real number})}{2}$, and

(iii) the interval between the seventh degree of heptatonic scale of *fr* in H and (the fourth degree of heptatonic scale of (the octave of *fr* in H) in $H) \neq \frac{(3 \textbf{ qua } \text{real number})}{2}$, and

(iv) the interval between the eight degree of heptatonic scale of fr in H and (the fifth degree of heptatonic scale of (the octave of fr in H) in H) $= \frac{(3 \textbf{ qua } \text{real number})}{2}$.

The theorem is a consequence of (107) and (113).

(116) Let us consider an euclidean heptatonic pythagorean score H, and elements f_1, f_2 of H. Then the interval between f_1 and $f_2 =$ (the Ratio of H)(f_1, f_2).

(117) Let us consider an euclidean heptatonic pythagorean score H, and an element fr of H. Then

(i) $\langle$(the heptatonic pythagorean scale of fr in H)(5), (the heptatonic pythagorean scale ascending of fr in H)(2)$\rangle$, $\langle$(the heptatonic pythagorean scale of fr in H)(6), (the heptatonic pythagorean scale ascending of fr in H)(3)$\rangle \in$ the set of fifth of H, and

(ii) $\langle$(the heptatonic pythagorean scale of fr in H)(7), (the heptatonic pythagorean scale ascending of fr in H)(4)$\rangle \notin$ the set of fifth of H.

The theorem is a consequence of (115), (24), and (116).

Let H be a space of music, n be a non zero, natural number, s_{10} be an element of (the carrier of $H)^n$, and i be a natural number. The functor $\#_i^{s_{10}}$ yielding an element of H is defined by the term

(Def. 91) $\begin{cases} s_{10}(i), & \textbf{if } i \in \operatorname{Seg} n, \\ \text{the element of } H, & \textbf{otherwise}. \end{cases}$

Assume s_{10} is heptatonic. We say that s_{10} is dorian if and only if

(Def. 92) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_1$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_2$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_1$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_1$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_1$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_2$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_1$.

Assume s_{10} is heptatonic. We say that s_{10} is hypodorian if and only if

(Def. 93) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_1$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_2$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_1$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_1$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_2$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_1$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_1$.

Assume s_{10} is heptatonic. We say that s_{10} is phrygian if and only if

(Def. 94) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_1$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_2$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_1$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_2$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_1$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_1$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_1$.

Assume s_{10} is heptatonic. We say that s_{10} is hypophrygian if and only if

(Def. 95) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_2$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_1$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_1$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_2$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_1$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_1$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_1$.

Assume s_{10} is heptatonic. We say that s_{10} is lydian if and only if

(Def. 96) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_1$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_1$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_2$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_1$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_1$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_2$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_1$.

Assume s_{10} is heptatonic. We say that s_{10} is hypolydian if and only if

(Def. 97) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_1$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_1$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_2$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_1$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_1$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_1$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_2$.

Assume s_{10} is heptatonic. We say that s_{10} is mixolydian if and only if

(Def. 98) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_1$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_1$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_2$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_1$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_1$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_2$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_1$.

Assume s_{10} is heptatonic. We say that s_{10} is hypomixolydian if and only if

(Def. 99) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_1$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_2$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_1$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_1$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_1$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_2$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_1$.

Assume s_{10} is heptatonic. We say that s_{10} is eolian if and only if

(Def. 100) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_1$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_2$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_1$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_1$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_2$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_1$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_1$.

Assume s_{10} is heptatonic. We say that s_{10} is hypoeolian if and only if

(Def. 101) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_2$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_1$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_1$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_1$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_2$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_1$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_1$.

Assume s_{10} is heptatonic. We say that s_{10} is ionan if and only if

(Def. 102) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_1$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_1$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_2$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_1$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_1$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_1$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_2$.

Assume s_{10} is heptatonic. We say that s_{10} is hypoionan if and only if

(Def. 103) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_1$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_1$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_2$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_1$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_1$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_2$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_1$.

Now we state the proposition:

(118) The heptatonic pythagorean scale of *fr* in H is ionan. The theorem is a consequence of (108), (107), and (111).

