

Zariski Topology

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Summary. We formalize in the Mizar System [5], [6], basic definitions of commutative ring theory such as prime spectrum, nilradical, Jacobson radical, local ring, and semi-local ring [9], [10], then formalize proofs of some relating theorems along with the first chapter of [1].

The article formalized the so-called Zariski topology as well. The set of all prime ideals of a commutative ring A is called the prime spectrum of A denoted by Spectrum A . A new a functor Spec is introduced and it generates Zariski topology to make Spectrum A a topological space. A different role is given to Spec as a map from a ring morphism of commutative rings to that of topological spaces by the following manner: for a ring homomorphism $h : A \rightarrow B$, we defined $(\text{Spec } h) : \text{Spec } B \rightarrow \text{Spec } A$ by $(\text{Spec } h)(\mathfrak{p}) = h^{-1}(\mathfrak{p})$ where $\mathfrak{p} \in \text{Spec } B$.

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1. PRELIMINARIES: SOME PROPERTIES OF IDEALS

From now on R denotes a commutative ring, A, B denote non degenerated, commutative rings, h denotes a function from A into B , I_0, I, I_1, I_2 denote ideals of A , J, J_1, J_2 denote proper ideals of A , p denotes a prime ideal of A , S, S_1 denote non empty subsets of A , E, E_1, E_2 denote subsets of A , a, b, f denote elements of A , n denotes a natural number, and x, o, o_1 denote objects.

Let us consider A and S . The functor $\text{Ideals}(A, S)$ yielding a subset of Ideals A is defined by the term

(Def. 1) $\{I, \text{ where } I \text{ is an ideal of } A : S \subseteq I\}$.

Let us observe that $\text{Ideals}(A, S)$ is non empty.

Now we state the proposition:

- (1) $\text{Ideals}(A, S) = \text{Ideals}(A, S\text{-ideal})$.

PROOF: $\text{Ideals}(A, S) \subseteq \text{Ideals}(A, S\text{-ideal})$ by [2, (57), (44)]. Consider y being an ideal of A such that $x = y$ and $S\text{-ideal} \subseteq y$. \square

Let A be a unital, non empty multiplicative loop with zero structure and a be an element of A . We say that a is nilpotent if and only if

- (Def. 2) there exists a non zero natural number k such that $a^k = 0_A$.

Let us note that 0_A is nilpotent and there exists an element of A which is nilpotent.

Let us consider A . Observe that 1_A is non nilpotent.

Let us consider f . The functor $\text{MultClSet}(f)$ yielding a subset of A is defined by the term

- (Def. 3) the set of all f^i where i is a natural number.

Let us observe that $\text{MultClSet}(f)$ is multiplicatively closed.

Now we state the propositions:

- (2) Let us consider a natural number n . Then $(1_A)^n = 1_A$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (1_A)^{\mathbb{S}^1} = 1_A$. For every natural number n , $\mathcal{P}[n]$ from [3, Sch. 2]. \square

- (3) $1_A \notin \sqrt{J}$. The theorem is a consequence of (2).

- (4) $\text{MultClSet}(1_A) = \{1_A\}$. The theorem is a consequence of (2).

Let us consider A, J , and f . The functor $\text{Ideals}(A, J, f)$ yielding a subset of $\text{Ideals } A$ is defined by the term

- (Def. 4) $\{I, \text{ where } I \text{ is a subset of } A : I \text{ is a proper ideal of } A \text{ and } J \subseteq I \text{ and } I \cap \text{MultClSet}(f) = \emptyset\}$.

Let us consider A, J , and f . Now we state the propositions:

- (5) If $f \notin \sqrt{J}$, then $J \in \text{Ideals}(A, J, f)$.

- (6) If $f \notin \sqrt{J}$, then $\text{Ideals}(A, J, f)$ has the upper Zorn property w.r.t. $\subseteq_{\text{Ideals}(A, J, f)}$.

PROOF: Set $S = \text{Ideals}(A, J, f)$. Set $P = \subseteq_S$. For every set Y such that $Y \subseteq S$ and $P \upharpoonright^2 Y$ is a linear order there exists a set x such that $x \in S$ and for every set y such that $y \in Y$ holds $\langle y, x \rangle \in P$ by (5), [8, (74)], [2, (19)], [12, (71)]. \square

Now we state the propositions:

- (7) If $f \notin \sqrt{J}$, then there exists a prime ideal m of A such that $f \notin m$ and $J \subseteq m$.

