FORMALIZED MATHEMATICS
Vol. 26, No. 4, Pages 269–272, 2018
DOI: 10.2478/forma-2018-0023

sciendo
https://www.sciendo.com/

Fundamental Properties of Fuzzy Implications

Adam Grabowski
Institute of Informatics
University of Białystok
Poland

Summary. In the article we continue in the Mizar system [1], [2] the formalization of fuzzy implications according to the book of Baczyński [???]. We develop a framework of Mizar attributes allowing us for a smooth proving of basic properties of impliactions. We also give a set of theorems about the ordering of these important fuzzy operators.

This work is a continuation of the development of fuzzy sets in Mizar [4]; it could be used to give a variety of more general operations on fuzzy sets.

MSC: 68T99 03B35

Keywords:

MML identifier: FUZIMPL2, version: 8.1.08 5.53.1335

1. Preliminaries

We introduce the notation I-LK as a synonym of the Łukasiewicz implication and I-GD as a synonym of the Gödel implication and I-RC as a synonym of the Reichenbach implication and I-KD as a synonym of the Kleene-Dienes implication and I-GG as a synonym of the Goguen implication and I-RS as a synonym of the Rescher implication and I-YG as a synonym of the Yager implication and I-WB as a synonym of the Weber implication and I-FD as a synonym of the Fodor implication.

From now on x, y denote elements of $[0, 1]$.

Now we state the propositions:

ISSN 1426–2630(Print), 1898-9934(Online)

(1) $\square^1 = (\mathrm{AffineMap}(1,0)) \restriction]0,+\infty[$.
PROOF: Set $f = \square^1$. Set $g = (\mathrm{AffineMap}(1,0)) \restriction]0,+\infty[$. For every object x such that $x \in \mathrm{dom}\, f$ holds $f(x) = g(x)$ by [5, (72)], [3, (49)]. $\square$

(2) Let us consider real numbers a, b. Then

(i) $\mathrm{AffineMap}(a,b)$ is differentiable on $\mathbb{R}$, and

(ii) for every real number x, $(\mathrm{AffineMap}(a,b))'(x) = a$.

(3) If $0 < x < 1$ and $0 < y < 1$, then $(\square^x + (\mathrm{AffineMap}(-x, x-1))) \restriction]0,1[$ is increasing.
PROOF: Set $f_1 = \square^x$. Set $f_2 = \mathrm{AffineMap}(-x, x-1)$. Reconsider $Y =]0,1[$ as an open subset of $\mathbb{R}$. Set $f = f_1 + f_2$. Set $A =]0,+\infty[$. f_2 is differentiable on A. $f_1 \restriction A$ is differentiable on A by [8, (21)], [7, (18), (17)], [3, (49)]. f_2 is differentiable on Y. For every real number y such that $y \in Y$ holds $0 < f'(y)$ by [6, (9), (13)], [8, (21)], (2). $\square$

(4) Let us consider a real number u. Suppose $u \in]0,1]$. Then $(\square^x + (\mathrm{AffineMap}(-x, x-1)))(u) = u^x - 1 + x - x \cdot u$.

2. The Ordering of Fuzzy Implications

Now we state the propositions:

(5) (i) if $x \leqslant y$, then $(\text{I-LK})(x,y) = 1$, and

(ii) if $x > y$, then $(\text{I-LK})(x,y) = 1 - x + y$.

(6) (i) if $x = 0$, then $(\text{I-GG})(x,y) = 1$, and

(ii) if $x > 0$, then $(\text{I-GG})(x,y) = \min(1, \frac{y}{x})$.

Now we state the propositions:

(7) I-KD $\leqslant$ I-RC $\leqslant$ I-LK $\leqslant$ I-WB.

(8) I-RS $\leqslant$ I-GD $\leqslant$ I-GG $\leqslant$ I-LK $\leqslant$ I-WB.

(9) I-RC $\leqslant$ I-LK $\leqslant$ I-WB.

(10) I-KD $\leqslant$ I-FD $\leqslant$ I-LK $\leqslant$ I-WB.

(11) I-RS $\leqslant$ I-GD $\leqslant$ I-FD $\leqslant$ I-LK $\leqslant$ I-WB.

3. Additional Properties of Fuzzy Implications

Let I be a binary operation on $[0,1]$. We say that I is satisfying(NP) if and only if

(Def. 1) for every element y of $[0,1]$, $I(1,y) = y$.

We say that I is satisfying(EP) if and only if

(Def. 2) for every elements x, y, z of $[0,1]$, $I(x, I(y,z)) = I(y, I(x,z))$.