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Fundamental Properties of Fuzzy Implications

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Summary. In the article we continue in the Mizar system [8], [2] the formalization of fuzzy implications according to the monograph of Baczyński and Jayaram *"Fuzzy Implications"* [1]. We develop a framework of Mizar attributes allowing us for a smooth proving of basic properties of these fuzzy connectives [9]. We also give a set of theorems about the ordering of nine fundamental implications: Łukasiewicz ($I_{\rm LK}$), Gödel ($I_{\rm GD}$), Reichenbach ($I_{\rm RC}$), Kleene-Dienes ($I_{\rm KD}$), Goguen ($I_{\rm GG}$), Rescher ($I_{\rm RS}$), Yager ($I_{\rm YG}$), Weber ($I_{\rm WB}$), and Fodor ($I_{\rm FD}$).

This work is a continuation of the development of fuzzy sets in Mizar [6]; it could be used to give a variety of more general operations on fuzzy sets [13]. The formalization follows [10], [5], and [4].

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0. Introduction

There are two fundamental aims of this Mizar article: first of all, I wanted to introduce in the Mizar Mathematical Library how nine basic fuzzy implications formally defined in [4] are ordered – and this result is given in Section 2 as a formal counterpart of Example 1.1.6, p. 3 of [1].

On the other hand, in the final section I prove the formal characterization of fundamental fuzzy implications in terms of four elementary properties [12] expressed in Table 1.4 of [1], p. 10 (note the absence of the continuity of the operators in our version of this presentation). Here

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- (NP) – the left neutrality property,
- (EP) – the exchange principle,
- (IP) – the identity principle,
- (OP) – the ordering property.

Actually, this is the part of Example 1.3.2, p. 9 from [1]:

Fuzzy implication	(NP)	(EP)	(IP)	(OP)
I_{LK}	+	+	+	+
I_{GD}	+	+	+	+
I_{RC}	+	+	−	−
I_{KD}	+	+	−	−
I_{GG}	+	+	+	+
I_{RS}	−	−	+	+
I_{YG}	+	+	−	−
I_{WB}	+	+	+	−
I_{FD}	+	+	+	+

Additionally, Section 4 contains registrations of clusters of adjectives allowing for further work in more automated framework within fuzzy sets [3] – this is the Mizar version of Lemma 1.3.3 and 1.3.4 from [1]. Such automatization can be especially useful in the hybridization of fuzzy and rough approaches [7].

1. Preliminaries

We introduce the notation I_{LK} as a synonym of the Łukasiewicz implication and I_{GD} as a synonym of the Gödel implication. We introduce I_{RC} as a synonym of the Reichenbach implication and I_{KD} as a synonym of the Kleene-Dienes implication.

We introduce I_{GG} as a synonym of the Goguen implication and I_{RS} as a synonym of the Rescher implication. We introduce I_{YG} as a synonym of the Yager implication and I_{WB} as a synonym of the Weber implication and I_{FD} as a synonym of the Fodor implication.

From now on x, y denote elements of $[0,1]$. Now we state the propositions:

(1) $\square^1 = (\mathrm{AffineMap}(1,0)) \restriction]0,+\infty[$.

PROOF: Set $f = \square^1$. Set $g = (\mathrm{AffineMap}(1,0)) \restriction]0,+\infty[$. For every object x such that $x \in \mathrm{dom}\, f$ holds $f(x) = g(x)$. $\square$

(2) Let us consider real numbers a, b. Then

(i) AffineMap(a, b) is differentiable on $\mathbb{R}$, and

(ii) for every real number x, $(\text{AffineMap}(a, b))'(x) = a$.

(3) If $0 < x < 1$ and $0 < y < 1$, then $(\square^x + (\text{AffineMap}(-x, x-1)))\restriction]0,1[$ is increasing.

PROOF: Set $f_1 = \square^x$. Set $f_2 = \text{AffineMap}(-x, x-1)$. Reconsider $Y =]0,1[$ as an open subset of $\mathbb{R}$. Set $f = f_1 + f_2$. Set $A =]0,+\infty[$. f_2 is differentiable on A. $f_1\restriction A$ is differentiable on A. f_2 is differentiable on Y. For every real number y such that $y \in Y$ holds $0 < f'(y)$ by [11, (21)], (2). $\square$

(4) Let us consider a real number u. Suppose $u \in]0,1]$.
Then $(\square^x + (\text{AffineMap}(-x, x-1)))(u) = u^x - 1 + x - x \cdot u$.

