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Continuity of Bounded Linear Operators on Normed Linear Spaces[1]

Kazuhisa Nakasho
Yamaguchi University
Yamaguchi, Japan

Yuichi Futa
Tokyo University of Technology
Tokyo, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article, using the Mizar system [1], [2], we discuss on the continuity of bounded linear operators on normed linear spaces. In the first section, it is discussed that bounded linear operators on normed linear spaces are uniformly continuous and Lipschitz continuous. Especially, a bounded linear operator on the dense subset of a complete normed linear space has a unique natural extension over the whole space. In the next section, several basic currying properties are formalized.

In the last section, we formalized that continuity of bilineaer operator is equivalent to both Lipschitz continuity and local continuity. We referred to [7], [21], and [6] in this formalization.

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1. Uniform Continuity and Lipschitz Continuity of Bounded Linear Operators

From now on S, T, W, Y denote real normed spaces, f, f_1, f_2 denote partial functions from S to T, Z denotes a subset of S, and i, n denote natural numbers.

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Now we state the propositions:

(1) Let us consider real normed spaces E, F, a subset E_1 of E, and a partial function f from E to F. Suppose E_1 is dense and F is complete and $\mathrm{dom}\, f = E_1$ and f is uniformly continuous on E_1. Then

(i) there exists a function g from E into F such that $g \restriction E_1 = f$ and g is uniformly continuous on the carrier of E and for every point x of E, there exists a sequence s_0 of E such that $\mathrm{rng}\, s_0 \subseteq E_1$ and s_0 is convergent and $\lim s_0 = x$ and $f_* s_0$ is convergent and $g(x) = \lim(f_* s_0)$ and for every point x of E and for every sequence s_0 of E such that $\mathrm{rng}\, s_0 \subseteq E_1$ and s_0 is convergent and $\lim s_0 = x$ holds $f_* s_0$ is convergent and $g(x) = \lim(f_* s_0)$, and

(ii) for every functions g_1, g_2 from E into F such that $g_1 \restriction E_1 = f$ and g_1 is continuous on the carrier of E and $g_2 \restriction E_1 = f$ and g_2 is continuous on the carrier of E holds $g_1 = g_2$.

PROOF: For every point x of E and for every sequence s_0 of E such that $\mathrm{rng}\, s_0 \subseteq E_1$ and s_0 is convergent for every real number s such that $0 < s$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $\|(f_* s_0)(m) - (f_* s_0)(n)\| < s$ by [17, (4)], [4, (4), (109)]. For every point x of E and for every sequence s_0 of E such that $\mathrm{rng}\, s_0 \subseteq E_1$ and s_0 is convergent holds $f_* s_0$ is convergent by [17, (5)]. For every point x of E and for every sequences s_1, s_2 of E such that $\mathrm{rng}\, s_1 \subseteq E_1$ and s_1 is convergent and $\lim s_1 = x$ and $\mathrm{rng}\, s_2 \subseteq E_1$ and s_2 is convergent and $\lim s_2 = x$ holds $\lim(f_* s_1) = \lim(f_* s_2)$ by [4, (11)], [11, (14)], [4, (4), (109)]. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a sequence s_0 of E such that $\mathrm{rng}\, s_0 \subseteq E_1$ and s_0 is convergent and $\lim s_0 = \$_1$ and $f_* s_0$ is convergent and $\$_2 = \lim(f_* s_0)$. For every element x of E, there exists an element y of F such that $\mathcal{P}[x, y]$ by [9, (14)]. Consider g being a function from E into F such that for every element x of E, $\mathcal{P}[x, g(x)]$ from [4, Sch. 3]. For every object x such that $x \in \mathrm{dom}\, f$ holds $f(x) = g(x)$ by [18, (7)], [9, (23)], [4, (11)], [13, (18)]. For every point x of E and for every sequence s_0 of E such that $\mathrm{rng}\, s_0 \subseteq E_1$ and s_0 is convergent and $\lim s_0 = x$ holds $f_* s_0$ is convergent and $g(x) = \lim(f_* s_0)$. For every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every points x_1, x_2 of E such that x_1, $x_2 \in$ the carrier of E and $\|x_1 - x_2\| < s$ holds $\|g_{/x_1} - g_{/x_2}\| < r$ by [20, (1)], [19, (28)], [15, (7)], [4, (4), (109)]. For every element x of E, $g_1(x) = g_2(x)$ by [9, (14)], [4, (117)], [13, (18)]. □

(2) Let us consider real normed spaces E, F, G, a point f of the real norm space of bounded linear operators from E into F, and a point g of the real

norm space of bounded linear operators from F into G. Then there exists a point h of the real norm space of bounded linear operators from E into G such that

(i) $h = g \cdot f$, and

(ii) $\|h\| \leqslant \|g\| \cdot \|f\|$.

