

# Some Remarks about Product Spaces

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**Summary.** This article covers some technical aspects about the product topology which are usually not given much of a thought in mathematics and standard literature like [19] and [18], not even by Bourbaki in [9].

Let  $\{\mathcal{T}_i\}_{i \in I}$  be a family of topological spaces. The prebasis of the product space  $\mathcal{T} = \prod_{i \in I} \mathcal{T}_i$  is defined in [17] as the set of all  $\pi_i^{-1}(V)$  with  $i \in I$  and  $V$  open in  $\mathcal{T}_i$ . Here it is shown that the basis generated by this prebasis consists exactly of the sets  $\prod_{i \in I} V_i$  with  $V_i$  open in  $\mathcal{T}_i$  and for all but finitely many  $i \in I$  holds  $V_i = \mathcal{T}_i$ . Given  $I = \{a\}$  we have  $\mathcal{T} \cong \mathcal{T}_a$ , given  $I = \{a, b\}$  with  $a \neq b$  we have  $\mathcal{T} \cong \mathcal{T}_a \times \mathcal{T}_b$ . Given another family of topological spaces  $\{\mathcal{S}_i\}_{i \in I}$  such that  $\mathcal{S}_i \cong \mathcal{T}_i$  for all  $i \in I$ , we have  $\mathcal{S} = \prod_{i \in I} \mathcal{S}_i \cong \mathcal{T}$ . If instead  $\mathcal{S}_i$  is a subspace of  $\mathcal{T}_i$  for each  $i \in I$ , then  $\mathcal{S}$  is a subspace of  $\mathcal{T}$ .

These results are obvious for mathematicians, but formally proven here by means of the Mizar system [8], [7].

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## 1. PRELIMINARIES

Now we state the propositions:

- (1) Let us consider a one-to-one function  $f$ , and an object  $y$ . Suppose  $\text{rng } f = \{y\}$ . Then  $\text{dom } f = \{(f^{-1})(y)\}$ .

PROOF: Consider  $x_0$  being an object such that  $x_0 \in \text{dom } f$  and  $f(x_0) = y$ .

For every object  $x$ ,  $x \in \text{dom } f$  iff  $x = (f^{-1})(y)$  by [10, (3), (34)].  $\square$

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- (2) Let us consider a one-to-one function  $f$ , and objects  $y_1, y_2$ . Suppose  $\text{rng } f = \{y_1, y_2\}$ . Then  $\text{dom } f = \{(f^{-1})(y_1), (f^{-1})(y_2)\}$ .

PROOF: Consider  $x_1$  being an object such that  $x_1 \in \text{dom } f$  and  $f(x_1) = y_1$ . Consider  $x_2$  being an object such that  $x_2 \in \text{dom } f$  and  $f(x_2) = y_2$ . For every object  $x$ ,  $x \in \text{dom } f$  iff  $x = (f^{-1})(y_1)$  or  $x = (f^{-1})(y_2)$  by [10, (3), (34)].  $\square$

Let  $X, Y$  be sets. Note that there exists a function which is empty,  $X$ -defined,  $Y$ -valued, and one-to-one.

Let  $T, S$  be sets,  $f$  be a function from  $T$  into  $S$ , and  $G$  be a finite family of subsets of  $T$ . Let us note that  $f^\circ G$  is finite.

Now we state the propositions:

- (3) Let us consider a set  $A$ , a family  $F$  of subsets of  $A$ , and a binary relation  $R$ . Then  $R^\circ(\cap F) \subseteq \cap \{R^\circ X, \text{ where } X \text{ is a subset of } A : X \in F\}$ .

- (4) Let us consider a set  $A$ , a family  $F$  of subsets of  $A$ , and a one-to-one function  $f$ . Then  $f^\circ(\cap F) = \cap \{f^\circ X, \text{ where } X \text{ is a subset of } A : X \in F\}$ .

PROOF: Set  $S = \{f^\circ X, \text{ where } X \text{ is a subset of } A : X \in F\}$ .  $\cap S \subseteq f^\circ(\cap F)$  by [21, (1)].  $f^\circ(\cap F) \subseteq \cap S$ .  $\square$

- (5) Let us consider a set  $X$ , a non empty set  $Y$ , and a function  $f$  from  $X$  into  $Y$ . Then  $\{f^{-1}(\{y\}), \text{ where } y \text{ is an element of } Y : y \in \text{rng } f\}$  is a partition of  $X$ .

PROOF: Set  $P = \{f^{-1}(\{y\}), \text{ where } y \text{ is an element of } Y : y \in \text{rng } f\}$ . For every object  $x$ ,  $x \in X$  iff there exists a set  $A$  such that  $x \in A$  and  $A \in P$  by [11, (4), (5), (38)]. For every subset  $A$  of  $X$  such that  $A \in P$  holds  $A \neq \emptyset$  and for every subset  $B$  of  $X$  such that  $B \in P$  holds  $A = B$  or  $A$  misses  $B$  by [11, (11), (38)], [14, (11)], [10, (71)].  $P \subseteq 2^X$ .  $\square$

- (6) Let us consider a non empty set  $X$ , and objects  $x, y$ . If  $X \mapsto x = X \mapsto y$ , then  $x = y$ .

- (7) Let us consider an object  $i$ , and a many sorted set  $J$  indexed by  $\{i\}$ . Then  $J = \{i\} \mapsto J(i)$ .

PROOF: For every object  $x$  such that  $x \in \text{dom } J$  holds  $J(x) = (\{i\} \mapsto J(i))(x)$  by [24, (72)].  $\square$

- (8) Let us consider a 2-element set  $I$ , and elements  $i, j$  of  $I$ . If  $i \neq j$ , then  $I = \{i, j\}$ .

PROOF: For every object  $x$ ,  $x = i$  or  $x = j$  iff  $x \in I$  by [2, (58)], [1, (11), (51)].  $\square$

- (9) Let us consider a 2-element set  $I$ , a many sorted set  $f$  indexed by  $I$ , and elements  $i, j$  of  $I$ . If  $i \neq j$ , then  $f = [i \mapsto f(i), j \mapsto f(j)]$ . The theorem is a consequence of (8).