2. The Ordering of Fuzzy Implications

Now we state the propositions:

(5) (i) if $x \leqslant y$, then $(I_{\text{LK}})(x, y) = 1$, and

(ii) if $x > y$, then $(I_{\text{LK}})(x, y) = 1 - x + y$.

(6) (i) if $x = 0$, then $(I_{\text{GG}})(x, y) = 1$, and

(ii) if $x > 0$, then $(I_{\text{GG}})(x, y) = \min(1, \frac{y}{x})$.

(7) $I_{\text{KD}} \leqslant I_{\text{RC}} \leqslant I_{\text{LK}} \leqslant I_{\text{WB}}$.

(8) $I_{\text{RS}} \leqslant I_{\text{GD}} \leqslant I_{\text{GG}} \leqslant I_{\text{LK}} \leqslant I_{\text{WB}}$.

(9) $I_{\text{RC}} \leqslant I_{\text{LK}} \leqslant I_{\text{WB}}$.

(10) $I_{\text{KD}} \leqslant I_{\text{FD}} \leqslant I_{\text{LK}} \leqslant I_{\text{WB}}$.

(11) $I_{\text{RS}} \leqslant I_{\text{GD}} \leqslant I_{\text{FD}} \leqslant I_{\text{LK}} \leqslant I_{\text{WB}}$.

3. Additional Properties of Fuzzy Implications

Let I be a binary operation on $[0,1]$. We say that I satisfies (NP) if and only if

(Def. 1) for every element y of $[0,1]$, $I(1, y) = y$.

We say that I satisfies (EP) if and only if

(Def. 2) for every elements x, y, z of $[0,1]$, $I(x, I(y, z)) = I(y, I(x, z))$.

We say that I satisfies (IP) if and only if

(Def. 3) for every element x of $[0,1]$, $I(x, x) = 1$.

We say that I satisfies (OP) if and only if

(Def. 4) for every elements x, y of $[0,1]$, $I(x, y) = 1$ iff $x \leqslant y$.

In the sequel I denotes a binary operation on $[0, 1]$.

Let I be a binary operation on $[0, 1]$. We introduce the notation I satisfies (NC) as a synonym of I is 01-dominant and I satisfies (I1) as a synonym of I is antitone w.r.t. 1st coordinate.

We introduce I satisfies (I2) as a synonym of I is isotone w.r.t. 2nd coordinate and I satisfies (I3) as a synonym of I is 00-dominant and I satisfies (I4) as a synonym of I is 11-dominant and I satisfies (I5) as a synonym of I is 10-weak.

4. Dependencies between Chosen Properties

Now we state the proposition:

(12) If I satisfies (LB), then I satisfies (I3) and (NC).

One can verify that every binary operation on $[0, 1]$ which satisfies (LB) satisfies also (I3) and (NC).

Now we state the proposition:

(13) If I satisfies (RB), then I satisfies (I4) and (NC).

One can check that every binary operation on $[0, 1]$ which satisfies (RB) satisfies also (I4) and (NC).

Now we state the proposition:

(14) If I satisfies (NP), then I satisfies (I4) and (I5).

Note that every binary operation on $[0, 1]$ which satisfies (NP) satisfies also (I4) and (I5).

Now we state the proposition:

(15) If I satisfies (IP), then I satisfies (I3) and (I4).

Let us note that every binary operation on $[0, 1]$ which satisfies (IP) satisfies also (I3) and (I4).

Now we state the proposition:

(16) If I satisfies (OP), then I satisfies (I3), (I4), (NC), (LB), (RB), and (IP).

One can verify that every binary operation on $[0, 1]$ which satisfies (OP) satisfies also (I3), (I4), (NC), (LB), (RB), and (IP).

Now we state the proposition:

(17) If I satisfies (EP) and (OP), then I satisfies (I1), (I3), (I4), (I5), (LB), (RB), (NC), (NP), and (IP).

