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Pythagorean Tuning: Pentatonic and Heptatonic Scale

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Summary. In this article, using the Mizar system [3], [4], we define a structure [1], [6] in order to build a Pythagorean pentatonic scale and a Pythagorean heptatonic scale[1] [5], [8].

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1. Preliminaries

Now we state the proposition:

(1) Let us consider an object r. Then $r \in \mathbb{R}_{+\cup\{0\}} \setminus \{0\}$ if and only if r is a positive real number.

Note that there exists a rational number which is positive.

The functor $\mathbb{Q}_+$ yielding a non empty subset of $\mathbb{R}_{+\cup\{0\}}$ is defined by the term

(Def. 1) the set of all r where r is a positive rational number.

Now we state the propositions:

(2) Let us consider an object r. Then r is an element of $\mathbb{Q}_+$ if and only if r is a positive rational number.

(3) $\mathbb{Q}_{+\cup\{0\}} \subseteq \mathbb{Q}$.

[1]`https://en.wikipedia.org/wiki/Pythagorean_tuning`

The functor $\mathbb{R}_{+}$ yielding a non empty subset of $\mathbb{R}_{+\cup\{0\}}$ is defined by the term

(Def. 2) $\mathbb{R}_{+\cup\{0\}} \setminus \{0\}$.

Now we state the propositions:

(4) $\mathbb{N}_{+} \subseteq \mathbb{Q}_{+}$.

(5) $\mathbb{N}_{+} \subseteq \mathbb{R}_{+}$. The theorem is a consequence of (1).

(6) $\mathbb{Q}_{+} \subseteq \mathbb{R}_{+}$. The theorem is a consequence of (2) and (1).

2. Real Frequency

We consider structures of music which extend 1-sorted structures and are systems

$$\langle \text{a carrier, an equidistance, a Ratio} \rangle$$

where the carrier is a set, the equidistance is a relation between (the carrier) $\times$ (the carrier) and (the carrier) $\times$ (the carrier), the Ratio is a function from (the carrier) $\times$ (the carrier) into the carrier.

Let S be a structure of music and a, b, c, d be elements of S. We say that $\overline{ab} \cong \overline{cd}$ if and only if

(Def. 3) $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the equidistance of S.

Let x, y be elements of $\mathbb{R}_{+}$. The functor $\mathbb{R}\text{-ratio}(x, y)$ yielding an element of $\mathbb{R}_{+}$ is defined by

(Def. 4) there exist positive real numbers r, s such that $x = r$ and $s = y$ and $it = \frac{s}{r}$.

Now we state the proposition:

(7) Let us consider elements a, b, c, d of $\mathbb{R}_{+}$. Then $\mathbb{R}\text{-ratio}(a, b) = \mathbb{R}\text{-ratio}(c, d)$ if and only if $\mathbb{R}\text{-ratio}(b, a) = \mathbb{R}\text{-ratio}(d, c)$.

The functor $\mathbb{R}$-ratio yielding a function from $\mathbb{R}_{+} \times \mathbb{R}_{+}$ into $\mathbb{R}_{+}$ is defined by

(Def. 5) for every element x of $\mathbb{R}_{+} \times \mathbb{R}_{+}$, there exist elements y, z of $\mathbb{R}_{+}$ such that $x = \langle y, z\rangle$ and $it(x) = \mathbb{R}\text{-ratio}(y, z)$.

The functor eq-$\mathbb{R}$-ratio yielding a relation between $\mathbb{R}_{+} \times \mathbb{R}_{+}$ and $\mathbb{R}_{+} \times \mathbb{R}_{+}$ is defined by

(Def. 6) for every elements x, y of $\mathbb{R}_{+} \times \mathbb{R}_{+}$, $\langle x, y\rangle \in it$ iff there exist elements a, b, c, d of $\mathbb{R}_{+}$ such that $x = \langle a, b\rangle$ and $y = \langle c, d\rangle$ and $\mathbb{R}\text{-ratio}(a, b) = \mathbb{R}\text{-ratio}(c, d)$.

The functor $\mathbb{R}$-music yielding a structure of music is defined by the term

(Def. 7) $\langle \mathbb{R}_{+}, \text{eq-}\mathbb{R}\text{-ratio}, \mathbb{R}\text{-ratio} \rangle$.

Now we state the propositions:

(8) (i) $\mathbb{R}$-music is not empty, and

(ii) the carrier of $\mathbb{R}$-music $\subseteq \mathbb{R}_+$, and

(iii) for every elements f_1, f_2, f_3, f_4 of $\mathbb{R}$-music, $\overline{f_1 f_2} \cong \overline{f_3 f_4}$ iff (the Ratio of $\mathbb{R}$-music)(f_1, f_2) = (the Ratio of $\mathbb{R}$-music)(f_3, f_4).

(9) Let us consider elements f_1, f_2, f_3 of $\mathbb{R}$-music. Suppose (the Ratio of $\mathbb{R}$-music)(f_1, f_2) = (the Ratio of $\mathbb{R}$-music)(f_1, f_3). Then $f_2 = f_3$.

(10) $\mathbb{N}_+ \subseteq$ the carrier of $\mathbb{R}$-music.

(11) Let us consider an element *fr* of $\mathbb{R}$-music, and a non zero natural number n. Then there exists an element h of $\mathbb{R}$-music such that $\langle fr, h\rangle \in [\langle 1, n\rangle]_\alpha$, where α is the equidistance of $\mathbb{R}$-music. The theorem is a consequence of (1) and (8).

(12) Let us consider elements f_1, f_2, f_3 of $\mathbb{R}$-music. Suppose (the Ratio of $\mathbb{R}$-music)(f_1, f_1) = (the Ratio of $\mathbb{R}$-music)(f_2, f_3). Then $f_2 = f_3$.

(13) Let us consider elements f_1, f_2, f_8, f_6, f_9, f_7 of $\mathbb{R}$-music, positive real numbers r_1, r_2, and non zero natural numbers n, m. Suppose $f_8 = n \cdot r_1$ and $f_6 = m \cdot r_1$ and $f_9 = n \cdot r_2$ and $f_7 = m \cdot r_2$. Then $\overline{f_8 f_6} \cong \overline{f_9 f_7}$. The theorem is a consequence of (8).

(14) Let us consider elements f_1, f_2, f_3, f_4 of $\mathbb{R}$-music. Then (the Ratio of $\mathbb{R}$-music)(f_1, f_2) = (the Ratio of $\mathbb{R}$-music)(f_3, f_4) if and only if (the Ratio of $\mathbb{R}$-music)(f_2, f_1) = (the Ratio of $\mathbb{R}$-music)(f_4, f_3). The theorem is a consequence of (7).

3. Rational Frequency

Let x, y be elements of $\mathbb{Q}_+$. The functor $\mathbb{Q}$-ratio(x, y) yielding an element of $\mathbb{Q}_+$ is defined by

(Def. 8) there exist positive rational numbers r, s such that $x = r$ and $s = y$ and $it = \frac{s}{r}$.

Now we state the proposition:

(15) Let us consider elements a, b, c, d of $\mathbb{Q}_+$. Then $\mathbb{Q}$-ratio(a, b) = $\mathbb{Q}$-ratio(c, d) if and only if $\mathbb{Q}$-ratio(b, a) = $\mathbb{Q}$-ratio(d, c).

The functor $\mathbb{Q}$-ratio yielding a function from $\mathbb{Q}_+ \times \mathbb{Q}_+$ into $\mathbb{Q}_+$ is defined by

(Def. 9) for every element x of $\mathbb{Q}_+ \times \mathbb{Q}_+$, there exist elements y, z of $\mathbb{Q}_+$ such that $x = \langle y, z\rangle$ and $it(x)$ = $\mathbb{Q}$-ratio(y, z).

The functor eq-$\mathbb{Q}$-ratio yielding a relation between $\mathbb{Q}_+ \times \mathbb{Q}_+$ and $\mathbb{Q}_+ \times \mathbb{Q}_+$ is defined by

(Def. 10) for every elements x, y of $\mathbb{Q}_+ \times \mathbb{Q}_+$, $\langle x, y\rangle \in it$ iff there exist elements a, b, c, d of $\mathbb{Q}_+$ such that $x = \langle a, b\rangle$ and $y = \langle c, d\rangle$ and $\mathbb{Q}$-ratio$(a, b) =$ $\mathbb{Q}$-ratio(c, d).

The functor $\mathbb{Q}$-music yielding a structure of music is defined by the term

(Def. 11) $\langle \mathbb{Q}_+$, eq-$\mathbb{Q}$-ratio, $\mathbb{Q}$-ratio$\rangle$.