(10) Let us consider objects  $a, b, c, d$ . If  $a \neq b$ , then  $[a \mapsto c, b \mapsto d] = [b \mapsto d, a \mapsto c]$ .

PROOF: For every object  $x$  such that  $x \in \text{dom}[a \mapsto c, b \mapsto d]$  holds  $[a \mapsto c, b \mapsto d](x) = [b \mapsto d, a \mapsto c](x)$  by [13, (63)].  $\square$

(11) Let us consider a function  $f$ , and objects  $i, j$ . If  $i, j \in \text{dom } f$ , then  $f = f + \cdot [i \mapsto f(i), j \mapsto f(j)]$ .

(12) Let us consider objects  $x, y, z$ . Then  $x \mapsto y + \cdot (x \mapsto z) = x \mapsto z$ .

Let us observe that there exists a function which is non non-empty.

Now we state the propositions:

(13) Let us consider non empty sets  $X, Y$ , and an element  $y$  of  $Y$ . Then  $X \mapsto y \in \prod(X \mapsto Y)$ .

PROOF: Set  $f = X \mapsto y$ . For every object  $x$  such that  $x \in \text{dom}(X \mapsto Y)$  holds  $f(x) \in (X \mapsto Y)(x)$  by [24, (7)].  $\square$

(14) Let us consider a non empty set  $X$ , a set  $Y$ , and a subset  $Z$  of  $Y$ . Then  $\prod(X \mapsto Z) \subseteq \prod(X \mapsto Y)$ .

(15) Let us consider a non empty set  $X$ , and an object  $i$ . Then  $\prod(\{i\} \mapsto X) = \{\{i\} \mapsto x, \text{ where } x \text{ is an element of } X\}$ .

PROOF: Set  $S = \{\{i\} \mapsto x, \text{ where } x \text{ is an element of } X\}$ . For every object  $z, z \in \prod(\{i\} \mapsto X)$  iff  $z \in S$  by [24, (13), (11), (72)], (13).  $\square$

(16) Let us consider a non empty set  $X$ , and objects  $i, f$ . Then  $f \in \prod(\{i\} \mapsto X)$  if and only if there exists an element  $x$  of  $X$  such that  $f = \{i\} \mapsto x$ . The theorem is a consequence of (15).

(17) Let us consider a non empty set  $X$ , an object  $i$ , and an element  $x$  of  $X$ . Then  $(\text{proj}(\{i\} \mapsto X, i))(\{i\} \mapsto x) = x$ . The theorem is a consequence of (13).

(18) Let us consider sets  $X, Y$ . Then  $X \neq \emptyset$  and  $Y = \emptyset$  if and only if  $\prod(X \mapsto Y) = \emptyset$ .

Let  $f$  be an empty function and  $x$  be an object. Let us note that  $\text{proj}(f, x)$  is trivial.

Now we state the proposition:

(19) Let us consider a trivial function  $f$ , and an object  $x$ . If  $x \in \text{dom } f$ , then  $\text{proj}(f, x)$  is one-to-one.

PROOF: Consider  $t$  being an object such that  $\text{dom } f = \{t\}$ . Set  $F = \text{proj}(f, x)$ . For every objects  $y, z$  such that  $y, z \in \text{dom } F$  and  $F(y) = F(z)$  holds  $y = z$  by [10, (2)].  $\square$

Let  $x, y$  be objects. Note that  $\text{proj}(x \mapsto y, x)$  is one-to-one.

Let  $I$  be a 1-element set,  $J$  be a many sorted set indexed by  $I$ , and  $i$  be an element of  $I$ . One can verify that  $\text{proj}(J, i)$  is one-to-one.

Now we state the propositions:

- (20) Let us consider a non empty set  $X$ , a subset  $Y$  of  $X$ , and an object  $i$ . Then  $(\text{proj}(\{i\} \mapsto X, i))^\circ(\prod(\{i\} \mapsto Y)) = Y$ . The theorem is a consequence of (16), (13), and (14).
- (21) Let us consider non-empty functions  $f, g$ , and objects  $i, x$ . Suppose  $x \in \prod f \cap \prod(f+g)$ . Then  $(\text{proj}(f, i))(x) = (\text{proj}(f+g, i))(x)$ .
- (22) Let us consider non-empty functions  $f, g$ , an object  $i$ , and a set  $A$ . Suppose  $A \subseteq \prod f \cap \prod(f+g)$ . Then  $(\text{proj}(f, i))^\circ A = (\text{proj}(f+g, i))^\circ A$ . The theorem is a consequence of (21).
- (23) Let us consider non-empty functions  $f, g$ . Suppose  $\text{dom } g \subseteq \text{dom } f$  and for every object  $i$  such that  $i \in \text{dom } g$  holds  $g(i) \subseteq f(i)$ . Then  $\prod(f+g) \subseteq \prod f$ .

Let us consider non-empty functions  $f, g$  and an object  $i$ . Now we state the propositions:

- (24) Suppose  $\text{dom } g \subseteq \text{dom } f$  and for every object  $i$  such that  $i \in \text{dom } g$  holds  $g(i) \subseteq f(i)$ . Then if  $i \in \text{dom } f \setminus \text{dom } g$ , then  $(\text{proj}(f, i))^\circ(\prod(f+g)) = f(i)$ . The theorem is a consequence of (23) and (22).
- (25) Suppose  $\text{dom } g \subseteq \text{dom } f$  and for every object  $i$  such that  $i \in \text{dom } g$  holds  $g(i) \subseteq f(i)$ . Then if  $i \in \text{dom } g$ , then  $(\text{proj}(f, i))^\circ(\prod(f+g)) = g(i)$ . The theorem is a consequence of (23) and (22).
- (26) Suppose  $\text{dom } g = \text{dom } f$  and for every object  $i$  such that  $i \in \text{dom } g$  holds  $g(i) \subseteq f(i)$ . Then if  $i \in \text{dom } g$ , then  $(\text{proj}(f, i))^\circ(\prod g) = g(i)$ . The theorem is a consequence of (25).