PROOF: Set $S = \text{Ideals}(A, J, f)$. Set $P = \subseteq_S$. Consider I being a set such that I is maximal in P . Consider p being a subset of A such that $p = I$

and p is a proper ideal of A and $J \subseteq p$ and $p \cap \text{MultClSet}(f) = \emptyset$. p is a quasi-prime ideal of A by [2, (74), (66), (73)]. \square

(8) There exists a maximal ideal m of A such that $J \subseteq m$.

PROOF: $1_A \notin \sqrt{J}$. Set $S = \text{Ideals}(A, J, 1_A)$. Set $P = \overset{\subseteq}{=} S$. Consider I being a set such that I is maximal in P . Consider p being a subset of A such that $p = I$ and p is a proper ideal of A and $J \subseteq p$ and $p \cap \text{MultClSet}(1_A) = \emptyset$. For every ideal q of A such that $p \subseteq q$ holds $q = p$ or q is not proper by (4), [2, (19)]. \square

(9) There exists a prime ideal m of A such that $J \subseteq m$. The theorem is a consequence of (8).

(10) If a is a non-unit of A , then there exists a maximal ideal m of A such that $a \in m$. The theorem is a consequence of (8).

2. SPECTRUM OF PRIME IDEALS (SPECTRUM) AND MAXIMAL IDEALS (M-SPECTRUM)

Let R be a commutative ring. The spectrum of R yielding a family of subsets of R is defined by the term

(Def. 5) $\begin{cases} \{I, \text{ where } I \text{ is an ideal of } R : I \text{ is quasi-prime and } I \neq \Omega_R\}, & \text{if } R \text{ is not degen} \\ \emptyset, & \text{otherwise.} \end{cases}$

Let us consider A . Observe that the spectrum of A yields a family of subsets of A and is defined by the term

(Def. 6) the set of all I where I is a prime ideal of A .

Observe that the spectrum of A is non empty.

Let us consider R . The functor **m-Spectrum(R)** yielding a family of subsets of R is defined by the term

(Def. 7) $\begin{cases} \{I, \text{ where } I \text{ is an ideal of } R : I \text{ is quasi-maximal and } I \neq \Omega_R\}, & \text{if } R \text{ is not degen} \\ \emptyset, & \text{otherwise.} \end{cases}$

Let us consider A . Observe that the functor **m-Spectrum(A)** yields a family of subsets of the carrier of A and is defined by the term

(Def. 8) the set of all I where I is a maximal ideal of A .

Observe that **m-Spectrum(A)** is non empty.

3. LOCAL AND SEMI-LOCAL RING

Let us consider A . We say that **A is local** if and only if

(Def. 9) there exists an ideal m of A such that **m-Spectrum(A)** = $\{m\}$.

We say that A is semi-local if and only if

(Def. 10) $\text{m-Spectrum}(A)$ is finite.

Now we state the propositions:

(11) If $x \in I$ and I is a proper ideal of A , then x is a non-unit of A .

(12) If for every objects m_1, m_2 such that $m_1, m_2 \in \text{m-Spectrum}(A)$ holds $m_1 = m_2$, then A is local.

(13) If for every x such that $x \in \Omega_A \setminus J$ holds x is a unit of A , then A is local.

The theorem is a consequence of (8), (11), and (12).

In the sequel m denotes a maximal ideal of A .

Now we state the propositions:

(14) If $a \in \Omega_A \setminus m$, then $\{a\}$ -ideal $+ m = \Omega_A$.

(15) If for every a such that $a \in m$ holds $1_A + a$ is a unit of A , then A is local.

PROOF: For every x such that $x \in \Omega_A \setminus m$ holds x is a unit of A by (14), [2, (64)], [13, (5)], [2, (13)]. \square

Let us consider R . Let E be a subset of R . The functor $\text{PrimeIdeals}(R, E)$ yielding a subset of the spectrum of R is defined by the term

(Def. 11) $\begin{cases} \{p, \text{ where } p \text{ is an ideal of } R : p \text{ is quasi-prime and } p \neq \Omega_R \text{ and } E \subseteq p\}, & \text{if } R \text{ is} \\ \text{otherwise.} \end{cases}$

Let us consider A . Let E be a subset of A . Let us note that the functor $\text{PrimeIdeals}(A, E)$ yields a subset of the spectrum of A and is defined by the term

(Def. 12) $\{p, \text{ where } p \text{ is a prime ideal of } A : E \subseteq p\}$.