Now we state the propositions:

(16) (i) $\mathbb{Q}$-music is not empty, and

(ii) the carrier of $\mathbb{Q}$-music $\subseteq \mathbb{R}_+$, and

(iii) for every elements f_1, f_2, f_3, f_4 of $\mathbb{Q}$-music, $\overline{f_1 f_2} \cong \overline{f_3 f_4}$ iff (the Ratio of $\mathbb{Q}$-music)$(f_1, f_2) =$ (the Ratio of $\mathbb{Q}$-music)(f_3, f_4).

The theorem is a consequence of (6).

(17) Let us consider elements f_1, f_2, f_3 of $\mathbb{Q}$-music. Suppose (the Ratio of $\mathbb{Q}$-music)$(f_1, f_2) =$ (the Ratio of $\mathbb{Q}$-music)(f_1, f_3). Then $f_2 = f_3$.

(18) $\mathbb{N}_+ \subseteq$ the carrier of $\mathbb{Q}$-music.

(19) Let us consider an element fr of $\mathbb{Q}$-music, and a non zero natural number n. Then there exists an element h of $\mathbb{Q}$-music such that $\langle fr, h\rangle \in [\langle 1, n\rangle]_\alpha$, where α is the equidistance of $\mathbb{Q}$-music. The theorem is a consequence of (2) and (16).

(20) Let us consider elements f_1, f_2, f_3 of $\mathbb{Q}$-music. Suppose (the Ratio of $\mathbb{Q}$-music)$(f_1, f_1) =$ (the Ratio of $\mathbb{Q}$-music)(f_2, f_3). Then $f_2 = f_3$.

(21) Let us consider an element fr of $\mathbb{Q}$-music. Then there exists a positive real number r such that

(i) $fr = r$, and

(ii) for every non zero natural number n, $n \cdot r$ is an element of $\mathbb{Q}$-music.

The theorem is a consequence of (2).

(22) Let us consider elements f_1, f_2, f_8, f_6, f_9, f_7 of $\mathbb{Q}$-music, positive rational numbers r_1, r_2, and non zero natural numbers n, m. Suppose $f_8 = n \cdot r_1$ and $f_6 = m \cdot r_1$ and $f_9 = n \cdot r_2$ and $f_7 = m \cdot r_2$. Then $\overline{f_8 f_6} \cong \overline{f_9 f_7}$. The theorem is a consequence of (16).

(23) Let us consider elements f_1, f_2, f_3, f_4 of $\mathbb{Q}$-music. Then (the Ratio of $\mathbb{Q}$-music)$(f_1, f_2) =$ (the Ratio of $\mathbb{Q}$-music)(f_3, f_4) if and only if (the Ratio of $\mathbb{Q}$-music)$(f_2, f_1) =$ (the Ratio of $\mathbb{Q}$-music)(f_4, f_3). The theorem is a consequence of (15).

4. Musical Structure and Some Axioms

Let S be a structure of music. We say that S is satisfying real if and only if

(Def. 12) the carrier of $S \subseteq \mathbb{R}_+$.

We say that S is equidistant-ratio equivalent if and only if

(Def. 13) for every elements f_1, f_2, f_3, f_4 of S, $\overline{f_1 f_2} \cong \overline{f_3 f_4}$ iff (the Ratio of S)(f_1, f_2) = (the Ratio of S)(f_3, f_4).

We say that S is satisfying interval if and only if

(Def. 14) for every elements f_1, f_2, f_3 of S such that (the Ratio of S)(f_1, f_2) = (the Ratio of S)(f_1, f_3) holds $f_2 = f_3$.

We say that S is unison-ratio stable if and only if

(Def. 15) for every elements f_1, f_2, f_3 of S such that (the Ratio of S)(f_1, f_1) = (the Ratio of S)(f_2, f_3) holds $f_2 = f_3$.

We say that S is ratio symmetric if and only if

(Def. 16) for every elements f_1, f_2, f_3, f_4 of S, (the Ratio of S)(f_1, f_2) = (the Ratio of S)(f_3, f_4) iff (the Ratio of S)(f_2, f_1) = (the Ratio of S)(f_4, f_3).

We say that S is natural if and only if

(Def. 17) $\mathbb{N}_+ \subseteq$ the carrier of S.

We say that S is harmonic closed if and only if

(Def. 18) for every element fr of S and for every non zero natural number n, there exists an element h of S such that $\langle fr, h\rangle \in [\langle 1, n\rangle]_\alpha$, where α is the equidistance of S.

Note that there exists a structure of music which is harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, and non empty.

Let us note that the functor $\mathbb{R}$-music yields a harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Observe that the functor $\mathbb{Q}$-music yields a harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Now we state the propositions:

(24) Let us consider a natural structure of music S. Then every non zero natural number is an element of S.

(25) Let us consider an equidistant-ratio equivalent structure of music M, and elements a, b of M. Then $\overline{ab} \cong \overline{ab}$.

(26) Let us consider an equidistant-ratio equivalent structure of music M, and elements a, b, c, d of M. Then $\overline{ab} \cong \overline{cd}$ if and only if $\overline{cd} \cong \overline{ab}$.

(27) Let us consider an equidistant-ratio equivalent structure of music M, and elements a, b, c, d, e, f of M. Suppose $\overline{ab} \cong \overline{cd}$ and $\overline{cd} \cong \overline{ef}$. Then $\overline{ab} \cong \overline{ef}$.

(28) Let us consider a satisfying interval, equidistant-ratio equivalent structure of music S, and elements a, b, c of S. Then $\overline{ab} \cong \overline{ac}$ if and only if $b = c$. The theorem is a consequence of (25).

From now on M denotes an equidistant-ratio equivalent structure of music and a, b, c, d, e, f denote elements of M.

Now we state the propositions:

(29) $\overline{aa} \cong \overline{aa}$.

(30) The equidistance of M is reflexive in (the carrier of M) $\times$ (the carrier of M). The theorem is a consequence of (25).

(31) Suppose M is not empty. Then

(i) the equidistance of M is reflexive, and

(ii) field(the equidistance of M) $=$ (the carrier of M) $\times$ (the carrier of M).

The theorem is a consequence of (30).

(32) The equidistance of M is symmetric in (the carrier of M) $\times$ (the carrier of M). The theorem is a consequence of (26).

(33) The equidistance of M is transitive in (the carrier of M) $\times$ (the carrier of M). The theorem is a consequence of (27).

(34) The equidistance of M is an equivalence relation of (the carrier of M) $\times$ (the carrier of M). The theorem is a consequence of (30), (32), and (33).

(35) Let us consider a ratio symmetric, equidistant-ratio equivalent structure of music M, and elements a, b, c, d of M. Then $\overline{ab} \cong \overline{cd}$ if and only if $\overline{ba} \cong \overline{dc}$.

(36) Let us consider a unison-ratio stable, equidistant-ratio equivalent structure of music S, and elements a, b, c of S. If $\overline{aa} \cong \overline{bc}$, then $b = c$.

Let S be a natural, satisfying interval, harmonic closed, equidistant-ratio equivalent structure of music, fr be an element of S, and n be a non zero natural number. The n-harmonic of fr in S yielding an element of S is defined by

(Def. 19) $\langle fr, it \rangle \in [\langle 1, n \rangle]_{\alpha}$, where α is the equidistance of S.

We say that S is harmonic linear if and only if

(Def. 20) for every element fr of S and for every non zero natural number n, there exists a positive real number f such that $fr = f$ and the n-harmonic of fr in $S = n \cdot f$.

Now we state the propositions:

(37) $\mathbb{R}$-music is harmonic linear. The theorem is a consequence of (1) and (24).

(38) $\mathbb{Q}$-music is harmonic linear. The theorem is a consequence of (2) and (24).

One can check that there exists a harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music which is harmonic linear.

One can check that the functor $\mathbb{R}$-music yields a harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Let us note that the functor $\mathbb{Q}$-music yields a harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Let M be a harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music. We say that M is harmonic stable if and only if

(Def. 21) for every elements f_1, f_2 of M and for every non zero natural numbers n, m, $\overline{\text{the } n\text{-harmonic of } f_1 \text{ in } M \text{the } m\text{-harmonic of } f_1 \text{ in } M} \cong \overline{\text{the } n\text{-harmonic of } f_2 \text{ i}}$

Now we state the propositions:

(39) $\mathbb{R}$-music is harmonic stable. The theorem is a consequence of (1) and (13).

(40) $\mathbb{Q}$-music is harmonic stable. The theorem is a consequence of (2) and (22).

Observe that there exists a harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music which is harmonic stable.