Now we state the propositions:

- (27) Let us consider a function  $f$ , sets  $X, Y$ , and an object  $i$ . Suppose  $X \subseteq Y$ . Then  $\prod(f+(i \mapsto X)) \subseteq \prod(f+(i \mapsto Y))$ .
- (28) Let us consider objects  $i, j$ , and sets  $A, B, C, D$ . Suppose  $A \subseteq C$  and  $B \subseteq D$ . Then  $\prod[i \mapsto A, j \mapsto B] \subseteq \prod[i \mapsto C, j \mapsto D]$ . The theorem is a consequence of (14).
- (29) Let us consider sets  $X, Y$ , and objects  $f, i, j$ . Suppose  $i \neq j$ . Then  $f \in \prod[i \mapsto X, j \mapsto Y]$  if and only if there exist objects  $x, y$  such that  $x \in X$  and  $y \in Y$  and  $f = [i \mapsto x, j \mapsto y]$ .

PROOF: If  $f \in \prod[i \mapsto X, j \mapsto Y]$ , then there exist objects  $x, y$  such that  $x \in X$  and  $y \in Y$  and  $f = [i \mapsto x, j \mapsto y]$  by [13, (62), (63), (66)]. Reconsider  $g = f$  as a function. For every object  $z$  such that  $z \in \text{dom}[i \mapsto X, j \mapsto Y]$  holds  $g(z) \in [i \mapsto X, j \mapsto Y](z)$  by [13, (62), (63)].  $\square$

- (30) Let us consider a non-empty function  $f$ , sets  $X, Y$ , objects  $i, j, x, y$ ,

and a function  $g$ . Suppose  $x \in X$  and  $y \in Y$  and  $i \neq j$  and  $g \in \prod f$ . Then  $g + \cdot [i \mapsto x, j \mapsto y] \in \prod (f + \cdot [i \mapsto X, j \mapsto Y])$ .

PROOF: For every object  $z$  such that  $z \in \text{dom}(f + \cdot [i \mapsto X, j \mapsto Y])$  holds  $(g + \cdot [i \mapsto x, j \mapsto y])(z) \in (f + \cdot [i \mapsto X, j \mapsto Y])(z)$  by [13, (62), (13), (63)].  $\square$

(31) Let us consider a function  $f$ , sets  $A, B, C, D$ , and objects  $i, j$ . Suppose  $A \subseteq C$  and  $B \subseteq D$ . Then  $\prod (f + \cdot [i \mapsto A, j \mapsto B]) \subseteq \prod (f + \cdot [i \mapsto C, j \mapsto D])$ . The theorem is a consequence of (27).

(32) Let us consider a function  $f$ , sets  $A, B$ , and objects  $i, j$ . Suppose  $i, j \in \text{dom } f$  and  $A \subseteq f(i)$  and  $B \subseteq f(j)$ . Then  $\prod (f + \cdot [i \mapsto A, j \mapsto B]) \subseteq \prod f$ . The theorem is a consequence of (11) and (31).

(33) Let us consider a set  $I$ , and many sorted sets  $f, g$  indexed by  $I$ . Then  $\prod f \cap \prod g = \prod (f \cap g)$ .

PROOF: For every object  $x$ ,  $x \in \prod f \cap \prod g$  iff there exists a function  $h$  such that  $h = x$  and  $\text{dom } h = \text{dom}(f \cap g)$  and for every object  $y$  such that  $y \in \text{dom}(f \cap g)$  holds  $h(y) \in (f \cap g)(y)$  by [3, (9)].  $\square$

(34) Let us consider a 2-element set  $I$ , a many sorted set  $f$  indexed by  $I$ , elements  $i, j$  of  $I$ , and an object  $x$ . Suppose  $i \neq j$ . Then

$$(i) \quad f + \cdot (i, x) = [i \mapsto x, j \mapsto f(j)], \text{ and}$$

$$(ii) \quad f + \cdot (j, x) = [i \mapsto f(i), j \mapsto x].$$

The theorem is a consequence of (10).

Let us consider a non-empty function  $f$ , a set  $X$ , and an object  $i$ . Now we state the propositions:

(35) If  $i \in \text{dom } f$ , then  $f + \cdot (i, X)$  is non-empty iff  $X$  is not empty.

PROOF: For every object  $x$  such that  $x \in \text{dom}(f + \cdot (i, X))$  holds  $(f + \cdot (i, X))(x)$  is not empty by [6, (30), (32), (31)].  $\square$

(36) If  $i \in \text{dom } f$ , then  $\prod (f + \cdot (i, X)) = \emptyset$  iff  $X$  is empty. The theorem is a consequence of (35).

Now we state the propositions:

(37) Let us consider a non-empty function  $f$ , a set  $X$ , objects  $i, x$ , and a function  $g$ . Suppose  $i \in \text{dom } f$  and  $x \in X$  and  $g \in \prod f$ . Then  $g + \cdot (i, x) \in \prod (f + \cdot (i, X))$ .

PROOF: For every object  $y$  such that  $y \in \text{dom}(f + \cdot (i, X))$  holds  $(g + \cdot (i, x))(y) \in (f + \cdot (i, X))(y)$  by [6, (30)], [3, (9)], [6, (31), (32)].  $\square$

(38) Let us consider a function  $f$ , sets  $X, Y$ , and an object  $i$ . Suppose  $i \in \text{dom } f$  and  $X \subseteq Y$ . Then  $\prod (f + \cdot (i, X)) \subseteq \prod (f + \cdot (i, Y))$ . The theorem is a consequence of (27).

- (39) Let us consider a function  $f$ , a set  $X$ , and an object  $i$ . Suppose  $i \in \text{dom } f$  and  $X \subseteq f(i)$ . Then  $\prod(f + \cdot (i, X)) \subseteq \prod f$ . The theorem is a consequence of (38).
- (40) Let us consider a non-empty function  $f$ , non empty sets  $X, Y$ , and objects  $i, j$ . Suppose  $i, j \in \text{dom } f$  and  $(X \not\subseteq f(i) \text{ or } f(j) \not\subseteq Y)$  and  $\prod(f + \cdot (i, X)) \subseteq \prod(f + \cdot (j, Y))$ . Then
- (i)  $i = j$ , and
  - (ii)  $X \subseteq Y$ .