One can verify that every binary operation on $[0, 1]$ which satisfies (EP) and (OP) satisfies also (I1), (I5), and (NP).

5. Properties of Nine Classical Fuzzy Implications

Let us note that I_{LK} satisfies (NP), (EP), (IP), and (OP).

I_{GD} satisfies (NP), (EP), (IP), and (OP).

I_{RC} satisfies (NP) and (EP) but does not satisfy (IP) and (OP).

I_{KD} satisfies (NP) and (EP) but does not satisfy (IP) and (OP).

I_{GG} satisfies (NP), (EP), (IP), and (OP).

Let us note that I_{RS} satisfies (IP) and (OP) but does not satisfy (NP) and (EP).

I_{YG} satisfies (NP) and (EP) but does not satisfy (IP) and (OP).

I_{WB} satisfies (NP), (EP), and (IP) but does not satisfy (OP).

I_{FD} satisfies (NP), (EP), (IP), and (OP).

I_0 satisfies (EP) but does not satisfy (NP), (IP), and (OP).

I_1 satisfies (EP) and (IP) but does not satisfy (NP) and (OP).

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Zariski Topology

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Summary. We formalize in the Mizar system [3], [4] basic definitions of commutative ring theory such as prime spectrum, nilradical, Jacobson radical, local ring, and semi-local ring [5], [6], then formalize proofs of some related theorems along with the first chapter of [1].

The article introduces the so-called Zariski topology. The set of all prime ideals of a commutative ring A is called the prime spectrum of A denoted by Spectrum A. A new functor Spec generates Zariski topology to make Spectrum A a topological space. A different role is given to Spec as a map from a ring morphism of commutative rings to that of topological spaces by the following manner: for a ring homomorphism $h : A \longrightarrow B$, we defined (Spec h) : Spec $B \longrightarrow$ Spec A by (Spec h)($\mathfrak{p}$) $= h^{-1}(\mathfrak{p})$ where $\mathfrak{p} \in$ Spec B.

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1. Preliminaries: Some Properties of Ideals

From now on R denotes a commutative ring, A, B denote non degenerated, commutative rings, h denotes a function from A into B, I, I_1, I_2 denote ideals of A, J, J_1, J_2 denote proper ideals of A, p denotes a prime ideal of A.

S denotes non empty subset of A, E, E_1, E_2 denote subsets of A, a, b, f denote elements of A, n denotes a natural number, and x denotes object.

Let us consider A and S. The functor Ideals(A, S) yielding a subset of Ideals A is defined by the term

(Def. 1) $\{I$, where I is an ideal of $A : S \subseteq I\}$.

Let us observe that Ideals(A, S) is non empty.

Now we state the proposition:

(1) Ideals(A, S) = Ideals$(A, S$–ideal$)$.

PROOF: Ideals$(A, S) \subseteq$ Ideals$(A, S$–ideal$)$. Consider y being an ideal of A such that $x = y$ and S–ideal $\subseteq y$. □

Let A be a unital, non empty multiplicative loop with zero structure and a be an element of A. We say that a is nilpotent if and only if

(Def. 2) there exists a non zero natural number k such that $a^k = 0_A$.

Let us note that 0_A is nilpotent and there exists an element of A which is nilpotent.

Let us consider A. Observe that 1_A is non nilpotent.

Let us consider f. The functor MultClSet(f) yielding a subset of A is defined by the term

(Def. 3) the set of all f^i where i is a natural number.

Let us observe that MultClSet(f) is multiplicatively closed.

Now we state the propositions:

(2) Let us consider a natural number n. Then $(1_A)^n = 1_A$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (1_A)^{\$_1} = 1_A$. For every natural number n, $\mathcal{P}[n]$. □

(3) $1_A \notin \sqrt{J}$. The theorem is a consequence of (2).

(4) MultClSet$(1_A) = \{1_A\}$. The theorem is a consequence of (2).

Let us consider A, J, and f. The functor Ideals(A, J, f) yielding a subset of Ideals A is defined by the term

(Def. 4) $\{I$, where I is a subset of A : I is a proper ideal of A and $J \subseteq I$ and $I \cap$ MultClSet$(f) = \emptyset\}$.