One can verify that the functor $\mathbb{R}$-music yields a harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Observe that the functor $\mathbb{Q}$-music yields a harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Let M be a harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music and fr be an element of M. The functors: the set of unison of fr in M, the set of octave of fr in M, the set of fifth of fr in M, the set of fourth of fr in M, and the set of major sixth of
yielding subsets of (the carrier of M) $\times$ (the carrier of M) are defined by terms

(Def. 22) $[\langle$the 1-harmonic of fr in M, the 1-harmonic of fr in $M\rangle]_{\alpha}$, where α is the equidistance of M, (Def. 23) $[\langle$the 1-harmonic of *fr* in M, the 2-harmonic of *fr* in $M\rangle]_{\alpha}$, where α is the equidistance of M,

(Def. 24) $[\langle$the 2-harmonic of *fr* in M, the 3-harmonic of *fr* in $M\rangle]_{\alpha}$, where α is the equidistance of M,

(Def. 25) $[\langle$the 3-harmonic of *fr* in M, the 4-harmonic of *fr* in $M\rangle]_{\alpha}$, where α is the equidistance of M,

(Def. 26) $[\langle$the 3-harmonic of *fr* in M, the 5-harmonic of *fr* in $M\rangle]_{\alpha}$, where α is the equidistance of M,

respectively. The functors: the set of major third of *fr* in M, the set of minor third of *f* the set of minor sixth of *fr* in M, the set of major tone of *fr* in M, and the set of mir yielding subsets of (the carrier of M) $\times$ (the carrier of M) are defined by terms

(Def. 27) $[\langle$the 4-harmonic of *fr* in M, the 5-harmonic of *fr* in $M\rangle]_{\alpha}$, where α is the equidistance of M,

(Def. 28) $[\langle$the 5-harmonic of *fr* in M, the 6-harmonic of *fr* in $M\rangle]_{\alpha}$, where α is the equidistance of M,

(Def. 29) $[\langle$the 5-harmonic of *fr* in M, the 8-harmonic of *fr* in $M\rangle]_{\alpha}$, where α is the equidistance of M,

(Def. 30) $[\langle$the 8-harmonic of *fr* in M, the 9-harmonic of *fr* in $M\rangle]_{\alpha}$, where α is the equidistance of M,

(Def. 31) $[\langle$the 9-harmonic of *fr* in M, the 10-harmonic of *fr* in $M\rangle]_{\alpha}$, where α is the equidistance of M,

respectively. The functors: the set of unison of M, the set of octave of M, the set of fif the set of fourth of M, and the set of major sixth of M yielding subsets of (the carrier of M) $\times$ (the carrier of M) are defined by terms

(Def. 32) $[\langle 1, 1\rangle]_{\alpha}$, where α is the equidistance of M,

(Def. 33) $[\langle 1, 2\rangle]_{\alpha}$, where α is the equidistance of M,

(Def. 34) $[\langle 2, 3\rangle]_{\alpha}$, where α is the equidistance of M,

(Def. 35) $[\langle 3, 4\rangle]_{\alpha}$, where α is the equidistance of M,

(Def. 36) $[\langle 3, 5\rangle]_{\alpha}$, where α is the equidistance of M,

respectively. The functors: the set of major third of M, the set of minor third of M, the set of minor sixth of M, the set of major tone of M, and the set of minor tone of yielding subsets of (the carrier of M) $\times$ (the carrier of M) are defined by terms

(Def. 37) $[\langle 4, 5\rangle]_{\alpha}$, where α is the equidistance of M,

(Def. 38) $[\langle 5, 6\rangle]_{\alpha}$, where α is the equidistance of M,

(Def. 39) $[\langle 5, 8\rangle]_{\alpha}$, where α is the equidistance of M,

(Def. 40) $[\langle 8, 9\rangle]_{\alpha}$, where α is the equidistance of M, (Def. 41) $[\langle 9, 10\rangle]_{\alpha}$, where α is the equidistance of M,

respectively. Let S be a harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music. We say that **S is fifth constructible** if and only if

(Def. 42) for every element fr of S, there exists an element q of S such that $\langle fr, q\rangle \in$ the set of fifth of S.

Now we state the propositions:

(41) Let us consider an element fr of $\mathbb{R}$-music. Then there exist positive real numbers f, q_1 such that

(i) $f = fr$, and

(ii) $q_1 = \frac{(3 \textbf{ qua } \text{real number})}{2} \cdot f$, and

(iii) $\langle f, q_1\rangle \in$ the set of fifth of $\mathbb{R}$-music.

The theorem is a consequence of (1) and (24).

(42) $\mathbb{R}$-music is fifth constructible. The theorem is a consequence of (41) and (1).

(43) Let us consider an element fr of $\mathbb{Q}$-music. Then there exist positive rational numbers f, q_1 such that

(i) $f = fr$, and

(ii) $q_1 = \frac{(3 \textbf{ qua } \text{rational number})}{2} \cdot f$, and

(iii) $\langle f, q_1\rangle \in$ the set of fifth of $\mathbb{Q}$-music.

The theorem is a consequence of (2) and (24).

(44) $\mathbb{Q}$-music is fifth constructible. The theorem is a consequence of (43) and (2).

Let us observe that there exists a harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music which is fifth constructible.

Let us note that the functor $\mathbb{R}$-music yields a fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Let us note that the functor $\mathbb{Q}$-music yields a fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Let M be a fifth constructible, harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music and fr be an element of M. **The fifth of fr in M** yielding an element of M is defined by

(Def. 43) $\langle fr, it \rangle \in$ the set of fifth of M.

Now we state the propositions:

(45) Let us consider a fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music M, and an element fr of M. Then the set of fifth of fr in $M =$ the set of fifth of M. The theorem is a consequence of (24) and (27).

(46) Let us consider an element fr of $\mathbb{R}$-music. Then there exists a positive real number f such that

(i) $fr = f$, and

(ii) the fifth of fr in $\mathbb{R}$-music $= \frac{(3 \textbf{ qua } \text{real number})}{2} \cdot f$.

The theorem is a consequence of (1) and (41).

(47) Let us consider an element fr of $\mathbb{Q}$-music. Then there exists a positive rational number f such that

(i) $fr = f$, and

(ii) the fifth of fr in $\mathbb{Q}$-music $= \frac{(3 \textbf{ qua } \text{rational number})}{2} \cdot f$.

The theorem is a consequence of (2) and (43).

Let M be a fifth constructible, harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music. We say that M is classical fifth if and only if

(Def. 44) for every element fr of M, there exists a positive real number f such that $fr = f$ and the fifth of fr in $M = \frac{(3 \textbf{ qua } \text{real number})}{2} \cdot f$.

One can verify that there exists a fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music which is classical fifth.

One can verify that the functor $\mathbb{R}$-music yields a classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. One can check that the functor $\mathbb{Q}$-music yields a classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music.

5. Harmonic

Now we state the propositions:

(48) Let us consider a harmonic closed, natural, unison-ratio stable, satisfying interval, equidistant-ratio equivalent structure of music M, and an element fr of M. Then the 1-harmonic of fr in $M = fr$. The theorem is a consequence of (36).

(49) Let us consider a harmonic stable, harmonic closed, natural, unison-ratio stable, satisfying interval, equidistant-ratio equivalent structure of music M, and elements a, b of M. Then $\overline{aa} \cong \overline{bb}$. The theorem is a consequence of (48).

(50) Let us consider a harmonic stable, harmonic linear, harmonic closed, natural, unison-ratio stable, satisfying interval, equidistant-ratio equivalent structure of music M, and an element fr of M. Then the set of octave of fr in $M =$ the set of octave of M. The theorem is a consequence of (48), (27), and (24).

(51) Let us consider a fifth constructible, harmonic closed, natural, satisfying interval, equidistant-ratio equivalent, non empty structure of music M, and an element fr of M. Then there exists a sequence s_{11} of M such that

(i) $s_{11}(0) = fr$, and

(ii) for every natural number n, $\langle s_{11}(n), s_{11}(n+1)\rangle \in$ the set of fifth of M.

Proof: Define $\mathcal{P}[\text{set}, \text{set}, \text{set}] \equiv$ there exist positive real numbers x, y such that $\langle \$_2, \$_3\rangle \in$ the set of fifth of M. For every natural number n and for every element x of M, there exists an element y of M such that $\mathcal{P}[n, x, y]$. Consider s_{11} being a sequence of M such that $s_{11}(0) = fr$ and for every natural number n, $\mathcal{P}[n, s_{11}(n), s_{11}(n+1)]$ from [7, Sch. 2]. □

Let M be a structure of music and a, b, c be elements of M. We say that b is between a and c if and only if

(Def. 45) there exist positive real numbers r_1, r_2, r_3 such that $a = r_1$ and $b = r_2$ and $c = r_3$ and $r_1 \leqslant r_2 < r_3$.

Let S be a harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music. We say that S is octave constructible if and only if

(Def. 46) for every element fr of S, there exists an element o of S such that $\langle fr, o\rangle \in$ the set of octave of S.

