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Tarski Geometry Axioms. Part IV – Right Angle

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Summary. In the article, we continue [2] the formalization of the work devoted to Tarski's geometry – the book "Metamathematische Methoden in der Geometrie" (SST for short) by W. Schwabhäuser, W. Szmielew, and A. Tarski [**? ? ?**]. We use the Mizar system [3], [1] to systematically formalize Chapter 8 ("Rechte Winkel – Right angle") of the SST book.

Note that in this case, the definition of the "right angle" is as follows:

a,b,c bilden einen *rechten winkel* (mit dem *Scheitel* b):

$$Rabc :\Leftrightarrow ac \equiv aS_b(c)$$

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1.

From now on S denotes a non empty Tarski plane satisfying seven Tarski's geometry axioms and a, b, c, d, c', x, y, z, p, q, q' denote points of S.

Let S be a non empty Tarski plane satisfying the axiom of congruence identity, the axiom of segment construction, the axiom of betweenness identity, and the axiom of Pasch and a, b be points of S. Let us note that the functor $\mathrm{Line}(a, b)$ is commutative.

Now we state the proposition:

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(1) Let us consider Tarski plane S satisfying the axiom of congruence symmetry, the axiom of congruence equivalence relation, and the axiom of congruence identity, and points a, b, c, d of S. Suppose $\overline{ab} \cong \overline{cd}$. Then

(i) $\overline{ab} \cong \overline{dc}$, and
(ii) $\overline{ba} \cong \overline{cd}$, and
(iii) $\overline{ba} \cong \overline{dc}$, and
(iv) $\overline{cd} \cong \overline{ab}$, and
(v) $\overline{dc} \cong \overline{ab}$, and
(vi) $\overline{cd} \cong \overline{ba}$, and
(vii) $\overline{dc} \cong \overline{ba}$.

Let us consider Tarski plane S satisfying the axiom of congruence symmetry, the axiom of congruence equivalence relation, and the axiom of congruence identity and points p, q, a, b, c, d of S. Now we state the propositions:

(2) Suppose ($\overline{pq} \cong \overline{ab}$ or $\overline{pq} \cong \overline{ba}$ or $\overline{qp} \cong \overline{ab}$ or $\overline{qp} \cong \overline{ba}$) and ($\overline{pq} \cong \overline{cd}$ or $\overline{pq} \cong \overline{dc}$ or $\overline{qp} \cong \overline{cd}$ or $\overline{qp} \cong \overline{dc}$). Then

(i) $\overline{ab} \cong \overline{dc}$, and
(ii) $\overline{ba} \cong \overline{cd}$, and
(iii) $\overline{ba} \cong \overline{dc}$, and
(iv) $\overline{cd} \cong \overline{ab}$, and
(v) $\overline{dc} \cong \overline{ab}$, and
(vi) $\overline{cd} \cong \overline{ba}$, and
(vii) $\overline{dc} \cong \overline{ba}$.

The theorem is a consequence of (1).

(3) Suppose ($\overline{pq} \cong \overline{ab}$ or $\overline{pq} \cong \overline{ba}$ or $\overline{qp} \cong \overline{ab}$ or $\overline{qp} \cong \overline{ba}$ or $\overline{ab} \cong \overline{pq}$ or $\overline{ba} \cong \overline{pq}$ or $\overline{ab} \cong \overline{qp}$ or $\overline{ba} \cong \overline{qp}$) and ($\overline{pq} \cong \overline{cd}$ or $\overline{pq} \cong \overline{dc}$ or $\overline{qp} \cong \overline{cd}$ or $\overline{qp} \cong \overline{dc}$ or $\overline{cd} \cong \overline{pq}$ or $\overline{dc} \cong \overline{pq}$ or $\overline{cd} \cong \overline{qp}$ or $\overline{dc} \cong \overline{qp}$). Then

(i) $\overline{ab} \cong \overline{dc}$, and
(ii) $\overline{ba} \cong \overline{cd}$, and
(iii) $\overline{ba} \cong \overline{dc}$, and
(iv) $\overline{cd} \cong \overline{ab}$, and
(v) $\overline{dc} \cong \overline{ab}$, and
(vi) $\overline{cd} \cong \overline{ba}$, and
(vii) $\overline{dc} \cong \overline{ba}$, and

(viii) $\overline{ab} \cong \overline{cd}$.