PROOF: Reconsider $L_1 = f$ as a Lipschitzian linear operator from E into F. Reconsider $L_2 = g$ as a Lipschitzian linear operator from F into G. Set $L_3 = L_2 \cdot L_1$. For every real number t such that $t \in \mathrm{PreNorms}(L_3)$ holds $t \leqslant \|g\| \cdot \|f\|$ by [4, (15)], [16, (32)]. □

(3) Let us consider real normed spaces E, F. Then every Lipschitzian linear operator from E into F is Lipschitzian on the carrier of E and uniformly continuous on the carrier of E.

PROOF: Consider K being a real number such that $0 \leqslant K$ and for every vector x of E, $\|L(x)\| \leqslant K \cdot \|x\|$. Set $r = K + 1$. Set $E_0 =$ the carrier of E. For every points x_1, x_2 of E such that $x_1, x_2 \in E_0$ holds $\|L_{/x_1} - L_{/x_2}\| \leqslant r \cdot \|x_1 - x_2\|$ by [19, (16)]. □

(4) Let us consider real normed spaces E, F, a subreal normal space E_1 of E, and a point f of the real norm space of bounded linear operators from E_1 into F. Suppose F is complete and there exists a subset E_0 of E such that $E_0 =$ the carrier of E_1 and E_0 is dense. Then

(i) there exists a point g of the real norm space of bounded linear operators from E into F such that $\operatorname{dom} g =$ the carrier of E and $g{\restriction}(\text{the carrier of } E_1) = f$ and $\|g\| = \|f\|$ and there exists a partial function L_1 from E to F such that $L_1 = f$ and for every point x of E, there exists a sequence s_0 of E such that $\operatorname{rng} s_0 \subseteq$ the carrier of E_1 and s_0 is convergent and $\lim s_0 = x$ and $L_{1*}s_0$ is convergent and $g(x) = \lim(L_{1*}s_0)$ and for every point x of E and for every sequence s_0 of E such that $\operatorname{rng} s_0 \subseteq$ the carrier of E_1 and s_0 is convergent and $\lim s_0 = x$ holds $L_{1*}s_0$ is convergent and $g(x) = \lim(L_{1*}s_0)$, and

(ii) for every points g_1, g_2 of the real norm space of bounded linear operators from E into F such that $g_1{\restriction}(\text{the carrier of } E_1) = f$ and $g_2{\restriction}(\text{the carrier of } E_1) = f$ holds $g_1 = g_2$.

PROOF: Consider E_0 being a subset of E such that $E_0 =$ the carrier of E_1 and E_0 is dense. Reconsider $L = f$ as a Lipschitzian linear operator from E_1 into F. Reconsider $L_1 = L$ as a partial function from E to F. Consider K being a real number such that $0 \leqslant K$ and for every vector x of E_1, $\|L(x)\| \leqslant K \cdot \|x\|$. Set $r = K + 1$. For every points x_1, x_2 of E such that $x_1, x_2 \in E_0$ holds $\|L_{1/x_1} - L_{1/x_2}\| \leqslant r \cdot \|x_1 - x_2\|$ by