Now we state the propositions:

(52) Let us consider an element *fr* of $\mathbb{R}$-music. Then there exist positive real numbers f, q_1 such that

(i) $f = fr$, and

(ii) $q_1 = 2 \cdot f$, and

(iii) $\langle f, q_1 \rangle \in$ the set of octave of $\mathbb{R}$-music.

The theorem is a consequence of (1) and (24).

(53) $\mathbb{R}$-music is octave constructible. The theorem is a consequence of (52) and (1).

(54) Let us consider an element *fr* of $\mathbb{Q}$-music. Then there exist positive rational numbers f, q_1 such that

(i) $f = fr$, and

(ii) $q_1 = 2 \cdot f$, and

(iii) $\langle f, q_1 \rangle \in$ the set of octave of $\mathbb{Q}$-music.

The theorem is a consequence of (2) and (24).

(55) $\mathbb{Q}$-music is octave constructible. The theorem is a consequence of (54) and (2).

Let us note that there exists a classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music which is octave constructible.

Let us observe that the functor $\mathbb{R}$-music yields an octave constructible, classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Let us note that the functor $\mathbb{Q}$-music yields an octave constructible, classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Let M be an octave constructible, harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music and *fr* be an element of M. The octave of *fr* in M yielding an element of M is defined by

(Def. 47) $\langle fr, it \rangle \in$ the set of octave of M.

Let M be a satisfying real, non empty structure of music and r be an element of M. The functor $^{@}r$ yielding a positive real number is defined by the term

(Def. 48) r.

Let M be an octave constructible, harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music. We say that M is classical octave if and only if

(Def. 49) for every element fr of M, there exists a positive real number f such that $fr = f$ and the octave of fr in $M = 2 \cdot f$.

Now we state the propositions:

(56) $\mathbb{R}$-music is classical octave. The theorem is a consequence of (52) and (1).

(57) $\mathbb{Q}$-music is classical octave. The theorem is a consequence of (54) and (2).

One can verify that there exists an octave constructible, classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music which is classical octave.

Observe that the functor $\mathbb{R}$-music yields a classical octave, octave constructible, classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Observe that the functor $\mathbb{Q}$-music yields a classical octave, octave constructible, classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Let M be an octave constructible, harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music. We say that M is octave descending constructible if and only if

(Def. 50) for every element fr of M, there exists an element o of M such that $\langle o, fr \rangle \in$ the set of octave of M.

Now we state the propositions:

(58) Let us consider an element fr of $\mathbb{R}$-music. Then there exist positive real numbers f, q_1 such that

   (i) $f = fr$, and

   (ii) $q_1 = \frac{(1 \textbf{ qua } \text{real number})}{2} \cdot f$, and

   (iii) $\langle q_1, f \rangle \in$ the set of octave of $\mathbb{R}$-music.

The theorem is a consequence of (1), (24), and (35).

(59) $\mathbb{R}$-music is octave descending constructible. The theorem is a consequence of (58) and (1).

(60) Let us consider an element fr of $\mathbb{Q}$-music. Then there exist positive rational numbers f, q_1 such that

(i) $f = fr$, and

(ii) $q_1 = \frac{(1 \textbf{ qua } \text{rational number})}{2} \cdot f$, and

(iii) $\langle q_1, f \rangle \in$ the set of octave of $\mathbb{Q}$-music.

The theorem is a consequence of (2), (24), and (35).

(61) $\mathbb{Q}$-music is octave descending constructible. The theorem is a consequence of (60) and (2).

One can verify that there exists a classical octave, octave constructible, classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music which is octave descending constructible.

One can verify that the functor $\mathbb{R}$-music yields an octave descending constructible, classical octave, octave constructible, classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Note that the functor $\mathbb{Q}$-music yields an octave descending constructible, classical octave, octave constructible, classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Let M be an octave descending constructible, octave constructible, fifth constructible, harmonic closed, natural, ratio symmetric, satisfying interval, equidistant-ratio equivalent structure of music and fr be an element of M. The octave descending of fr in M yielding an element of M is defined by

(Def. 51) $\langle it, fr \rangle \in$ the set of octave of M.

Now we state the propositions:

(62) Let us consider an octave descending constructible, classical octave, octave constructible, fifth constructible, harmonic closed, natural, ratio symmetric, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music M, and an element fr of M. Then there exists a positive real number r such that

(i) $fr = r$, and

(ii) the octave descending of fr in $M = \frac{r}{2}$.

The theorem is a consequence of (1).

(63) Let us consider classical octave, octave constructible, classical fifth, fifth constructible, harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structures of music M_1, M_2, an element f_1 of M_1, and an element f_2 of M_2. Suppose $f_1 = f_2$. Then

(i) the fifth of f_1 in M_1 = the fifth of f_2 in M_2, and

(ii) the octave of f_1 in M_1 = the octave of f_2 in M_2.

(64) Let us consider octave descending constructible, classical octave, octave constructible, fifth constructible, harmonic closed, natural, ratio symmetric, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structures of music M_1, M_2, an element fr_1 of M_1, and an element fr_2 of M_2. Suppose $fr_1 = fr_2$. Then the octave descending of fr_1 in M_1 = the octave descending of fr_2 in M_2. The theorem is a consequence of (62).

Let M be an octave descending constructible, octave constructible, fifth constructible, harmonic closed, natural, ratio symmetric, satisfying interval, equidistant-ratio equivalent structure of music and f_{10}, fr be elements of M. The reduct fifth of the fr with fundamental frequency f_{10} in M yielding an element of M is defined by the term

(Def. 52) $\begin{cases} \text{the fifth of } fr \text{ in } M, & \textbf{if} \text{ the fifth of } fr \text{ in } M \text{ is between } f_{10} \text{ and the octave of } f \\ \textbf{otherwise}. \end{cases}$

Now we state the propositions:

(65) Let us consider octave descending constructible, classical octave, octave constructible, classical fifth, fifth constructible, harmonic closed, natural, ratio symmetric, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structures of music M_1, M_2, elements fr_1, f_{11} of M_1, and elements fr_2, f_{12} of M_2. Suppose $fr_1 = fr_2$ and $f_{11} = f_{12}$. Then the reduct fifth of the fr_1 with fundamental frequency f_{11} in M_1 = the reduct fifth of the fr_2 with fundamental frequency f_{12} in M_2. The theorem is a consequence of (63) and (64).

(66) Let us consider a classical fifth, fifth constructible, harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music M, and an element fr of M. Then there exist positive real numbers r, s such that

(i) $r = fr$, and

(ii) $s = \frac{(3 \textbf{ qua } \text{real number})}{2} \cdot r$, and

(iii) the fifth of fr in $M = s$.

(67) Let us consider an octave descending constructible, classical octave, octave constructible, classical fifth, fifth constructible, harmonic closed, natural, satisfying interval, equidistant-ratio equivalent structure of music M, and elements f_{10}, fr of M. Suppose fr is between f_{10} and the octave of f_{10} in M. Then there exist positive real numbers r_1, r_2, r_3 such that

(i) $f_{10} = r_1$, and

(ii) $fr = r_2$, and

(iii) the octave of f_{10} in $M = 2 \cdot r_1$, and

(iv) $r_1 \leqslant r_2 \leqslant 2 \cdot r_1$.

(68) Let us consider an octave descending constructible, classical octave, octave constructible, classical fifth, fifth constructible, harmonic closed, natural, ratio symmetric, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music M, and elements f_{10}, fr of M. Suppose fr is between f_{10} and the octave of f_{10} in M. Then the reduct fifth of the fr with fundamental frequency f_{10} in M is between f_{10} and the octave of f_{10} in M. The theorem is a consequence of (67) and (62).

A space of music is an octave descending constructible, classical octave, octave constructible, classical fifth, fifth constructible, harmonic stable, harmonic linear, harmonic closed, natural, ratio symmetric, unison-ratio stable, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music. Now we state the propositions:

(69) $\mathbb{R}$-music is a space of music.

(70) $\mathbb{Q}$-music is a space of music.

6. Spiral of Fifths

Now we state the proposition:

(71) Let us consider an octave descending constructible, octave constructible, fifth constructible, harmonic closed, natural, ratio symmetric, satisfying interval, equidistant-ratio equivalent, non empty structure of music M, and elements f_{10}, fr of M. Then there exists a sequence s_{11} of M such that

(i) $s_{11}(0) = fr$, and

(ii) for every natural number n, $s_{11}(n+1) =$ the reduct fifth of the $s_{11}(n)$ with fundamental frequency f_{10} in M.