PROOF:  $f + \cdot (i, X)$  is non-empty and  $f + \cdot (j, Y)$  is non-empty.  $i = j$  by [6, (30)], [4, (1)], [6, (31), (32)]. Set  $g =$  the element of  $\prod f$ .  $g + \cdot (i, x) \in \prod(f + \cdot (i, X))$ .  $\square$

- (41) Let us consider a non-empty function  $f$ , a set  $X$ , and an object  $i$ . Suppose  $i \in \text{dom } f$  and  $\prod(f + \cdot (i, X)) \subseteq \prod f$ . Then  $X \subseteq f(i)$ . The theorem is a consequence of (37).
- (42) Let us consider a non-empty function  $f$ , non empty sets  $X, Y$ , and objects  $i, j$ . Suppose  $i, j \in \text{dom } f$  and  $(X \neq f(i) \text{ or } Y \neq f(j))$  and  $\prod(f + \cdot (i, X)) = \prod(f + \cdot (j, Y))$ . Then
- (i)  $i = j$ , and
  - (ii)  $X = Y$ .

PROOF:  $f + \cdot (i, X)$  is non-empty and  $f + \cdot (j, Y)$  is non-empty.  $i = j$  by [6, (31), (32)].  $\square$

- (43) Let us consider a non-empty function  $f$ , a set  $X$ , and an object  $i$ . Suppose  $i \in \text{dom } f$  and  $X \subseteq f(i)$ . Then  $(\text{proj}(f, i))^\circ(\prod(f + \cdot (i, X))) = X$ . The theorem is a consequence of (25).
- (44) Let us consider objects  $x, y, z$ . Then  $x \mapsto y + \cdot (x, z) = x \mapsto z$ . The theorem is a consequence of (12).

Let  $I$  be a non empty set and  $J$  be a 1-sorted yielding, nonempty many sorted set indexed by  $I$ . Let us observe that the support of  $J$  is non-empty.

## 2. REMARKS ABOUT PRODUCT SPACES

Now we state the propositions:

- (45) Let us consider topological spaces  $T, S$ , and a function  $f$  from  $T$  into  $S$ . Then  $f$  is open if and only if there exists a basis  $B$  of  $T$  such that for every subset  $V$  of  $T$  such that  $V \in B$  holds  $f^\circ V$  is open.

- (46) Let us consider non empty topological spaces  $T_1, T_2, S_1, S_2$ , a function  $f$  from  $T_1$  into  $S_1$ , and a function  $g$  from  $T_2$  into  $S_2$ . If  $f$  is open and  $g$  is open, then  $f \times g$  is open.

PROOF: There exists a basis  $B$  of  $T_1 \times T_2$  such that for every subset  $P$  of  $T_1 \times T_2$  such that  $P \in B$  holds  $(f \times g)^\circ P$  is open by [5, (40)], [12, (72)], [23, (6)].  $\square$

Let us consider non empty topological spaces  $S, T$  and a function  $f$  from  $S$  into  $T$ . Now we state the propositions:

- (47) If  $f$  is bijective and there exists a basis  $K$  of  $S$  and there exists a basis  $L$  of  $T$  such that  $f^\circ K = L$ , then  $f$  is a homeomorphism.

PROOF: For every subset  $W$  of  $T$  such that  $W \in L$  holds  $f^{-1}(W)$  is open by [10, (94)]. For every subset  $V$  of  $S$  such that  $V \in K$  holds  $f^\circ V$  is open.  $f$  is open.  $\square$

- (48) If  $f$  is bijective and there exists a prebasis  $K$  of  $S$  and there exists a prebasis  $L$  of  $T$  such that  $f^\circ K = L$ , then  $f$  is a homeomorphism.

PROOF: Reconsider  $K_0 = \text{FinMeetCl}(K)$  as a basis of  $S$ . Reconsider  $L_0 = \text{FinMeetCl}(L)$  as a basis of  $T$ . For every subset  $W$  of  $T$ ,  $W \in L_0$  iff there exists a subset  $V$  of  $S$  such that  $V \in K_0$  and  $f^\circ V = W$  by [10, (85), (94), (77)], (4).  $\square$

Let us consider topological spaces  $S, T$ . Now we state the propositions:

- (49) If there exists a basis  $K$  of  $S$  and there exists a basis  $L$  of  $T$  such that  $K = L \pitchfork \{\Omega_S\}$ , then  $S$  is a subspace of  $T$ .

PROOF: For every subset  $A$  of  $S$ ,  $A \in$  the topology of  $S$  iff there exists a subset  $B$  of  $T$  such that  $B \in$  the topology of  $T$  and  $A = B \cap \Omega_S$  by [16, (19)], [21, (25)]. Consider  $B$  being a subset of  $T$  such that  $B \in$  the topology of  $T$  and the carrier of  $S = B \cap \Omega_S$ .  $\square$

- (50) Suppose  $\Omega_S \subseteq \Omega_T$  and there exists a prebasis  $K$  of  $S$  and there exists a prebasis  $L$  of  $T$  such that  $K = L \pitchfork \{\Omega_S\}$ . Then  $S$  is a subspace of  $T$ .

PROOF: Reconsider  $K_0 = \text{FinMeetCl}(K)$  as a basis of  $S$ . Reconsider  $L_0 = \text{FinMeetCl}(L)$  as a basis of  $T$ . For every object  $x$ ,  $x \in K_0$  iff  $x \in L_0 \pitchfork \{\Omega_S\}$  by [15, (8)], [10, (3)].  $\square$

- (51) If there exists a prebasis  $K$  of  $S$  and there exists a prebasis  $L$  of  $T$  such that  $\Omega_S \in K$  and  $K = L \pitchfork \{\Omega_S\}$ , then  $S$  is a subspace of  $T$ . The theorem is a consequence of (50).

Now we state the proposition:

- (52) Let us consider a non empty set  $I$ , a topological structure yielding, non-empty many sorted set  $J$  indexed by  $I$ , and an element  $i$  of  $I$ . Then  $\text{rng proj}(J, i) =$  the carrier of  $J(i)$ .