We say that I is satisfying(IP) if and only if

(Def. 3) for every element x of $[0,1]$, $I(x,x) = 1$.

We say that I is satisfying(OP) if and only if

(Def. 4) for every elements x, y of $[0,1]$, $I(x,y) = 1$ iff $x \leqslant y$.

In the sequel I denotes a binary operation on $[0,1]$.

Let I be a binary operation on $[0,1]$. We introduce the notation I is satisfying(NC) as a synonym of I is 01-dominant and I is satisfying(I1) as a synonym of I is antitone w.r.t. 1st coordinate and I is satisfying(I2) as a synonym of I is isotone w.r.t. 2nd coordinate and I is satisfying(I3) as a synonym of I is 00-dominant and I is satisfying(I4) as a synonym of I is 11-dominant and I is satisfying(I5) as a synonym of I is 10-weak.

Now we state the proposition:

(12) If I satisfies (LB), then I is satisfying(I3) and satisfying(NC).

One can check that every binary operation on $[0,1]$ which satisfies (LB) is also satisfying(I3) and satisfying(NC).

Now we state the proposition:

(13) If I satisfies (RB), then I is satisfying(I4) and satisfying(NC).

Note that every binary operation on $[0,1]$ which satisfies (RB) is also satisfying(I4) and satisfying(NC).

Now we state the proposition:

(14) If I is satisfying(NP), then I is satisfying(I4) and satisfying(I5).

Let us note that every binary operation on $[0,1]$ which is satisfying(NP) is also satisfying(I4) and satisfying(I5).

Now we state the proposition:

(15) If I is satisfying(IP), then I is satisfying(I3) and satisfying(I4).

Observe that every binary operation on $[0,1]$ which is satisfying(IP) is also satisfying(I3) and satisfying(I4).

Now we state the proposition:

(16) If I is satisfying(OP), then I is satisfying(I3), satisfying(I4), satisfying(NC), and satisfying(IP) and satisfies (LB) and (RB).

One can check that every binary operation on $[0,1]$ which is satisfying(OP) is also satisfying(I3), satisfying(I4), satisfying(NC), and satisfying(IP) and satisfies also (LB) and (RB).

Now we state the proposition:

(17) Suppose I is satisfying(EP) and satisfying(OP). Then I is satisfying(I1), satisfying(I3), satisfying(I4), satisfying(I5), satisfying(NC), satisfying(NP),

and satisfying(IP) and satisfies (LB) and (RB).

One can check that every binary operation on $[0, 1]$ which is satisfying(EP) and satisfying(OP) is also satisfying(I1), satisfying(I5), and satisfying(NP) and I-LK is satisfying(NP), satisfying(EP), satisfying(IP), and satisfying(OP) and I-GD is satisfying(NP), satisfying(EP), satisfying(IP), and satisfying(OP) and I-RC is satisfying(NP), satisfying(EP), non satisfying(IP), and non satisfying(OP) and I-KD is satisfying(NP), satisfying(EP), non satisfying(IP), and non satisfying(OP) and I-GG is satisfying(NP), satisfying(EP), satisfying(IP), and satisfying(OP) and I-RS is non satisfying(NP), non satisfying(EP), satisfying(IP), and satisfying(OP) and I-YG is satisfying(NP), satisfying(EP), non satisfying(IP), and non satisfying(OP) and I-WB is satisfying(NP), satisfying(EP), satisfying(IP), and non satisfying(OP) and I-FD is satisfying(NP), satisfying(EP), satisfying(IP), and satisfying(OP) and I_0 is non satisfying(NP), satisfying(EP), non satisfying(IP), and non satisfying(OP) and I_1 is non satisfying(NP), satisfying(EP), satisfying(IP), and non satisfying(OP).

References

[1] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.

[2] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, and Karol Pąk. The role of the Mizar Mathematical Library for interactive proof development in Mizar. *Journal of Automated Reasoning*, 61(1):9–32, 2018. doi:10.1007/s10817-017-9440-6.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**): 55–65, 1990.

[4] Adam Grabowski. On the computer certification of fuzzy numbers. In M. Ganzha, L. Maciaszek, and M. Paprzycki, editors, *2013 Federated Conference on Computer Science and Information Systems (FedCSIS)*, Federated Conference on Computer Science and Information Systems, pages 51–54, 2013.

[5] Konrad Raczkowski. Integer and rational exponents. *Formalized Mathematics*, 2(**1**):125–130, 1991.

[6] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Formalized Mathematics*, 1(**4**):797–801, 1990.

[7] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.

[8] Yasunari Shidama. The Taylor expansions. *Formalized Mathematics*, 12(**2**):195–200, 2004.

Accepted September 29, 2018