Proof: Define $\mathcal{P}[\text{set}, \text{set}, \text{set}] \equiv$ there exist elements x, y of M such that $x = \$_2$ and $y = \$_3$ and $y =$ the reduct fifth of the x with fundamental frequency f_{10} in M. For every natural number n and for every element x of M, there exists an element y of M such that $\mathcal{P}[n, x, y]$. Consider s_{11} being a sequence of M such that $s_{11}(0) = fr$ and for every natural number n, $\mathcal{P}[n, s_{11}(n), s_{11}(n+1)]$ from [7, Sch. 2]. $\square$

Let M be an octave descending constructible, octave constructible, fifth constructible, harmonic closed, natural, ratio symmetric, satisfying interval, equidistant-ratio equivalent, non empty structure of music and f_{10}, fr be elements of M. The spiral of fifths of fr with fundamental frequency f_{10} in M yielding a sequence of M is defined by

(Def. 53) $it(0) = fr$ and for every natural number n, $it(n+1) =$ the reduct fifth of the $it(n)$ with fundamental frequency f_{10} in M.

From now on M denotes an octave descending constructible, classical octave, octave constructible, classical fifth, fifth constructible, harmonic closed, natural, ratio symmetric, satisfying interval, equidistant-ratio equivalent, satisfying real, non empty structure of music and f_{10}, fr denote elements of M.

Now we state the propositions:

(72) Suppose fr is between f_{10} and the octave of f_{10} in M. Let us consider a natural number n. Then (the spiral of fifths of fr with fundamental frequency f_{10} in M)(n) is between f_{10} and the octave of f_{10} in M.
Proof: Define $\mathcal{P}$[natural number] $\equiv$ (the spiral of fifths of fr with fundamental frequency f_{10} in M)($\$_1$) is between f_{10} and the octave of f_{10} in M. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$ from [2, Sch. 2]. □

(73) (The spiral of fifths of f_{10} with fundamental frequency f_{10} in M)(1) $= \frac{(3 \textbf{ qua } \text{real number})}{2} \cdot ({}^{@}f_{10})$. The theorem is a consequence of (66).

(74) (The spiral of fifths of f_{10} with fundamental frequency f_{10} in M)(2) $= \frac{(9 \textbf{ qua } \text{real number})}{8} \cdot ({}^{@}f_{10})$. The theorem is a consequence of (73), (66), and (62).

(75) (The spiral of fifths of f_{10} with fundamental frequency f_{10} in M)(3) $= \frac{(27 \textbf{ qua } \text{real number})}{16} \cdot ({}^{@}f_{10})$. The theorem is a consequence of (74) and (66).

(76) (The spiral of fifths of f_{10} with fundamental frequency f_{10} in M)(4) $= \frac{(81 \textbf{ qua } \text{real number})}{64} \cdot ({}^{@}f_{10})$. The theorem is a consequence of (75), (66), and (62).

(77) (The spiral of fifths of f_{10} with fundamental frequency f_{10} in M)(5) $= \frac{(243 \textbf{ qua } \text{real number})}{128} \cdot ({}^{@}f_{10})$. The theorem is a consequence of (76) and (66).

(78) $\frac{{}^{@}(\text{the spiral of fifths of } fr \text{ with fundamental frequency } fr \text{ in } M)(2)}{{}^{@}fr} = \frac{(3 \cdot 3 \textbf{ qua } \text{real number})}{2 \cdot 2 \cdot 2}$.
The theorem is a consequence of (74).

(79) $\frac{{}^{@}(\text{the spiral of fifths of } fr \text{ with fundamental frequency } fr \text{ in } M)(4)}{{}^{@}(\text{the spiral of fifths of } fr \text{ with fundamental frequency } fr \text{ in } M)(2)} = \frac{(3 \cdot 3 \textbf{ qua } \text{real number})}{2 \cdot 2 \cdot 2}$.
The theorem is a consequence of (74) and (76).

(80) $\frac{{}^{@}(\text{the spiral of fifths of } fr \text{ with fundamental frequency } fr \text{ in } M)(1)}{{}^{@}(\text{the spiral of fifths of } fr \text{ with fundamental frequency } fr \text{ in } M)(4)} = \frac{(32 \textbf{ qua } \text{real number})}{27}$.
The theorem is a consequence of (73) and (76).

(81) $\frac{^{@}(\text{the spiral of fifths of } fr \text{ with fundamental frequency } fr \text{ in } M)(3)}{^{@}(\text{the spiral of fifths of } fr \text{ with fundamental frequency } fr \text{ in } M)(1)} = \frac{(9 \textbf{ qua } \text{real number})}{8}$.
The theorem is a consequence of (73) and (75).

(82) $\frac{^{@}(\text{the octave of } fr \text{ in } M)}{^{@}(\text{the spiral of fifths of } fr \text{ with fundamental frequency } fr \text{ in } M)(3)} = \frac{(32 \textbf{ qua } \text{real number})}{27}$.
The theorem is a consequence of (75).

Let M be a space of music and s_{10} be an element of (the carrier of $M)^2$. We say that s_{10} is monotonic if and only if

(Def. 54) there exists an element fr of M and there exist positive real numbers r_1, r_2 such that $s_{10}(1) = fr$ and $s_{10}(1) = r_1$ and $s_{10}(2) = r_2$ and $r_1 < r_2$ and $s_{10}(2) =$ the octave of fr in M.

Let s_{10} be an element of (the carrier of $M)^3$. We say that s_{10} is ditonic if and only if

(Def. 55) there exists an element fr of M and there exist positive real numbers r_1, r_2, r_3 such that $s_{10}(1) = fr$ and $s_{10}(1) = r_1$ and $s_{10}(2) = r_2$ and $s_{10}(3) = r_3$ and $r_1 < r_2 < r_3$ and $s_{10}(3) =$ the octave of fr in M.

Let s_{10} be an element of (the carrier of $M)^4$. We say that s_{10} is tritonic if and only if

(Def. 56) there exists an element fr of M and there exist positive real numbers r_1, r_2, r_3, r_4 such that $s_{10}(1) = fr$ and $s_{10}(1) = r_1$ and $s_{10}(2) = r_2$ and $s_{10}(3) = r_3$ and $s_{10}(4) = r_4$ and $r_1 < r_2 < r_3$ and $r_3 < r_4$ and $s_{10}(4) =$ the octave of fr in M.

Let s_{10} be an element of (the carrier of $M)^5$. We say that s_{10} is tetratonic if and only if

(Def. 57) there exists an element fr of M and there exist positive real numbers r_1, r_2, r_3, r_4, r_5 such that $s_{10}(1) = fr$ and $s_{10}(1) = r_1$ and $s_{10}(2) = r_2$ and $s_{10}(3) = r_3$ and $s_{10}(4) = r_4$ and $s_{10}(5) = r_5$ and $r_1 < r_2 < r_3$ and $r_3 < r_4 < r_5$ and $s_{10}(5) =$ the octave of fr in M.

Let n be a natural number and s_{10} be an element of (the carrier of $M)^n$. We say that s_{10} is pentatonic if and only if

(Def. 58) $n = 6$ and there exists an element fr of M and there exist positive real numbers r_1, r_2, r_3, r_4, r_5, r_6 such that $s_{10}(1) = fr$ and $s_{10}(1) = r_1$ and $s_{10}(2) = r_2$ and $s_{10}(3) = r_3$ and $s_{10}(4) = r_4$ and $s_{10}(5) = r_5$ and $s_{10}(6) = r_6$ and $r_1 < r_2 < r_3$ and $r_3 < r_4 < r_5$ and $r_5 < r_6$ and $s_{10}(6) =$ the octave of fr in M.

Let s_{10} be an element of (the carrier of $M)^7$. We say that s_{10} is hexatonic if and only if

(Def. 59) there exists an element fr of M and there exist positive real numbers r_1, r_2, r_3, r_4, r_5, r_6, r_7 such that $s_{10}(1) = fr$ and $s_{10}(1) = r_1$ and $s_{10}(2) = r_2$ and $s_{10}(3) = r_3$ and $s_{10}(4) = r_4$ and $s_{10}(5) = r_5$ and $s_{10}(6) = r_6$ and

$s_{10}(7) = r_7$ and $r_1 < r_2 < r_3$ and $r_3 < r_4 < r_5$ and $r_5 < r_6 < r_7$ and $s_{10}(7) =$ the octave of *fr* in M.

Let n be a natural number and s_{10} be an element of (the carrier of M)^n. We say that s_{10} is heptatonic if and only if

(Def. 60) $n = 8$ and there exists an element *fr* of M and there exist positive real numbers r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8 such that $s_{10}(1) =$ *fr* and $s_{10}(1) = r_1$ and $s_{10}(2) = r_2$ and $s_{10}(3) = r_3$ and $s_{10}(4) = r_4$ and $s_{10}(5) = r_5$ and $s_{10}(6) = r_6$ and $s_{10}(7) = r_7$ and $s_{10}(8) = r_8$ and $r_1 < r_2 < r_3$ and $r_3 < r_4 < r_5$ and $r_5 < r_6 < r_7$ and $r_7 < r_8$ and $s_{10}(8) =$ the octave of *fr* in M.