Let  $X$  be a set and  $T$  be a topological structure. Observe that  $X \mapsto T$  is topological structure yielding.

Let  $F$  be a binary relation. We say that  $F$  is topological space yielding if and only if

(Def. 1) for every object  $x$  such that  $x \in \text{rng } F$  holds  $x$  is a topological space.

Note that every binary relation which is topological space yielding is also topological structure yielding and every function which is topological space yielding is also 1-sorted yielding.

Let  $X$  be a set and  $T$  be a topological space. One can verify that  $X \mapsto T$  is topological space yielding.

Let  $I$  be a set. One can verify that there exists a many sorted set indexed by  $I$  which is topological space yielding and nonempty.

Let  $I$  be a non empty set,  $J$  be a topological space yielding, nonempty many sorted set indexed by  $I$ , and  $i$  be an element of  $I$ . Let us note that the functor  $J(i)$  yields a non empty topological space. Let  $f$  be a function. The functor  $\text{ProjMap } f$  yielding a many sorted function indexed by  $\text{dom } f$  is defined by

(Def. 2) for every object  $x$  such that  $x \in \text{dom } f$  holds  $it(x) = \text{proj}(f, x)$ .

Let  $f$  be an empty function. One can verify that  $\text{ProjMap } f$  is empty.

Let  $f$  be a non-empty function. Note that  $\text{ProjMap } f$  is non-empty.

Let  $f$  be a non non-empty function. Let us note that  $\text{ProjMap } f$  is empty yielding.

Let  $I$  be a non empty set and  $J$  be a topological structure yielding, nonempty many sorted set indexed by  $I$ . The functor  $\text{ProjMap } J$  yielding a many sorted set indexed by  $I$  is defined by the term

(Def. 3)  $\text{ProjMap}(\text{the support of } J)$ .

Observe that  $\text{ProjMap } J$  is function yielding, non empty, and non-empty.

Now we state the proposition:

(53) Let us consider a non empty set  $I$ , a topological structure yielding, nonempty many sorted set  $J$  indexed by  $I$ , and an element  $i$  of  $I$ . Then  $(\text{ProjMap } J)(i) = \text{proj}(J, i)$ .

Let  $I$  be a non empty set,  $J$  be a topological structure yielding, nonempty many sorted set indexed by  $I$ , and  $f$  be a one-to-one,  $I$ -valued function. The functor  $\text{ProdBasSel}(J, f)$  yielding a many sorted set indexed by  $\text{rng } f$  is defined by the term

(Def. 4)  $(\text{ProjMap } J) \circ (I\text{-indexing } f^{-1}) \upharpoonright \text{rng } f$ .

Let  $f$  be an empty, one-to-one,  $I$ -valued function. Note that  $\text{ProdBasSel}(J, f)$  is empty.

Now we state the propositions:



- (54) Let us consider a non empty set  $I$ , a topological structure yielding, nonempty many sorted set  $J$  indexed by  $I$ , a one-to-one,  $I$ -valued function  $f$ , and an element  $i$  of  $I$ . Suppose  $i \in \text{rng } f$ . Then  $(\text{ProdBasSel}(J, f))(i) = (\text{proj}(J, i))^\circ(f^{-1})(i)$ . The theorem is a consequence of (53).
- (55) Let us consider a non empty set  $I$ , a topological structure yielding, nonempty many sorted set  $J$  indexed by  $I$ , and a one-to-one,  $I$ -valued function  $f$ . Suppose  $f^{-1}$  is non-empty and  $\text{dom } f \subseteq 2\Pi^\alpha$ . Then  $\text{ProdBasSel}(J, f)$  is non-empty, where  $\alpha$  is the support of  $J$ . The theorem is a consequence of (54).
- (56) Let us consider a non empty set  $I$ , and a topological space yielding, nonempty many sorted set  $J$  indexed by  $I$ . Then  $\emptyset \in$  the product prebasis for  $J$ . The theorem is a consequence of (36).
- (57) Let us consider a non empty set  $I$ , a topological structure yielding, nonempty many sorted set  $J$  indexed by  $I$ , and a non empty subset  $P$  of  $\Pi$ (the support of  $J$ ). Suppose  $P \in$  the product prebasis for  $J$ . Then there exists an element  $i$  of  $I$  such that

- (i)  $(\text{proj}(J, i))^\circ P$  is open, and
- (ii) for every element  $j$  of  $I$  such that  $j \neq i$  holds  $(\text{proj}(J, j))^\circ P = \Omega_{J(j)}$ .

PROOF: Consider  $i$  being a set,  $T$  being a topological structure,  $V$  being a subset of  $T$  such that  $i \in I$  and  $V$  is open and  $T = J(i)$  and  $P = \Pi((\text{the support of } J) + \cdot (i, V))$ .  $\text{rng } \text{proj}(J, i) =$  the carrier of  $J(i)$ . For every object  $x$ ,  $x \in (\text{proj}(J, j))^\circ P$  iff  $x \in \Omega_{J(j)}$  by [6, (30)], [22, (8)], [6, (32)], [20, (7)].  $\square$

- (58) Let us consider a non empty set  $I$ , a topological space yielding, nonempty many sorted set  $J$  indexed by  $I$ , and a non empty subset  $P$  of  $\Pi$ (the support of  $J$ ). Suppose  $P \in$  the product prebasis for  $J$ . Then
  - (i) for every element  $j$  of  $I$ ,  $(\text{proj}(J, j))^\circ P$  is open, and
  - (ii) there exists an element  $i$  of  $I$  such that for every element  $j$  of  $I$  such that  $j \neq i$  holds  $(\text{proj}(J, j))^\circ P = \Omega_{J(j)}$ .

The theorem is a consequence of (57).