The theorem is a consequence of (1) and (2).

Now we state the propositions:

(4) Let us consider Tarski plane S satisfying the axiom of congruence identity, the axiom of segment construction, the axiom of betweenness identity, and the axiom of Pasch, and points a, b of S. Then

(i) a, b and b are collinear, and

(ii) b, b and a are collinear, and

(iii) b, a and b are collinear.

(5) Let us consider a non empty Tarski plane S satisfying seven Tarski's geometry axioms, and points p, q, r of S. Suppose $p \neq q$ and $p \neq r$ and (p, q and r are collinear or q, r and p are collinear or r, p and q are collinear or p, r and q are collinear or q, p and r are collinear or r, q and p are collinear). Then

(i) $\mathrm{Line}(p, q) = \mathrm{Line}(p, r)$, and

(ii) $\mathrm{Line}(p, q) = \mathrm{Line}(r, p)$, and

(iii) $\mathrm{Line}(q, p) = \mathrm{Line}(p, r)$, and

(iv) $\mathrm{Line}(q, p) = \mathrm{Line}(r, p)$.

(6) Let us consider a Tarski plane S, and points a, b, c of S. Suppose $\mathrm{Middle}(a, b, c)$ or b lies between a and c. Then a, b and c are collinear.

(7) Let us consider Tarski plane S satisfying the axiom of congruence identity, the axiom of segment construction, the axiom of betweenness identity, and the axiom of Pasch, and points a, b, c of S. Suppose $\mathrm{Middle}(a, b, c)$ or b lies between a and c. Then

(i) a, b and c are collinear, and

(ii) b, c and a are collinear, and

(iii) c, a and b are collinear, and

(iv) c, b and a are collinear, and

(v) b, a and c are collinear, and

(vi) a, c and b are collinear.

The theorem is a consequence of (6).

(8) Let us consider a non empty Tarski plane S satisfying seven Tarski's geometry axioms, and points a, b, c, d of S. Suppose $a \neq b$ and a, b and c are collinear and a, b and d are collinear. Then a, c and d are collinear. The theorem is a consequence of (4) and (5).

(9) Let us consider a non empty Tarski plane S satisfying seven Tarski's geometry axioms, and points a, b of S. Suppose $\mathrm{Middle}(a, a, b)$ or $\mathrm{Middle}(a, b, b)$ or $\mathrm{Middle}(a, b, a)$. Then $a = b$.

(10) Suppose ($\mathrm{Middle}(a, b, c)$ or $\mathrm{Middle}(c, b, a)$) and ($a \neq b$ or $b \neq c$). Then

(i) $\mathrm{Line}(b, a) = \mathrm{Line}(b, c)$, and

(ii) $\mathrm{Line}(a, b) = \mathrm{Line}(b, c)$, and

(iii) $\mathrm{Line}(a, b) = \mathrm{Line}(c, b)$, and

(iv) $\mathrm{Line}(b, a) = \mathrm{Line}(c, b)$.

The theorem is a consequence of (9).

(11) Suppose $a \neq b$ and $c \neq c'$ and ($c \in \mathrm{Line}(a, b)$ or $c \in \mathrm{Line}(b, a)$) and ($c' \in \mathrm{Line}(a, b)$ or $c' \in \mathrm{Line}(b, a)$). Then

(i) $\mathrm{Line}(c, c') = \mathrm{Line}(a, b)$, and

(ii) $\mathrm{Line}(c, c') = \mathrm{Line}(b, a)$, and

(iii) $\mathrm{Line}(c', c) = \mathrm{Line}(b, a)$, and

(iv) $\mathrm{Line}(c', c) = \mathrm{Line}(a, b)$.