Let s_{10} be an element of (the carrier of M)^9. We say that s_{10} is octatonic if and only if

(Def. 61) there exists an element *fr* of M and there exist positive real numbers r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9 such that $s_{10}(1) =$ *fr* and $s_{10}(1) = r_1$ and $s_{10}(2) = r_2$ and $s_{10}(3) = r_3$ and $s_{10}(4) = r_4$ and $s_{10}(5) = r_5$ and $s_{10}(6) = r_6$ and $s_{10}(7) = r_7$ and $s_{10}(8) = r_8$ and $s_{10}(9) = r_9$ and $r_1 < r_2 < r_3$ and $r_3 < r_4 < r_5$ and $r_5 < r_6 < r_7$ and $r_7 < r_8 < r_9$ and $s_{10}(9) =$ the octave of *fr* in M.

7. PENTATONIC PYTHAGOREAN SCALE

Let M be a space of music and *fr* be an element of M. The pentatonic pythagorean sca
yielding an element of (the carrier of M)^6 is defined by

(Def. 62) $it(1) =$ *fr* and $it(2) =$ (the spiral of fifths of *fr* with fundamental frequency *fr* in M)(2) and $it(3) =$ (the spiral of fifths of *fr* with fundamental frequency *fr* in M)(4) and $it(4) =$ (the spiral of fifths of *fr* with fundamental frequency *fr* in M)(1) and $it(5) =$ (the spiral of fifths of *fr* with fundamental frequency *fr* in M)(3) and $it(6) =$ the octave of *fr* in M.

From now on M denotes a space of music and f_{10}, *fr*, f_1, f_2 denote elements of M.

Now we state the proposition:

(83) The pentatonic pythagorean scale of *fr* in M is pentatonic. The theorem is a consequence of (74), (76), (73), and (75).

Let M be a space of music and f_1, f_2 be elements of M. The interval between f_1 and f_2
yielding a positive real number is defined by

(Def. 63) there exist positive real numbers r_1, r_2 such that $r_1 = f_1$ and $r_2 = f_2$ and $it = \frac{r_2}{r_1}$.

The pythagorean tone yielding a positive real number is defined by the term

(Def. 64) $\frac{(9 \textbf{ qua} \text{ real number})}{8}$.

The pythagorean semiditone yielding a positive real number is defined by the term

(Def. 65) $\frac{(32 \textbf{ qua} \text{ real number})}{27}$.

The pythagorean major third yielding a positive real number is defined by the term

(Def. 66) (the pythagorean tone) $\cdot$ (the pythagorean tone).

The pythagorean pure major third yielding a positive real number is defined by the term

(Def. 67) $\frac{(5 \textbf{ qua} \text{ real number})}{4}$.

The syntonic comma yielding a positive real number is defined by the term

(Def. 68) $\frac{\text{the pythagorean major third}}{\text{the pythagorean pure major third}}$.

Now we state the propositions:

(84) The syntonic comma $= \frac{(81 \textbf{ qua} \text{ real number})}{80}$.

(85) The pythagorean tone $<$ the pythagorean semiditone.

(86) (The pythagorean tone)$\cdot$(the pythagorean tone)$\cdot$(the pythagorean semiditone)$\cdot$ (the pythagorean tone) $\cdot$ (the pythagorean semiditone) $= 2$.

Let M be a space of music and fr be an element of M. The functors: the first degree of pentatonic scale of fr in M, the second degree of pentatonic scale of the third degree of pentatonic scale of fr in M, the fourth degree of pentatonic scale of and the fifth degree of pentatonic scale of fr in M yielding elements of M are defined by terms

(Def. 69) (the pentatonic pythagorean scale of fr in M)(1),

(Def. 70) (the pentatonic pythagorean scale of fr in M)(2),

(Def. 71) (the pentatonic pythagorean scale of fr in M)(3),

(Def. 72) (the pentatonic pythagorean scale of fr in M)(4),

(Def. 73) (the pentatonic pythagorean scale of fr in M)(5),

respectively. The octave of pentatonic scale of fr in M yielding an element of M is defined by the term

(Def. 74) the octave of fr in M.

Now we state the propositions:

(87) There exist elements r_1, r_2 of $\mathbb{R}_+$ such that the interval between f_1 and $f_2 = \mathbb{R}$-ratio(r_1, r_2).

(88) Let us consider positive real numbers r_1, r_2, r_3, r_4, r_5, r_6. Suppose (the pentatonic pythagorean scale of *fr* in $M)(1) = r_1$ and (the pentatonic pythagorean scale of *fr* in $M)(2) = r_2$ and (the pentatonic pythagorean scale of *fr* in $M)(3) = r_3$ and (the pentatonic pythagorean scale of *fr* in $M)(4) = r_4$ and (the pentatonic pythagorean scale of *fr* in $M)(5) = r_5$ and (the pentatonic pythagorean scale of *fr* in $M)(6) = r_6$. Then

(i) $\frac{r_2}{r_1} = \frac{(9 \textbf{ qua } \text{real number})}{8}$, and

(ii) $\frac{r_3}{r_2} = \frac{(9 \textbf{ qua } \text{real number})}{8}$, and

(iii) $\frac{r_4}{r_3} = \frac{(32 \textbf{ qua } \text{real number})}{27}$, and

(iv) $\frac{r_5}{r_4} = \frac{(9 \textbf{ qua } \text{real number})}{8}$, and

(v) $\frac{r_6}{r_5} = \frac{(32 \textbf{ qua } \text{real number})}{27}$.

The theorem is a consequence of (83), (78), (79), (80), (81), and (82).

(89) There exist positive real numbers r_1, r_2, r_3, r_4, r_5, r_6 such that

(i) (the pentatonic pythagorean scale of *fr* in $M)(1) = r_1$, and

(ii) (the pentatonic pythagorean scale of *fr* in $M)(2) = r_2$, and

(iii) (the pentatonic pythagorean scale of *fr* in $M)(3) = r_3$, and

(iv) (the pentatonic pythagorean scale of *fr* in $M)(4) = r_4$, and

(v) (the pentatonic pythagorean scale of *fr* in $M)(5) = r_5$, and

(vi) (the pentatonic pythagorean scale of *fr* in $M)(6) = r_6$, and

(vii) $\frac{r_2}{r_1} = \frac{(9 \textbf{ qua } \text{real number})}{8}$, and

(viii) $\frac{r_3}{r_2} = \frac{(9 \textbf{ qua } \text{real number})}{8}$, and

(ix) $\frac{r_4}{r_3} = \frac{(32 \textbf{ qua } \text{real number})}{27}$, and

(x) $\frac{r_5}{r_4} = \frac{(9 \textbf{ qua } \text{real number})}{8}$, and

(xi) $\frac{r_6}{r_5} = \frac{(32 \textbf{ qua } \text{real number})}{27}$.

The theorem is a consequence of (1) and (88).

(90) $\frac{(9 \textbf{ qua } \text{real number})}{8} = \frac{(9 \textbf{ qua } \text{rational number})}{8}$.

(91) (i) the interval between the first degree of pentatonic scale of *fr* in M and (the second degree of pentatonic scale of *fr* in M) = the pythagorean tone, and

(ii) the interval between the second degree of pentatonic scale of *fr* in M and (the third degree of pentatonic scale of *fr* in M) = the pythagorean tone, and

(iii) the interval between the third degree of pentatonic scale of *fr* in M and (the fourth degree of pentatonic scale of M, *fr*)) = the pythagorean semiditone, and

(iv) the interval between the fourth degree of pentatonic scale of M, *fr*) and (the fifth degree of pentatonic scale of *fr* in M) = the pythagorean tone, and

(v) the interval between the fifth degree of pentatonic scale of *fr* in M and (the octave of pentatonic scale of *fr* in M) = the pythagorean semiditone.

The theorem is a consequence of (89).

(92) the fifth of *fr* in M is between *fr* and the octave of *fr* in M.

Let us consider positive real numbers r_1, r_2. Now we state the propositions:

(93) Suppose $f_1 = r_1$ and $f_2 = r_2$ and $r_2 = \frac{(4 \textbf{ qua } \text{real number})}{3} \cdot r_1$. Then

(i) the fifth of f_2 in $M = 2 \cdot r_1$, and

(ii) the fifth of f_2 in M is not between f_1 and the octave of f_1 in M.

(94) Suppose $f_1 = r_1$ and $f_2 = r_2$ and $r_2 = \frac{(4 \textbf{ qua } \text{real number})}{3} \cdot r_1$. Then

(i) if the fifth of f_2 in M is between f_{10} and the octave of f_{10} in M, then the octave descending of (the reduct fifth of the f_2 with fundamental frequency f_{10} in M) in $M = f_1$, and

(ii) if the fifth of f_2 in M is not between f_{10} and the octave of f_{10} in M, then the reduct fifth of the f_2 with fundamental frequency f_{10} in $M = f_1$.