Let us consider J . Observe that $\text{PrimeIdeals}(A, J)$ is non empty.

From now on p denotes a prime ideal of A and k denotes a non zero natural number.

Now we state the proposition:

(16) If $a \notin p$, then $a^k \notin p$.

PROOF: $a^k \notin p$ by [11, (8), (10)]. \square

4. NILRADICAL AND JACOBSON RADICAL

Let us consider A . The functor $\text{nilrad}(A)$ yielding a subset of A is defined by the term

(Def. 13) the set of all a where a is a nilpotent element of A .

Now we state the proposition:

(17) $\text{nilrad}(A) = \sqrt{\{0_A\}}$.

Let us consider A . One can verify that $\text{nilrad}(A)$ is non empty and $\text{nilrad}(A)$ is closed under addition as a subset of A and $\text{nilrad}(A)$ is left and right ideal as a subset of A .

Now we state the propositions:

- (18) $\sqrt{J} = \bigcap \text{PrimeIdeals}(A, J)$. The theorem is a consequence of (16), (7), and (9).
- (19) $\text{nilrad}(A) = \bigcap (\text{the spectrum of } A)$. The theorem is a consequence of (17) and (18).
- (20) $I \subseteq \sqrt{I}$.
- (21) If $I \subseteq J$, then $\sqrt{I} \subseteq \sqrt{J}$.

PROOF: Consider s_1 being an element of A such that $s_1 = s$ and there exists an element n of \mathbb{N} such that $s_1^n \in I$. Consider n_1 being an element of \mathbb{N} such that $s_1^{n_1} \in I$. $n_1 \neq 0$ by [11, (8)], [2, (19)]. \square

Let us consider A . The functor $\text{J-Rad}(A)$ yielding a subset of A is defined by the term

(Def. 14) $\bigcap \text{m-Spectrum}(A)$.

5. CONSTRUCTION OF ZARISKI TOPOLOGY OF THE PRIME SPECTRUM OF A

Now we state the propositions:

- (22) $\text{PrimeIdeals}(A, S) \subseteq \text{Ideals}(A, S)$.
 - (23) $\text{PrimeIdeals}(A, S) = \text{Ideals}(A, S) \cap (\text{the spectrum of } A)$. The theorem is a consequence of (22).
 - (24) $\text{PrimeIdeals}(A, S) = \text{PrimeIdeals}(A, S\text{-ideal})$. The theorem is a consequence of (23) and (1).
 - (25) If $I \subseteq p$, then $\sqrt{I} \subseteq p$.
- PROOF: Consider s_1 being an element of A such that $s_1 = s$ and there exists an element n of \mathbb{N} such that $s_1^n \in I$. Consider n_1 being an element of \mathbb{N} such that $s_1^{n_1} \in I$. $n_1 \neq 0$ by [11, (8)], [2, (19)]. \square
- (26) If $\sqrt{I} \subseteq p$, then $I \subseteq p$. The theorem is a consequence of (20).
 - (27) $\text{PrimeIdeals}(A, \sqrt{S\text{-ideal}}) = \text{PrimeIdeals}(A, S\text{-ideal})$. The theorem is a consequence of (26) and (25).
 - (28) If $E_2 \subseteq E_1$, then $\text{PrimeIdeals}(A, E_1) \subseteq \text{PrimeIdeals}(A, E_2)$.
 - (29) $\text{PrimeIdeals}(A, J_1) = \text{PrimeIdeals}(A, J_2)$ if and only if $\sqrt{J_1} = \sqrt{J_2}$. The theorem is a consequence of (18) and (27).
 - (30) If $I_1 * I_2 \subseteq p$, then $I_1 \subseteq p$ or $I_2 \subseteq p$.

PROOF: If it is not true that $I_1 \subseteq p$ or $I_2 \subseteq p$, then $I_1 * I_2 \not\subseteq p$ by [11, (3)], [4, (40)]. \square

(31) $\text{PrimeIdeals}(A, \{1_A\}) = \emptyset$.