- (59) Let us consider a non empty set  $I$ , a topological structure yielding, nonempty many sorted set  $J$  indexed by  $I$ , a one-to-one,  $I$ -valued function  $f$ , and a family  $X$  of subsets of  $\Pi$ (the support of  $J$ ). Suppose  $X \subseteq$  the product prebasis for  $J$  and  $\text{dom } f = X$  and  $f^{-1}$  is non-empty and for every subset  $A$  of  $\Pi$ (the support of  $J$ ) such that  $A \in X$  holds  $(\text{proj}(J, f/A))^\circ A$  is open. Let us consider an element  $i$  of  $I$ . Then
  - (i) if  $i \notin \text{rng } f$ , then  $(\text{proj}(J, i))^\circ(\Pi((\text{the support of } J) + \cdot \text{ProdBasSel}(J, f))) = \Omega_{J(i)}$ , and

- (ii) if  $i \in \text{rng } f$ , then  $(\text{proj}(J, i))^\circ(\prod((\text{the support of } J)+ \cdot \text{ProdBasSel}(J, f)))$  is open.

PROOF: Set  $g = \text{ProdBasSel}(J, f)$ . Set  $P = \prod((\text{the support of } J)+ \cdot g)$ .  $g$  is non-empty. If  $i \notin \text{rng } f$ , then  $(\text{proj}(J, i))^\circ P = \Omega_{J(i)}$  by (24), [20, (7)].  $\square$

- (60) Let us consider a non empty set  $I$ , a topological space yielding, nonempty many sorted set  $J$  indexed by  $I$ , a one-to-one,  $I$ -valued function  $f$ , and a family  $X$  of subsets of  $\prod(\text{the support of } J)$ . Suppose  $X \subseteq$  the product prebasis for  $J$  and  $\text{dom } f = X$  and  $f^{-1}$  is non-empty and for every subset  $A$  of  $\prod(\text{the support of } J)$  such that  $A \in X$  holds  $(\text{proj}(J, f/A))^\circ A$  is open. Let us consider an element  $i$  of  $I$ . Then

- (i)  $(\text{proj}(J, i))^\circ(\prod((\text{the support of } J)+ \cdot \text{ProdBasSel}(J, f)))$  is open, and
- (ii) if  $i \notin \text{rng } f$ , then  $(\text{proj}(J, i))^\circ(\prod((\text{the support of } J)+ \cdot \text{ProdBasSel}(J, f))) = \Omega_{J(i)}$ .

The theorem is a consequence of (59).

- (61) Let us consider a non empty set  $I$ , a topological space yielding, nonempty many sorted set  $J$  indexed by  $I$ , and a subset  $P$  of  $\prod(\text{the support of } J)$ . Then  $P \in \text{FinMeetCl}(\text{the product prebasis for } J)$  if and only if there exists a family  $X$  of subsets of  $\prod(\text{the support of } J)$  and there exists a one-to-one,  $I$ -valued function  $f$  such that  $X \subseteq$  the product prebasis for  $J$  and  $X$  is finite and  $P = \text{Intersect}(X)$  and  $\text{dom } f = X$  and  $P = \prod((\text{the support of } J)+ \cdot \text{ProdBasSel}(J, f))$ .

Let us consider a non empty set  $I$ , a topological space yielding, nonempty many sorted set  $J$  indexed by  $I$ , and a non empty subset  $P$  of  $\prod(\text{the support of } J)$ . Now we state the propositions:

- (62) Suppose  $P \in \text{FinMeetCl}(\text{the product prebasis for } J)$ . Then there exists a family  $X$  of subsets of  $\prod(\text{the support of } J)$  and there exists a one-to-one,  $I$ -valued function  $f$  such that  $X \subseteq$  the product prebasis for  $J$  and  $X$  is finite and  $P = \text{Intersect}(X)$  and  $\text{dom } f = X$  and for every element  $i$  of  $I$ ,  $(\text{proj}(J, i))^\circ P$  is open and if  $i \notin \text{rng } f$ , then  $(\text{proj}(J, i))^\circ P = \Omega_{J(i)}$ .

PROOF: Consider  $X$  being a family of subsets of  $\prod(\text{the support of } J)$ ,  $f$  being a one-to-one,  $I$ -valued function such that  $X \subseteq$  the product prebasis for  $J$  and  $X$  is finite and  $P = \text{Intersect}(X)$  and  $\text{dom } f = X$  and  $P = \prod((\text{the support of } J)+ \cdot \text{ProdBasSel}(J, f))$ .  $f^{-1}$  is non-empty by [10, (33)], [21, (4)].  $\square$

- (63) Suppose  $P \in \text{FinMeetCl}(\text{the product prebasis for } J)$ . Then there exists a finite subset  $I_0$  of  $I$  such that for every element  $i$  of  $I$ ,  $(\text{proj}(J, i))^\circ P$  is open and if  $i \notin I_0$ , then  $(\text{proj}(J, i))^\circ P = \Omega_{J(i)}$ . The theorem is a consequence of (62).

Now we state the propositions:

- (64) Let us consider a 1-element set  $I$ , a topological structure yielding, nonempty many sorted set  $J$  indexed by  $I$ , an element  $i$  of  $I$ , and a subset  $P$  of  $\prod$ (the support of  $J$ ). Then  $P \in$  the product prebasis for  $J$  if and only if there exists a subset  $V$  of  $J(i)$  such that  $V$  is open and  $P = \prod(\{i\} \mapsto V)$ . The theorem is a consequence of (7) and (44).
- (65) Let us consider a 1-element set  $I$ , and a topological space yielding, nonempty many sorted set  $J$  indexed by  $I$ . Then the topology of  $\prod J =$  the product prebasis for  $J$ .
- (66) Let us consider a 1-element set  $I$ , a topological space yielding, nonempty many sorted set  $J$  indexed by  $I$ , an element  $i$  of  $I$ , and a subset  $P$  of  $\prod J$ . Then  $P$  is open if and only if there exists a subset  $V$  of  $J(i)$  such that  $V$  is open and  $P = \prod(\{i\} \mapsto V)$ . The theorem is a consequence of (65) and (64).

Let  $I$  be a non empty set,  $J$  be a topological structure yielding, nonempty many sorted set indexed by  $I$ , and  $i$  be an element of  $I$ . Note that  $\text{proj}(J, i)$  is continuous and onto.

Let  $J$  be a topological space yielding, nonempty many sorted set indexed by  $I$ . Note that  $\text{proj}(J, i)$  is open.