The theorem is a consequence of (62).

(95) Suppose $f_1 = r_1$ and $f_2 = r_2$ and $r_2 = \frac{(4 \textbf{ qua } \text{real number})}{3} \cdot r_1$. Then the reduct fifth of the f_2 with fundamental frequency f_1 in $M = f_1$. The theorem is a consequence of (94) and (93).

8. Heptatonic Pythagorean Scale

Let S be a space of music. We say that S is fourth constructible if and only if

(Def. 75) for every element *fr* of S, there exists an element q of S such that $\langle fr, q\rangle \in$ the set of fourth of S.

Now we state the propositions:

(96) Let us consider a space of music M. Suppose $M = \mathbb{R}$-music. Let us consider an element *fr* of M. Then there exist positive real numbers f, q_1 such that

(i) $f = fr$, and

(ii) $q_1 = \frac{(4 \textbf{ qua } \text{real number})}{3} \cdot f$, and

(iii) $\langle f, q_1 \rangle \in$ the set of fourth of M.

The theorem is a consequence of (1) and (24).

(97) $\mathbb{R}$-music is fourth constructible. The theorem is a consequence of (96) and (1).

One can verify that there exists a space of music which is fourth constructible.

Let M be a fourth constructible space of music and fr be an element of M. The fourth of fr in M yielding an element of M is defined by

(Def. 76) $\langle fr, it \rangle \in$ the set of fourth of M.

We say that M is classical fourth if and only if

(Def. 77) for every element fr of M, there exists a positive real number f such that $fr = f$ and the fourth of fr in $M = \frac{(4 \textbf{ qua } \text{real number})}{3} \cdot f$.

Now we state the proposition:

(98) Let us consider a fourth constructible space of music M. Suppose $M = \mathbb{R}$-music. Let us consider an element fr of M. Then there exists a positive real number f such that

(i) $fr = f$, and

(ii) the fourth of fr in $M = \frac{(4 \textbf{ qua } \text{real number})}{3} \cdot f$.

The theorem is a consequence of (1) and (96).

Let us note that there exists a fourth constructible space of music which is classical fourth.

Let M be a satisfying real, non empty structure of music. We say that M is euclidean if and only if

(Def. 78) for every elements f_1, f_2 of M, (the Ratio of M)$(f_1, f_2) = \frac{@f_2}{@f_1}$.

One can verify that there exists a satisfying real, non empty structure of music which is euclidean and every satisfying real, non empty structure of music which is euclidean is also satisfying interval and every satisfying real, non empty structure of music which is euclidean is also unison-ratio stable and every satisfying real, non empty structure of music which is euclidean is also ratio symmetric and there exists a classical fourth, fourth constructible space of music which is euclidean.

A heptatonic pythagorean score is a classical fourth, fourth constructible space of music. From now on H denotes a heptatonic pythagorean score and fr denotes an element of H.

Let H be a heptatonic pythagorean score and *fr* be an element of H. The heptatonic pythagorean scale of *fr* in H yielding an element of (the carrier of $H)^8$ is defined by

(Def. 79) $it(1)$ = (the spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(1)$ and $it(2)$ = (the spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(3)$ and $it(3)$ = (the spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(5)$ and $it(4)$ = the fourth of *fr* in H and $it(5)$ = (the spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(2)$ and $it(6)$ = (the spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(4)$ and $it(7)$ = (the spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(6)$ and $it(8)$ = the octave of (the spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(1)$ in H.

Now we state the propositions:

(99) the fourth of *fr* in H is between *fr* and the octave of *fr* in H.

(100) Let us consider a natural number n. Then (the spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(n)$ is between *fr* and the octave of *fr* in H.

(101) (The spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(1) = fr$. The theorem is a consequence of (66) and (62).

(102) (The spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(2) = \frac{(3 \textbf{ qua } \text{real number})}{2} \cdot ({}^{@}fr)$. The theorem is a consequence of (101) and (66).

(103) (The spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(3) = \frac{(9 \textbf{ qua } \text{real number})}{8} \cdot ({}^{@}fr)$. The theorem is a consequence of (102), (66), and (62).

(104) (The spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(4) = \frac{(27 \textbf{ qua } \text{real number})}{16} \cdot ({}^{@}fr)$. The theorem is a consequence of (103) and (66).

(105) (The spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(5) = \frac{(81 \textbf{ qua } \text{real number})}{64} \cdot ({}^{@}fr)$. The theorem is a consequence of (104), (66), and (62).

(106) (The spiral of fifths of (the fourth of *fr* in H) with fundamental frequency *fr* in $H)(6) = \frac{(243 \textbf{ qua } \text{real number})}{128} \cdot ({}^{@}fr)$. The theorem is a consequence of (105) and (66).

(107) (i) (the heptatonic pythagorean scale of *fr* in $H)(1) = 1 \cdot ({}^{@}fr)$, and

(ii) (the heptatonic pythagorean scale of *fr* in H)(2) $= \frac{(9 \textbf{ qua} \text{ real number})}{8} \cdot$ $({}^{@}\mathit{fr})$, and

(iii) (the heptatonic pythagorean scale of *fr* in H)(3) $= \frac{(81 \textbf{ qua} \text{ real number})}{64} \cdot$ $({}^{@}\mathit{fr})$, and

(iv) (the heptatonic pythagorean scale of *fr* in H)(4) $= \frac{(4 \textbf{ qua} \text{ real number})}{3} \cdot$ $({}^{@}\mathit{fr})$, and

(v) (the heptatonic pythagorean scale of *fr* in H)(5) $= \frac{(3 \textbf{ qua} \text{ real number})}{2} \cdot$ $({}^{@}\mathit{fr})$, and

(vi) (the heptatonic pythagorean scale of *fr* in H)(6) $= \frac{(27 \textbf{ qua} \text{ real number})}{16} \cdot$ $({}^{@}\mathit{fr})$, and

(vii) (the heptatonic pythagorean scale of *fr* in H)(7) $= \frac{(243 \textbf{ qua} \text{ real number})}{128} \cdot$ $({}^{@}\mathit{fr})$, and

(viii) (the heptatonic pythagorean scale of *fr* in H)(8) $= 2 \cdot ({}^{@}\mathit{fr})$.

The theorem is a consequence of (101), (103), (105), (102), (104), and (106).

(108) The heptatonic pythagorean scale of *fr* in H is heptatonic. The theorem is a consequence of (107).

The pythagorean semitone yielding a positive real number is defined by the term

(Def. 80) $\frac{(256 \textbf{ qua} \text{ real number})}{243}$.

We introduce the notation LimmaPyth as a synonym of the pythagorean semitone.

We introduce the notation DiatonicTone as a synonym of the pythagorean tone.

Now we state the propositions:

(109) $\frac{\text{the pythagorean tone}}{2} <$ the pythagorean semitone.

(110) (The pythagorean tone)·(the pythagorean tone)·(the pythagorean semitone)· (the pythagorean tone) · (the pythagorean tone) · (the pythagorean tone) · (the pythagorean semitone) = 2.

Let H be a heptatonic pythagorean score and *fr* be an element of H. The functors: the first degree of heptatonic scale of *fr* in H, the second degree of heptatonic the third degree of heptatonic scale of *fr* in H, the fourth degree of heptatonic scale of and the fifth degree of heptatonic scale of *fr* in H yielding elements of H are defined by terms

(Def. 81) (the heptatonic pythagorean scale of *fr* in H)(1),

(Def. 82) (the heptatonic pythagorean scale of *fr* in H)(2), (Def. 83) (the heptatonic pythagorean scale of *fr* in H)(3),

(Def. 84) (the heptatonic pythagorean scale of *fr* in H)(4),

(Def. 85) (the heptatonic pythagorean scale of *fr* in H)(5),

respectively. The functors: the sixth degree of heptatonic scale of *fr* in H, the seventh d and the eight degree of heptatonic scale of *fr* in H yielding elements of H are defined by terms

(Def. 86) (the heptatonic pythagorean scale of *fr* in H)(6),

(Def. 87) (the heptatonic pythagorean scale of *fr* in H)(7),

(Def. 88) the octave of *fr* in H,

respectively. Now we state the proposition:

(111) (i) the interval between the first degree of heptatonic scale of *fr* in H and (the second degree of heptatonic scale of *fr* in H) = the pythagorean tone, and

(ii) the interval between the second degree of heptatonic scale of *fr* in H and (the third degree of heptatonic scale of *fr* in H) = the pythagorean tone, and

(iii) the interval between the third degree of heptatonic scale of *fr* in H and (the fourth degree of heptatonic scale of *fr* in H) = the pythagorean semitone, and

(iv) the interval between the fourth degree of heptatonic scale of *fr* in H and (the fifth degree of heptatonic scale of *fr* in H) = the pythagorean tone, and

(v) the interval between the fifth degree of heptatonic scale of *fr* in H and (the sixth degree of heptatonic scale of *fr* in H) = the pythagorean tone, and

(vi) the interval between the sixth degree of heptatonic scale of *fr* in H and (the seventh degree of heptatonic scale of *fr* in H) = the pythagorean tone, and

(vii) the interval between the seventh degree of heptatonic scale of *fr* in H and (the eight degree of heptatonic scale of *fr* in H) = the pythagorean semitone.