(12) $\mathrm{Middle}(\mathrm{S}_p(c), \mathrm{S}_p(b), \mathrm{S}_p((\mathrm{S}_b(c))))$.

2.

Let S be Tarski plane satisfying the axiom of congruence identity, the axiom of congruence symmetry, the axiom of congruence equivalence relation, the axiom of segment construction, the axiom of betweenness identity, and the axiom of SAS and a, b, c be points of S. We say that $\mathrm{RightAngle}(a, b, c)$ if and only if

(Def. 1) $\overline{ac} \cong \overline{a\mathrm{S}_b(c)}$.

From now on S denotes Tarski plane satisfying seven Tarski's geometry axioms and a, a', b, b', c, c' denote points of S.

Now we state the propositions:

(13) 8.2 Satz:
If $\mathrm{RightAngle}(a, b, c)$, then $\mathrm{RightAngle}(c, b, a)$.

(14) $\mathrm{S}_a(a) = a$.

Now we state the propositions:

(15) 8.3 Satz:
If $\mathrm{RightAngle}(a, b, c)$ and $a \neq b$ and b, a and a' are collinear, then $\mathrm{RightAngle}(a', b, c)$.
The theorem is a consequence of (14).

(16) If $\mathrm{RightAngle}(a, b, c)$, then $\mathrm{RightAngle}(a, b, \mathrm{S}_b(c))$.

Now we state the proposition:

(17) 8.5 SATZ:
RightAngle(a, b, b). The theorem is a consequence of (14).

Now we state the proposition:

(18) 8.6 SATZ:
If RightAngle(a, b, c) and RightAngle(a', b, c) and c lies between a and a', then $b = c$.

Now we state the proposition:

(19) 8.7 SATZ:
If RightAngle(a, b, c) and RightAngle(a, c, b), then $b = c$. The theorem is a consequence of (13), (17), (1), (7), (15), and (18).

Now we state the proposition:

(20) 8.8 SATZ:
If RightAngle(a, b, a), then $a = b$. The theorem is a consequence of (13), (17), and (19).

Now we state the proposition:

(21) 8.9 SATZ:
If RightAngle(a, b, c) and a, b and c are collinear, then $a = b$ or $c = b$. The theorem is a consequence of (15) and (20).

Now we state the proposition:

(22) 8.10 SATZ:
If RightAngle(a, b, c) and $\triangle abc \cong \triangle a'b'c'$, then RightAngle$(a', b', c')$. The theorem is a consequence of (17), (1), and (3).

Let S be a non empty Tarski plane satisfying seven Tarski's geometry axioms, A, A' be subsets of S, and x be a point of S. We say that $A \perp_x A'$ if and only if

(Def. 2) A is a line and A' is a line and $x \in A$ and $x \in A'$ and for every points u, v of S such that $u \in A$ and $v \in A'$ holds RightAngle(u, x, v).

We say that $A \perp A'$ if and only if

(Def. 3) there exists a point x of S such that $A \perp_x A'$.

Let A be a subset of S and x, c, d be points of S. We say that $\overline{A, x} \perp \overline{c, d}$ if and only if

(Def. 4) $c \neq d$ and $A \perp_x$ Line(c, d).

Let a, b, x, c, d be points of S. We say that $\overline{a, b} \perp_x \overline{c, d}$ if and only if

(Def. 5) $a \neq b$ and $c \neq d$ and Line$(a, b) \perp_x$ Line(c, d).

Let a, b, c, d be points of S. We say that $\overline{a, b} \perp \overline{c, d}$ if and only if

(Def. 6) $a \neq b$ and $c \neq d$ and $\mathrm{Line}(a, b) \perp \mathrm{Line}(c, d)$.