[9, (28)], [19, (16)]. there exists a function P_3 from E into F such that $P_3 \restriction E_0 = L_1$ and P_3 is uniformly continuous on the carrier of E and for every point x of E, there exists a sequence s_0 of E such that $\operatorname{rng} s_0 \subseteq E_0$ and s_0 is convergent and $\lim s_0 = x$ and $L_{1*}s_0$ is convergent and $P_3(x) = \lim(L_{1*}s_0)$ and for every point x of E and for every sequence s_0 of E such that $\operatorname{rng} s_0 \subseteq E_0$ and s_0 is convergent and $\lim s_0 = x$ holds $L_{1*}s_0$ is convergent and $P_3(x) = \lim(L_{1*}s_0)$ and for every functions P_1, P_2 from E into F such that $P_1 \restriction E_0 = L_1$ and P_1 is continuous on the carrier of E and $P_2 \restriction E_0 = L_1$ and P_2 is continuous on the carrier of E holds $P_1 = P_2$. Consider P_3 being a function from E into F such that $P_3 \restriction E_0 = L_1$ and P_3 is uniformly continuous on the carrier of E and for every point x of E, there exists a sequence s_0 of E such that $\operatorname{rng} s_0 \subseteq E_0$ and s_0 is convergent and $\lim s_0 = x$ and $L_{1*}s_0$ is convergent and $P_3(x) = \lim(L_{1*}s_0)$ and for every point x of E and for every sequence s_0 of E such that $\operatorname{rng} s_0 \subseteq E_0$ and s_0 is convergent and $\lim s_0 = x$ holds $L_{1*}s_0$ is convergent and $P_3(x) = \lim(L_{1*}s_0)$ and for every point x of E, there exists a sequence s_0 of E such that $\operatorname{rng} s_0 \subseteq E_0$ and s_0 is convergent and $\lim s_0 = x$ and $L_{1*}s_0$ is convergent and $P_3(x) = \lim(L_{1*}s_0)$ and for every point x of E and for every sequence s_0 of E such that $\operatorname{rng} s_0 \subseteq E_0$ and s_0 is convergent and $\lim s_0 = x$ holds $L_{1*}s_0$ is convergent and $P_3(x) = \lim(L_{1*}s_0)$. For every points x, y of E, $P_3(x + y) = P_3(x) + P_3(y)$ by [4, (113), (4)], [9, (28)], [15, (19), (25)]. For every point x of E and for every real number a, $P_3(a \cdot x) = a \cdot P_3(x)$ by [4, (113), (4)], [9, (28)], [15, (22), (28)]. Reconsider $g = P_3$ as a point of the real norm space of bounded linear operators from E into F. For every real number t such that $t \in \operatorname{PreNorms}(L)$ holds $t \leqslant \|g\|$ by [9, (28)], [16, (27)], [3, (49)]. For every real number t such that $t \in \operatorname{PreNorms}(P_3)$ holds $t \leqslant \|f\|$ by [16, (20)], [15, (23)], [8, (7)], [4, (4), (109)]. For every points g_1, g_2 of the real norm space of bounded linear operators from E into F such that $g_1 \restriction$(the carrier of E_1) $= f$ and $g_2 \restriction$(the carrier of E_1) $= f$ holds $g_1 = g_2$ by (3), [12, (7)], (1). □

2. Basic Properties of Currying

Now we state the propositions:

(5) Let us consider non empty sets E, F, G, a function f from $E \times F$ into G, and an object x. If $x \in E$, then $(\operatorname{curry} f)(x)$ is a function from F into G.

(6) Let us consider non empty sets E, F, G, a function f from $E \times F$ into G, and an object y. If $y \in F$, then $(\operatorname{curry}' f)(y)$ is a function from E into G.

Let us consider non empty sets E, F, G, a function f from $E \times F$ into G, and objects x, y. Now we state the propositions:

(7) If $x \in E$ and $y \in F$, then $(\text{curry}\, f)(x)(y) = f(x, y)$.

(8) If $x \in E$ and $y \in F$, then $(\text{curry}'\, f)(y)(x) = f(x, y)$.

Let E, F, G be real linear spaces and f be a function from (the carrier of E) $\times$ (the carrier of F) into the carrier of G. We say that f is bilinear if and only if

(Def. 1) for every point v of E such that $v \in \text{dom}(\text{curry}\, f)$ holds $(\text{curry}\, f)(v)$ is an additive, homogeneous function from F into G and for every point v of F such that $v \in \text{dom}(\text{curry}'\, f)$ holds $(\text{curry}'\, f)(v)$ is an additive, homogeneous function from E into G.

3. Equivalence of Some Definitions of Continuity of Bilinear Operators

Now we state the proposition:

(9) Let us consider real linear spaces E, F, G. Then (the carrier of E) $\times$ (the carrier of F) $\longmapsto 0_G$ is bilinear.
Proof: Set $f =$ (the carrier of E) $\times$ (the carrier of F) $\longmapsto 0_G$. For every point x of E, $(\text{curry}\, f)(x)$ is an additive, homogeneous function from F into G by (7), [5, (87)], [18, (7)], [19, (4), (10)]. For every point x of F such that $x \in \text{dom}(\text{curry}'\, f)$ holds $(\text{curry}'\, f)(x)$ is an additive, homogeneous function from E into G by (8), [5, (87)], [18, (7)], [19, (4), (10)]. □

Let E, F, G be real linear spaces. Observe that there exists a function from (the carrier of E) $\times$ (the carrier of F) into the carrier of G which is bilinear.