Let us consider A, J, and f. Now we state the propositions:

(5) If $f \notin \sqrt{J}$, then $J \in$ Ideals(A, J, f).

(6) If $f \notin \sqrt{J}$, then Ideals(A, J, f) has the upper Zorn property w.r.t. $\subseteq_{\text{Ideals}(A,J,f)}$.

PROOF: Set $S =$ Ideals(A, J, f). Set $P = \subseteq_S$. For every set Y such that $Y \subseteq S$ and $P\,|^2\, Y$ is a linear order there exists a set x such that $x \in S$ and for every set y such that $y \in Y$ holds $\langle y, x \rangle \in P$. □

(7) If $f \notin \sqrt{J}$, then there exists a prime ideal m of A such that $f \notin m$ and $J \subseteq m$.

PROOF: Set $S =$ Ideals(A, J, f). Set $P = \subseteq_S$. Consider I being a set such that I is maximal in P. Consider p being a subset of A such that $p = I$ and p is a proper ideal of A and $J \subseteq p$ and $p \cap$ MultClSet$(f) = \emptyset$. p is a quasi-prime ideal of A. □

(8) There exists a maximal ideal m of A such that $J \subseteq m$.
Proof: $1_A \notin \sqrt{J}$. Set $S = \mathrm{Ideals}(A, J, 1_A)$. Set $P = \subseteq_S$. Consider I being a set such that I is maximal in P. Consider p being a subset of A such that $p = I$ and p is a proper ideal of A and $J \subseteq p$ and $p \cap \mathrm{MultClSet}(1_A) = \emptyset$. For every ideal q of A such that $p \subseteq q$ holds $q = p$ or q is not proper. □

(9) There exists a prime ideal m of A such that $J \subseteq m$. The theorem is a consequence of (8).

(10) If a is a non-unit of A, then there exists a maximal ideal m of A such that $a \in m$. The theorem is a consequence of (8).

2. Spectrum of Prime Ideals (Spectrum) and Maximal Ideals (m-Spectrum)

Let R be a commutative ring. The spectrum of R yielding a family of subsets of R is defined by the term

(Def. 5) $\begin{cases} \{I, \text{ where } I \text{ is an ideal of } R : I \text{ is quasi-prime and } I \neq \Omega_R\}, \\ \quad \textbf{if } R \text{ is not degenerated}, \\ \emptyset, \ \textbf{otherwise}. \end{cases}$

Let us consider A. Observe that the spectrum of A yields a family of subsets of A and is defined by the term

(Def. 6) the set of all I where I is a prime ideal of A.

Observe that the spectrum of A is non empty.

Let us consider R. The functor m-Spectrum(R) yielding a family of subsets of R is defined by the term

(Def. 7) $\begin{cases} \{I, \text{ where } I \text{ is an ideal of } R : I \text{ is quasi-maximal and } I \neq \Omega_R\}, \\ \quad \textbf{if } R \text{ is not degenerated}, \\ \emptyset, \ \textbf{otherwise}. \end{cases}$

Let us consider A. Observe that the functor m-Spectrum(A) yields a family of subsets of the carrier of A and is defined by the term

(Def. 8) the set of all I where I is a maximal ideal of A.

Observe that m-Spectrum(A) is non empty.

3. Local and Semi-Local Ring

Let us consider A. We say that A is local if and only if

(Def. 9) there exists an ideal m of A such that m-Spectrum$(A) = \{m\}$.

We say that A is semi-local if and only if

(Def. 10) m-Spectrum(A) is finite.

Now we state the propositions:

(11) If $x \in I$ and I is a proper ideal of A, then x is a non-unit of A.

(12) If for every objects m_1, m_2 such that m_1, $m_2 \in$ m-Spectrum(A) holds $m_1 = m_2$, then A is local.

(13) If for every x such that $x \in \Omega_A \setminus J$ holds x is a unit of A, then A is local. The theorem is a consequence of (8), (11), and (12).

In the sequel m denotes a maximal ideal of A. Now we state the propositions:

(14) If $a \in \Omega_A \setminus m$, then $\{a\}$–ideal $+ m = \Omega_A$.

(15) If for every a such that $a \in m$ holds $1_A + a$ is a unit of A, then A is local.