(32) The spectrum of $A = \text{PrimeIdeals}(A, \{0_A\})$.

(33) Let us consider non empty subsets E_1, E_2 of A . Then there exists a non empty subset E_3 of A such that $\text{PrimeIdeals}(A, E_1) \cup \text{PrimeIdeals}(A, E_2) = \text{PrimeIdeals}(A, E_3)$.

PROOF: Set $I_1 = E_1$ -ideal. Set $I_2 = E_2$ -ideal. Reconsider $I_3 = I_1 * I_2$ as an ideal of A . $\text{PrimeIdeals}(A, E_1) = \text{PrimeIdeals}(A, I_1)$. $\text{PrimeIdeals}(A, I_3) \subseteq \text{PrimeIdeals}(A, I_1) \cup \text{PrimeIdeals}(A, I_2)$. $\text{PrimeIdeals}(A, I_1) \cup \text{PrimeIdeals}(A, I_2) \subseteq \text{PrimeIdeals}(A, I_3)$ by [2, (79)], (28). $\text{PrimeIdeals}(A, I_3) = \text{PrimeIdeals}(A, E_1) \cup \text{PrimeIdeals}(A, E_2)$. \square

(34) Let us consider a family G of subsets of the spectrum of A . Suppose for every set S such that $S \in G$ there exists a non empty subset E of A such that $S = \text{PrimeIdeals}(A, E)$. Then there exists a non empty subset F of A such that $\text{Intersect}(G) = \text{PrimeIdeals}(A, F)$. The theorem is a consequence of (28).

Let us consider A . The functor $\text{Spec}(A)$ yielding a strict topological space is defined by

(Def. 15) the carrier of it = the spectrum of A and for every subset F of it , F is closed iff there exists a non empty subset E of A such that $F = \text{PrimeIdeals}(A, E)$.

Note that $\text{Spec}(A)$ is non empty.

Now we state the proposition:

(35) Let us consider points P, Q of $\text{Spec}(A)$. Suppose $P \neq Q$. Then there exists a subset V of $\text{Spec}(A)$ such that

(i) V is open, and

(ii) $P \in V$ and $Q \notin V$ or $Q \in V$ and $P \notin V$.

Note that there exists a commutative ring which is degenerated.

Let R be a degenerated, commutative ring. Let us observe that ADTS(the spectrum of R) is T_0 .

Let us consider A . Observe that $\text{Spec}(A)$ is T_0 .

6. CONTINUOUS MAP OF ZARISKI TOPOLOGY ASSOCIATED WITH A RING HOMOMORPHISM

From now on M_0 denotes an ideal of B .

Now we state the proposition:

(36) If h inherits ring homomorphism, then $h^{-1}(M_0)$ is an ideal of A .

In the sequel M_0 denotes a prime ideal of B .

Now we state the proposition:

- (37) If h inherits ring homomorphism, then $h^{-1}(M_0)$ is a prime ideal of A .

PROOF: For every elements x, y of A such that $x \cdot y \in h^{-1}(M_0)$ holds $x \in h^{-1}(M_0)$ or $y \in h^{-1}(M_0)$. $h^{-1}(M_0) \neq$ the carrier of A by [2, (19)]. \square

Let us consider A, B , and h . Assume h inherits ring homomorphism. The functor $\text{Spec}(h)$ yielding a function from $\text{Spec}(B)$ into $\text{Spec}(A)$ is defined by

(Def. 16) for every point x of $\text{Spec}(B)$, $it(x) = h^{-1}(x)$.

Now we state the propositions:

- (38) If h inherits ring homomorphism, then $(\text{Spec}(h))^{-1}(\text{PrimeIdeals}(A, E)) = \text{PrimeIdeals}(B, h^\circ E)$.

PROOF: $(\text{Spec}(h))^{-1}(\text{PrimeIdeals}(A, E)) \subseteq \text{PrimeIdeals}(B, h^\circ E)$ by [14, (123)], [7, (75)]. Consider q being a prime ideal of B such that $x = q$ and $h^\circ E \subseteq q$. $h^{-1}(q)$ is a prime ideal of A . \square

- (39) If h inherits ring homomorphism, then $\text{Spec}(h)$ is continuous. The theorem is a consequence of (38).

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