Now we state the proposition:

(10) Let us consider real linear spaces E, F, G, and a function L from (the carrier of E) $\times$ (the carrier of F) into the carrier of G. Then L is bilinear if and only if for every points x_1, x_2 of E and for every point y of F, $L(x_1 + x_2, y) = L(x_1, y) + L(x_2, y)$ and for every point x of E and for every point y of F and for every real number a, $L(a \cdot x, y) = a \cdot L(x, y)$ and for every point x of E and for every points y_1, y_2 of F, $L(x, y_1 + y_2) = L(x, y_1) + L(x, y_2)$ and for every point x of E and for every point y of F and for every real number a, $L(x, a \cdot y) = a \cdot L(x, y)$. The theorem is a consequence of (8) and (7).

Let E, F, G be real linear spaces and f be a function from $E \times F$ into G. We say that f is bilinear if and only if

(Def. 2) there exists a function g from (the carrier of E) $\times$ (the carrier of F) into the carrier of G such that $f = g$ and g is bilinear.

One can verify that there exists a function from $E \times F$ into G which is bilinear.

Let f be a function from $E \times F$ into G, x be a point of E, and y be a point of F. Note that the functor $f(x, y)$ yields a point of G. Now we state the proposition:

(11) Let us consider real linear spaces E, F, G, and a function L from $E \times F$ into G. Then L is bilinear if and only if for every points x_1, x_2 of E and for every point y of F, $L(x_1 + x_2, y) = L(x_1, y) + L(x_2, y)$ and for every point x of E and for every point y of F and for every real number a, $L(a \cdot x, y) = a \cdot L(x, y)$ and for every point x of E and for every points y_1, y_2 of F, $L(x, y_1 + y_2) = L(x, y_1) + L(x, y_2)$ and for every point x of E and for every point y of F and for every real number a, $L(x, a \cdot y) = a \cdot L(x, y)$.

Let E, F, G be real linear spaces.

A bilinear operator from $E \times F$ into G is a bilinear function from $E \times F$ into G. Let E, F, G be real normed spaces and f be a function from $E \times F$ into G. We say that f is bilinear if and only if

(Def. 3) there exists a function g from (the carrier of E) $\times$ (the carrier of F) into the carrier of G such that $f = g$ and g is bilinear.

Let us note that there exists a function from $E \times F$ into G which is bilinear.

A bilinear operator from $E \times F$ into G is a bilinear function from $E \times F$ into G. From now on E, F, G denote real normed spaces, L denotes a bilinear operator from $E \times F$ into G, x denotes an element of E, and y denotes an element of F.

Let E, F, G be real normed spaces, f be a function from $E \times F$ into G, x be a point of E, and y be a point of F. Note that the functor $f(x, y)$ yields a point of G. Now we state the propositions:

(12) Let us consider real normed spaces E, F, G, and a function L from $E \times F$ into G. Then L is bilinear if and only if for every points x_1, x_2 of E and for every point y of F, $L(x_1 + x_2, y) = L(x_1, y) + L(x_2, y)$ and for every point x of E and for every point y of F and for every real number a, $L(a \cdot x, y) = a \cdot L(x, y)$ and for every point x of E and for every points y_1, y_2 of F, $L(x, y_1 + y_2) = L(x, y_1) + L(x, y_2)$ and for every point x of E and for every point y of F and for every real number a, $L(x, a \cdot y) = a \cdot L(x, y)$.

(13) Let us consider real normed spaces E, F, G, and a bilinear operator f from $E \times F$ into G. Then

(i) f is continuous on the carrier of $E \times F$ iff f is continuous in $0_{E \times F}$, and

(ii) f is continuous on the carrier of $E \times F$ iff there exists a real number K such that $0 \leqslant K$ and for every point x of E and for every point y

of F, $\|f(x,y)\| \leqslant K \cdot \|x\| \cdot \|y\|$.

PROOF: If f is continuous in $0_{E \times F}$, then there exists a real number K such that $0 \leqslant K$ and for every point x of E and for every point y of F, $\|f(x,y)\| \leqslant K \cdot \|x\| \cdot \|y\|$ by [13, (7)], [10, (22)], [14, (18)], [19, (13)]. If there exists a real number K such that $0 \leqslant K$ and for every point x of E and for every point y of F, $\|f(x,y)\| \leqslant K \cdot \|x\| \cdot \|y\|$, then f is continuous on the carrier of $E \times F$ by [14, (18)], [20, (1)], [19, (28), (16)]. □

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