PROOF: For every x such that $x \in \Omega_A \setminus m$ holds x is a unit of A. □

Let us consider R. Let E be a subset of R. The functor PrimeIdeals(R, E) yielding a subset of the spectrum of R is defined by the term

(Def. 11) $\begin{cases} \{p, \text{ where } p \text{ is an ideal of } R : p \text{ is quasi-prime and } p \neq \Omega_R \text{ and } E \subseteq p\}, \\ \quad \textbf{if } R \text{ is not degenerated,} \\ \emptyset, \ \textbf{otherwise}. \end{cases}$

Let us consider A. Let E be a subset of A. Let us note that the functor PrimeIdeals(A, E) yields a subset of the spectrum of A and is defined by the term

(Def. 12) $\{p$, where p is a prime ideal of $A : E \subseteq p\}$.

Let us consider J. Observe that PrimeIdeals(A, J) is non empty.

From now on p denotes a prime ideal of A and k denotes a non zero natural number. Now we state the proposition:

(16) If $a \notin p$, then $a^k \notin p$.

4. Nilradical and Jacobson Radical

Let us consider A. The functor nilrad(A) yielding a subset of A is defined by the term

(Def. 13) the set of all a where a is a nilpotent element of A.

Now we state the proposition:

(17) nilrad(A) $= \sqrt{\{0_A\}}$.

Let us consider A. One can verify that nilrad(A) is non empty and nilrad(A) is closed under addition as a subset of A and nilrad(A) is left and right ideal as a subset of A.

Now we state the propositions:

(18) $\sqrt{J} = \bigcap \mathrm{PrimeIdeals}(A, J)$. The theorem is a consequence of (16), (7), and (9).

(19) $\mathrm{nilrad}(A) = \bigcap(\text{the spectrum of } A)$. The theorem is a consequence of (17) and (18).

(20) $I \subseteq \sqrt{I}$.

(21) If $I \subseteq J$, then $\sqrt{I} \subseteq \sqrt{J}$.

Proof: Consider s_1 being an element of A such that $s_1 = s$ and there exists an element n of $\mathbb{N}$ such that ${s_1}^n \in I$. Consider n_1 being an element of $\mathbb{N}$ such that ${s_1}^{n_1} \in I$. $n_1 \neq 0$ by [7, (8)], [2, (19)]. $\square$

Let us consider A. The functor J-Rad(A) yielding a subset of A is defined by the term

(Def. 14) $\bigcap \text{m-Spectrum}(A)$.

5. Construction of Zariski Topology of the Prime Spectrum of A

Now we state the propositions:

(22) $\mathrm{PrimeIdeals}(A, S) \subseteq \mathrm{Ideals}(A, S)$.

(23) $\mathrm{PrimeIdeals}(A, S) = \mathrm{Ideals}(A, S) \cap (\text{the spectrum of } A)$. The theorem is a consequence of (22).

(24) $\mathrm{PrimeIdeals}(A, S) = \mathrm{PrimeIdeals}(A, S\text{–ideal})$. The theorem is a consequence of (23) and (1).

(25) If $I \subseteq p$, then $\sqrt{I} \subseteq p$.

Proof: Consider s_1 being an element of A such that $s_1 = s$ and there exists an element n of $\mathbb{N}$ such that ${s_1}^n \in I$. Consider n_1 being an element of $\mathbb{N}$ such that ${s_1}^{n_1} \in I$. $n_1 \neq 0$. $\square$

(26) If $\sqrt{I} \subseteq p$, then $I \subseteq p$. The theorem is a consequence of (20).

(27) $\mathrm{PrimeIdeals}(A, \sqrt{S\text{–ideal}}) = \mathrm{PrimeIdeals}(A, S\text{–ideal})$. The theorem is a consequence of (26) and (25).

(28) If $E_2 \subseteq E_1$, then $\mathrm{PrimeIdeals}(A, E_1) \subseteq \mathrm{PrimeIdeals}(A, E_2)$.

(29) $\mathrm{PrimeIdeals}(A, J_1) = \mathrm{PrimeIdeals}(A, J_2)$ if and only if $\sqrt{J_1} = \sqrt{J_2}$. The theorem is a consequence of (18) and (27).