From now on S denotes a non empty Tarski plane satisfying seven Tarski's geometry axioms, A, A' denote subsets of S, and x, y, z, a, b, c, c', d, u, p, q, q' denote points of S.

Now we state the proposition:

(23) 8.12 Satz:
$A \perp_x A'$ if and only if $A' \perp_x A$.

Now we state the proposition:

(24) 8.13 Satz:
$A \perp_x A'$ if and only if A is a line and A' is a line and $x \in A$ and $x \in A'$ and there exist points u, v of S such that $u \in A$ and $v \in A'$ and $u \neq x$ and $v \neq x$ and $\mathrm{RightAngle}(u, x, v)$. The theorem is a consequence of (15) and (13).

Now we state the proposition:

(25) 8.14 (i) Satz:
If $A \perp A'$, then $A \neq A'$. The theorem is a consequence of (24) and (21).

Let S be a non empty Tarski plane, A, B be subsets of S, and x be a point of S. We say that A, B intersect at x if and only if

(Def. 7) A is a line and B is a line and $A \neq B$ and $x \in A$ and $x \in B$.

Now we state the proposition:

(26) 8.14 (ii) Satz:
$A \perp_x A'$ if and only if $A \perp A'$ and A, A' intersect at x. The theorem is a consequence of (25).

Now we state the propositions:

(27) 8.14 (iii) Satz:
If $A \perp_x A'$ and $A \perp_y A'$, then $x = y$. The theorem is a consequence of (25) and (26).

(28) If a, b and x are collinear and $\overline{a, b} \perp \overline{c, x}$, then $\overline{a, b} \perp_x \overline{c, x}$. The theorem is a consequence of (25) and (26).

Now we state the proposition:

(29) 8.15 Satz:
If $a \neq b$ and a, b and x are collinear, then $\overline{a, b} \perp \overline{c, x}$ iff $\overline{a, b} \perp_x \overline{c, x}$. The theorem is a consequence of (28).

Now we state the proposition:

(30) 8.16 Satz:
Suppose $a \neq b$ and a, b and x are collinear and a, b and u are collinear and $u \neq x$. Then $\overline{a, b} \perp \overline{c, x}$ if and only if a, b and c are not collinear and

RightAngle(c, x, u). The theorem is a consequence of (29), (13), (21), and (24).

Let S be a non empty Tarski plane satisfying seven Tarski's geometry axioms and a, b, c, x be points of S. We say that x is perpendicular foot of a, b, c if and only if

(Def. 8) a, b and x are collinear and $\overline{a, b} \perp \overline{c, x}$.

Now we state the propositions:

(31) 8.18 Satz - Uniqueness:
If x is perpendicular foot of a, b, c and y is perpendicular foot of a, b, c, then $x = y$. The theorem is a consequence of (29), (13), and (19).

(32) Suppose a, b and c are not collinear and a lies between b and y and $a \neq y$ and y lies between a and z and $\overline{yz} \cong \overline{yp}$ and $y \neq p$ and $q' = \mathrm{S}_z(q)$ and Middle(c, x, c') and $c \neq y$ and y lies between q' and c' and Middle(y, p, c) and y lies between p and q and $q \neq q'$. Then $x \neq y$. The theorem is a consequence of (10) and (11).

In the sequel S denotes a non empty Tarski plane satisfying Lower Dimension Axiom and seven Tarski's geometry axioms and a, b, c, p, q, x, y, z, t denote points of S.