Let us consider a 1-element set  $I$ , a topological space yielding, nonempty many sorted set  $J$  indexed by  $I$ , and an element  $i$  of  $I$ . Now we state the propositions:

- (67)  $\text{proj}(J, i)$  is a homeomorphism. The theorem is a consequence of (7).
- (68)  $\prod J$  and  $J(i)$  are homeomorphic. The theorem is a consequence of (67).

Let us consider a 2-element set  $I$ , a topological space yielding, nonempty many sorted set  $J$  indexed by  $I$ , elements  $i, j$  of  $I$ , and a subset  $P$  of  $\prod$ (the support of  $J$ ). Now we state the propositions:

- (69) Suppose  $i \neq j$ . Then  $P \in$  the product prebasis for  $J$  if and only if there exists a subset  $V$  of  $J(i)$  such that  $V$  is open and  $P = \prod[i \mapsto V, j \mapsto \Omega_{J(j)}]$  or there exists a subset  $W$  of  $J(j)$  such that  $W$  is open and  $P = \prod[i \mapsto \Omega_{J(i)}, j \mapsto W]$ . The theorem is a consequence of (34).
- (70) Suppose  $i \neq j$ . Then  $P \in \text{FinMeetCl}$ (the product prebasis for  $J$ ) if and only if there exists a subset  $V$  of  $J(i)$  and there exists a subset  $W$  of  $J(j)$  such that  $V$  is open and  $W$  is open and  $P = \prod[i \mapsto V, j \mapsto W]$ .

PROOF: There exists a family  $Y$  of subsets of  $\prod$ (the support of  $J$ ) such that  $Y \subseteq$  the product prebasis for  $J$  and  $Y$  is finite and  $P = \text{Intersect}(Y)$  by [20, (7)], (39), [14, (32)], (34).  $\square$

Now we state the propositions:

- (71) Let us consider a non empty set  $I$ , a topological space yielding, nonempty many sorted set  $J$  indexed by  $I$ , and elements  $i, j$  of  $I$ . Then  $\langle \text{proj}(J, i), \text{proj}(J, j) \rangle$  is a function from  $\prod J$  into  $J(i) \times J(j)$ .
- (72) Let us consider a non empty set  $I$ , a topological space yielding, nonempty many sorted set  $J$  indexed by  $I$ , a subset  $P$  of  $\prod(\text{the support of } J)$ , and elements  $i, j$  of  $I$ . Suppose  $i \neq j$  and there exists a many sorted set  $F$  indexed by  $I$  such that  $P = \prod F$  and for every element  $k$  of  $I$ ,  $F(k) \subseteq (\text{the support of } J)(k)$ . Then  $\langle \text{proj}(J, i), \text{proj}(J, j) \rangle^\circ P = (\text{proj}(J, i))^\circ P \times (\text{proj}(J, j))^\circ P$ . The theorem is a consequence of (26), (30), and (11).
- (73) Let us consider a non empty set  $I$ , a topological space yielding, nonempty many sorted set  $J$  indexed by  $I$ , elements  $i, j$  of  $I$ , and a function  $f$  from  $\prod J$  into  $J(i) \times J(j)$ . Suppose  $i \neq j$  and  $f = \langle \text{proj}(J, i), \text{proj}(J, j) \rangle$ . Then  $f$  is onto and open.

PROOF: For every element  $k$  of  $I$ ,  $(\text{proj}(J, k))^\circ(\Omega_{\prod \alpha}) = \text{the carrier of } J(k)$ , where  $\alpha$  is the support of  $J$  by [25, (113)], (52). There exists a basis  $B$  of  $\prod J$  such that for every subset  $P$  of  $\prod J$  such that  $P \in B$  holds  $f^\circ P$  is open by [5, (23)], (63), [13, (13)], (54).  $\square$

- (74) Let us consider a 2-element set  $I$ , a topological space yielding, nonempty many sorted set  $J$  indexed by  $I$ , elements  $i, j$  of  $I$ , and a function  $f$  from  $\prod J$  into  $J(i) \times J(j)$ . Suppose  $i \neq j$  and  $f = \langle \text{proj}(J, i), \text{proj}(J, j) \rangle$ . Then  $f$  is a homeomorphism.

PROOF:  $f$  is onto and open. For every objects  $x_1, x_2$  such that  $x_1, x_2 \in \text{dom } f$  and  $f(x_1) = f(x_2)$  holds  $x_1 = x_2$  by [22, (8)], (9), (29), [13, (63)].  $\square$

- (75) Let us consider a 2-element set  $I$ , a topological space yielding, nonempty many sorted set  $J$  indexed by  $I$ , and elements  $i, j$  of  $I$ . If  $i \neq j$ , then  $\prod J$  and  $J(i) \times J(j)$  are homeomorphic. The theorem is a consequence of (74).

Let  $I_1, I_2$  be non empty sets,  $J$  be a topological space yielding, nonempty many sorted set indexed by  $I_2$ , and  $f$  be a function from  $I_1$  into  $I_2$ . One can check that  $J \cdot f$  is topological space yielding and nonempty.

Let  $J_1$  be a topological space yielding, nonempty many sorted set indexed by  $I_1$ ,  $J_2$  be a topological space yielding, nonempty many sorted set indexed by  $I_2$ , and  $p$  be a function from  $I_1$  into  $I_2$ . Assume  $p$  is bijective and for every element  $i$  of  $I_1$ ,  $J_1(i)$  and  $(J_2 \cdot p)(i)$  are homeomorphic.

**A product homeomorphism of  $J_1, J_2$  and  $p$**  is a function from  $\prod J_1$  into  $\prod J_2$  defined by

- (Def. 5) there exists a many sorted function  $F$  indexed by  $I_1$  such that for every element  $i$  of  $I_1$ , there exists a function  $f$  from  $J_1(i)$  into  $(J_2 \cdot p)(i)$  such that  $F(i) = f$  and  $f$  is a homeomorphism and for every element  $g$  of  $\prod J_1$

and for every element  $i$  of  $I_1$ ,  $(it(g))(p(i)) = F(i)(g(i))$ .