(30) If $I_1 * I_2 \subseteq p$, then $I_1 \subseteq p$ or $I_2 \subseteq p$.

Proof: If it is not true that $I_1 \subseteq p$ or $I_2 \subseteq p$, then $I_1 * I_2 \not\subseteq p$. $\square$

(31) $\mathrm{PrimeIdeals}(A, \{1_A\}) = \emptyset$.

(32) The spectrum of $A = \mathrm{PrimeIdeals}(A, \{0_A\})$.

(33) Let us consider non empty subsets E_1, E_2 of A. Then there exists a non empty subset E_3 of A such that $\mathrm{PrimeIdeals}(A, E_1) \cup \mathrm{PrimeIdeals}(A, E_2) = \mathrm{PrimeIdeals}(A, E_3)$.

PROOF: Set $I_1 = E_1$–ideal. Set $I_2 = E_2$–ideal. Reconsider $I_3 = I_1 * I_2$ as an ideal of A. PrimeIdeals(A, E_1) = PrimeIdeals(A, I_1). PrimeIdeals(A, I_3) $\subseteq$ PrimeIdeals$(A, I_1)\cup$PrimeIdeals(A, I_2). PrimeIdeals$(A, I_1)\cup$PrimeIdeals $(A, I_2) \subseteq$ PrimeIdeals(A, I_3). PrimeIdeals(A, I_3) = PrimeIdeals$(A, E_1) \cup$ PrimeIdeals(A, E_2). □

(34) Let us consider a family G of subsets of the spectrum of A. Suppose for every set S such that $S \in G$ there exists a non empty subset E of A such that S = PrimeIdeals(A, E). Then there exists a non empty subset F of A such that Intersect(G) = PrimeIdeals(A, F). The theorem is a consequence of (28).

Let us consider A. The functor Spec(A) yielding a strict topological space is defined by

(Def. 15) the carrier of *it* = the spectrum of A and for every subset F of *it*, F is closed iff there exists a non empty subset E of A such that F = PrimeIdeals(A, E).

Note that Spec(A) is non empty. Now we state the proposition:

(35) Let us consider points P, Q of Spec(A). Suppose $P \neq Q$. Then there exists a subset V of Spec(A) such that

(i) V is open, and

(ii) $P \in V$ and $Q \notin V$ or $Q \in V$ and $P \notin V$.

Note that there exists a commutative ring which is degenerated. Let R be a degenerated, commutative ring. Let us observe that ADTS(the spectrum of R) is T_0. Let us consider A. Observe that Spec(A) is T_0.

6. Continous Map of Zariski Topology Associated with a Ring Homomorphism

From now on M_0 denotes an ideal of B. Now we state the proposition:

(36) If h inherits ring homomorphism, then $h^{-1}(M_0)$ is an ideal of A.

In the sequel M_0 denotes a prime ideal of B.

(37) If h inherits ring homomorphism, then $h^{-1}(M_0)$ is a prime ideal of A.
PROOF: For every elements x, y of A such that $x \cdot y \in h^{-1}(M_0)$ holds $x \in h^{-1}(M_0)$ or $y \in h^{-1}(M_0)$. $h^{-1}(M_0) \neq$ the carrier of A. □

Let us consider A, B, and h. Assume h inherits ring homomorphism. The functor Spec(h) yielding a function from Spec(B) into Spec(A) is defined by

(Def. 16) for every point x of Spec(B), $it(x) = h^{-1}(x)$.

Now we state the propositions:

(38) If h inherits ring homomorphism, then $\mathrm{Spec}(h)^{-1}\,\mathrm{PrimeIdeals}(A, E) = \mathrm{PrimeIdeals}(B, h^{\circ}E)$.
Proof: $\mathrm{Spec}(h)^{-1}\,\mathrm{PrimeIdeals}(A, E) \subseteq \mathrm{PrimeIdeals}(B, h^{\circ}E)$. Consider q being a prime ideal of B such that $x = q$ and $h^{\circ}E \subseteq q$. $h^{-1}(q)$ is a prime ideal of A. □

(39) If h inherits ring homomorphism, then $\mathrm{Spec}(h)$ is continuous. The theorem is a consequence of (38).

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