Now we state the proposition:

(33) 8.18 Satz - Existence:
If a, b and c are not collinear, then there exists x such that x is perpendicular foot of a, b, c.
Proof: Consider y such that a lies between b and y and $\overline{ay} \cong \overline{ac}$. Consider p such that Middle(y, p, c). Consider z such that y lies between a and z and $\overline{yz} \cong \overline{yp}$. Consider q such that y lies between p and q and $\overline{yq} \cong \overline{ya}$. Set $q' = \mathrm{S}_z(q)$. Consider c' such that y lies between q' and c' and $\overline{yc'} \cong \overline{yc}$. $a \neq y$ by [2, (4), (46), (45)]. RightAngle(q, z, y). RightAngle(y, z, q). Consider x such that Middle(c, x, c'). $y \neq p$ by [2, (4), (45)]. $c \neq y$ by [2, (45)]. $q \neq q'$ by [2, (97), (82), (45)]. $c \neq x$ by [2, (104), (4)], (9), [2, (82), (45)]. □

Now we state the propositions:

(34) 8.20 Lemma:
If RightAngle(a, b, c) and Middle$(\mathrm{S}_a(c), p, \mathrm{S}_b(c))$, then RightAngle$(b, a, p)$ and if $b \neq c$, then $a \neq p$.
Proof: Set $d = \mathrm{S}_b(c)$. Set $b' = \mathrm{S}_a(b)$. Set $c' = \mathrm{S}_a(c)$. Set $d' = \mathrm{S}_a(d)$. Set $p' = \mathrm{S}_a(p)$. RightAngle(b', b, c) by [2, (104)], (13), (17), (15). $\overline{b'b} \cong \overline{bb'}$ by [2, (105), (101)]. $\overline{b'c} \cong \overline{bc'}$ by [2, (105), (101)]. $\triangle b'bc \cong \triangle bb'c'$ by [2, (105)]. RightAngle(b, b', c'). $\mathrm{S}_{b'}(c') = d'$. IFS $\binom{c', p, d, b}{d', p', c, b}$ by [2, (14), (106), (101)]. If $b \neq c$, then $a \neq p$ by [2, (101), (97)]. □

(35) Suppose a, b and c are not collinear. Then there exists p and there exists t such that $\overline{a,b} \perp \overline{p,a}$ and a, b and t are collinear and t lies between c and p. The theorem is a consequence of (33), (29), (34), and (24).

Now we state the propositions:

(36) 8.21 Satz:
If $a \neq b$, then there exists p and there exists t such that $\overline{a,b} \perp \overline{p,a}$ and a, b and t are collinear and t lies between c and p. The theorem is a consequence of (35).

(37) If $a \neq b$ and $a \neq p$ and RightAngle(b, a, p) and RightAngle(a, b, q), then p, a and q are not collinear. The theorem is a consequence of (13), (15), and (19).

(38) Let us consider a non empty Tarski plane S satisfying Lower Dimension Axiom and seven Tarski's geometry axioms, and points a, b, p, q, t of S. Suppose $a, p \leqslant b, q$ and $\overline{a,b} \perp \overline{q,b}$ and $\overline{a,b} \perp \overline{p,a}$ and a, b and t are collinear and t lies between q and p. Then there exists a point x of S such that Middle(a, x, b).

Proof: Consider r being a point of S such that r lies between b and q and $\overline{ap} \cong \overline{br}$. Consider x being a point of S such that x lies between t and b and x lies between r and p. a, b and x are collinear by [2, (46), (45), (82)]. Consider x' being a point of S such that Line$(a, b) \perp_{x'}$ Line(q, b). Consider y being a point of S such that Line$(a, b) \perp_{y}$ Line(p, a). RightAngle(q, b, a) and $q \neq b$ and b, q and r are collinear. RightAngle(r, b, a). b, a and p are not collinear and a, b and q are not collinear. □

Now we state the proposition:

(39) 8.22 Satz:
Let us consider a non empty Tarski plane S satisfying Lower Dimension Axiom and seven Tarski's geometry axioms, and points a, b of S. Then there exists a point x of S such that Middle(a, x, b). The theorem is a consequence of (36) and (38).