The theorem is a consequence of (107).

From now on H denotes a heptatonic pythagorean score and *fr* denotes an element of H.

Let M be a space of music, n be a natural number, and s_{10} be an element of (the carrier of M)^n. Assume s_{10} is heptatonic. We say that s_{10} is perfect fifth if and only if

(Def. 89) $\langle s_{10}(1), s_{10}(5)\rangle$, $\langle s_{10}(2), s_{10}(6)\rangle$, $\langle s_{10}(3), s_{10}(7)\rangle$, $\langle s_{10}(4), s_{10}(8)\rangle \in$ the set of fifth of M.

Now we state the proposition:

(112) Let us consider an euclidean heptatonic pythagorean score H, and an element *fr* of H. Then the heptatonic pythagorean scale of *fr* in H is perfect fifth. The theorem is a consequence of (108), (107), and (24).

Let H be a heptatonic pythagorean score and *fr* be an element of H. The heptatonic pythagorean scale ascending of *fr* in H yielding an element of (the carrier of $H)^8$ is defined by the term

(Def. 90) the heptatonic pythagorean scale of (the octave of *fr* in H) in H.

Now we state the propositions:

(113) (i) (the heptatonic pythagorean scale ascending of *fr* in H)(1) $= 2 \cdot (^{@}fr)$, and

(ii) (the heptatonic pythagorean scale ascending of *fr* in H)(2) $= \frac{(9 \textbf{ qua } \text{real number})}{4} \cdot (^{@}fr)$, and

(iii) (the heptatonic pythagorean scale ascending of *fr* in H)(3) $= \frac{(81 \textbf{ qua } \text{real number})}{32}$ $(^{@}fr)$, and

(iv) (the heptatonic pythagorean scale ascending of *fr* in H)(4) $= \frac{(8 \textbf{ qua } \text{real number})}{3} \cdot (^{@}fr)$, and

(v) (the heptatonic pythagorean scale ascending of *fr* in H)(5) $= (3 \textbf{ qua } \text{real number}) \cdot (^{@}fr)$, and

(vi) (the heptatonic pythagorean scale ascending of *fr* in H)(6) $= \frac{(27 \textbf{ qua } \text{real number})}{8}$ $(^{@}fr)$, and

(vii) (the heptatonic pythagorean scale ascending of *fr* in H)(7) $= \frac{(243 \textbf{ qua } \text{real number}}{64}$ $(^{@}fr)$, and

(viii) (the heptatonic pythagorean scale ascending of *fr* in H)(8) $= 4 \cdot (^{@}fr)$.

The theorem is a consequence of (107).

(114) (The heptatonic pythagorean scale of *fr* in H)(8) = (the heptatonic pythagorean scale ascending of *fr* in H)(1). The theorem is a consequence of (107) and (113).

(115) (i) the interval between the fifth degree of heptatonic scale of *fr* in H and (the second degree of heptatonic scale of (the octave of *fr* in H) in H) $= \frac{(3 \textbf{ qua } \text{real number})}{2}$, and

(ii) the interval between the sixth degree of heptatonic scale of *fr* in H and (the third degree of heptatonic scale of (the octave of *fr* in H) in H) $= \frac{(3 \textbf{ qua } \text{real number})}{2}$, and

(iii) the interval between the seventh degree of heptatonic scale of *fr* in H and (the fourth degree of heptatonic scale of (the octave of *fr* in H) in H) $\neq \frac{(3 \textbf{ qua } \text{real number})}{2}$, and

(iv) the interval between the eight degree of heptatonic scale of *fr* in H and (the fifth degree of heptatonic scale of (the octave of *fr* in H) in H) $= \frac{(3 \textbf{ qua } \text{real number})}{2}$.

The theorem is a consequence of (107) and (113).

(116) Let us consider an euclidean heptatonic pythagorean score H, and elements f_1, f_2 of H. Then the interval between f_1 and f_2 = (the Ratio of H)(f_1, f_2).

(117) Let us consider an euclidean heptatonic pythagorean score H, and an element *fr* of H. Then

(i) $\langle$(the heptatonic pythagorean scale of *fr* in H)(5), (the heptatonic pythagorean scale ascending of *fr* in H)(2)$\rangle$, $\langle$(the heptatonic pythagorean scale of *fr* in H)(6), (the heptatonic pythagorean scale ascending of *fr* in H)(3)$\rangle$ $\in$ the set of fifth of H, and

(ii) $\langle$(the heptatonic pythagorean scale of *fr* in H)(7), (the heptatonic pythagorean scale ascending of *fr* in H)(4)$\rangle$ $\notin$ the set of fifth of H.

The theorem is a consequence of (115), (24), and (116).

Let H be a space of music, n be a non zero, natural number, s_{10} be an element of (the carrier of H)^n, and i be a natural number. The functor $\#_i^{s_{10}}$ yielding an element of H is defined by the term

(Def. 91) $\begin{cases} s_{10}(i), & \textbf{if } i \in \operatorname{Seg} n, \\ \text{the element of } H, & \textbf{otherwise}. \end{cases}$

Assume s_{10} is heptatonic. We say that s_{10} is dorian if and only if

(Def. 92) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_1$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_2$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_1$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_1$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_1$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_2$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_1$.

Assume s_{10} is heptatonic. We say that s_{10} is hypodorian if and only if

(Def. 93) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_1$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_2$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_1$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_1$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_2$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_1$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_1$.

Assume s_{10} is heptatonic. We say that **s_{10} is phrygian** if and only if

(Def. 94) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_1$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_2$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_1$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_2$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_1$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_1$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_1$.

Assume s_{10} is heptatonic. We say that **s_{10} is hypophrygian** if and only if

(Def. 95) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_2$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_1$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_1$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_2$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_1$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_1$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_1$.

Assume s_{10} is heptatonic. We say that **s_{10} is lydian** if and only if

(Def. 96) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_1$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_1$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_2$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_1$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_1$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_2$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_1$.

Assume s_{10} is heptatonic. We say that **s_{10} is hypolydian** if and only if

(Def. 97) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_1$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_1$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_2$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_1$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_1$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_1$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_2$.

Assume s_{10} is heptatonic. We say that **s_{10} is mixolydian** if and only if

(Def. 98) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_1$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_1$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_2$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_1$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_1$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_2$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_1$.

Assume s_{10} is heptatonic. We say that **s_{10} is hypomixolydian** if and only if

(Def. 99) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_1$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_2$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_1$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_1$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_1$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_2$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_1$.

Assume s_{10} is heptatonic. We say that s_{10} is eolian if and only if

(Def. 100) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_1$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_2$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_1$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_1$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_2$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_1$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_1$.

Assume s_{10} is heptatonic. We say that s_{10} is hypoeolian if and only if

(Def. 101) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_2$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_1$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_1$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_1$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_2$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_1$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_1$.

Assume s_{10} is heptatonic. We say that s_{10} is ionan if and only if

(Def. 102) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_1$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_1$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_2$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_1$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_1$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_1$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_2$.

Assume s_{10} is heptatonic. We say that s_{10} is hypoionan if and only if

(Def. 103) there exist positive real numbers t_1, t_2 such that $t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_1 \cdot t_2 \cdot t_2 = 2$ and the interval between $\#_1^{s_{10}}$ and $\#_2^{s_{10}} = t_1$ and the interval between $\#_2^{s_{10}}$ and $\#_3^{s_{10}} = t_1$ and the interval between $\#_3^{s_{10}}$ and $\#_4^{s_{10}} = t_2$ and the interval between $\#_4^{s_{10}}$ and $\#_5^{s_{10}} = t_1$ and the interval between $\#_5^{s_{10}}$ and $\#_6^{s_{10}} = t_1$ and the interval between $\#_6^{s_{10}}$ and $\#_7^{s_{10}} = t_2$ and the interval between $\#_7^{s_{10}}$ and $\#_8^{s_{10}} = t_1$.

Now we state the proposition:

(118) The heptatonic pythagorean scale of *fr* in H is ionan. The theorem is a consequence of (108), (107), and (111).

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