Now we state the proposition:

- (76) Let us consider non empty sets  $I_1, I_2$ , a topological space yielding, nonempty many sorted set  $J_1$  indexed by  $I_1$ , a topological space yielding, nonempty many sorted set  $J_2$  indexed by  $I_2$ , a function  $p$  from  $I_1$  into  $I_2$ , a product homeomorphism  $H$  of  $J_1, J_2$  and  $p$ , and a many sorted function  $F$  indexed by  $I_1$ . Suppose  $p$  is bijective and for every element  $i$  of  $I_1$ , there exists a function  $f$  from  $J_1(i)$  into  $(J_2 \cdot p)(i)$  such that  $F(i) = f$  and  $f$  is a homeomorphism and for every element  $g$  of  $\prod J_1$  and for every element  $i$  of  $I_1$ ,  $(H(g))(p(i)) = F(i)(g(i))$ . Let us consider an element  $i$  of  $I_1$ , and a subset  $U$  of  $J_1(i)$ . Then  $H^\circ(\prod((\text{the support of } J_1) + \cdot (i, U))) = \prod((\text{the support of } J_2) + \cdot (p(i), F(i)^\circ U))$ .

PROOF: Reconsider  $j = p(i)$  as an element of  $I_2$ . Consider  $f$  being a function from  $J_1(i)$  into  $(J_2 \cdot p)(i)$  such that  $F(i) = f$  and  $f$  is a homeomorphism. For every object  $y$ ,  $y \in H^\circ(\prod((\text{the support of } J_1) + \cdot (i, U)))$  iff  $y \in \prod((\text{the support of } J_2) + \cdot (j, F(i)^\circ U))$  by [10, (3)], [3, (9)], [6, (30)], [11, (11)].  $\square$

Let us consider non empty sets  $I_1, I_2$ , a topological space yielding, nonempty many sorted set  $J_1$  indexed by  $I_1$ , a topological space yielding, nonempty many sorted set  $J_2$  indexed by  $I_2$ , a function  $p$  from  $I_1$  into  $I_2$ , and a product homeomorphism  $H$  of  $J_1, J_2$  and  $p$ . Now we state the propositions:

- (77) If  $p$  is bijective and for every element  $i$  of  $I_1$ ,  $J_1(i)$  and  $(J_2 \cdot p)(i)$  are homeomorphic, then  $H$  is bijective.

PROOF: Consider  $F$  being a many sorted function indexed by  $I_1$  such that for every element  $i$  of  $I_1$ , there exists a function  $f$  from  $J_1(i)$  into  $(J_2 \cdot p)(i)$  such that  $F(i) = f$  and  $f$  is a homeomorphism and for every element  $g$  of  $\prod J_1$  and for every element  $i$  of  $I_1$ ,  $(H(g))(p(i)) = F(i)(g(i))$ . For every objects  $x_1, x_2$  such that  $x_1, x_2 \in \text{dom } H$  and  $H(x_1) = H(x_2)$  holds  $x_1 = x_2$  by [3, (9)], [20, (7)], [10, (2)]. Set  $i_0 =$  the element of  $I_1$ . Consider  $f_0$  being a function from  $J_1(i_0)$  into  $(J_2 \cdot p)(i_0)$  such that  $F(i_0) = f_0$  and  $f_0$  is a homeomorphism.  $\square$

- (78) If  $p$  is bijective and for every element  $i$  of  $I_1$ ,  $J_1(i)$  and  $(J_2 \cdot p)(i)$  are homeomorphic, then  $H$  is a homeomorphism.

PROOF: Consider  $F$  being a many sorted function indexed by  $I_1$  such that for every element  $i$  of  $I_1$ , there exists a function  $f$  from  $J_1(i)$  into  $(J_2 \cdot p)(i)$  such that  $F(i) = f$  and  $f$  is a homeomorphism and for every element  $g$  of  $\prod J_1$  and for every element  $i$  of  $I_1$ ,  $(H(g))(p(i)) = F(i)(g(i))$ .  $H$  is bijective. There exists a prebasis  $K$  of  $\prod J_1$  and there exists a prebasis  $L$  of  $\prod J_2$  such that  $H^\circ K = L$  by [11, (11)], [20, (7)], (39), [10, (13)].  $\square$

Now we state the propositions:

- (79) Let us consider non empty sets  $I_1, I_2$ , a topological space yielding, nonempty many sorted set  $J_1$  indexed by  $I_1$ , a topological space yielding, nonempty many sorted set  $J_2$  indexed by  $I_2$ , and a function  $p$  from  $I_1$  into  $I_2$ . Suppose  $p$  is bijective and for every element  $i$  of  $I_1$ ,  $J_1(i)$  and  $(J_2 \cdot p)(i)$  are homeomorphic. Then  $\prod J_1$  and  $\prod J_2$  are homeomorphic. The theorem is a consequence of (78).
- (80) Let us consider a non empty set  $I$ , topological space yielding, nonempty many sorted sets  $J_1, J_2$  indexed by  $I$ , and a permutation  $p$  of  $I$ . Suppose for every element  $i$  of  $I$ ,  $J_1(i)$  and  $(J_2 \cdot p)(i)$  are homeomorphic. Then  $\prod J_1$  and  $\prod J_2$  are homeomorphic.
- (81) Let us consider a non empty set  $I$ , a topological space yielding, nonempty many sorted set  $J$  indexed by  $I$ , and a permutation  $p$  of  $I$ . Then  $\prod J$  and  $\prod J \cdot p$  are homeomorphic. The theorem is a consequence of (79).
- (82) Let us consider a non empty set  $I$ , and topological space yielding, nonempty many sorted sets  $J_1, J_2$  indexed by  $I$ . Suppose for every element  $i$  of  $I$ ,  $J_1(i)$  is a subspace of  $J_2(i)$ . Then  $\prod J_1$  is a subspace of  $\prod J_2$ .

PROOF: There exists a prebasis  $K_1$  of  $\prod J_1$  and there exists a prebasis  $K_2$  of  $\prod J_2$  such that  $\Omega_{\prod J_1} \in K_1$  and  $K_1 = K_2 \smallfrown \{\Omega_{\prod J_1}\}$  by [6, (35)], [20, (7)], [6, (30), (31), (32)].  $\square$

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