Now we state the proposition:

(40) 8.24 Lemma:
Let us consider a non empty Tarski plane S satisfying Lower Dimension Axiom and seven Tarski's geometry axioms, and points a, b, p, q, r, t of S. Suppose $\overline{p,a} \perp \overline{a,b}$ and $\overline{q,b} \perp \overline{a,b}$ and a, b and t are collinear and t lies between p and q and r lies between b and q and $\overline{ap} \cong \overline{br}$. Then there exists a point x of S such that

(i) Middle(a, x, b), and

(ii) Middle(p, x, r).

Proof: Consider x being a point of S such that x lies between t and b and x lies between r and p. a, b and x are collinear by [2, (46), (45), (82)]. Consider x' being a point of S such that $\mathrm{Line}(a,b) \perp_{x'} \mathrm{Line}(q,b)$. Consider y being a point of S such that $\mathrm{Line}(a,b) \perp_{y} \mathrm{Line}(p,a)$. $\mathrm{RightAngle}(q,b,a)$ and $q \neq b$ and b, q and r are collinear. $\mathrm{RightAngle}(r,b,a)$. b, a and p are not collinear and a, b and q are not collinear. □

3. Additional Lemmas Needed by Otter: Chapter 8A

Now we state the proposition:

(41) ExtCol2:

Let us consider points a, b, c, d, x, p, q of S. Suppose c, $d \in \mathrm{Line}(a,b)$ and $a \neq b$ and $c \neq d$. Then $\mathrm{Line}(a,b) = \mathrm{Line}(c,d)$.

Now we state the proposition:

(42) ExtPerp:

Let us consider points a, b, c, d, x, p, q of S. Suppose c, $d \in \mathrm{Line}(a,b)$ and $c \neq d$ and $\overline{a,b} \perp_x \overline{p,q}$. Then $\overline{c,d} \perp_x \overline{p,q}$. The theorem is a consequence of (11).

Now we state the proposition:

(43) ExtPerp2:

Let us consider points a, b, c, d, p, q of S. Suppose p, $q \in \mathrm{Line}(a,b)$ and $a \neq b$ and $\overline{p,q} \perp \overline{c,d}$. Then $\overline{a,b} \perp \overline{c,d}$. The theorem is a consequence of (11).

Now we state the proposition:

(44) ExtPerp3:

Let us consider points a, b, c, d of S. Suppose $a \neq b$ and $b \neq c$ and $c \neq d$ and $a \neq c$ and $a \neq d$ and $b \neq d$ and $\overline{b,a} \perp \overline{a,c}$ and a, c and d are collinear. Then $\overline{b,a} \perp \overline{a,d}$. The theorem is a consequence of (11).

Now we state the proposition:

(45) ExtPerp4:

Let us consider points a, b, p, q of S. If $\overline{a,b} \perp \overline{p,q}$, then $\overline{a,b} \perp \overline{q,p}$.

Now we state the proposition:

(46) ExtPerp5:

Let us consider points a, b, c, d, p, q of S. Suppose p, $q \in \mathrm{Line}(a,b)$ and $p \neq q$ and $\overline{a,b} \perp \overline{c,d}$. Then $\overline{p,q} \perp \overline{c,d}$. The theorem is a consequence of (11).

Now we state the proposition:

(47) ExtPerp5A:

Let us consider points a, b, c, d, p, q of S. Suppose a, b and p are collinear and a, b and q are collinear and $p \neq q$ and $\overline{a, b} \perp \overline{c, d}$. Then $\overline{p, q} \perp \overline{c, d}$. The theorem is a consequence of (46).

Now we state the proposition:

(48) ExtPerp6:

Let us consider points a, b, c, d, p, q of S. Suppose p, $q \in \mathrm{Line}(a, b)$ and $p \neq q$ and $a \neq b$ and $\overline{c, d} \perp \overline{p, q}$. Then $\overline{c, d} \perp \overline{a, b}$. The theorem is a consequence of (11).

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